Infinite Games

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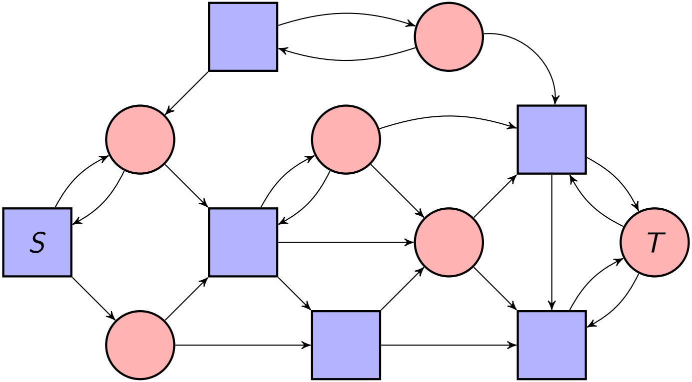
Let's Play



You move at circles and want to reach T from S.

Motivation

- Model-checking and satisfiability for fixed-point logics, e.g., the modal μ-calculus, CTL, CTL*.
- Automata emptiness often expressible in terms of games.
- Semantics of alternating automata in terms of games.
- Synthesis of correct-by-construction controllers for reactive systems (non-terminating, interacting with antagonistic environment).

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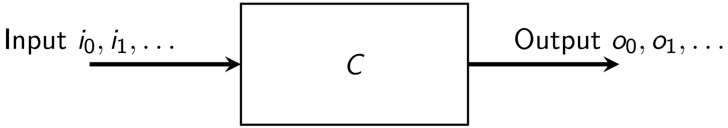
Earliest appearance: **Church's problem (1957)**

Given requirement φ on input-output behavior of boolean circuits, compute a circuit C that satisfies φ (or prove that none exists).



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Game theoretic formulation:

- Player 0 generates infinite stream of input bits.
- Player 1 has to answer each input bit by output bit.
- Player 1 wins, if combination of streams satisfies φ.

Church's Problem: Example

φ is conjunction of following properties:

1. Whenever the input bit is 1, then the output bit is 1, too.
2. If there are infinitely many 0's in the input stream, then there are infinitely many 0's in the output stream.
3. At least one out of every three consecutive output bits is a 1.

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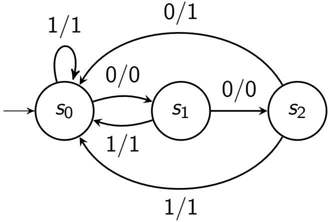
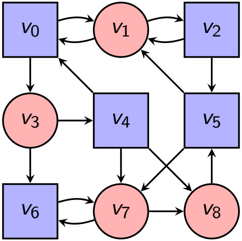


Outline

Arenas and Games

- An *arena* $\mathcal{A} = (V, V_0, V_1, E)$ consists of
  - a finite set V of vertices,
  - a set $V_0 \subseteq V$ of vertices owned by Player 0 (circles),
  - the set $V_1 = V \setminus V_0$ of vertices owned by Player 1 (squares),
  - a directed edge-relation $E \subseteq V \times V$.



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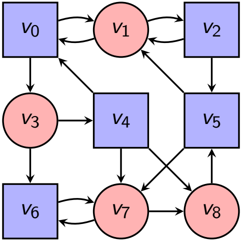

- A play is an infinite path through $\mathcal{A}$.

Strategies

- A strategy for Player i in $\mathcal{A}$ is a mapping $\sigma \colon V^* V_i \to V$ satisfying $(v_n, \sigma(v_0 \cdots v_n)) \in E$ (only legal moves).

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- Positional strategies: $\sigma(v_0 \cdots v_n) = \sigma(v_n)$ for all $v_0 \cdots v_n$: move only depends on position the token is at at the moment.

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- Finite-state strategies: implemented by DFA with output reading play prefix $v_0 \cdots v_n$ and outputting $\sigma(v_0 \cdots v_n)$.

Winning

- A game $\mathcal{G} = (\mathcal{A}, \mathrm{Win})$ consists of an arena $\mathcal{A}$ and a set $\mathrm{Win} \subseteq V^\omega$ of winning plays for Player 0.
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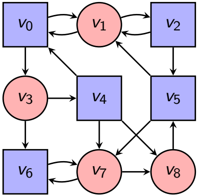

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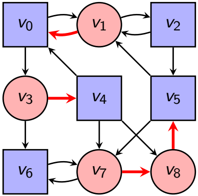

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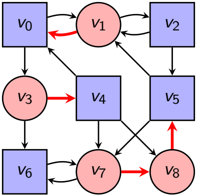

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Player 0 wins from every vertex with positional strategies.

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- Strategy σ for Player i is winning strategy from v, if every play that starts in v and is consistent with σ is winning for him.

- Winning region $W_i(\mathcal{G})$: set of vertices from which Player i has a winning strategy.
- Always: $W_0(\mathcal{G}) \cap W_1(\mathcal{G}) = \emptyset$.
- $\mathcal{G}$ determined, if $W_0(\mathcal{G}) \cup W_1(\mathcal{G}) = V$.
- Solving a game: determine the winning regions and winning strategies.

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There are many other winning conditions.

What Are We Interested in?

Given a type of winning condition (e.g., reachability, parity, Muller),..

- .. are games with this condition always determined?
- .. what kind of strategy do the players need (e.g., positional, finite-state)?
- .. if finite-state strategies are necessary, how large do they have to be?
- How hard is it to solve the game?

Outline

1. Definitions
2. **Reachability Games**
3. Parity Games
4. Muller Games
5. Outlook

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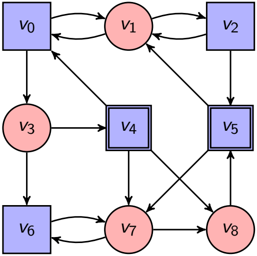

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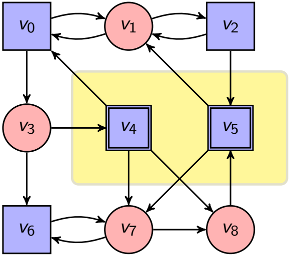

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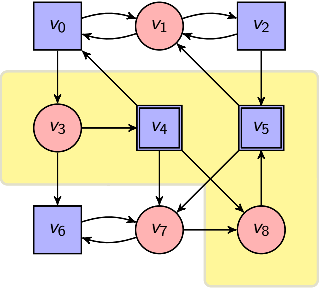

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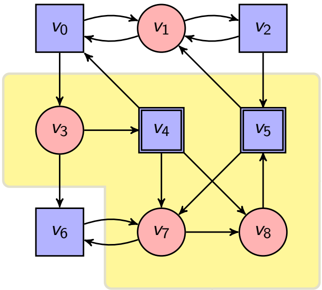

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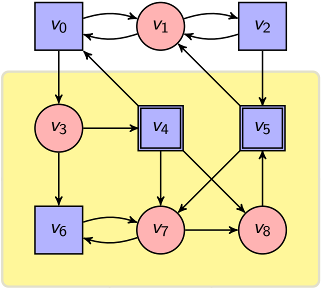

Attractor Construction

$\mathrm{Attr}_i^{\mathcal{A}}(R) = \bigcup_{n \in \mathbb{N}} A_n$ where $A_0 = R$ and

$$A_{j+1} = A_j \cup \{v \in V_i \mid \exists (v, v') \in E \text{ s.t. } v' \in A_j\}$$
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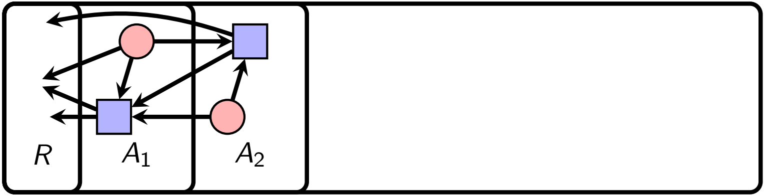

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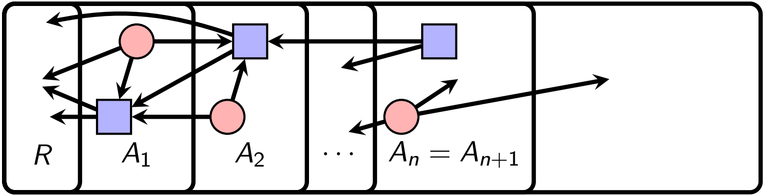

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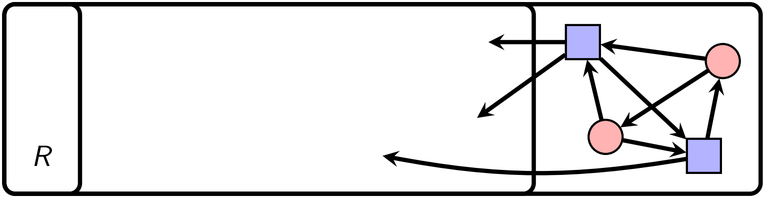

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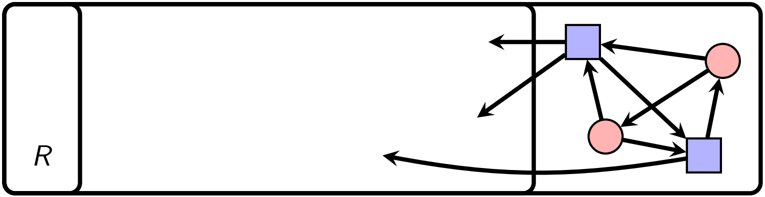

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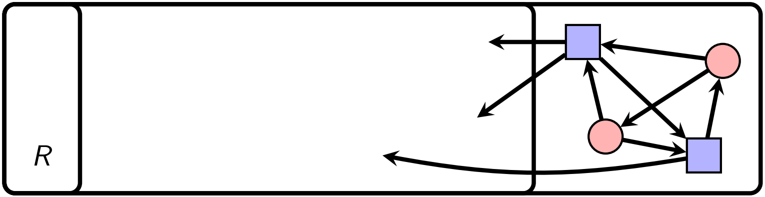

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Remark: Attractors can be computed in linear time in $|E|$.

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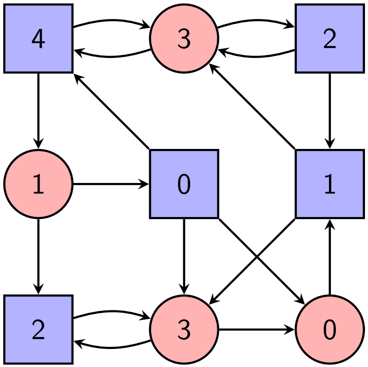

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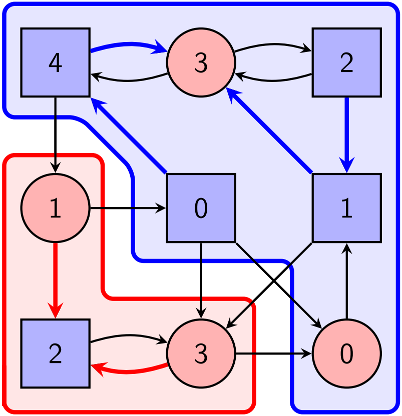

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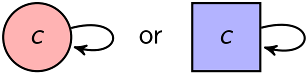

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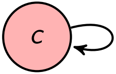
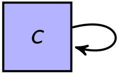
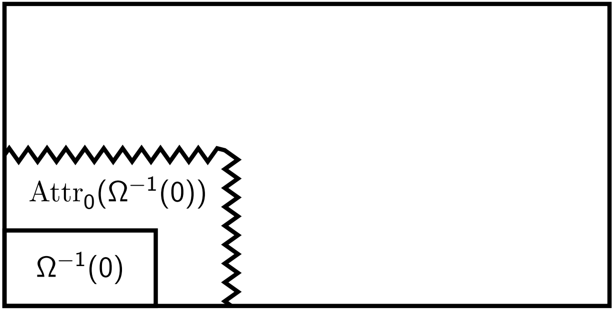


or


Player i wins iff $\mathrm{Par}(c) = i$

Proof Sketch

Now $n > 1$ and $\min \Omega(V) = 0$.



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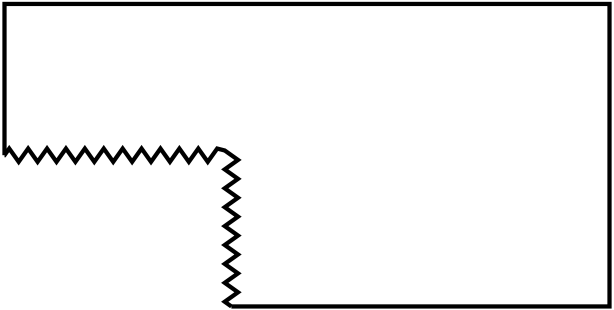
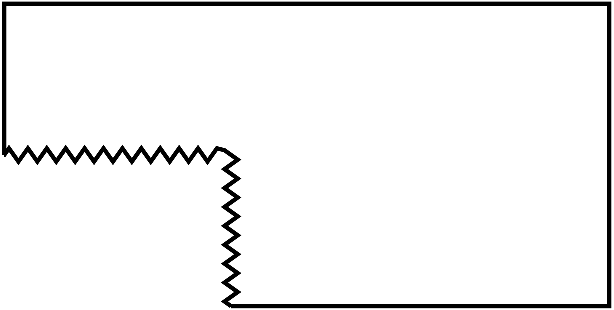


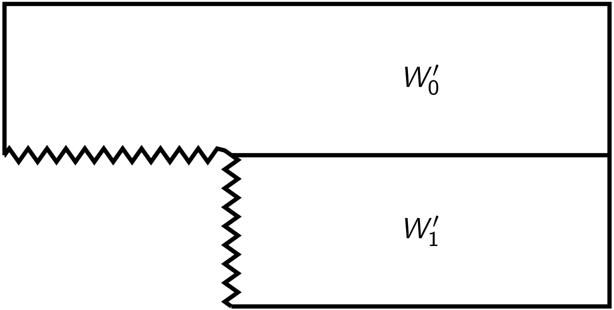
Proof Sketch

Induction hypothesis applicable..



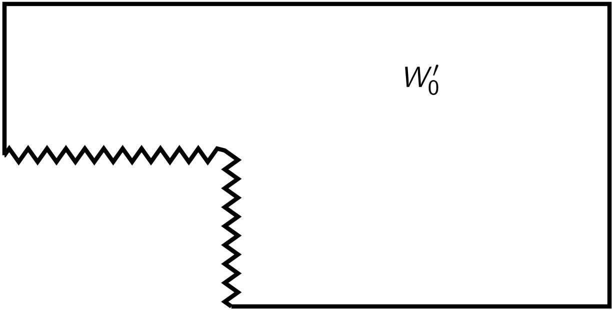
Proof Sketch

.. yields winning regions W_i' and positional strategies σ', τ'.



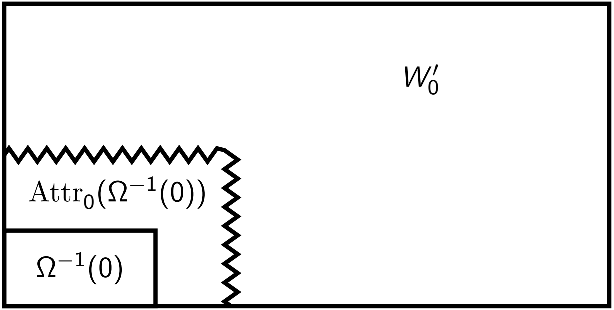
Proof Sketch

W_1' empty:



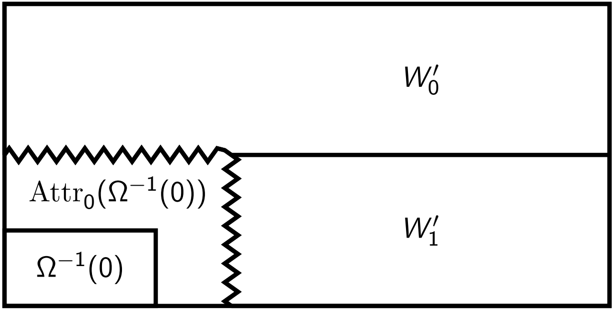
Proof Sketch

W_1' empty: Player 0 wins from everywhere.
Winning strategy: combine σ' and attractor strategy, play arbitrarily at $\Omega^{-1}(0)$.



Proof Sketch

W_1' non-empty:



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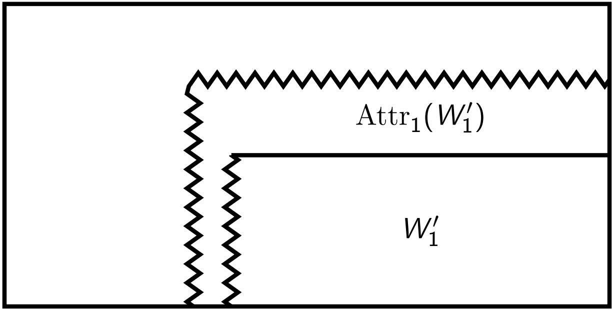

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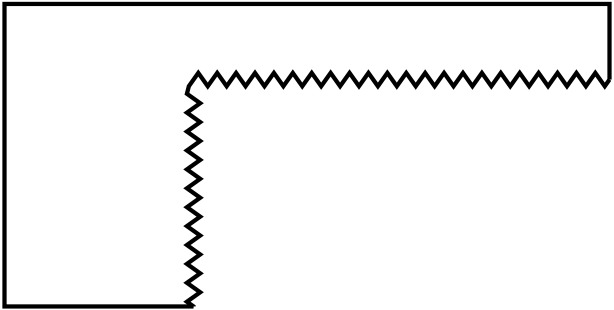

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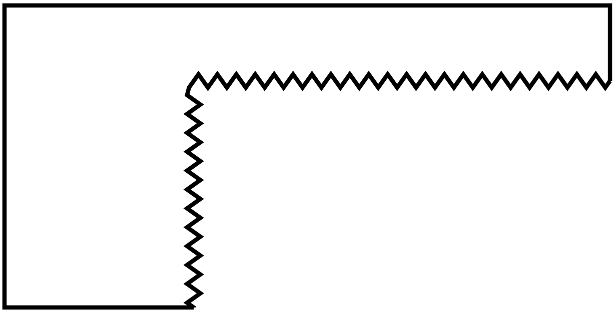

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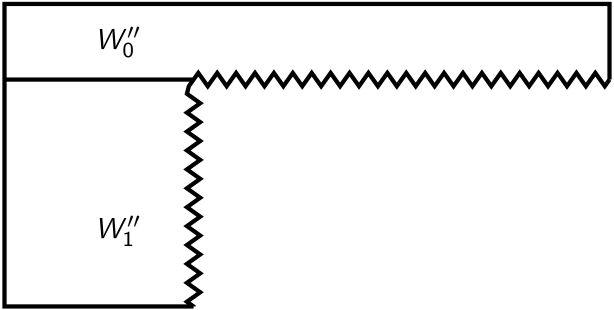

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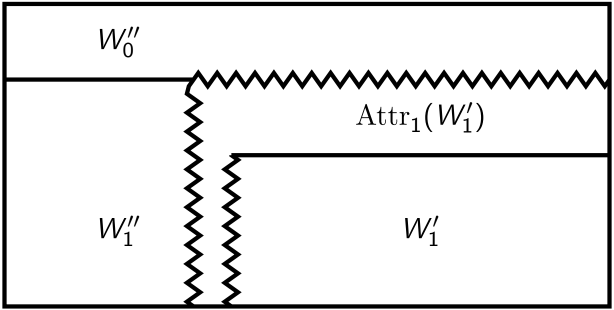
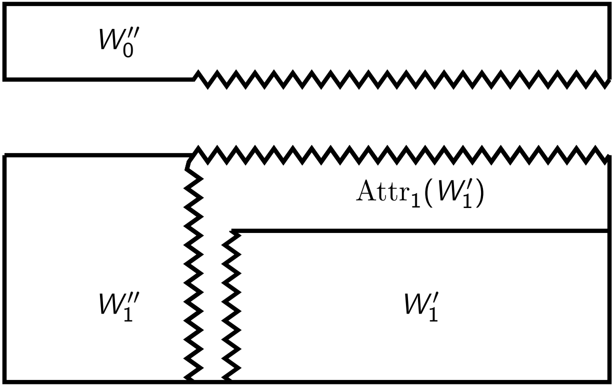


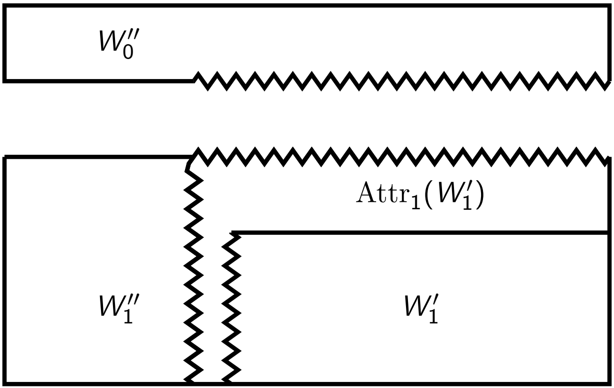
Proof Sketch

W_1' non-empty: Player 0 wins from W_0'' with σ''.



Proof Sketch

W_1' non-empty: Player 1 wins from $W_1'' \cup \mathrm{Attr}_1(W_1')$.
Winning strategy: combine τ', τ'', and attractor strategy.



Algorithms for Parity Games

- Determinacy proof yields recursive algorithm with exponential running time.
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- **Open problem:** is solving parity games in polynomial time?

Outline

Muller Games

Muller games: for $\mathcal{F} \subseteq 2^V$ define

$$\textsc{Muller}(\mathcal{F}) = \{\, \rho \in V^\omega \mid \text{set of vertices seen infinitely often during } \rho \text{ is in } \mathcal{F} \,\}$$

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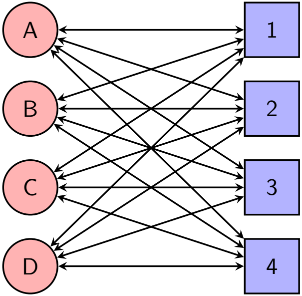

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iff

$|F \cap \{A, B, C, D\}| = \max(F \cap \{1, 2, 3, 4\})$

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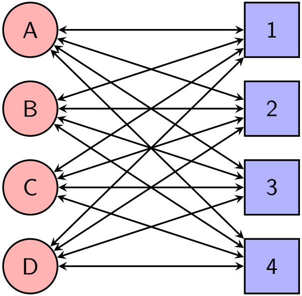
in general: DJW_n
here: DJW_4



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Need to *estimate* set of vertices in $\{A, B, C, D\}$ visited infinitely often during the play:

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C (A B C D #)

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B   B A # C D

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C	A B C D #
4	
B	B A # C D
2	
D	D B A C #

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4		3	
B	B A # C D		
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B	B A # C D	C	C A D B #
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4			

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4		3	
B	B A # C D	C	C A D B #
2		4	
D	D B A C #	C	C # A D B
4		1	

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C	A B C D #	A	A D B # C	A	A C # D B
4		3			
B	B A # C D	C	C A D B #		
2		4			
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2		4			
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4		3		2	
B	B A # C D	C	C A D B #	C	C A # D B
2		4			
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4		3		2	
B	B A # C D	C	C A D B #	C	C A # D B
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4		3		2	
B	B A # C D	C	C A D B #	C	C A # D B
2		4		2	
D	D B A C #	C	C # A D B	A	A C # D B
4		1		2	

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C	A B C D #	A	A D B # C	A	A C # D B
4		3		2	
B	B A # C D	C	C A D B #	C	C A # D B
2		4		2	
D	D B A C #	C	C # A D B	A	A C # D B
4		1		2	

- From some point onwards only vertices that are visited infinitely often are in front of #, and
- infinitely often exactly the set of vertices that are visited infinitely often is in front of #.

Muller Games

- Bookkeeping works in general (use permutations over V).
- Product of arena and LAR-structure can be turned into *equivalent* parity game from which finite-state strategies can be derived ("Muller games are reducible to parity games").

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Muller games are determined with finite-state strategies of size $n \cdot n!$.

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Theorem

Muller games are determined with finite-state strategies of size $n \cdot n!$.

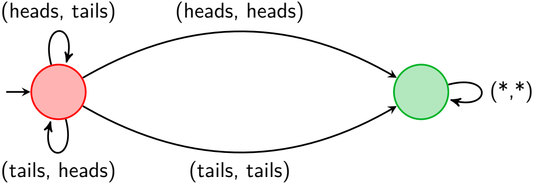
- Matching lower bounds via DJW_n games.
- Complexity depends on encoding of $\mathcal{F}$:
  - P, if $\mathcal{F}$ is given as list of sets.
  - $\mathrm{NP} \cap \mathrm{Co\text{-}NP}$, if $\mathcal{F}$ is encoded by a tree.
  - PSPACE-complete, if $\mathcal{F}$ is encoded by circuit or boolean formula (with variables V).

Outline

Concurrent Games

- Both players choose their moves simultaneously

Matching pennies:



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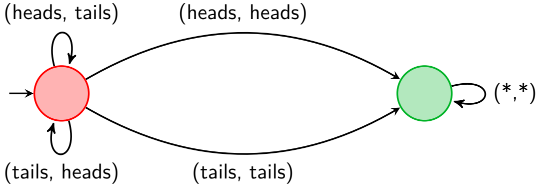

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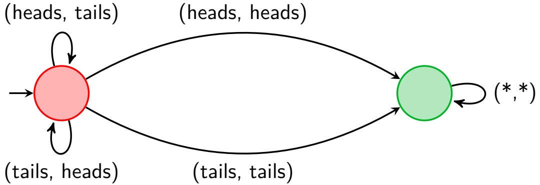
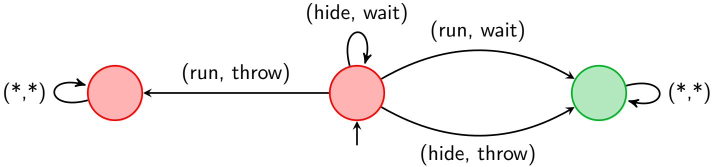


The "Snowball Game":



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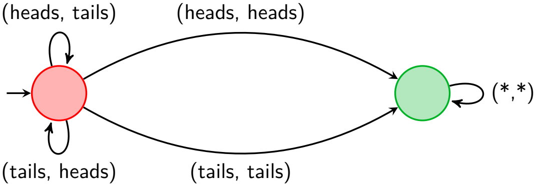
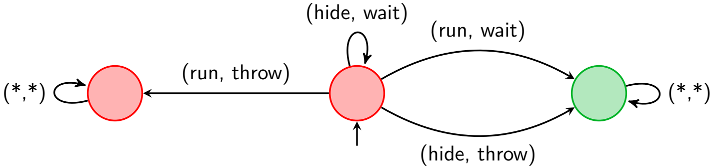
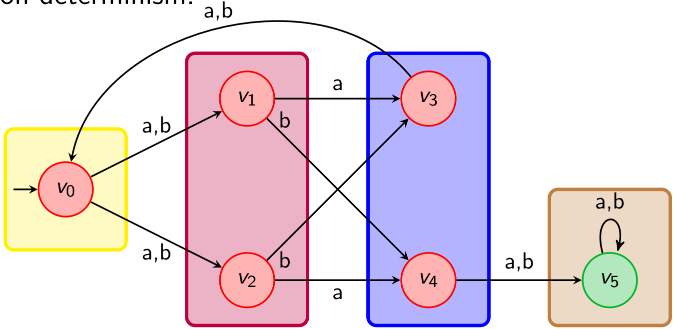


The "Snowball Game": for every ε, randomized strategy winning with probability $1 - \varepsilon$.



Games of Imperfect Information

- Players do not observe sequence of states, but sequence of non-unique observations (yellow, purple, blue, brown).
- Player 0 picks action (a or b), Player 1 resolves non-determinism.



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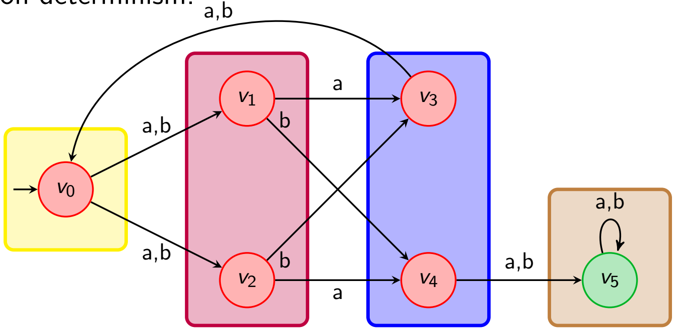
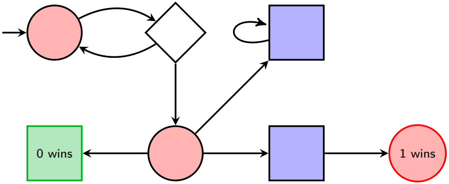


No winning strategy for Player 0: every fixed choice of actions to pick at ( )*( ) can be countered by going to v_1 or v_2.

(Simple) Stochastic Games

- Enter a new player ($\Diamond$), it flips a coin to pick a successor.



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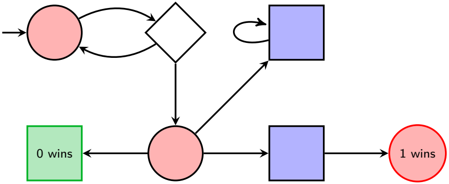

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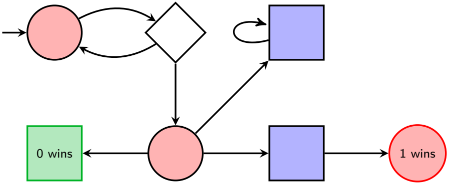

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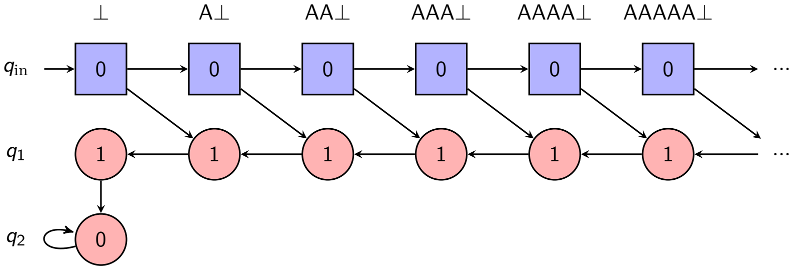
More formally: Value of the game

$$\max_{\sigma} \min_{\tau} p_{\sigma,\tau}$$

where $p_{\sigma,\tau}$ is the probability that Player 0 wins when using strategy σ and Player 1 uses strategy τ.

Pushdown Games

Use configuration graphs of pushdown machines as arena (in general infinite).



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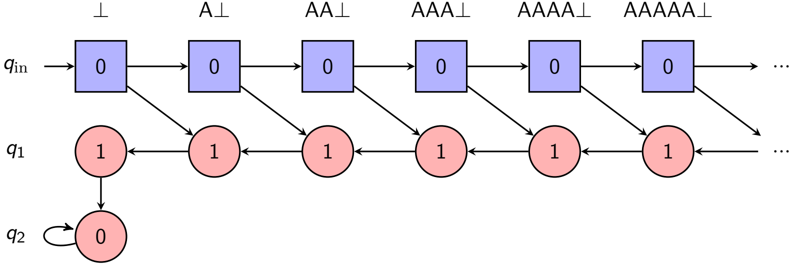
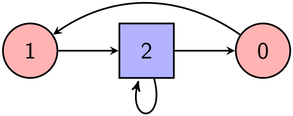


- Positional determinacy still holds, but positional strategies are infinite objects!
- Solution: winning strategies implemented by pushdown machines with output.

Quantitative Winning Conditions

- Parity game: Player 0 wins from everywhere, but it takes arbitrarily long two “answer” 1 by 0.



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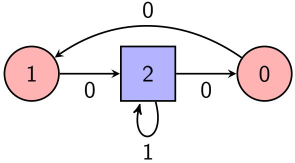

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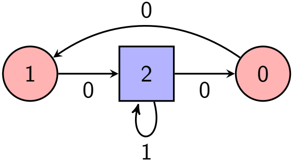

- Add edge-costs: Player 0 wins if there is a bound b and a position n such that every odd color after n is followed by a smaller even color with cost $\leq b$ in between
- Player 1 wins example from everywhere (stay at 2 longer and longer).

Many other variants

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- Games on infinite arenas beyond pushdown graphs.
- Games on timed automata: uncountable arenas.
- Play even longer: games of ordinal length.
- Games with delay: Player 0 is allowed to skip some moves to obtain lookahead on Player 1's moves. Basic question: what kind of lookahead is necessary to win.

Many other variants

- Many more winning conditions.
- Games on infinite arenas beyond pushdown graphs.
- Games on timed automata: uncountable arenas.
- Play even longer: games of ordinal length.
- Games with delay: Player 0 is allowed to skip some moves to obtain lookahead on Player 1's moves. Basic question: what kind of lookahead is necessary to win.
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And: any combination of extensions discussed above.

Literature

- Lecture notes "Infinite Games" (*hidden* in the Teaching section)

  `www.react.uni-saarland.de/teaching/infinite-games-13-14`
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