Down the Borel Hierarchy: Solving Muller Games via Safety Games

Joint work with Daniel Neider and Roman Rabinovich (RWTH Aachen University)

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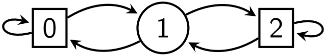
Muller Games

Running example



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- $\mathcal{F}_1 = 2^V \setminus \mathcal{F}_0$

Player 0 has a winning strategy from every vertex: alternate between 0 and 2.

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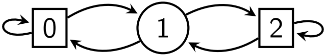

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Muller game $(\mathcal{A}, \mathcal{F}_0, \mathcal{F}_1)$:

- Arena $\mathcal{A}$ and partition $\{\mathcal{F}_0, \mathcal{F}_1\}$ of the power set of vertices.
- Player i wins ρ iff $\mathrm{Inf}(\rho) = \{v \mid \exists^\omega n \text{ s.t. } \rho_n = v\} \in \mathcal{F}_i$.

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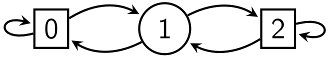

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Theorem

1. *Muller games are determined with finite-state strategies of size $n!$.*
2. *Muller games cannot be reduced to safety games.*

Scoring Functions for Muller Games

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Lemma (Fearnley, Z. 2010)

In every Muller game, Player 0 has a winning strategy that bounds the scores for all $F \in \mathcal{F}_1$ by two.

Corollary

Player 0 wins Muller game from $v \Leftrightarrow$ she is able to bound the scores for all $F \in \mathcal{F}_1$ by two (safety condition).

Reducing Muller Games to Safety Games

Theorem

For every Muller game $\mathcal{G}$, we can construct a safety game $\mathcal{S}$ and a mapping $f\colon V(\mathcal{G}) \to V(\mathcal{S})$ such that

1. *Player i wins $\mathcal{G}$ from v iff she wins $\mathcal{S}$ from $f(v)$.*
2. *Player 0's winning region in $\mathcal{S}$ can be used as memory to implement a finite-state winning strategy for her in $\mathcal{G}$.*
3. $|V(\mathcal{S})| \leq (|V(\mathcal{G})|!)^3$.

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Remarks:

- Size of parity game in LAR-reduction $|V(\mathcal{G})|!$. But: simpler algorithms for safety games.
- 2. does not hold for Player 1.

Conclusion

Reducing Muller games to safety games via scoring functions:

- "Simple" algorithm for Muller games.
- New memory structure: keep track of scores up to value three (size can be improved by only taking maximal elements).
- Permissive strategies: most general non-deterministic strategy that prevents opponent from reaching a score of three.
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Further research:

- Progress measures for Muller games?
- Determine influence of safety game algorithms on memory for Muller games obtained in our reduction.
- Understand tradeoff between size and quality of a strategy.