The First-order Logic of Hyperproperties

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LTL vs. First-order Logic

Theorem (Kamp '68, Gabbay et al. '80)

LTL and FO$[<]$ *are expressively equivalent.*

HyperLTL

A new logic:

$$\forall\pi\forall\pi'.\, \mathbf{F}\, \mathrm{on}_\pi \leftrightarrow \mathrm{on}_{\pi'}$$

- Extend LTL by trace quantifiers to express security, privacy, and information flow properties

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Is there a first-order logic that is expressively equivalent to HyperLTL?

An Example

Fix $\mathrm{AP} = \{a\}$ and consider the conjunction φ of

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$$\{a\} \quad \emptyset \quad \emptyset \quad \emptyset \quad \emptyset \quad \emptyset \quad \emptyset \quad \emptyset \quad \cdots$$

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$\{a\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\cdots$
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$$
\begin{array}{cccccccccc}
\{a\} & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \cdots \\
\emptyset & \{a\} & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \cdots \\
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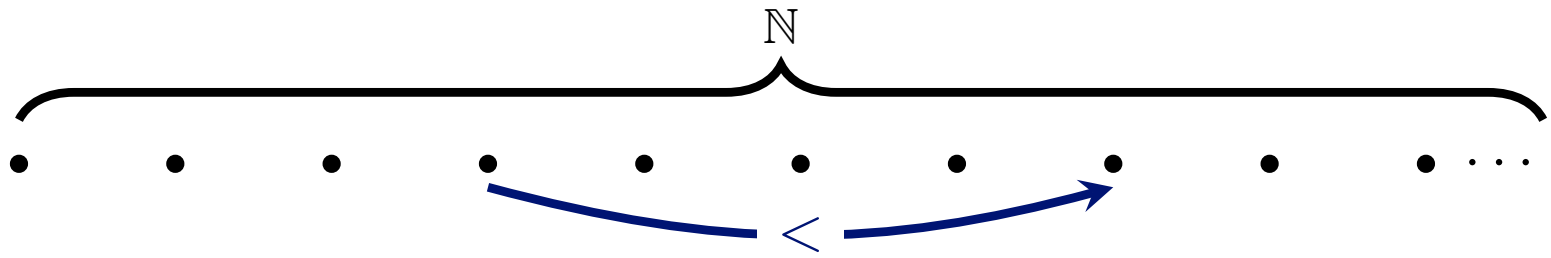
$$

The unique model of φ is $\{\emptyset^n\,\{a\}\,\emptyset^\omega \mid n \in \mathbb{N}\}$.

First-order Logic for Hyperproperties



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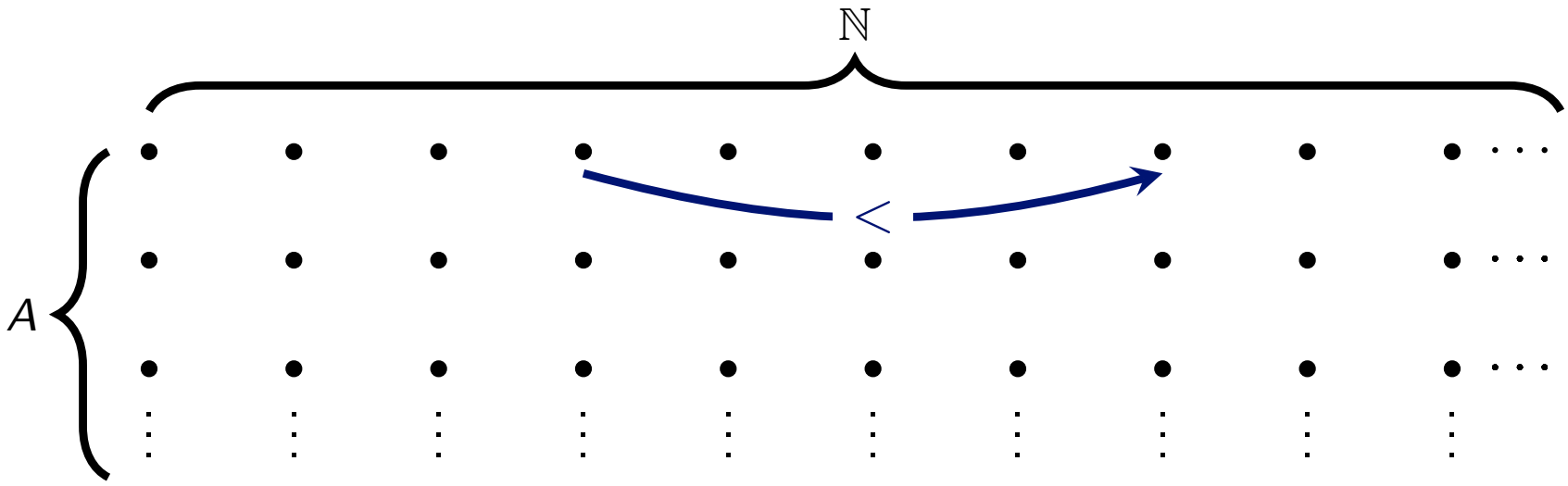

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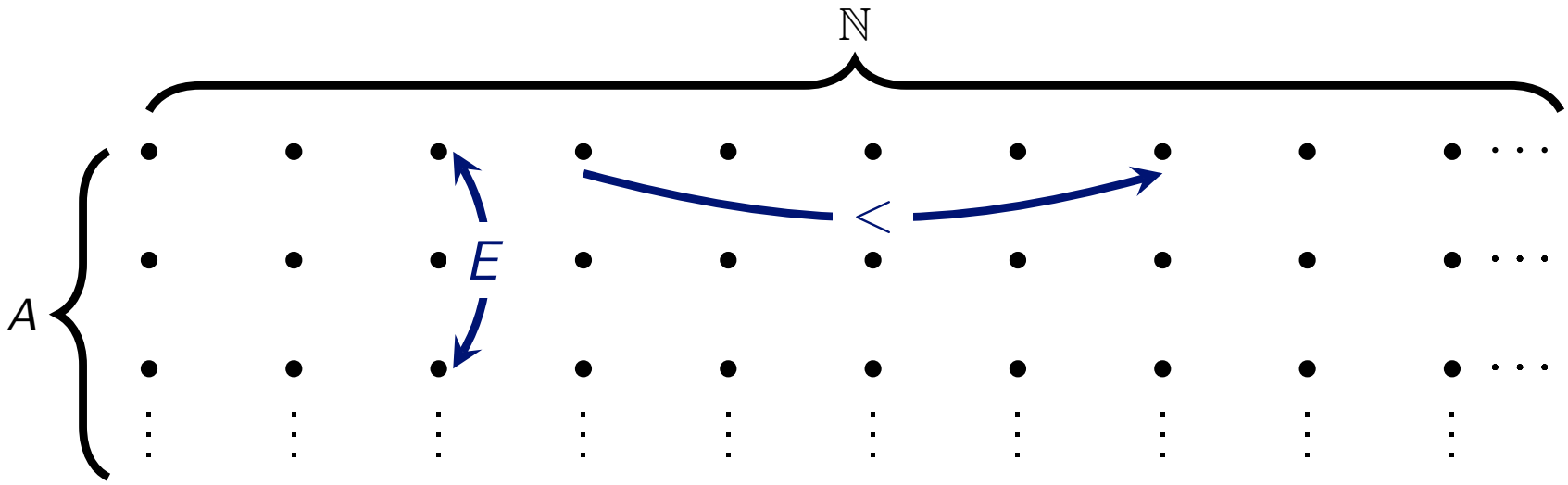

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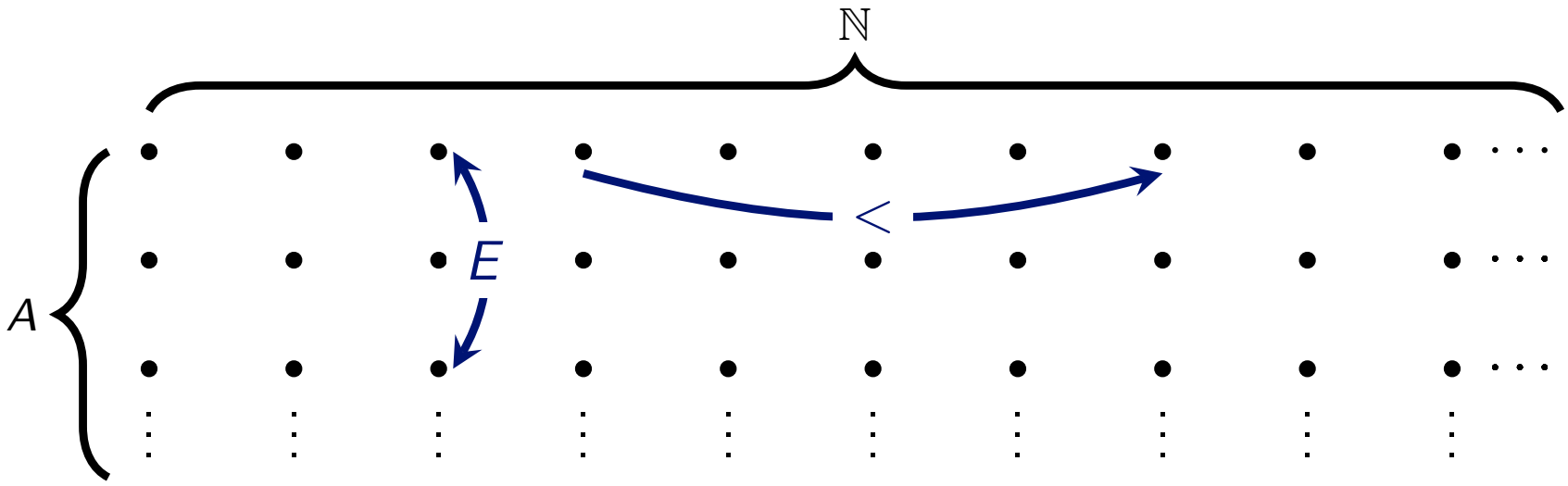

- FO[$<$, E]: first-order logic with equality over the signature $\{<, E\} \cup \{P_a \mid a \in \mathrm{AP}\}$ over structures with universe $A \times \mathbb{N}$.

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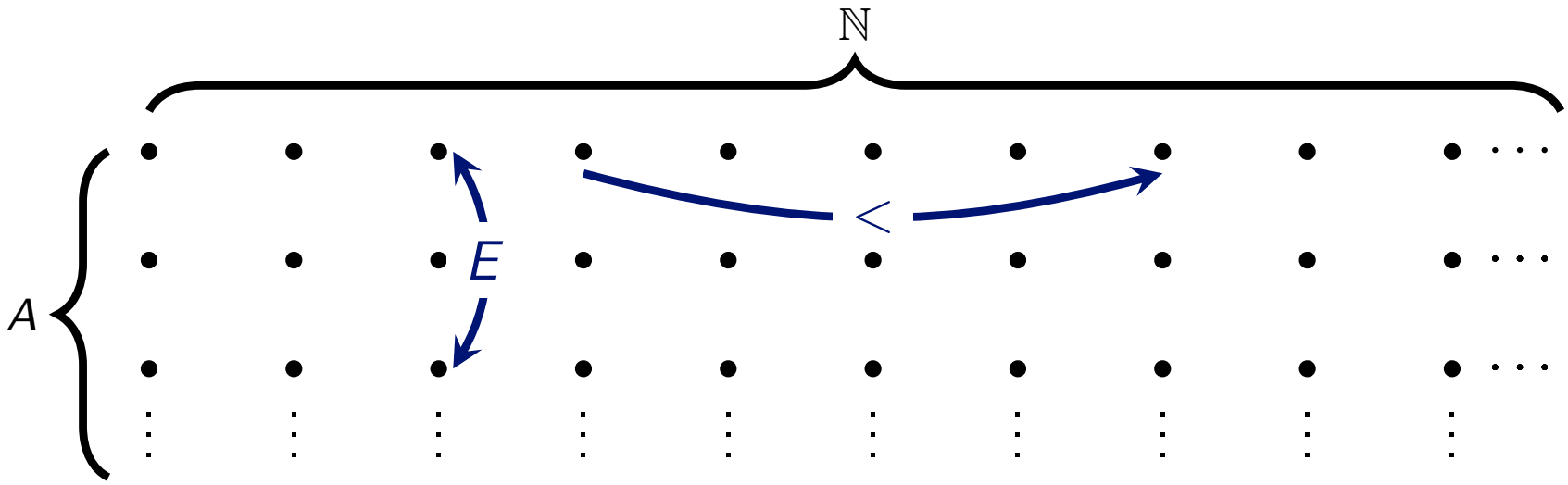

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Proposition

For every HyperLTL sentence there is an equivalent FO[$<$, E] *sentence.*

A Setback

- Let φ be the following property of sets $T \subseteq (2^{\{a\}})^\omega$:

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- But, φ is easily expressible in FO$[<, E]$:

$$\exists x \, \forall y \, E(x, y) \to \neg P_a(y)$$

Corollary

FO$[<, E]$ strictly subsumes HyperLTL.

HyperFO

- $\exists^I x$ and $\forall^I x$: quantifiers restricted to initial positions.
- $\exists^G y \geq x$ and $\forall^G y \geq x$: if x is initial, then quantifiers restricted to positions on the same trace as x.

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HyperFO:

$$\forall^I x_1 \, \forall^I x_2 \, \forall^G y_1 \geq x_1 \, \forall^G y_2 \geq x_2 \, E(y_1, y_2) \rightarrow (P_{\mathrm{on}}(y_1) \leftrightarrow P_{\mathrm{on}}(y_2))$$

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Conclusion

Theorem

HyperLTL and HyperFO are equally expressive.