Delay Games with WMSO+U Winning Conditions

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March 6th, 2015

AVACS Meeting, Freiburg, Germany

Introduction

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  - Type of interaction: one player may delay her moves.
  - Type of winning conditions: quantitative instead of qualitative.

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Weak MSO with the unbounding quantifier:

- quantitative extension of (weak) MSO
- able to express many high-level quantitative specification languages, e.g., parameterized LTL, finitary parity conditions, etc.

Outline

1. **WMSO with the Unbounding Quantifier**
2. Delay Games
3. WMSO+U Delay Games w.r.t. Constant Lookahead
4. Constant Lookahead is not Sufficient
5. Conclusion

Monadic Second-order Logic

- Monadic Second-order Logic (MSO)
  - Existential/universal quantification of elements: $\exists x$, $\forall x$.
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  - Unary predicates P_a for every $a \in \Sigma$.
  - Order relation $<$ and successor relation S.

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Theorem (Büchi '62)

The following are (effectively) equivalent:

1. *L MSO-definable.*
2. *L WMSO-definable.*
3. *L recognized by Büchi automaton.*

The Unbounding Quantifier

Bojańczyk: Let's add a new quantifier

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L defined by

$$\forall x \exists y(y > x \land P_b(y)) \land$$
$$UX\ \ [\forall x \forall y \forall z(x < y < z \land x \in X \land z \in X \rightarrow y \in X)$$
$$\land \forall x(x \in X \rightarrow P_a(x))\ ]$$

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Theorem (Bojańczyk et al. '14)

There is no algorithm that decides MSO+U on infinite trees and has a correctness proof using the axioms of ZFC.

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Theorem (Bojańczyk et al. '15)

MSO+U on infinite words is undecidable.

WMSO+U

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Corollary

Games with WMSO+U winning conditions are decidable.

Max-Automata

Equivalent automaton model for WMSO+U on infinite words:

- Deterministic finite automata with counters
- counter actions: `incr`, `reset`, `max`
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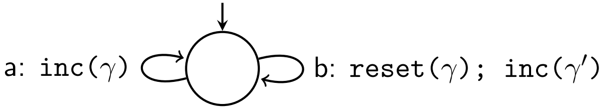

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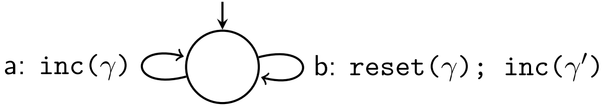

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Delay Games

The delay game $\Gamma_f(L)$:

- Delay function: $f : \mathbb{N} \rightarrow \mathbb{N}_+$.
- ω-language $L \subseteq (\Sigma_I \times \Sigma_O)^\omega$.
- Two players: Input (I) vs. Output (O).

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  - I picks word $u_i \in \Sigma_I^{f(i)}$ (building $\alpha = u_0 u_1 \cdots$).
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Definition: f is constant, if $f(i) = 1$ for every $i > 0$.

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Proof idea:

- Winning condition recognized by some automaton $\mathcal{A}$.
- Encode game as parity game in countable arena. States store:
  - Current lookahead (queue over Σ_I)
  - state $\mathcal{A}$ reaches on current play prefix.
  - Current counter values after this run prefix.
  - Maximal counter values seen thus far.
  - Flag marking whether maximum was increased during last transition.
- Thus: counter γ unbounded if corresponding flag is raised infinitely often $\Rightarrow$ parity condition.

Capturing Finite Runs of Max-Automata

Theorem (Z. '14)

The following problem is decidable: given a max-automaton $\mathcal{A}$, does Player O win $\Gamma_f(L(\mathcal{A}))$ for some constant delay function f.

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Proof Idea:

- Adapt technique for parity automata to max-automata.
- Capture behavior of $\mathcal{A}$, i.e., evolution of counter values:
  - Transfers from counter γ to γ'.
  - Existence of increments, but not how many.
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Lemma

Let $(x_i)_{i\in\mathbb{N}}$ and $(x_i')_{i\in\mathbb{N}}$ be two sequences of words over Σ^* with $\sup_i |x_i| < \infty$, $\sup_i |x_i'| < \infty$, and $x_i \equiv x_i'$ for all i. Then, $x = x_0x_1x_2\cdots \in L(\mathcal{A})$ if and only if $x' = x_0'x_1'x_2'\cdots \in L(\mathcal{A})$.

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Resulting game is delay-free with WMSO+U winning condition.

- Can be solved effectively by a reduction to a satisfiability problem for WMSO+U with path quantifiers over infinite trees.
- Doubly-exponential upper bound on necessary constant lookahead.

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Constant Lookahead is not Sufficient

- $\Sigma_I = \{0, 1, \#\}$ and $\Sigma_O = \{0, 1, *\}$.
- Input block: $\#w$ with $w \in \{0, 1\}^+$. Length: $|w|$.
- Output block:

$$\binom{\#}{\alpha(n)}\binom{\alpha(1)}{*}\binom{\alpha(2)}{*}\cdots\binom{\alpha(n-1)}{*}\binom{\alpha(n)}{\alpha(n)} \in (\Sigma_I \times \Sigma_O)^+$$

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Define language L: if infinitely many $\#$ and arbitrarily long input blocks, then arbitrarily long output blocks.

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Theorem (Z. '14)

I wins $\Gamma_f(L)$, if f is a constant delay function, O if f is unbounded.

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2. Let f be unbounded:

- If Player I produces arbitrarily long input blocks, then the lookahead will contain arbitrarily long input blocks.
- Thus, Player O can produce arbitrarily long output blocks.

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Current work:

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Conjecture

The following are equivalent for L definable in WMSO+U:

1. *Player O wins $\Gamma_f(L)$ for some f.*
2. *Player O wins $\Gamma_f(L)$ for every unbounded f.*

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- Matching bounds on necessary lookahead for the case of constant delay functions.
- A general determinacy theorem.