Time-optimal Winning Strategies for Poset Games

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Introduction

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- Solution to the synthesis problem for reactive systems.
- Well-developed theory with nice results.

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- From linearly ordered objectives to partially ordered objectives.
- Quantitative analysis of winning strategies: synthesize optimal winning strategies.

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- Well-developed theory with nice results.

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- From linearly ordered objectives to partially ordered objectives.
- Quantitative analysis of winning strategies: synthesize optimal winning strategies.

Outline:

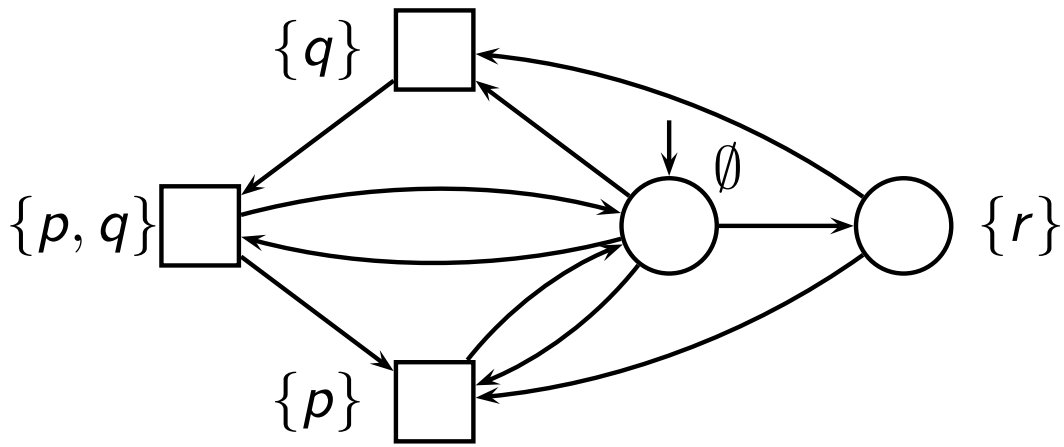
- Definitions and related work
- Poset games
- Time-optimal strategies for poset games

Outline

Definitions

An (initialized and labeled) arena $G = (V, V_0, V_1, E, s_0, l_G)$ consists of

- a finite directed graph (V, E) without dead-ends,
- a partition $\{V_0, V_1\}$ of V denoting the positions of Player 0 (circles) and Player 1 (squares),
- an initial vertex $s_0 \in V$,
- a labeling function $l_G : V \to 2^P$ for some set P of atomic propositions.



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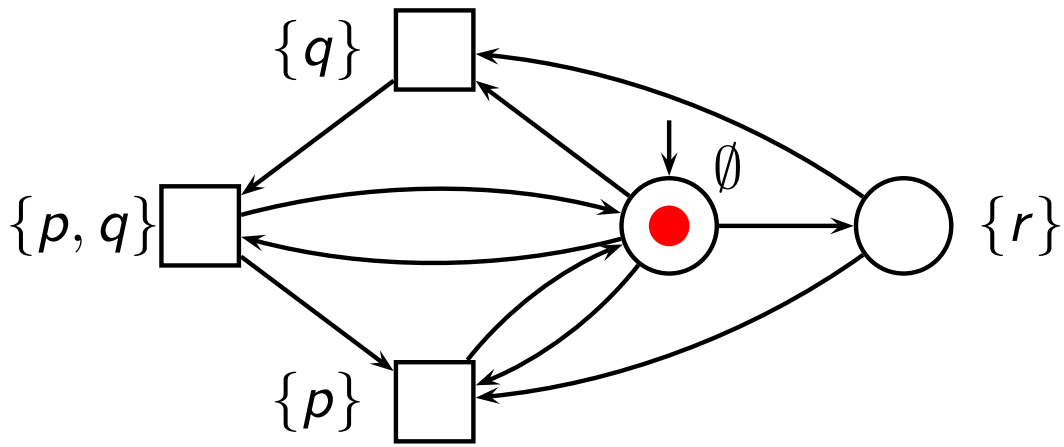

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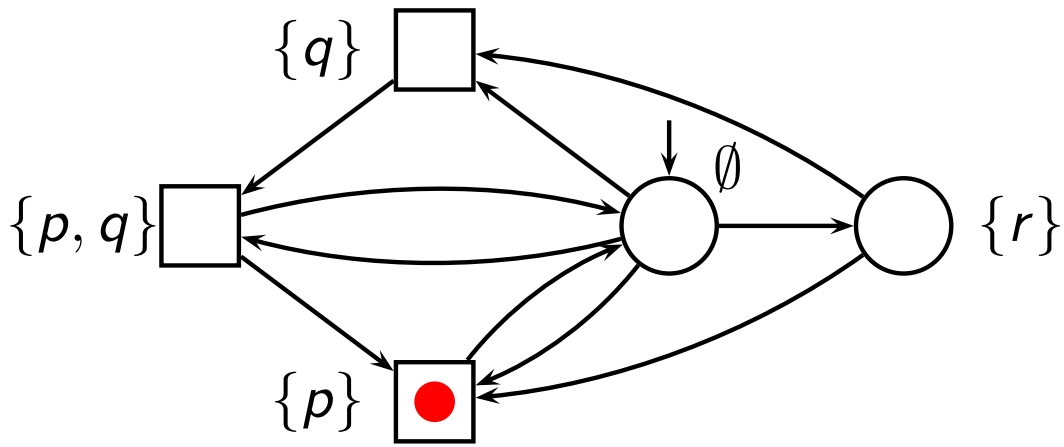

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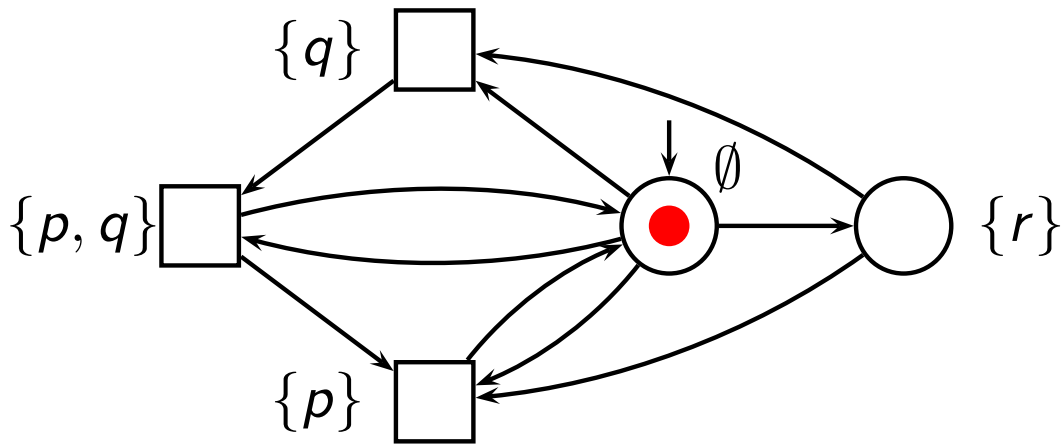

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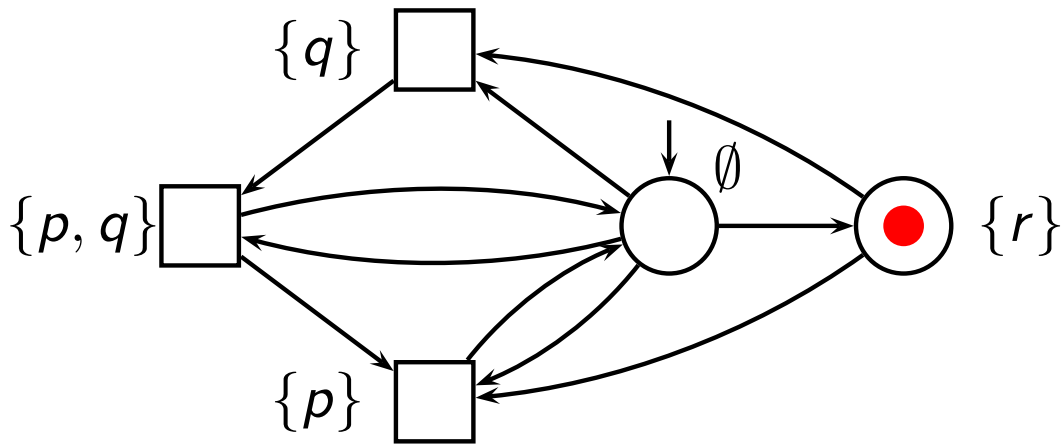

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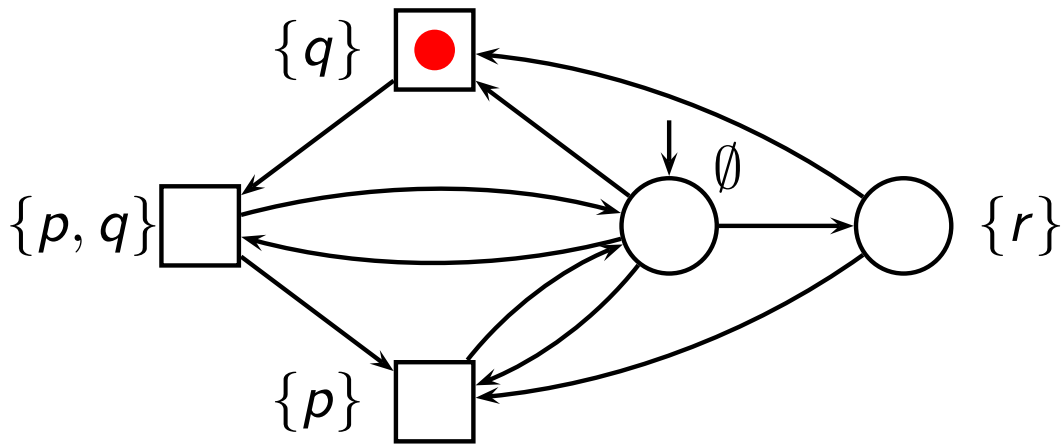

Definitions

- Play in G: infinite path $\rho_0\rho_1\rho_2\ldots$ starting in s_0.
- Strategy for Player i: (partial) mapping $\sigma : V^*V_i \to V$ such that $(s, \sigma(ws)) \in E$; σ is finite-state, if it is computable by a transducer.
- $\rho_0\rho_1\rho_2\ldots$ is consistent with σ: $\rho_{n+1} = \sigma(\rho_0\ldots\rho_n)$ for all n such that $\rho_n \in V_i$.
- Winning plays for Player 0: $\mathrm{Win} \subseteq V^\omega$.
- σ is a winning strategy for Player 0: every play that is consistent with σ is in Win (dual definition for Player 1).

Optimal Strategies

Idea: quantitative analysis of winning strategies.

- The outcome of a play is still binary: win or lose.
- But the (winning) plays are measured according to some evaluation.
- Task: determine optimal (w.r.t. given measure) winning strategies for Player 0.

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Natural measures for some winning conditions:

- Reachability games: time until goal state is visited.
- Büchi games: waiting times between visits of goal states.
- Co-Büchi games: time until goal states are reached for good.

The classical attractor-based algorithms compute optimal strategies for these games.

An Example: Request-Response games

Request-response game: $(G, (Q_j, P_j)_{j=1,\dots,k})$ where $Q_j, P_j \subseteq V$.

- Player 0 wins a play if every visit to Q_j (request) is responded by a later visit to P_j.

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- Waiting times: start a clock for every request that is stopped as soon as it is responded (and ignore subsequent requests).
- Accumulated waiting time: sum up the clock values of a play prefix (quadratic influence of open requests).
- Value of a play: limit superior of the average accumulated waiting time.
- Value of a strategy: value of the worst play consistent with the strategy.

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Theorem (Horn, Thomas, Wallmeier)

If Player 0 has a winning strategy for an RR-game, then she also has an optimal winning strategy, which is finite-state and effectively computable.

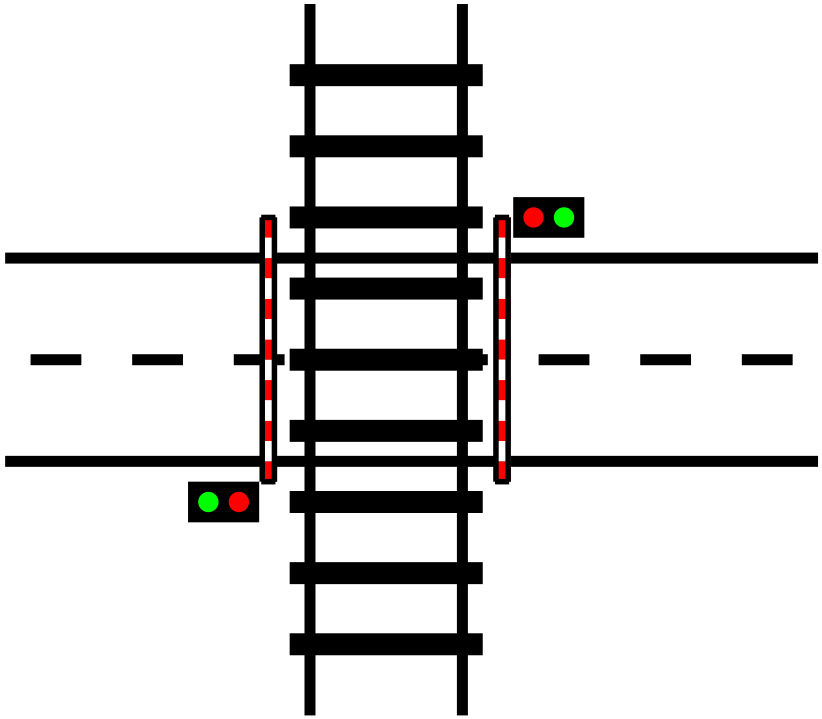
Outline

1. Definitions and Related Work

2. **Poset Games**

3. Time-optimal Strategies for Poset Games

Motivation for Poset Games



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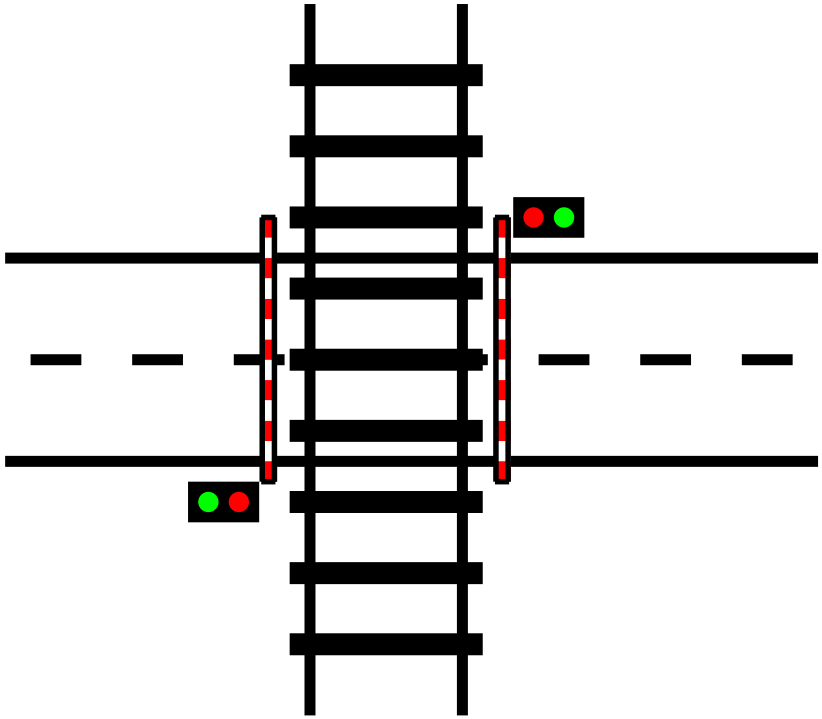

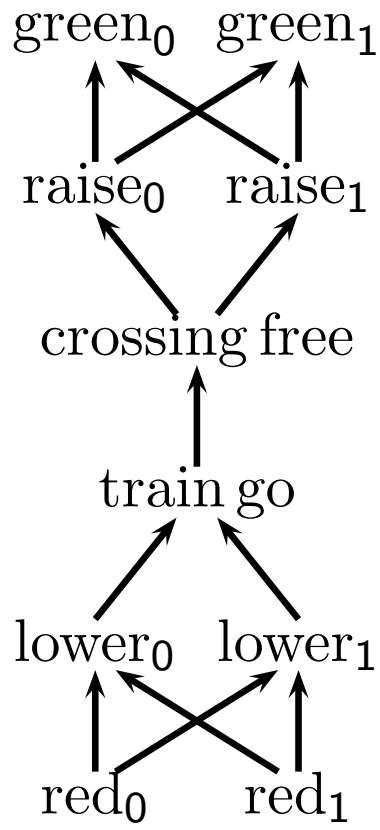

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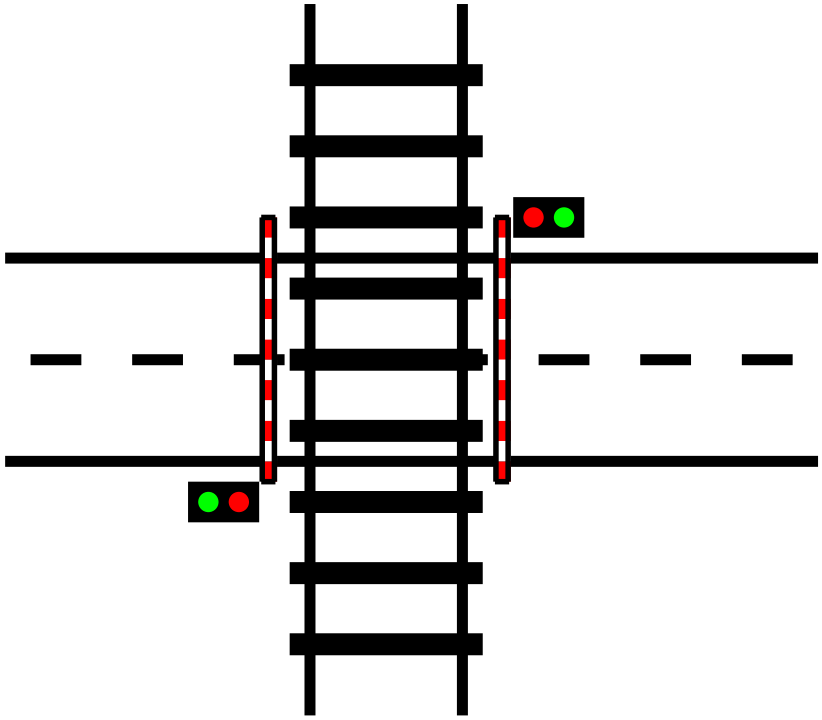

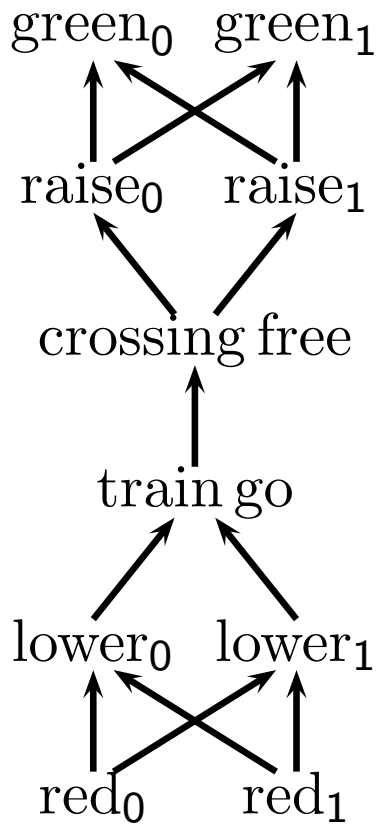

Generalize RR-games to express more complicated conditions, but retain notion of time-optimality.

Request: still a singular event.

Response: partially ordered set of events.

Poset Games

Poset game: $(G, (q_j, \mathcal{P}_j)_{j=1,\ldots,k})$ where

- G arena (labeled with $l_G : V \to 2^P$),
- $q_j \in P$ request,
- $\mathcal{P}_j = (D_j, \preceq_j)$ poset, where $D_j \subseteq P$.

Remember: P set of atomic propositions

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Embedding of $\mathcal{P}_j$ in $\rho_0\rho_1\rho_2\ldots$: function $f : D_j \to \mathbb{N}$ such that

- $d \in l_G(\rho_{f(d)})$ for all $d \in D_j$,
- $d \preceq_j d'$ implies $f(d) \leq f(d')$ for all $d, d' \in D_j$.

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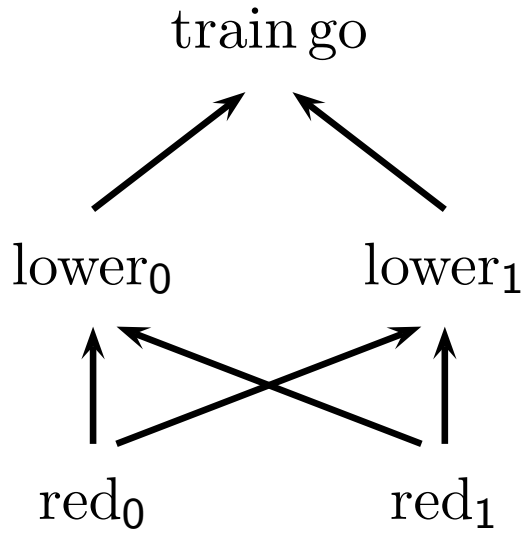
Player 0 wins $\rho_0\rho_1\rho_2\dots$ if

$$\forall j \forall n \, (q_j \in l_G(\rho_n) \implies \rho_n\rho_{n+1}\dots \text{ allows embedding of } \mathcal{P}_j) \quad .$$

"Every request q_j is responded by a later embedding of $\mathcal{P}_j$."

A Play



$\{\overline{\text{req}}\}$

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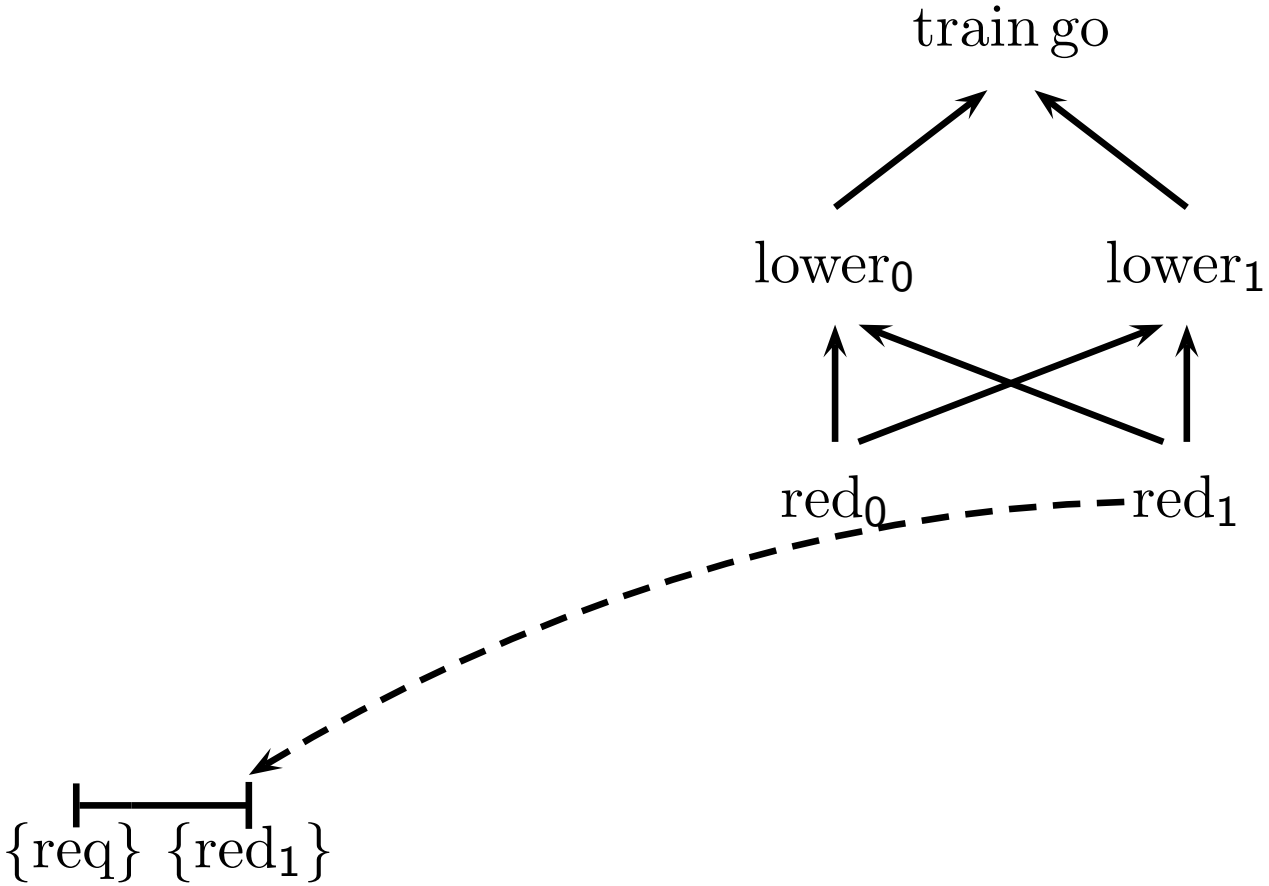

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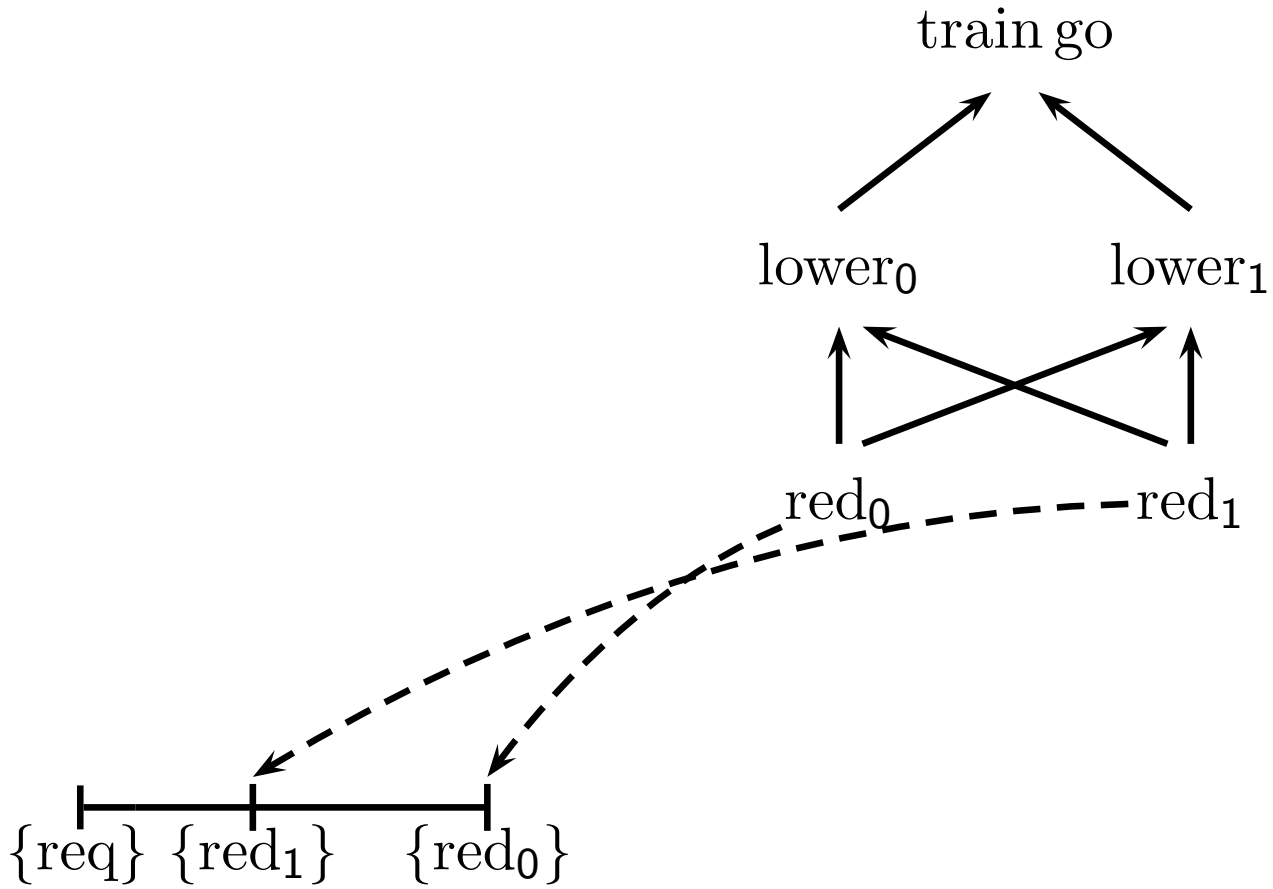

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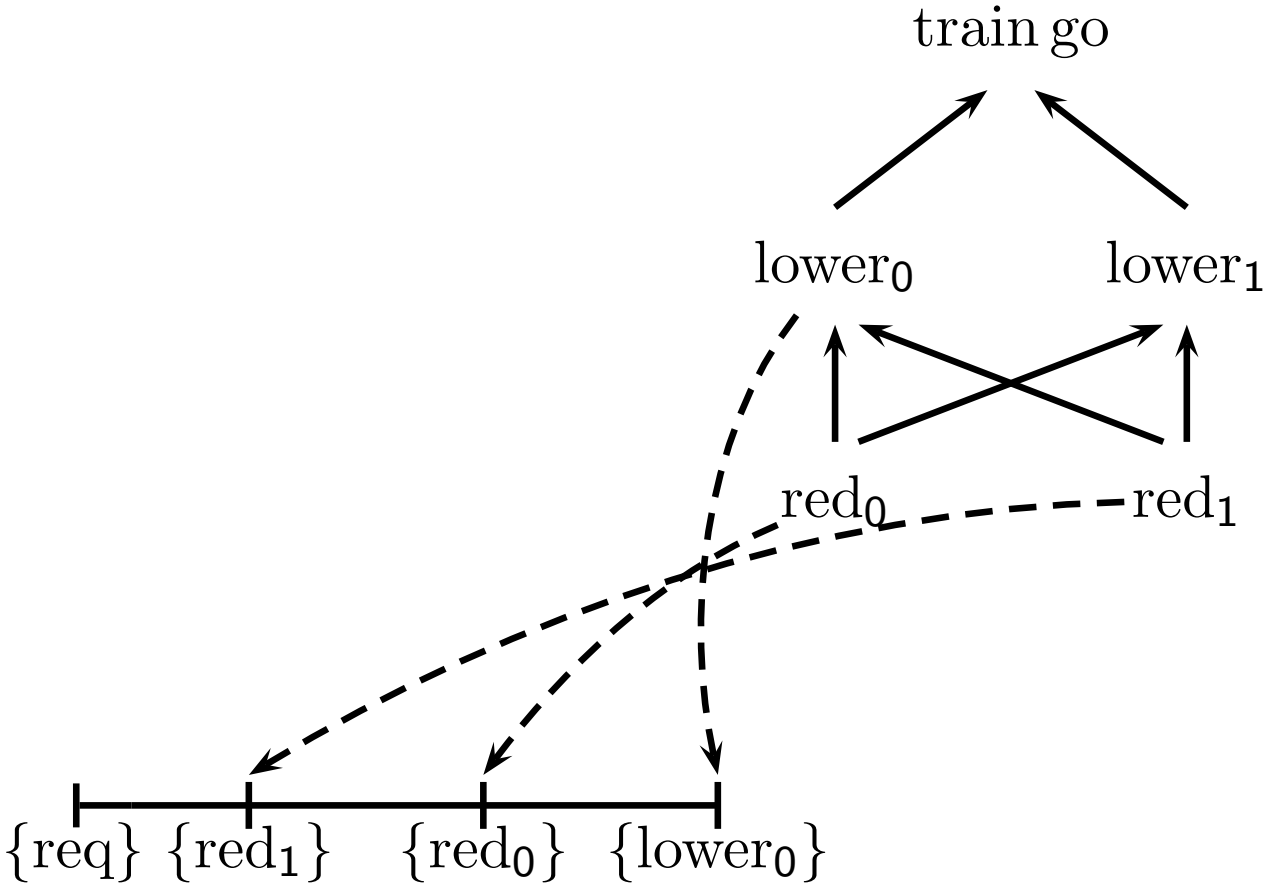

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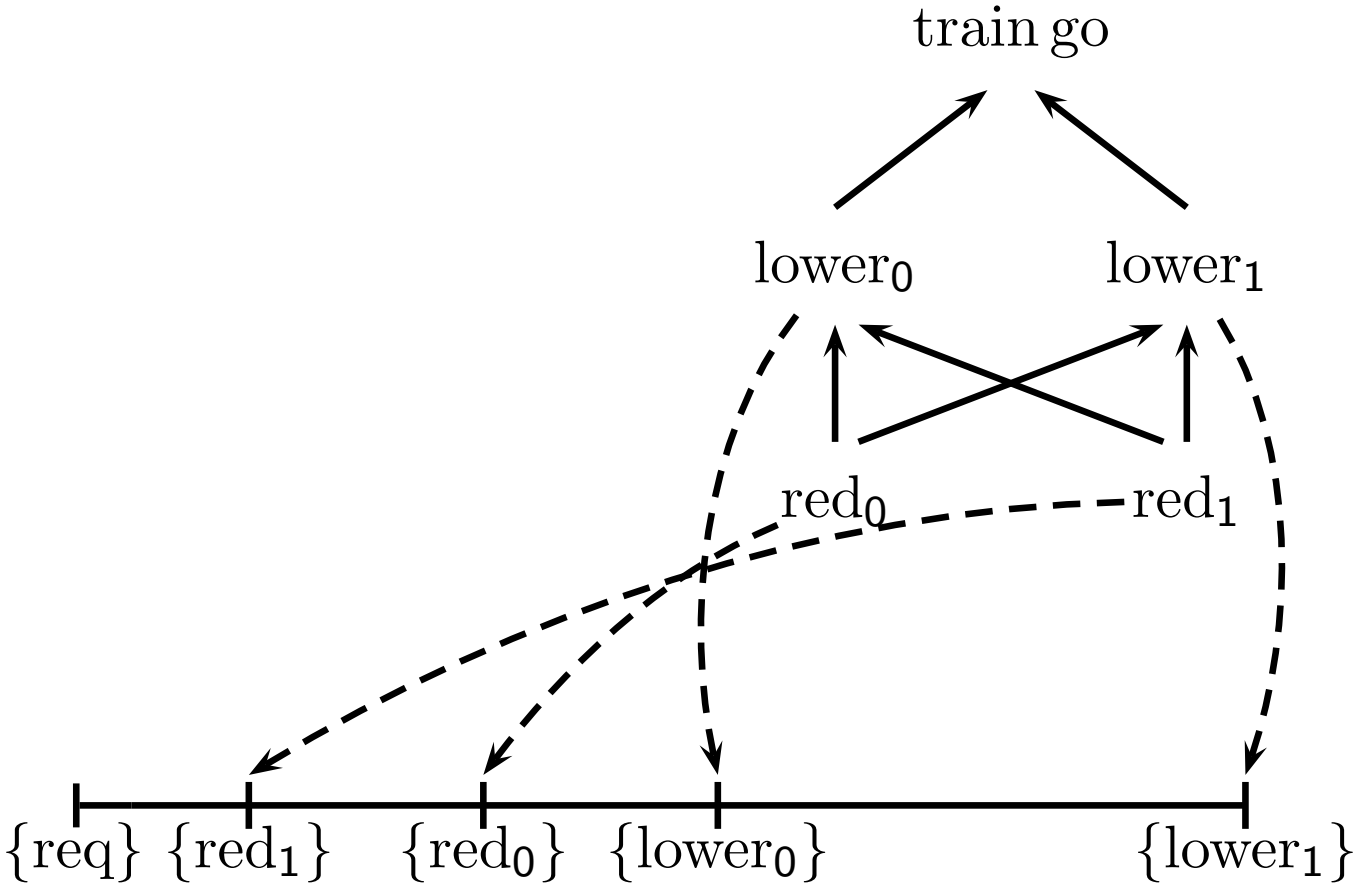
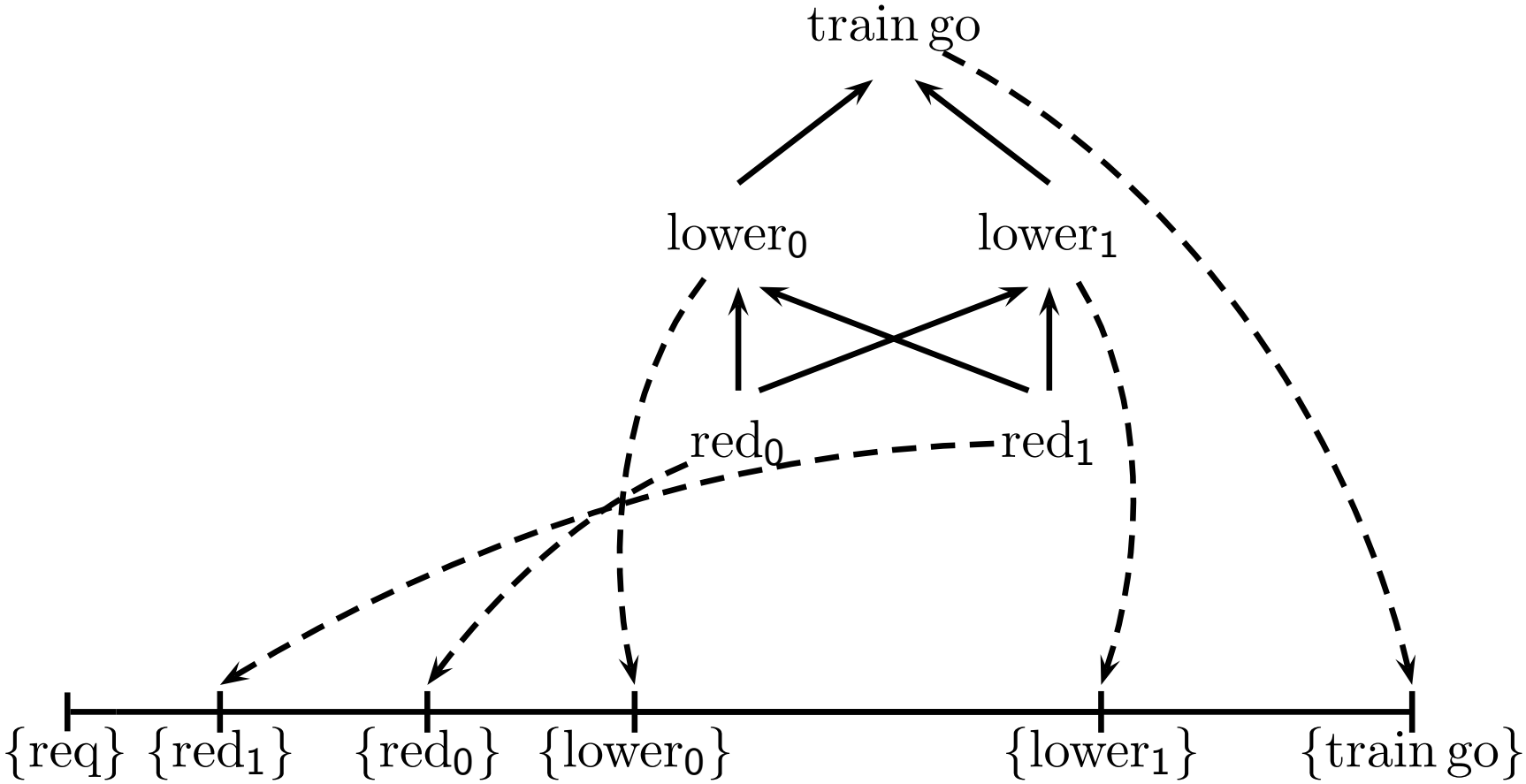

train go
lower0
lower1
red0
red1
{req}
{red1}
{red0}
{lower0}
{lower1}

A Play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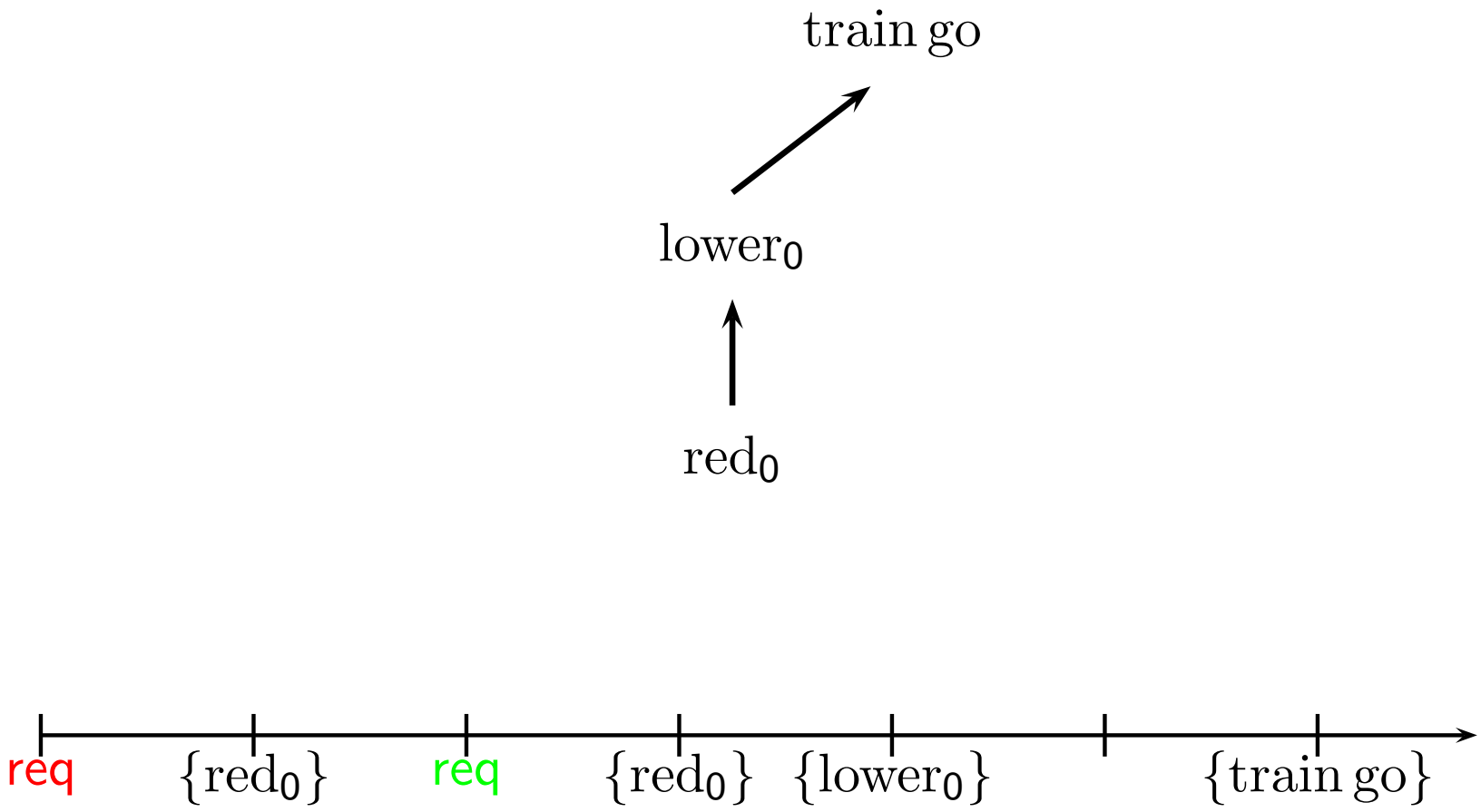
{train go}

Another Play



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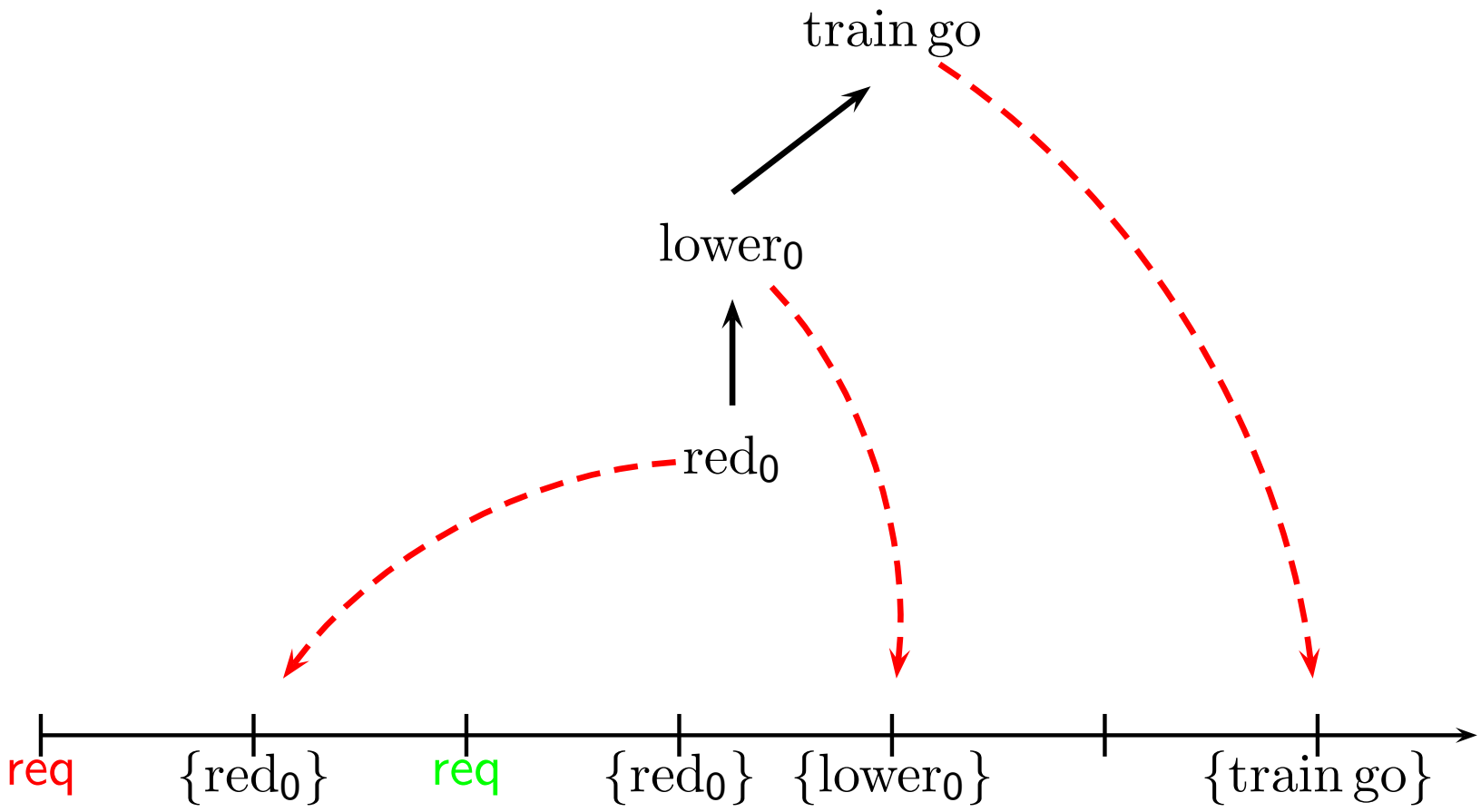

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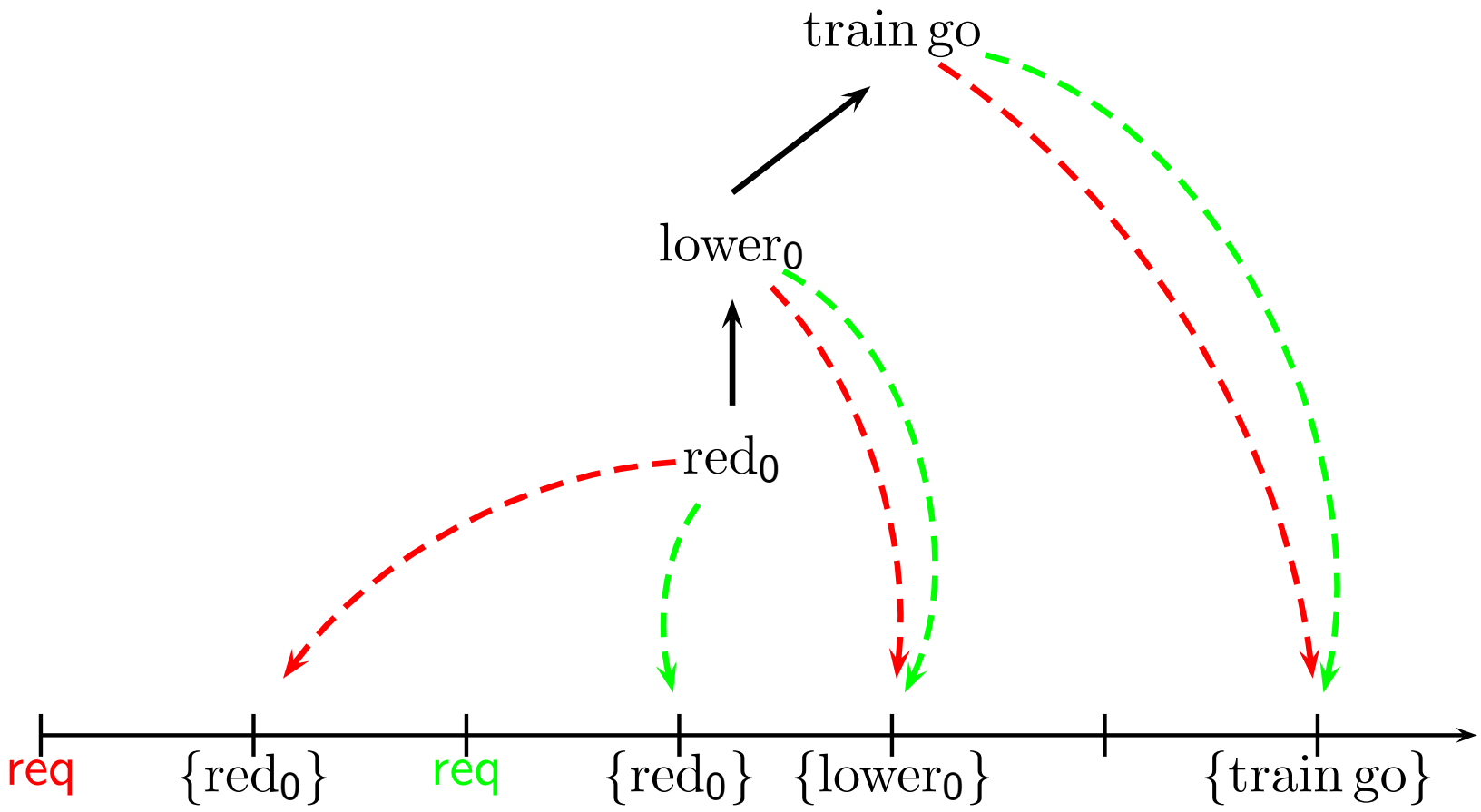

Embeddings might overlap. Hence, we cannot ignore requests that occur while another request is still being responded.

Solving Poset Games

Theorem

Poset games are determined with finite-state strategies, i.e., in every poset games, one of the players has a finite-state winning strategy.

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Proof:
Reduction to Büchi games; memory is used

- to store elements of the posets that still have to be embedded,
- to deal with overlapping embeddings,
- to implement a cyclic counter to ensure that every request is responded by an embedding.

Size of the memory: exponential in the size of the posets $\mathcal{P}_j$.

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2. Poset Games

3. **Time-optimal Strategies for Poset Games**

Waiting Times

As desired, a natural definition of waiting times is retained:

- Start a clock if a request is encountered...
- ... that is stopped as soon as the embedding is completed.
- Need a clock for every request (even if another request is already open).

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- Need a clock for every request (even if another request is already open).
- Value of a play: limit superior of the average accumulated waiting time.
- Value of a strategy: value of the worst play consistent with the strategy.
- Corresponding notion of optimal strategies.

The Main Theorem

Theorem

If Player 0 *has a winning strategy for a poset game* $\mathcal{G}$*, then she also has an optimal winning strategy, which is finite-state and effectively computable.*

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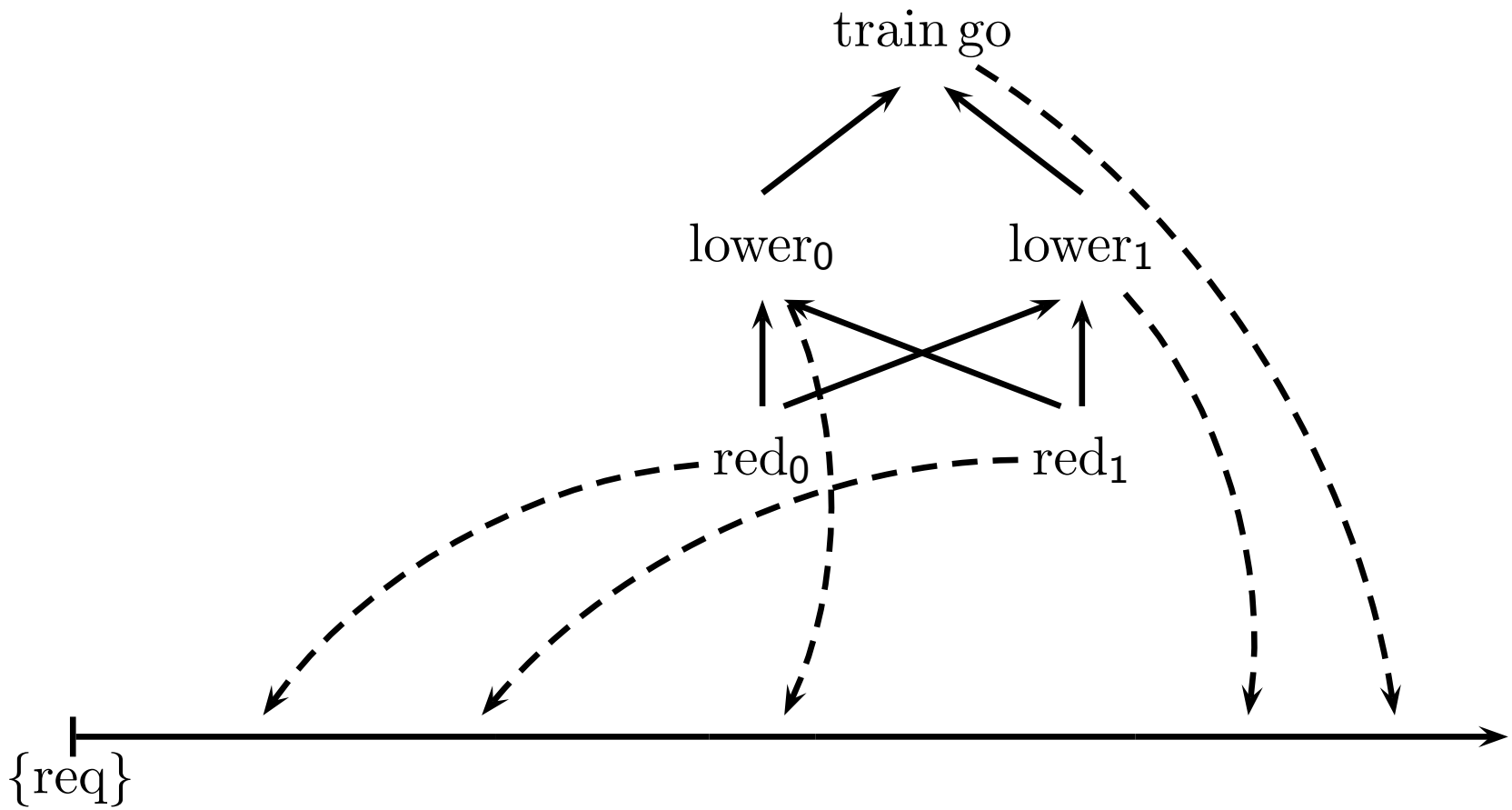
Proof:

- If Player 0 has a winning strategy, then she also has one of value less than a certain constant c (from reduction). This bounds the value of an optimal strategy, too.
- For every strategy of value $\leq c$ there is another strategy of smaller or equal value, that also bounds all waiting times and bounds the number of open requests.
- If the waiting times and the number of open requests are bounded, then $\mathcal{G}$ can be reduced to a mean-payoff game.

Proof: Bounding the Waiting Times



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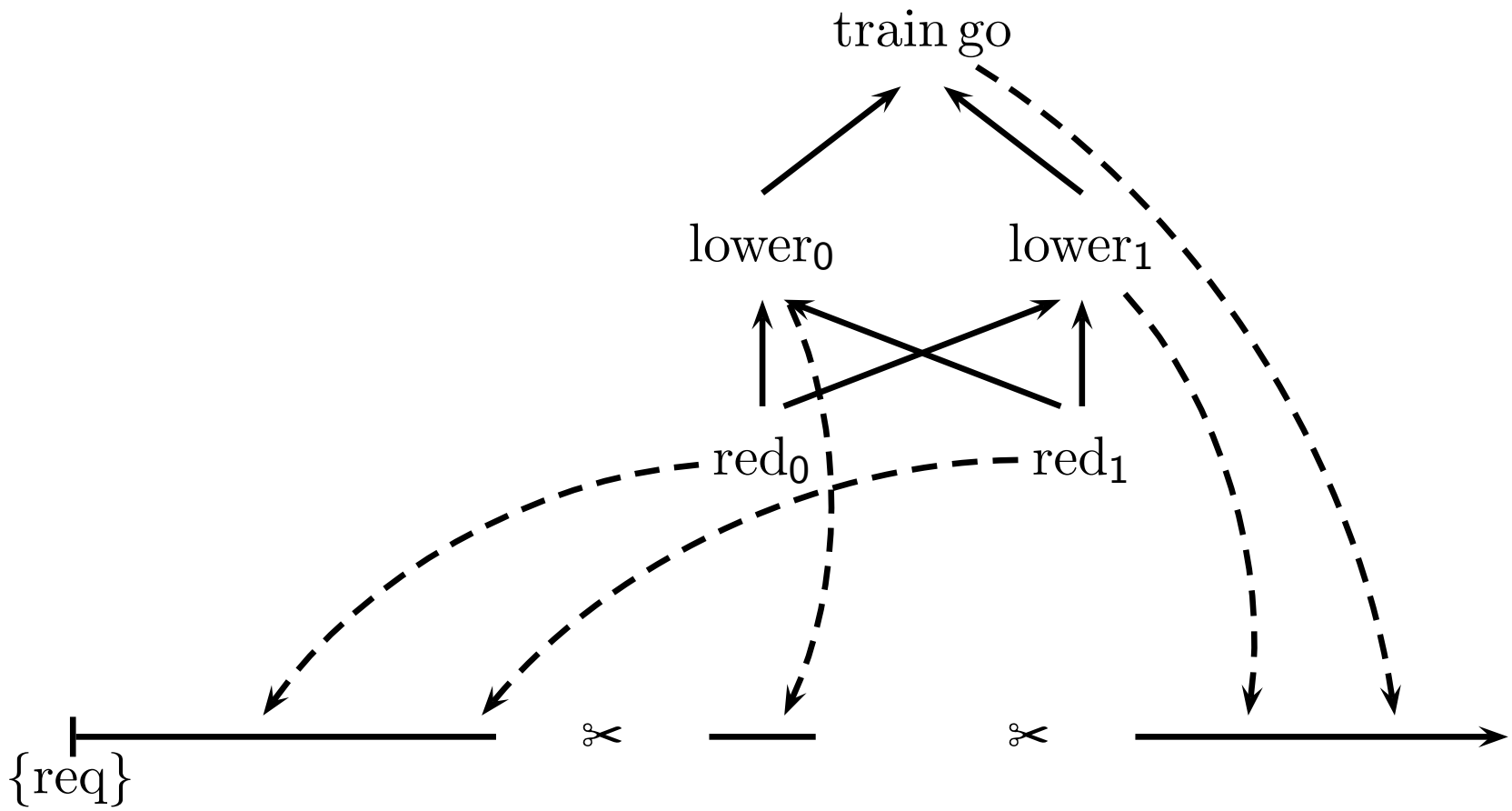

Skip loops, but pay attention to other overlapping embeddings!

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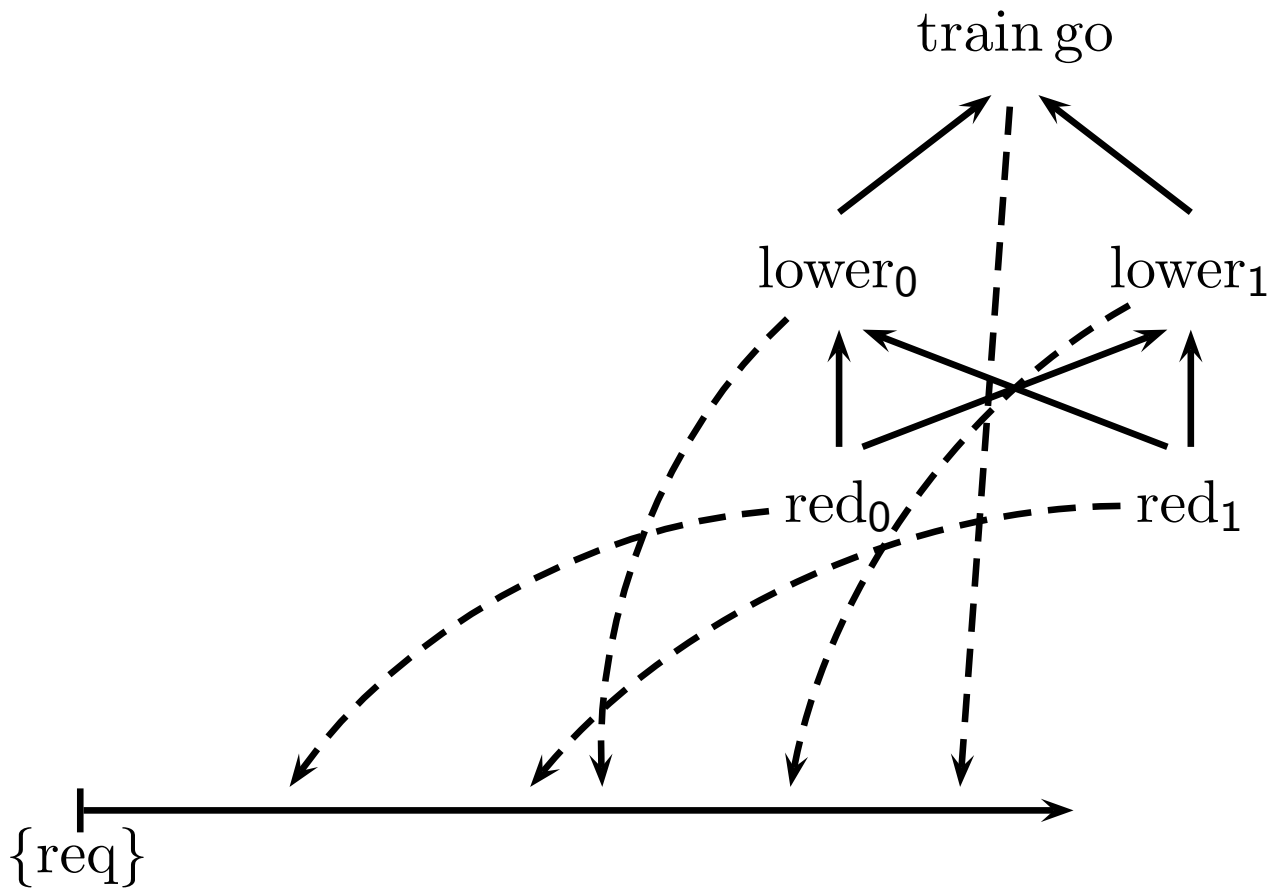

Repeating this leads to bounded waiting times and a bounded number of open requests.

Proof: Reduction to Mean-Payoff Games

Mean-payoff game: (G, l) where the edges of the arena G are labeled by $l : E \to \mathbb{Z}$.

- goal for Player 0: maximize the limit inferior of the average accumulated edge labels.
- goal for Player 1: minimize the limit superior of the average accumulated edge labels.

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Theorem (Ehrenfeucht, Mycielski / Zwick, Paterson)

In a mean-payoff game, both players have optimal strategies, which are positional and effectively computable.

Proof: Reduction to Mean Payoff Games

From a poset game $\mathcal{G}$ with bounded waiting times and a bounded number of open requests, construct a mean-payoff game $\mathcal{G}'$ such that

- the memory keeps track of the waiting times, and
- the value of a play in $\mathcal{G}$ and the payoff for Player 1 for the corresponding play in $\mathcal{G}'$ are equal.

Then: an optimal strategy for Player 1 in $\mathcal{G}'$ induces an optimal strategy for Player 0 in $\mathcal{G}$.

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Size of the mean-payoff game: super-exponential in the size of the poset game (holds already for RR-games).

Conclusion

We introduced a new winning condition for infinite games

- that extends the request-response condition,
- is well-suited to model real-life requirements,
- but retains a natural definition of waiting times and optimal strategies.

We showed the existence of optimal strategies for poset games, which are finite-state and effectively computable.

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Further research:

- Avoid the detour via mean-payoff games and directly compute (approximatively) optimal strategies, to overcome the atrocious complexity.
- Understand the trade-off between the size and value of a strategy.
- Consider optimal strategies for other winning conditions.