Delay Games with WMSO+U Winning Conditions

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Introduction

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  Interaction: one player may delay her moves.
  Winning condition: quantitative instead of qualitative.

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Outline

1. **WMSO with the Unbounding Quantifier**
2. Delay Games
3. WMSO+U Delay Games w.r.t. Constant Lookahead
4. Constant Lookahead is not Sufficient
5. Conclusion

The Unbounding Quantifier

Bojańczyk: Let's add a new quantifier to (weak) monadic second order logic (WMSO/MSO)

- $UX \varphi(X)$ holds, if there are arbitrarily large finite sets X such that $\varphi(X)$ holds.

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L defined by

$$\forall x \exists y (y > x \land P_b(y)) \land$$
$$UX\ [\forall x \forall y \forall z (x < y < z \land x \in X \land z \in X \rightarrow y \in X)$$
$$\land \forall x (x \in X \rightarrow P_a(x))\ ]$$

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$$\wedge \forall x (x \in X \rightarrow P_a(x)) \ ]$$

Theorem (Bojańczyk '14)

Games with WMSO+U winning conditions are decidable.

Max-Automata

Equivalent automaton model for WMSO+U on infinite words:

- Deterministic finite automata with counters
- counter actions: `incr`, `reset`, `max`
- acceptance: boolean combination of "counter γ is bounded".

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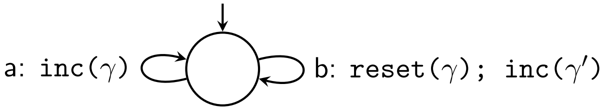

Acceptance condition: γ and γ' unbounded.

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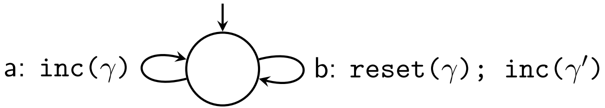

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Theorem (Bojańczyk '09)

The following are (effectively) equivalent:

1. *L WMSO+U-definable.*
2. *L recognized by max-automaton.*

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Delay Games

The delay game $\Gamma_f(L)$:

- Delay function: $f\colon \mathbb{N} \rightarrow \mathbb{N}_+$.
- ω-language $L \subseteq (\Sigma_I \times \Sigma_O)^\omega$.
- Two players: Input (I) vs. Output (O).

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  - I picks word $u_i \in \Sigma_I^{f(i)}$ (building $\alpha = u_0 u_1 \cdots$).
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Definition:

- f is constant, if $f(i) = 1$ for every $i > 0$.
- f is unbounded, if $f(i) > 1$ for infinitely many i.

Example

- $\Sigma_I = \{0, 1, \#\}$ and $\Sigma_O = \{0, 1, *\}$.
- Input block: $\#w$ with $w \in \{0, 1\}^+$. Length: $|w|$.
- Output block:

$$\binom{\#}{\alpha(n)}\binom{\alpha(1)}{*}\binom{\alpha(2)}{*}\cdots\binom{\alpha(n-1)}{*}\binom{\alpha(n)}{\alpha(n)}$$

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Define language L_0: if infinitely many $\#$ and arbitrarily long input blocks, then arbitrarily long output blocks.

O wins $\Gamma_f(L_0)$ for every unbounded f:

- If I produces arbitrarily long input blocks, then the lookahead will contain arbitrarily long input blocks.
- Thus, O can produce arbitrarily long output blocks.

Previous Results

For ω-regular L (given by deterministic parity automaton):

Theorem (Hosch & Landweber '72)

"Given L, does O win $\Gamma_f(L)$ for some constant f?" is decidable.

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Theorem (Holtmann, Kaiser & Thomas '10)

1. *O wins $\Gamma_f(L)$ for some $f \Leftrightarrow O$ wins $\Gamma_f(L)$ for some constant f.*
2. *Decision problem in 2EXPTIME.*
3. *Doubly-exponential upper bound on necessary (constant) lookahead.*

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Theorem (Klein, Z. '15)

1. *Decision problem ExpTime-complete.*
2. *Tight exponential bounds on necessary (constant) lookahead.*

Previous Results

For ω-context-free L (given by ω-pushdown automaton):

Theorem (Fridman, Löding & Z. '11)

1. *Decision problem is undecidable.*
2. *Constant lookahead not enough: lookahead has to grow non-elementarily.*

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Theorem (Klein, Z. '15)

Delay games with Borel winning conditions are determined.

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Determinacy

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Proof idea:

- Winning condition recognized by some automaton $\mathcal{A}$.
- Encode game as parity game in countable arena. States store:
  - Current lookahead (queue over Σ_I)
  - state $\mathcal{A}$ reaches on current play prefix.
  - Current counter values after this run prefix.
  - Maximal counter values seen thus far.
  - Flag marking whether maximum was increased during last transition.
- Thus: counter γ unbounded if, and only if, corresponding flag is raised infinitely often $\Rightarrow$ parity condition.

Capturing Finite Runs of Max-Automata

Theorem

The following problem is decidable: given a max-automaton $\mathcal{A}$, does O win $\Gamma_f(L(\mathcal{A}))$ for some constant delay function f.

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Capture behavior of $\mathcal{A}$, i.e., state changes and evolution of counter values:

- Transfers from counter γ to γ'.
- Existence of increments, but not how many.

$\Rightarrow$ equivalence relation $\equiv$ over Σ^* of exponential index.

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Lemma

Let $(x_i)_{i\in\mathbb{N}}$ and $(x_i')_{i\in\mathbb{N}}$ be two sequences of words over Σ^* with $\sup_i |x_i| < \infty$, $\sup_i |x_i'| < \infty$, and $x_i \equiv x_i'$ for all i. Then,

$$x_0x_1x_2\cdots \in L(\mathcal{A}) \Leftrightarrow x_0'x_1'x_2'\cdots \in L(\mathcal{A}).$$

Removing Delay

- In $\mathcal{A}$, project away Σ_O and construct equivalence $\equiv$ over Σ_I^*.
- Define abstract game $\mathcal{G}(\mathcal{A})$:
  - I picks equivalence classes,
  - O constructs run on representatives (always one step behind to account for delay).
  - O wins, if run is accepting.

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$\mathcal{G}(\mathcal{A})$ is delay-free with WMSO+U winning condition.

- Can be solved effectively by reduction to satisfiability problem for WMSO+U with path quantifiers over infinite trees.
- Doubly-exponential upper bound on necessary constant lookahead.

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Recall: O wins $\Gamma_f(L_0)$ for every unbounded f.

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I: $\#$ 0 0 $\cdots$ 0

Constant Lookahead is not Sufficient

Recall: O wins $\Gamma_f(L_0)$ for every unbounded f.

- Input block: $\#w$ with $w \in \{0,1\}^+$.
- Output block: $\binom{\#}{\alpha(n)}\binom{\alpha(1)}{*}\binom{\alpha(2)}{*}\cdots\binom{\alpha(n-1)}{*}\binom{\alpha(n)}{\alpha(n)}$
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O:	0					

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$$\begin{array}{rcccccc} I\!: & \# & 0 & 0 & \cdots & 0 & 1 \\ O\!: & 0 & * & & & & \end{array}$$

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$$\begin{array}{lllllllll} I\colon & \# & 0 & 0 & \cdots & 0 & 1 & 1 & 1 & \cdots \\ O\colon & 0 & * & * & \cdots \end{array}$$

Constant Lookahead is not Sufficient

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Claim: I wins $\Gamma_f(L_0)$ for every constant f.

$$\begin{array}{rcccccccc} I: & \# & 0 & 0 & \cdots & 0 & 1 & 1 & 1 & \cdots \\ O: & 0 & * & * & \cdots \end{array}$$

- Lookahead contains only input blocks of length $f(0)$.
- I can react to O's declaration at beginning of an output block to bound size of output blocks while producing arbitrarily large input blocks.

Outline

1. WMSO with the Unbounding Quantifier
2. Delay Games
3. WMSO+U Delay Games w.r.t. Constant Lookahead
4. Constant Lookahead is not Sufficient
5. **Conclusion**

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- Delay games with WMSO+U winning conditions w.r.t. constant delay functions are decidable.
- Doubly-exponential upper bound on necessary constant lookahead.
- But constant delay is not always sufficient.

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Conjecture

The following are equivalent for L definable in WMSO+U:

1. *O wins $\Gamma_f(L)$ for some f.*
2. *O wins $\Gamma_f(L)$ for every unbounded f s.t. $f(0)$ is "large enough".*