What are Strategies in Delay Games? Borel Determinacy for Games with Lookahead

Joint work with Felix Klein (Saarland University)

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Introduction

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The Delay Game $\Gamma_f(L)$

- Delay function: $f \colon \mathbb{N} \to \mathbb{N}_+$.
- ω-language $L \subseteq (\Sigma_I \times \Sigma_O)^\omega$.
- Two players: Input (I) vs. Output (O).

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  - I picks word $u_i \in \Sigma_I^{f(i)}$ (building $\alpha = u_0 u_1 \cdots$).
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- O wins iff $\binom{\alpha(0)}{\beta(0)}\binom{\alpha(1)}{\beta(1)} \cdots \in L$.

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Special case:

- (delay-free) Gale-Stewart games: pick $f(i) = 1$ for all i.
- Notation: $\Gamma(L)$.

Strategies in Delay Games

Fix some f.

- A strategy for I in $\Gamma_f(L)$ is a mapping $\tau\colon \Sigma_O^* \rightarrow \Sigma_I^+$ s.t. $|\tau(w)| = f(|w|)$.
- A strategy for O in $\Gamma_f(L)$ is a mapping $\sigma\colon \Sigma_I^* \rightarrow \Sigma_O$.

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So, both definitions depend on f.

Borel Hierarchy and Determinacy

A game is determined, if one of the players has a winning strategy.

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Borel hierarchy: levels levels $\boldsymbol{\Sigma}_\alpha$ and $\boldsymbol{\Pi}_\alpha$ for every countable ordinal $\alpha > 0$:

- $\boldsymbol{\Sigma}_1 = \{L \subseteq \Sigma^\omega \mid L = K \cdot \Sigma^\omega \text{ for some } K \subseteq \Sigma^*\}$,
- $\boldsymbol{\Pi}_\alpha = \{\Sigma^\omega \setminus L \mid L \in \boldsymbol{\Sigma}_\alpha\}$ for every α, and
- $\boldsymbol{\Sigma}_\alpha = \{\bigcup_{i \in \mathbb{N}} L_i \mid L_i \in \boldsymbol{\Pi}_{\alpha_i} \text{ with } \alpha_i < \alpha \text{ for every } i\}$ for every $\alpha > 1$.

Borel Hierarchy and Determinacy

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Our goal: Borel determinacy for delay games.

Borel Determinacy for Delay Games

Theorem

Let L be Borel and let f be a delay function. Then, $\Gamma_f(L)$ is determined.

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Proof.

- $\triangleright$: fresh skip-symbol not in Σ_O.
- $\mathrm{shift}_f(\beta) = \triangleright^{f(0)-1}\,\beta(0)\,\triangleright^{f(1)-1}\,\beta(1)\,\triangleright^{f(2)-1}\,\beta(2)\cdots$
- $\mathrm{shift}_f(L) = \{\binom{\alpha}{\mathrm{shift}_f(\beta)} \mid \binom{\alpha}{\beta} \in L\}$

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- $\mathrm{shift}_f(L) = \{\binom{\alpha}{\mathrm{shift}_f(\beta)} \mid \binom{\alpha}{\beta} \in L\}$

Lemma

1. $\mathrm{shift}_f(L)$ *is Borel.*
2. *I wins* $\Gamma(\mathrm{shift}_f(L)) \Rightarrow$ *I wins* $\Gamma_f(L)$.
3. *O wins* $\Gamma(\mathrm{shift}_f(L)) \Rightarrow$ *O wins* $\Gamma_f(L)$. □

Omnipotent Strategies

Example

$L_0 \subseteq (\{a,b,c\} \times \{b,c\})^\omega$ with $\binom{\alpha(0)}{\beta(0)}\binom{\alpha(1)}{\beta(1)}\binom{\alpha(2)}{\beta(2)} \cdots \in L_0$ if

- $\alpha(n) = a$ for every $n \in \mathbb{N}$, or
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I wins $\Gamma_f(L_0)$ for every f:

- $\tau(\varepsilon) = a^{f(0)}$, and
- $\tau(w_0 \cdots w_{n-1}) = w_0^{f(n)}$.

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- $\tau(w_0 \cdots w_{n-1}) = w_0^\omega$.

We call such a strategy omnipotent for L_0.

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2. *lookahead-counting* strategy: $\tau \colon \Sigma_O^* \times \mathbb{N} \to \Sigma_I^\omega$.

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   - Access to both player's move.
   - Cannot reconstruct when O's moves where made.

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4. *history-tracking* strategy: $\tau\colon (\Sigma_O \cup \{\triangleright\})^* \to \Sigma_I^\omega$.

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4. *history-tracking* strategy: $\tau\colon (\Sigma_O \cup \{\triangleright\})^* \to \Sigma_I^\omega$.
   - Can reconstruct moves of each round.

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3. *input-output-tracking* strategy: $\tau \colon \Sigma_O^* \times \Sigma_I^* \to \Sigma_I^\omega$.
4. *history-tracking* strategy: $\tau \colon (\Sigma_O \cup \{\triangleright\})^* \to \Sigma_I^\omega$.

These notions form a hierarchy, the first three can be separated:

Theorem

1. *Every output-tracking strategy is a lookahead-counting one.*
2. *Every lookahead-counting strategy is an input-output tracking one.*
3. *Every input-output tracking strategy is a history tracking one.*

Output-Tracking vs. Lookahead-Counting

Theorem

Let $L_1 = \{\binom{\alpha}{\beta} \mid \alpha \neq (ab)^\omega\}$. I has an omnipotent lookahead-counting strategy for L_1, but no omnipotent output-tracking strategy.

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Proof.

Assume τ is omnipotent output-tracking strategy:

- We have $\tau(\varepsilon) = (ab)^\omega$.
- Assume $\tau(c)$ starts with a. Then, τ is losing for every f with odd $f(0)$ (other case dual).

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The following lookahead-counting strategy is omnipotent:

$$\tau(x, n) = \begin{cases} (ab)^\omega & n \text{ even}, \\ (ba)^\omega & n \text{ odd}. \end{cases}$$

Output-Tracking vs. Lookahead-Counting

Theorem

There is a winning condition L_2 such that I has an omnipotent input-output-tracking strategy for L_2, but no omnipotent lookahead-counting strategy.

Output-Tracking vs. Lookahead-Counting

Theorem

There is a winning condition L_2 such that I has an omnipotent input-output-tracking strategy for L_2, but no omnipotent lookahead-counting strategy.

Open question:

Are omnipotent history-tracking strategies stronger than omnipotent input-output-tracking strategies?

Omnipotent Strategies for O

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Either, I wins $\Gamma_f(L)$ for some f or O has an omnipotent round-counting strategy for L.

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- (Un)decidability results, e.g., it is decidable whether I has an omnipotent strategy for a given ω-regular L.