Cost-Parity and Cost-Streett Games

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Introduction

Boundedness problems in automata theory

- Star-height problem, finite power problem
- Automata with counters: BS-automata, max-automata, R-automata
- Logics with bounds: MSO+U, Cost-MSO

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- Finitary games: bounds between requests and responses
- Consumption and energy games: resources are consumed and recharged along edges
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Here: an extension of ω-regular and finitary games

Outline

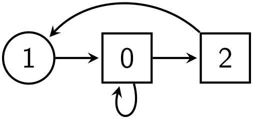
1. **Cost-Parity Games**

2. Cost-Streett Games

3. Conclusion

Parity Games and Extensions

Games are played in arena G colored by $\Omega \colon V \to \mathbb{N}$



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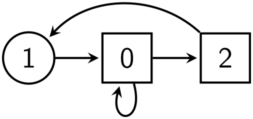

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Equivalently:

- Request: vertex of odd color
- Response: vertex of larger even color
- Parity condition: almost all requests are answered

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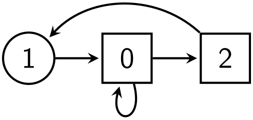

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cost-parity	"$\geq \mathbf{NP} \cap \mathbf{coNP}$"	"$\geq$ positional"	infinite

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Another example

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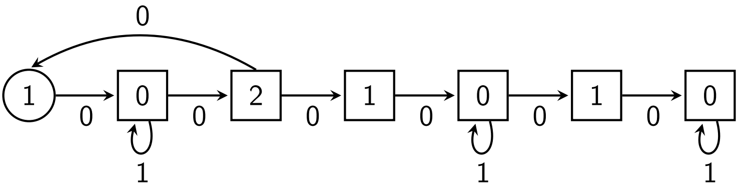

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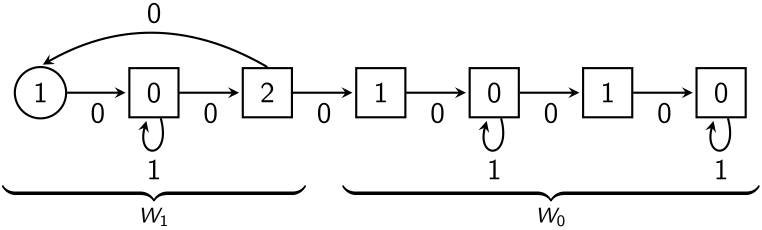

- Player 0 wins since only finitely many requests are seen
- Player 1 wins since he can stay longer and longer in loop

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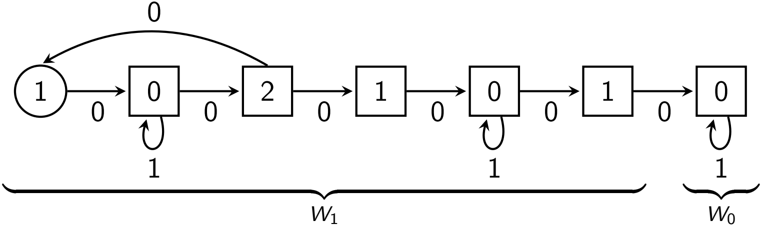

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Lemma

Let $\mathcal{C} = (G, \mathrm{CostParity}(\Omega))$ *and let* $\mathcal{B} = (G, \mathrm{BndCostParity}(\Omega))$.

1. $W_0(\mathcal{B}) \subseteq W_0(\mathcal{C})$.
2. *If* $W_0(\mathcal{B}) = \emptyset$, *then* $W_0(\mathcal{C}) = \emptyset$.

Corollary

"To solve cost-parity games, it suffices to solve bounded cost-parity games."

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- $\mathrm{FinCost}$: plays with finite cost
- $\mathrm{RR}(\Omega)$: plays in which every request is answered

$$\mathrm{PFRR}(\Omega) = \big(\mathrm{Parity}(\Omega) \cap \mathrm{FinCost}\big) \cup \mathrm{RR}(\Omega)$$

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Let $\mathcal{B} = (G, \mathrm{BndCostParity}(\Omega))$, and let $\mathcal{P} = (G, \mathrm{PFRR}(\Omega))$. Then, $W_i(\mathcal{B}) = W_i(\mathcal{P})$ for $i \in \{0, 1\}$.

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- $\mathrm{PFRR}(\Omega)$ is ω-regular
- $\mathcal{P}$ can be reduced to parity game using small memory
- Thus, small finite-state winning strategies for both players in $\mathcal{P}$

Computational Complexity

- n: number of vertices
- m: number of edges
- d: number of colors

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Given an algorithm that solves parity games in time $T(n, m, d)$, there is an algorithm that solves cost-parity games in time $O(n \cdot T(d \cdot n, d \cdot m, d + 2))$.

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The following problem is in $\mathbf{NP} \cap \mathbf{coNP}$*: given a cost-parity game $\mathcal{G}$ and a vertex v, has Player 0 a winning strategy from v?*

Half-positional Determinacy

Recall: Player 0 has finite state winning strategy σ in (bounded) cost-parity game

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Idea: use quality measure $\mathrm{Sh}\colon V^+ \to (D, \leq)$ for play prefixes with:

- $(D, \leq)$ is total order
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Positional winning strategy: always play like you are in the worst situation possible that is consistent with σ

Outline

Cost-Streett Games

- Requests: sets of vertices Q_i for $i = 1, \ldots, d$
- Responses: sets of vertices P_i for $i = 1, \ldots, d$
- Cost functions for every pair (Q_i, P_i)
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Overview of Results

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Tackle stronger winning conditions:

- Max-automata: deterministic automata, with multiple counters than can be incremented and reset, acceptance condition is boolean combination of boundedness requirements
- Equivalent to WMSO+U