Parametric LTL Games

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Motivation

We consider infinite games with winning conditions in linear temporal logic (LTL). Advantages of LTL as specification language are

- compact, variable-free syntax,
- intuitive semantics,
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- compact, variable-free syntax,
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- successfully employed in model checking tools.

However, LTL lacks capabilities to express timing constraints. There are many extensions of LTL that deal with this. Here, we consider two of them:

- PLTL: Parametric LTL (Alur et. al., '99)
- $\mathrm{PROMPT}-\mathrm{LTL}$ (Kupferman et. al., '07)

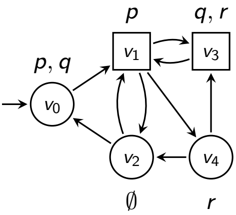
Outline

1. **Introduction**
2. PROMPT LTL
3. Parametric LTL
4. Conclusion

Infinite Games

An arena $\mathcal{A} = (V, V_0, V_1, E, v_0, l)$ consists of

- a finite, directed graph (V, E),
- a partition $\{V_0, V_1\}$ of V,
- an initial vertex v_0,
- a labeling $l \colon V \to 2^P$ for some set P of atomic propositions.



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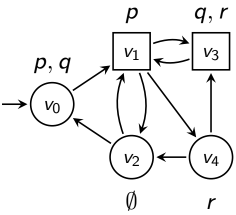


Winning conditions are expressed in extensions of LTL over P.

Theorem (Pnueli, Rosner '89)

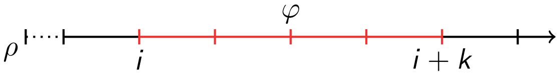
Determining the winner of an LTL game is **2EXPTIME**-*complete. Finite-state strategies suffice to win an LTL game.*

Outline

PROMPT-LTL Games

Add prompt-eventually $\mathbf{F_P}$ to LTL. Semantics defined w.r.t. free, but fixed bound k:

$(\rho, i, k) \models \mathbf{F_P}\varphi$:



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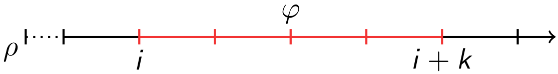

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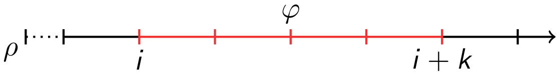

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Example

$\varphi = \mathbf{G}(q \to \mathbf{F_P}\, p)$. For some k, Player 0 has to answer every request q within k steps by seeing p.

Note: k may not depend on a single play.

PROMPT-LTL Games: Results

Theorem

Deciding whether Player 0 *has a winning strategy in a* $\mathrm{PROMPT-LTL}$ *game is* **2EXPTIME** *complete.*

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Proof

2EXPTIME algorithm: apply *alternating-color technique* of Kupferman et al.: reduce $\mathcal{G}$ to an LTL game $\mathcal{G}'$ such that a finite-state winning strategy for $\mathcal{G}'$ can be transformed into a finite-state winning strategy for $\mathcal{G}$ which bounds the waiting times. Player 0 wins $\mathcal{G}'$ only if she can ensure a bound on the prompt-eventualities in $\mathcal{G}$.

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2EXPTIME hardness follows from **2EXPTIME** hardness of solving LTL games.

Outline

Parametric LTL

Let $\mathcal{X}$ and $\mathcal{Y}$ be two disjoint sets of variables. PLTL adds bounded temporal operators to LTL:

- $\mathbf{F}_{\leq x}$ for $x \in \mathcal{X}$,
- $\mathbf{G}_{\leq y}$ for $y \in \mathcal{Y}$.

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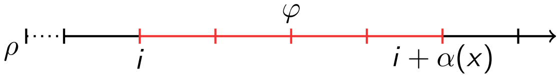
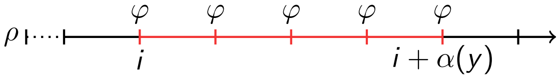
Semantics defined w.r.t. variable valuation $\alpha \colon \mathcal{X} \cup \mathcal{Y} \to \mathbb{N}$.

$(\rho, i, \alpha) \models \mathbf{F}_{\leq x}\varphi$:



$(\rho, i, \alpha) \models \mathbf{G}_{\leq y}\varphi$:



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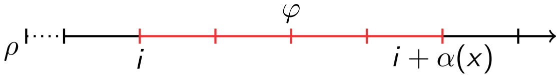

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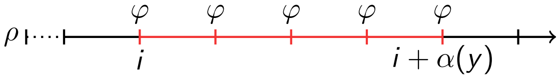

The operators $\mathbf{U}_{\leq x}$, $\mathbf{R}_{\leq y}$, $\mathbf{F}_{>y}$, $\mathbf{G}_{>x}$, $\mathbf{U}_{>y}$, and $\mathbf{R}_{>x}$ (with the obvious semantics) are syntactic sugar, and will be ignored.

Parametric LTL Games

PLTL game $(\mathcal{A}, \varphi)$:

- σ is a winning strategy for Player 0 w.r.t. α iff for all plays ρ consistent with σ: $(\rho, 0, \alpha) \models \varphi$.
- τ is a winning strategy for Player 1 w.r.t. α iff for all plays ρ consistent with τ: $(\rho, 0, \alpha) \not\models \varphi$.

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The set of winning valuations for Player i is

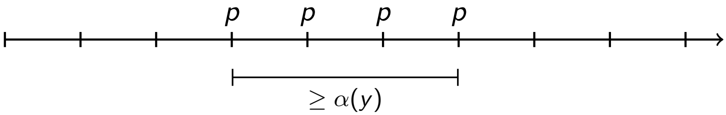
$$\mathcal{W}_{\mathcal{G}}^{i} = \{\alpha \mid \text{Player } i \text{ has winning strategy for } \mathcal{G} \text{ w.r.t. } \alpha\} \ .$$

We are interested in the emptiness, finiteness, and universality problem for $\mathcal{W}_{\mathcal{G}}^{i}$ and in finding optimal valuations in $\mathcal{W}_{\mathcal{G}}^{i}$.

PLTL Games: Examples

Winning condition $\mathbf{FG}_{\leq y} p$:

- Player 0's goal: eventually satisfy p for at least $\alpha(y)$ steps.



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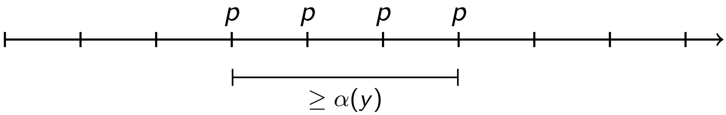
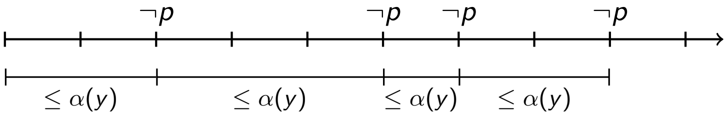


- Player 1's goal: reach vertex with $\neg p$ at least every $\alpha(y)$ steps.



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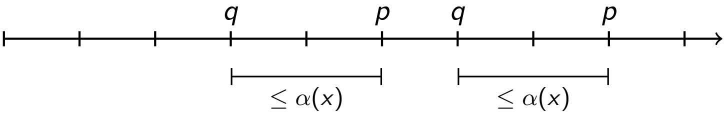

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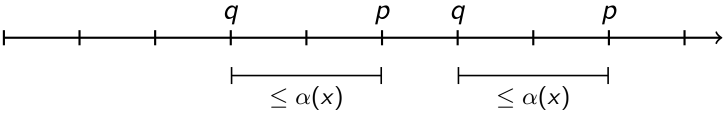
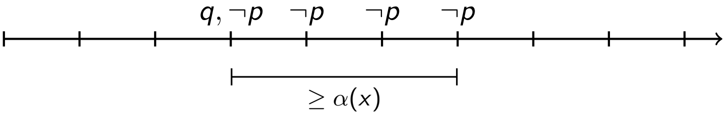


- Player 1's goal: enforce waiting time greater than $\alpha(x)$.



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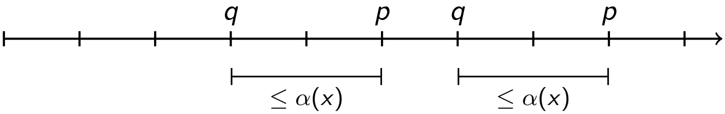

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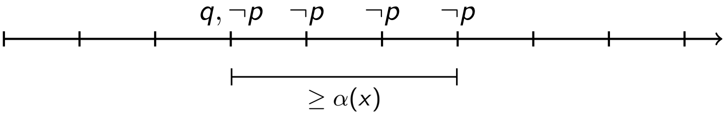

Note: both winning conditions induce an optimization problem (for Player 0): maximize $\alpha(y)$ respectively minimize $\alpha(x)$.

PLTL: Results

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For the proof, use:

- Duality of $\mathbf{F}_{\leq x}$ and $\mathbf{G}_{\leq y}$, i.e., $\neg\mathbf{G}_{\leq z}\neg\varphi \equiv \mathbf{F}_{\leq z}\varphi$.
- Monotonicity of $\mathbf{F}_{\leq x}$ and $\mathbf{G}_{\leq y}$, i.e., if $\alpha(z) \leq \beta(z)$, then $(\rho, i, \alpha) \models \mathbf{F}_{\leq z}\varphi$ implies $(\rho, i, \beta) \models \mathbf{F}_{\leq z}\varphi$ and $(\rho, i, \beta) \models \mathbf{G}_{\leq z}\varphi$ implies $(\rho, i, \alpha) \models \mathbf{G}_{\leq z}\varphi$.

PLTL: Proof Ideas

Proof

2EXPTIME algorithms: first consider formulae with only $\mathbf{F}_{\leq x}$:

- Emptiness: reduction to $\mathrm{PROMPT}-\mathrm{LTL}$ games.
- Universality: $\mathcal{W}_{\mathcal{G}}^{0}$ is universal iff it contains the valuation which maps every variable to 0.
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2EXPTIME hardness follows from **2EXPTIME** hardness of solving LTL games.

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Let $\varphi_{\mathbf{F}}$ *be* $\mathbf{G}_{\leq y}$*-free and* $\varphi_{\mathbf{G}}$ *be* $\mathbf{F}_{\leq x}$*-free, let* $\mathcal{G}_{\mathbf{F}} = (\mathcal{A}, \varphi_{\mathbf{F}})$ *and* $\mathcal{G}_{\mathbf{G}} = (\mathcal{A}, \varphi_{\mathbf{G}})$*. Then, the following values (and realizing strategies) are computable:*

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$$\min_{\alpha \in \mathcal{W}^0_{\mathcal{G}_{\mathbf{F}}}} \max_{x \in \mathrm{var}(\varphi_{\mathbf{F}})} \alpha(x) = \min_{\alpha \in \mathcal{W}^0_{\mathcal{G}'_{\mathbf{F}}}} \alpha(z) \ ,$$

by the monotonicity of $\mathbf{F}_{\leq x}$.

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$\varphi'_\mathbf{F}$ has a single variable, hence can be transformed into a $\mathrm{PROMPT-LTL}$ formula $\varphi_{\mathbf{F_P}}$ by replacing every $\mathbf{F}_{\leq z}$ by $\mathbf{F_P}$. Solving $(\mathcal{A}, \varphi_{\mathbf{F_P}})$ gives an (double-exponential) upper bound k on $\min_{\alpha \in \mathcal{W}^0_{\mathcal{G}'_\mathbf{F}}} \alpha(z)$. Using binary search in the interval $(0, k)$, the exact value can be found.

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For the other optimization problems, analogous techniques exist.

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Conclusion

We considered infinite games with winning conditions in extensions of LTL with bounded temporal operators.

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Further research:

- Better algorithms for the optimization problems.
- Hardness results for the optimization problems.
- Tradeoff between size and quality of a finite-state strategy.
- Time-optimal winning strategies for other winning conditions.