Playing Muller Games in a Hurry

Joint work with John Fearnley, University of Warwick

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Motivation

Robert McNaughton: *Playing Infinite Games in Finite Time.* In: *A Half-Century of Automata Theory*, World Scientific (2000).

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McNaughton suggests a method of keeping score to declare a winner such that

> *.. if the play were to continue with each [player] playing forever as he has so far, then the player declared to be the winner would be the winner of the infinite play of the game.*

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Questions:

- Is there an equivalent finite-duration version of a Muller game?
- How long do finite plays have to be?
- Do short finite plays lead to faster algorithms?
- Can we turn winning strategies for finite games into (small) finite-state winning strategies for infinite games?

A first idea

Consider an infinite game $\mathcal{G}$ played on finite graph.

- Stop a play as soon as a cycle is closed. The winner of the induced infinite play is declared to win the finite play.
- If $\mathcal{G}$ is positionally determined, then the winning regions of both games coincide.

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Drawbacks (assuming $\mathcal{G}$ is a Muller game with n vertices):

- maximal play length: $n!$.
- need to remember $n!$ memory states.

Our goal: improve both bounds.

Outline

1. **Muller Games and Scoring Functions**
2. Finite-time Muller Games
3. Conclusion

Muller Games

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- Player i wins play ρ iff $\mathrm{Inf}(\rho) = \{v \mid \exists^\omega j \text{ s.t. } \rho_j = v\} \in \mathcal{F}_i$.

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- Winning region of Player i:

$$W_i = \{v \in V \mid \exists \sigma \in \Pi_i \forall \tau \in \Pi_{1-i} : \mathrm{Play}(v, \sigma, \tau) \text{ won by Player } i\}$$

Scoring Functions

For $F \subseteq V$ define $\mathrm{Sc}_F \colon V^+ \to \mathbb{N}$:

$$\mathrm{Sc}_F(w) = \max\{k \mid \text{exists suffix } x_1 \cdots x_k \text{ of } w \text{ s.t. } x_i \in V^+ \text{ and } \mathrm{Occ}(x_i) = F \text{ for all } i\}$$

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Example:

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$\mathrm{Sc}_{\{a,b,c\}}$	0	0	0	0	0	0	0	1	1	1	2	2	2

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For $\mathcal{F} \subseteq 2^V$ define $\mathsf{MaxSc}_{\mathcal{F}} \colon V^+ \cup V^\omega \to \mathbb{N} \cup \{\infty\}$:

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$\mathcal{F} = \{\{a\}, \{a, b\}, \{a, b, c\}\}$:

$$\mathsf{MaxSc}_{\mathcal{F}}(w) = 3$$

Outline

Results about Scoring

Lemma

Every $w \in V^*$ *with* $|w| \geq k^{|V|}$ *satisfies* $\mathrm{MaxSc}_{2^V}(w) \geq k$.

“If you play long enough, some score value will be high”

Lower bound: there are words w_k of length $k^{|V|} - 1$ with $\mathrm{MaxSc}_{2^V}(w_k) < k$.

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Lemma (McNaughton 2000)

Let $k, m \geq 2$, let $F, H \subseteq V$, let $w \in V^*$ and $s \in V$ such that $\mathrm{Sc}_F(w) < k$ and $\mathrm{Sc}_H(w) < m$. If $\mathrm{Sc}_F(ws) = k$ and $\mathrm{Sc}_H(ws) = m$, then $F = H$.

"At most one score value can increase at a time"

Finite-time Muller Games

- Finite-time Muller game: $(G, \mathcal{F}_0, \mathcal{F}_1, k)$ with threshold $k \geq 2$.
- Play: path $w = w_1 \cdots w_n$ with $\mathrm{MaxSc}_{2^V}(w_0 \cdots w_n) = k$, but $\mathrm{MaxSc}_{2^V}(w_1 \cdots w_{n-1}) < k$.
- Previous Lemma yields unique $F \subseteq V$ such that $\mathrm{Sc}_F(w) = k$. Player i wins w iff $F \in \mathcal{F}_i$.
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McNaughton considered a different definition of a finite- time Muller game: stop play when some Sc_F reaches $|F|! + 1$.

Theorem (McNaughton 2000)

The winning regions in a Muller game and in McNaughton's finite-time Muller game coincide.

Main Theorem

Theorem

The winning regions in a Muller game $(G, \mathcal{F}_0, \mathcal{F}_1)$ *and in the finite-time Muller game* $(G, \mathcal{F}_0, \mathcal{F}_1, 3)$ *coincide.*

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We prove a stronger statement, which implies the theorem.

Lemma

Player i has a strategy σ for a Muller game $(G, \mathcal{F}_0, \mathcal{F}_1)$ such that $\mathrm{MaxSc}_{\mathcal{F}_{1-i}}(\mathrm{Play}(v, \sigma, \tau)) \leq 2$ for every $v \in W_i$ and every $\tau \in \Pi_{1-i}$.

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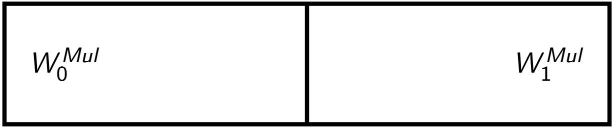

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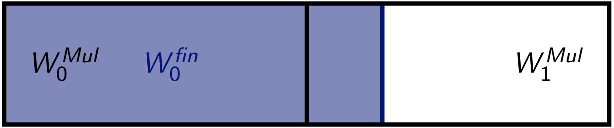

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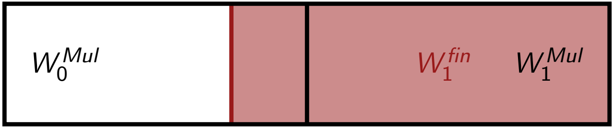

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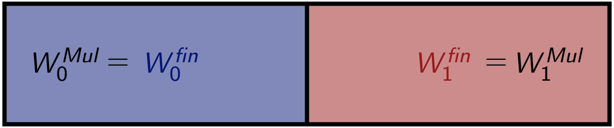

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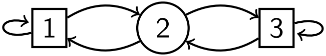
The bound 2 in the lemma is optimal: Player 0 has a winning strategy, but cannot avoid score values of 2 for Player 1.



- $\mathcal{F}_0 = \{\{1, 2, 3\}, \{1\}, \{3\}\}$
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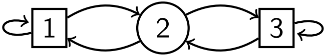

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Consequence:
To show that the finite-time Muller game with threshold 2 is equivalent, we need other proof techniques.

Outline

Conclusion

We have presented a finite-duration version of a Muller game that is equivalent to the original game.

- Reachability game on a tree; hence, simple algorithms are available.
- Maximal play length: 3^n;
- Space requirement $\mathcal{O}(3^n)$, where $n = |G|$.
- Our strategies are eager: they do not spend more time in "bad" loops than they have to.

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Open questions:

- Is the finite-time Muller game with threshold 2 equivalent to the original Muller game?
- Given a winning strategy for a finite-time Muller game, can we turn it into a winning strategy for the Muller game?