On Dynamic Bin Packing: An Improved Lower Bound and Resource Augmentation Analysis

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Abstract

We study the dynamic bin packing problem introduced by Coffman, Garey and Johnson. This problem is a generalization of the bin packing problem in which items may arrive and depart from the packing dynamically. The main result in this paper is a lower bound of 2.5 on the achievable competitive ratio, improving the best known 2.428 lower bound, and revealing that packing items of restricted form like unit fractions (i.e., of size $1/k$ for some integer k), for which a 2.4985-competitive algorithm is known, is indeed easier.

We also investigate the resource augmentation version of the problem where the on-line algorithm can use bins of size b (> 1) times that of the optimal off-line algorithm. An interesting result is that we prove $b = 2$ is both necessary and sufficient for the on-line algorithm to match the performance of the optimal off-line algorithm, i.e., achieve 1-competitiveness. Further analysis gives a trade-off between the bin size multiplier $1 < b \leq 2$ and the achievable competitive ratio.

1 Introduction

Bin packing is a classical combinatorial optimization problem (see the surveys [5, 8, 11]). The objective is to pack a set of items into a minimum number of unit-size bins such that the total size of the items in a bin does not exceed the bin capacity. The on-line version of the problem assumes that items may arrive at arbitrary time and no advance information is known about the items not yet arrived. *Dynamic bin packing* (DBP) was introduced as a generalization

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of the on-line bin packing by Coffman, Garey and Johnson [7]. In this generalization, items may also depart at arbitrary time and both on-line and off-line algorithms are not allowed to move items from one bin to another. The goal is to minimize the maximum number of bins used over all time.

The performance of an on-line algorithm $\mathcal{A}$ is generally measured by its *competitive ratio* [2]. For our problem where a sequence of item arrivals and departures is given, the competitive ratio c is the worst case ratio between the maximum number of bins used by $\mathcal{A}$ over all time and the maximum number of bins used by the optimal off-line algorithm (which knows the whole sequence in advance) over all time. Algorithm $\mathcal{A}$ is said to be *c-competitive*. The infimum over all values c such that $\mathcal{A}$ is c-competitive is called the *competitive ratio* of $\mathcal{A}$.

Coffman, Garey and Johnson [7] proved that the lower bound on the competitive ratio of any on-line algorithm on dynamic bin packing is 2.388 [1]. They also showed that a modified version of first-fit is 2.788-competitive [7]. Chan, Lam and Wong [3] improved the lower bound to 2.428 by considering only *unit fraction items*, where a unit fraction item is an item of size $1/k$ for some integer k. They also showed that for packing unit fraction items only, first-fit is 2.4985-competitive [3]. A natural question arises: Is packing items of restricted form, such as unit fraction items, as difficult as packing items of general form? Another aspect, *resource augmentation analysis* [16] has been studied in the context of on-line bin packing [12, 13], in which an on-line algorithm can use bins of size b (> 1) times that of the optimal off-line algorithm. To our knowledge, there is no previous work on resource augmentation analysis for dynamic bin packing. We address the above questions in this paper.

Our contributions. This paper presents the following results on DBP.

1. We push up the lower bound on competitive ratio from 2.428 [3] to 2.5 [2], giving a negative answer to the question that packing unit fraction items is as difficult as packing general items because packing unit fraction items attains a competitive ratio 2.4985 [3] (< 2.5).

2. We investigate the resource augmentation version, showing an interesting result that doubling the bin size for the on-line algorithm is both necessary and sufficient to match the performance of the optimal off-line algorithm, i.e., to attain 1-competitiveness. Further analysis is made to give a trade-off between the bin size multiplier b (for $1 < b \leq 2$) and the achievable competitive ratio.

Related work. There is a long history of results for the classical bin packing problem and its variants [5, 8, 11]. The off-line bin packing problem is NP-hard [14]. The best upper bound

[1] A variant of the problem is to assume a stronger off-line algorithm that can repack the current set of items into the minimum possible number of bins each time a new item arrives, in which case a stronger lower bound of 2.5 is achieved [7].

[2] There was a 2.5 lower bound [7] when the off-line algorithm can repack. Our result achieves the same bound even when the off-line algorithm does not repack.

and lower bound on the competitive ratio for on-line bin packing to date are 1.58889 [17] and 1.54014 [18], respectively. The upper bound reveals that dynamic bin packing is more difficult than on-line bin packing. For both dynamic and on-line bin packing, items of various restricted forms have been studied, which include unit fraction items [1, 3], items of divisible sizes [6] (where each possible item size can be divided by the next smaller item size), and items of discrete sizes [4, 9, 10] (where possible item sizes are $\{1/k, 2/k, \cdots, j/k\}$ for some $1 \leq j \leq k$).

Bar-Noy et al. [1] gave an off-line algorithm for unit fraction bin packing (with permanent items) that uses $H + 1$ bins, where $H = \sum_i 1/w_i$ and w_i is the size of the i-th item. They also gave an on-line algorithm that uses $H + O(\sqrt{H})$ bins and a lower bound of $H + \Omega(\ln H)$. Chan et al. [3] studied dynamic bin packing of unit fraction items giving an upper bound of 2.4985 and a lower bound of 2.428. For items of divisible sizes, Coffman et al. [6] showed that First Fit Decreasing produces optimal packing. Discrete sized items [4, 9, 10] are mainly studied in the context where the item sizes is chosen randomly from $\{1/k, 2/k, \cdots, j/k\}$. The expected performance of various algorithms like first fit was stuided.

Resource augmentation analysis for on-line bin packing has been studied [12, 13]; matching upper and lower bounds (up to an additive constant) are given for bounded space bin packing [12] in which there is a limit on the number of opened bins that can be used at any time; and a better upper bound has been derived for (unbounded space) on-line bin packing [13]. Ivkovic and Lloyd studied the *fully dynamic bin packing problem* [15], which is a variant of dynamic bin packing that allows repacking of items for each item arrival or departure. They gave a 1.25-competitive on-line algorithm for the problem [15].

Notations. We now give a precise definition of the problem and the necessary notations for further discussion. In dynamic bin packing, items arrive and depart at arbitrary time. Each item comes with a size. We denote by *s-item* an item of size s. When an item arrives, it must be assigned to a unit-sized bin immediately without exceeding the capacity of the assigned bin. At any time, the *load* of a bin is the total size of items currently assigned to that bin that have not yet departed. We denote by *ℓ-bin* a bin of load ℓ. Migration is not allowed, i.e., once an item is assigned to a bin, it cannot be moved to another bin. The objective is to minimize the maximum number of bins used over all time.

In the resource augmentation analysis (Sections 3 and 4), an on-line algorithm $\mathcal{A}$ is given size-b bins with $1 \leq b \leq 2$, while the optimal off-line algorithm uses size-1 bins. Consider any input sequence σ. Let $\mathcal{A}_b(\sigma, t)$ denote the number of size-b bins used at time t by $\mathcal{A}$, similarly, we have $\mathcal{O}_1(\sigma, t)$ for the optimal off-line algorithm. $\mathcal{A}$ is said to be c-competitive if there exists a constant k such that for any input sequence σ, $\max_t \mathcal{A}_b(\sigma, t) \leq c \cdot \max_t \mathcal{O}_1(\sigma, t) + k$. The infimum over all values c such that $\mathcal{A}$ is c-competitive is called the *competitive ratio* of $\mathcal{A}$.

Organization of the paper. In Section 2, we present the 2.5 lower bound. In Section 3, we show that doubling the bin size is both necessary and sufficient to achieve 1-competitiveness. In Section 4, we study the trade-off between bin size and competitive ratio. Finally, we give some concluding remarks in Section 5.

2 A 2.5 lower bound

In this section, we prove that no on-line algorithm can be better than 2.5-competitive. Consider any on-line algorithm $\mathcal{A}$. Let $\epsilon = \frac{1}{18k}$ for some large positive even integer k. The adversary works in stages using items of various sizes including ϵ, $\frac{1}{6}$, $\frac{1}{3}$, $\frac{1}{2}-\frac{\epsilon}{4}$, $\frac{1}{2}$, $\frac{1}{2}+\frac{\epsilon}{4}$, $\frac{2}{3}$ and 1. Roughly speaking, the adversary first releases some items of a particular size. Depending on how $\mathcal{A}$ packs the items, the adversary lets some items depart and further releases some other items such that the total size of items present at any time is always the same (with some minor difference). The choices of items to be departed ensures that the space released from some departed items cannot be reused for newly arrived items, thus, forcing $\mathcal{A}$ to use more new bins.

Recall that for any $s > 0$, $\ell > 0$, we denote by *s-item* an item of size s, and by *ℓ-bin* a bin of load ℓ. When we discuss how items are packed into bins, we use the following notations:

- Item configuration ψ: y_{*z} describes a load y consisting of $\frac{y}{z}$ items of size z, e.g., $\frac{1}{2}_{*\epsilon}$ means a load $\frac{1}{2}$ consisting of $\frac{1}{2\epsilon}$ items of size ϵ. We skip the subscript when $y = z$.
- Bin configuration π: $(\psi_1, \psi_2, \cdots)$, e.g., $(\frac{1}{3}, \frac{1}{2}_{*\epsilon})$ means a bin is packed with a $\frac{1}{3}$-item and an addition load $\frac{1}{2}$ with ϵ-items, i.e., the load of this bin is $\frac{5}{6}$. In some cases, it is clearer to state the bin configuration in other ways, e.g., $(\frac{1}{2}, \frac{1}{2})$, instead of $1_{*\frac{1}{2}}$. Similarly, we will use $6 \times \frac{1}{6}$ instead of $1_{*\frac{1}{6}}$.
- Packing configuration ρ: $\{x_1{:}\pi_1, x_2{:}\pi_2, \cdots\}$ refers to a packing where there are x_1 bins with bin configuration π_1, x_2 bins with π_2, and so on. E.g., $\{2k{:}1_{*\epsilon}, k{:}(\frac{1}{3}, \frac{1}{2}_{*\epsilon})\}$ means $2k$ bins are each packed with load 1 with ϵ-items and another k bins are each packed with a $\frac{1}{3}$-item and an addition load $\frac{1}{2}$ with ϵ-items.

The adversary releases items in stages such that $\mathcal{A}$ uses $45k$ bins at some point in time while the total size of items at any time is no more than $18k+2$. The item sizes are chosen carefully to allow the optimal off-line algorithm to use at most $18k+2$ bins. Then we have the following theorem.

Theorem 1. *For any online algorithm $\mathcal{A}$ for the dynamic bin packing problem, $\mathcal{A}$ is not c-competitive for any $c < 2.5$.*

The rest of this section is devoted to proving Theorem 1. We give the adversary and prove that $\mathcal{A}$ uses $45k$ bins at the end of the (adversarial) sequence while the optimal off-line algorithm $\mathcal{O}$ uses at most $18k+2$ bins. The sequence is divided into stages. In the first stage, some small items arrived. In the following stages, we first let some items depart based on how $\mathcal{A}$ packs the items in the previous stage, then more items arrived forcing $\mathcal{A}$ to use new bins. Let n_i be the number of new bins used by $\mathcal{A}$ in Stage i.

1. In Stage 1, $\frac{18k+2}{\epsilon}$ items of size ϵ are released, $\mathcal{A}$ needs $18k+2$ bins, thus, $n_1 \geq 18k+2 > 18k$. As for $\mathcal{O}$, all items are also packed into $18k+2$ bins, with those non-departing items packed together (details will be given later).

We distinguish between three cases: $n_1 \geq 24k$, $24k > n_1 \geq 21k$, and $21k > n_1 \geq 18k$.

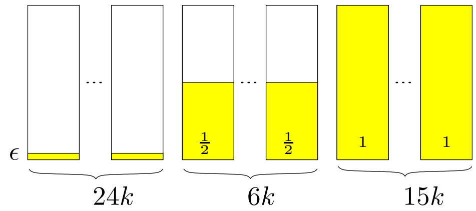


Figure 1: The final configuration of $\mathcal{A}$ achieved by the adversary in Case 1.

$\mathcal{O}$: # bins	**2**	$\mathbf{3k}$	$\mathbf{15k}$	**total**
Stage 1	-	-	-	0
	$\mathbf{1}_{*\epsilon}$	$\mathbf{1}_{*\epsilon}$	$\mathbf{1}_{*\epsilon}$	$18k{+}2$
Stage 2	$\frac{24}{36}{}_{*\epsilon}$	-	-	$24k\epsilon$
	$\frac{24}{36}{}_{*\epsilon}$	$\mathbf{\frac{1}{2}, \frac{1}{2}}$	$\mathbf{\frac{1}{2}, \frac{1}{2}}$	$18k{+}24k\epsilon$
Stage 3	$\frac{24}{36}{}_{*\epsilon}$	$\frac{1}{2}, \frac{1}{2}$	-	$3k{+}24k\epsilon$
(Final)	$\frac{24}{36}{}_{*\epsilon}$	$\frac{1}{2}, \frac{1}{2}$	**1**	$18k{+}24k\epsilon$

Table 1: The optimal schedule for Case 1. For each stage, the first row is the configuration just before items arrival and the second row is the configuration at the end of the stage. The very last row is the final configuration. Bolded entries are new items arrived in the corresponding stage. The notation y_{*z} means packing a bin with a load y of z-items. The last column shows the total load of all the bins.

Case 1: $n_1 \geq 24k$.

This is a simpler case.

2. In Stage 2, we keep one ϵ-item in $24k$ bins in $\mathcal{A}$ and let all other items depart such that the packing configuration of $\mathcal{A}$ is as follows. A total size of items departed is $18k + 12k\epsilon$. The packing configuration of $\mathcal{O}$ is chosen in a much better way.

# bins	$\mathbf{24k}$
$\mathcal{A}$	ϵ
total size: $24k\epsilon$	
# bins: $24k$	

# bins	**2**
$\mathcal{O}$	$\frac{24}{36}{}_{*\epsilon}$
total size: $24k\epsilon$	
# bins: 2	

Then $36k$ items of size $\frac{1}{2}$ are released. Note that the total size of items becomes $24k\epsilon + 18k < 18k + 2$. At most one new item can be packed into each existing bin of $\mathcal{A}$, thus, $n_2 \geq (36k - 24k)/2 = 6k$. On the other hand, the configuration of $\mathcal{O}$ is

# bins	**2**	$\mathbf{18k}$
$\mathcal{O}$	$\frac{24}{36}{}_{*\epsilon}$	$\frac{1}{2}, \frac{1}{2}$
total size: $18k + 24k\epsilon$		
# bins: $18k + 2$		

$\mathcal{A}$: # bins	$24k$	$6k$	$15k$	total
Stage 1	-	-	-	0
Stage 2	ϵ	-	-	$24k\epsilon$
Stage 3	ϵ	$\frac{1}{2}$	-	$3k{+}24k\epsilon$
Final	ϵ	$\frac{1}{2}$	1	$18k{+}24k\epsilon$

Table 2: The schedule of $\mathcal{A}$ for Case 1 right before items arrival in the corresponding stages. The last row shows the final configuration. The notation y_{*z} means packing a bin with a load y of z-items. The last column shows the total load of all the bins.

3. In Stage 3, we keep one ϵ-item in $24k$ bins and one $\frac{1}{2}$-item in $6k$ bins of $\mathcal{A}$, and let $30k$ items of size $\frac{1}{2}$ depart. The packing configuration of $\mathcal{A}$ and $\mathcal{O}$ becomes

# bins	$24k$	$6k$
$\mathcal{A}$	ϵ	$\frac{1}{2}$
total size: $3k + 24k\epsilon$		
# bins: $30k$		

# bins	2	$3k$
$\mathcal{O}$	$\frac{24}{36}{}_{*\epsilon}$	$\frac{1}{2}, \frac{1}{2}$
total size: $3k + 24k\epsilon$		
# bins: $3k + 2$		

We then release $15k$ items of size 1, none of which can be packed into an existing bin of $\mathcal{A}$, thus, $n_3 = 15k$. The total number of bins used by $\mathcal{A}$ equals $45k$. The packing configuration of $\mathcal{A}$ and $\mathcal{O}$ becomes:

# bins	$24k$	$6k$	$15k$
$\mathcal{A}$	ϵ	$\frac{1}{2}$	1
total size: $18k + 24k\epsilon$			
# bins: $45k$			

# bins	2	$3k$	$15k$
$\mathcal{O}$	$\frac{24}{36}{}_{*\epsilon}$	$\frac{1}{2}, \frac{1}{2}$	1
total size: $18k + 24k\epsilon$			
# bins: $18k + 2$			

Table 1 gives a summary of the packing of $\mathcal{O}$ while Table 2 shows the configuration of $\mathcal{A}$ right before items arrival in each stage with the final target also shown in Figure 1. Note that we do not show the configuration of $\mathcal{A}$ right after items arrival because it is the number of new bins that matters, not the exact configuration.

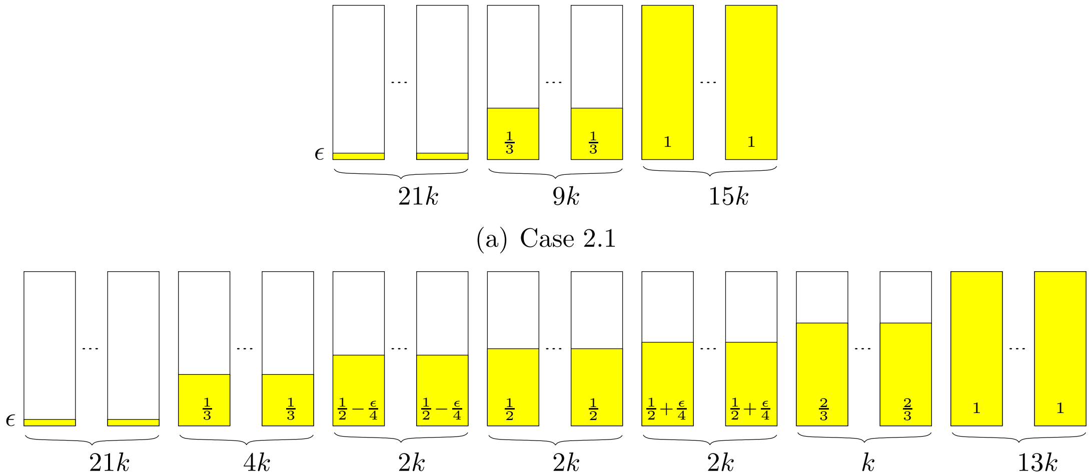
Case 2: $24k > n_1 \geq 21k$.

This case requires more stages. We make the following observations on the load of bins used by $\mathcal{A}$.

Observation 2. *If $24k > n_1 \geq 21k$, then $\mathcal{A}$ uses at least $6k$ bins of load at least $\frac{2}{3} + \epsilon$, $12k$ bins at least $\frac{1}{2} + \epsilon$, and $15k$ bins at least $\frac{1}{3} + \epsilon$ at the end of Stage 1.*

Proof. Assume that there are less than $6k$ bins of load at least $\frac{2}{3}+\epsilon$. The remaining bins has a maximum load of $\frac{2}{3}$. Then the maximum load that has packed is $< 6k+(24k-6k)(\frac{2}{3}) = 18k$, contradicting that a total load of $18k+2$ has been released. The other cases are similar. □

Stage 2 is divided into two phases.



(a) Case 2.1

(b) Case 2.2

Figure 2: The final configuration of $\mathcal{A}$ achieved by the adversary in Case 2.

- In Phase 1 of Stage 2, we let a total size of $10k + 15k\epsilon$ of ϵ-items depart until the packing configuration of $\mathcal{A}$ is shown below. Note that a total size of $8k + 21k\epsilon$ of ϵ-items remained.

# bins	$\boldsymbol{6k}$	$\boldsymbol{6k}$	$\boldsymbol{3k}$	$\boldsymbol{6k}$
$\mathcal{A}$	$(\frac{2}{3}+\epsilon)_{*\epsilon}$	$(\frac{1}{2}+\epsilon)_{*\epsilon}$	$(\frac{1}{3}+\epsilon)_{*\epsilon}$	ϵ
total size: $8k + 21k\epsilon$				
# bins: $21k$				

  We then release $30k$ items of size $\frac{1}{3}$; the total size of items becomes $18k + 21k\epsilon$.

- In Phase 2, we further let some ϵ-items depart depending how $\mathcal{A}$ has packed the $\frac{1}{3}$-items. If $\mathcal{A}$ packs a $\frac{1}{3}$-item into some bin B of load $\frac{1}{2}+\epsilon$, we depart a total size of $\frac{1}{6}$ of ϵ-item from B, making its load become $\frac{2}{3}+\epsilon$. For every two such bins, we further release one $\frac{1}{3}$-item. Repeat departing groups of ϵ-items of size $\frac{1}{6}$ and releasing items of size $\frac{1}{3}$ as long as $\mathcal{A}$ packs a $\frac{1}{3}$-item into a bin of load $\frac{1}{2}+\epsilon$. This process must terminate because once $\mathcal{A}$ packs a $\frac{1}{3}$-item into a bin, its load becomes $\frac{2}{3}+\epsilon$, meaning that it cannot accommodate another $\frac{1}{3}$-item.

  Let x and y be the number of $(\frac{1}{2}+\epsilon)$- and $(\frac{1}{3}+\epsilon)$-bins, respectively, that have packed a $\frac{1}{3}$-item at the end of Stage 2. Suppose $x = 2p + q$, for some integers p, q and $q = 0$ or 1. In Phase 2, the total size of ϵ-items departed is $\frac{x}{6} = \frac{2p+q}{6}$, and the total number of $\frac{1}{3}$-items arrived is p.

At the end of Stage 2, the total size of items is $18k + 21k\epsilon - \frac{q}{6}$, of it, $8k + 21k\epsilon - \frac{2p+q}{6}$ is ϵ-items and the number of $\frac{1}{3}$-items is $30k + p$.

We now consider the new bins used by $\mathcal{A}$ at the end of Stage 2. Let a_1 and a_2 be the number of new bins that have packed exactly one $\frac{1}{3}$-item and at least two $\frac{1}{3}$-items, respectively, so, $n_2 = a_1 + a_2$.

Observation 3. *The number of new bins used by $\mathcal{A}$ in Stage 2, $n_2 \geq 4k$.*

$\mathcal{O}$: # bins	**2**	$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{8k-p-q}$	$\boldsymbol{3k}$	$\boldsymbol{7k}$	**total**
Stage 1	-	-	-	-	-	-	0
	$\mathbf{1}_{*\epsilon}$	$\mathbf{1}_{*\epsilon}$	$\mathbf{1}_{*\epsilon}$	$\mathbf{1}_{*\epsilon}$	$\mathbf{1}_{*\epsilon}$	$\mathbf{1}_{*\epsilon}$	$18k+2$
Stage 2*	$\frac{21}{36}{}_{*\epsilon}$	$1_{*\epsilon}$	$1_{*\epsilon}$	$1_{*\epsilon}$	-	-	$8k+21k\epsilon$
	$\frac{21}{36}{}_{*\epsilon}$	$\frac{2}{3}{}_{*\epsilon}$, $\mathbf{\frac{1}{3}}$	$\frac{5}{6}{}_{*\epsilon}$	$1_{*\epsilon}$	$\mathbf{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}}$	$\mathbf{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}}$	$18k+21k\epsilon-\frac{q}{6}$
Stage 3	$\frac{21}{36}{}_{*\epsilon}$	-	-	-	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$	-	$3k+21k\epsilon$
(Final)	$\frac{21}{36}{}_{*\epsilon}$	**1**	**1**	**1**	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$	**1**	$18k+21k\epsilon$

Table 3: The optimal schedule for Case 2.1. For each stage, the first row is the configuration just before items arrival and the second row is the configuration at the end of the stage. The very last row is the final configuration. Bolded entries are new items arrived in the corresponding stage. The notation y_{*z} means packing a bin with a load y of z-items. * Note that in Stage 2, there are two phases: the first row shows the configuration right before the items arrival in Phase 1 and the second row the end of both phases (in Phase 2, ϵ-items of a size of $\frac{2p+q}{6}$ departed).

$\mathcal{A}$: # bins	$\boldsymbol{6k}$	$\boldsymbol{6k}$	$\boldsymbol{3k}$	$\boldsymbol{6k}$	$\boldsymbol{9k}$	$\boldsymbol{15k}$	**total**
Stage 1	-	-	-	-	-	-	0
Stage 2	$(\frac{2}{3}+\epsilon)_{*\epsilon}$	$(\frac{1}{2}+\epsilon)_{*\epsilon}$	$(\frac{1}{3}+\epsilon)_{*\epsilon}$	ϵ	-	-	$8k+21k\epsilon$
Stage 3	ϵ	ϵ	ϵ	ϵ	$\frac{1}{3}$	-	$3k+21k\epsilon$
Final	ϵ	ϵ	ϵ	ϵ	$\frac{1}{3}$	1	$18k+21k\epsilon$

Table 4: The schedule of $\mathcal{A}$ for Case 2.1 right before items arrival in the corresponding stages. The last row shows the final configuration. The notation y_{*z} means packing a bin with a load y of z-items. The last column shows the total load of all the bins.

Proof. The maximum possible load of all bins is $\leq (6k + x)(\frac{2}{3} + \epsilon) + (6k - x)(\frac{1}{2} + \epsilon) + y(\frac{2}{3} + \epsilon) + (3k - y)(\frac{1}{3} + \epsilon) + 6k(\frac{2}{3} + \epsilon) + a_2 + \frac{a_1}{3}$. This quantity must be $\geq 18k + 21k\epsilon - \frac{q}{6}$ because this is the total load of items at the end of Stage 2. Simplifying the inequality gives $\frac{y}{3} + \frac{x}{6} + a_2 + \frac{a_1}{3} \geq 6k - \frac{q}{6}$. Using the fact that $x \leq 6k - q$ and $y \leq 3k$, we can derive that $a_1 + a_2 \geq \frac{a_1}{3} + a_2 \geq 4k$. $\square$

We further consider two sub-cases: $a_1 + a_2 \geq 9k$ and $9k > a_1 + a_2 \geq 4k$.

Case 2.1: $\boldsymbol{a_1 + a_2 \geq 9k}$.

Before we move on to how the adversary continues, we first show how $\mathcal{O}$ packs items in Stage 2. Recall that in Stage 2, a total size of $10k + 15k\epsilon + \frac{2p+q}{6}$ of ϵ-items departed. A total number of $30k+p$ of $\frac{1}{3}$-items arrived in this stage, which will be packed by $\mathcal{O}$ as follows.

# bins	2	p	q	$8k-p-q$	$10k$
$\mathcal{O}$	$\frac{21}{36}_{*\epsilon}$	$\frac{2}{3}_{*\epsilon}, \frac{1}{3}$	$\frac{5}{6}_{*\epsilon}$	$1_{*\epsilon}$	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$

total size: $18k + 21k\epsilon - \frac{q}{6}$
 # bins: $18k + 2$

We then proceed to Stage 3. Table 3 gives a summary of the packing of $\mathcal{O}$ while Table 4 shows the configuration of $\mathcal{A}$ right before items arrival in each stage with the final target also shown in Figure 2(a).

- In Stage 3, we keep $21k$ items of size ϵ and $9k$ items of size $\frac{1}{3}$ and let all other items depart such that the configuration of $\mathcal{A}$ is as follows. The corresponding configuration of $\mathcal{O}$ is also shown below.

# bins	$21k$	$9k$
$\mathcal{A}$	ϵ	$\frac{1}{3}$

total size: $3k + 21k\epsilon$
 # bins: $30k$

# bins	2	$3k$
$\mathcal{O}$	$\frac{21}{36}_{*\epsilon}$	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$

total size: $3k + 21k\epsilon$
 # bins: $3k + 2$

- Finally, we release $15k$ items of size 1. All these items require a new bin, thus, $n_3 = 15k$. The total number of bins used by $\mathcal{A}$ becomes $21k + 9k + 15k = 45k$. The configuration of $\mathcal{A}$ and $\mathcal{O}$ is as follows.

# bins	$21k$	$9k$	$15k$
$\mathcal{A}$	ϵ	$\frac{1}{3}$	1

total size: $18k + 21k\epsilon$
 # bins: $45k$

# bins	2	$3k$	$15k$
$\mathcal{O}$	$\frac{21}{36}_{*\epsilon}$	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$	1

total size: $18k + 21k\epsilon$
 # bins: $18k + 2$

Case 2.2: $9k > a_1 + a_2 \geq 4k$.

Before we move on to how the adversary continues, we first show how $\mathcal{O}$ packs items in Stage 2. Recall that in Stage 2, a total size of $10k + 15k\epsilon + \frac{2p+q}{6}$ of ϵ-items departed. A total number of $30k+p$ of $\frac{1}{3}$-items arrived in this stage, which will be packed by $\mathcal{O}$ as follows.

# bins	2	k	$k+p$	q	$11k-p-q$	$5k$
$\mathcal{O}$	$\frac{33}{36}_{*\epsilon}$	$\frac{1}{3}_{*\epsilon}, \frac{1}{3}, \frac{1}{3}$	$(\frac{1}{3} - \epsilon)_{*\epsilon}, \frac{1}{3}, \frac{1}{3}$	$(\frac{1}{2} - \epsilon)_{*\epsilon}, \frac{1}{3}$	$(\frac{2}{3} - \epsilon)_{*\epsilon}, \frac{1}{3}$	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$

total size: $18k + 21k\epsilon - \frac{q}{6}$
 # bins: $18k + 2$

We then proceed with the adversary. Table 5 gives a summary of the packing of $\mathcal{O}$ while Table 6 shows the configuration of $\mathcal{A}$ right before items arrival in each stage with the final target also shown in Figure 2(b). Recall that a_1 and a_2 denote the number of new bins used by $\mathcal{A}$ in Stage 2 that have packed exactly one $\frac{1}{3}$-item and at least two $\frac{1}{3}$-items, respectively. We observe a further bound on a_2.

Observation 4. *For Case 2.2, we have $a_2 > k$.*

Proof. In the proof of Observation 3, we have shown that $\frac{a_1}{3} + a_2 \geq 4k$, i.e., $\frac{2a_2}{3} \geq 4k - \frac{a_1+a_2}{3}$. Since $a_1 + a_2 < 9k$, we have $a_2 \geq \frac{2a_2}{3} > 4k - 3k = k$. □

$\mathcal{O}$: # bins	**2**	$\boldsymbol{k}$	$\boldsymbol{k+p}$	$\boldsymbol{q}$	$\boldsymbol{p}$	$\boldsymbol{11k-2p-q}$	$\boldsymbol{k}$	$\boldsymbol{k}$	$\boldsymbol{2k}$	$\boldsymbol{k}$	**total**
Stage 1	-	-	-	-	-	-	-	-	-	-	0
	$\mathbf{1}_{*\epsilon}$	$\mathbf{1}_{*\epsilon}$	$\mathbf{1}_{*\epsilon}$	$\mathbf{1}_{*\epsilon}$	$\mathbf{1}_{*\epsilon}$	$\mathbf{1}_{*\epsilon}$	$\mathbf{1}_{*\epsilon}$	$\mathbf{1}_{*\epsilon}$	$\mathbf{1}_{*\epsilon}$	$\mathbf{1}_{*\epsilon}$	$18k+2$
Stage 2*	$\frac{33}{36}_{*\epsilon}$	$\frac{1}{3}_{*\epsilon}$	k bins: $(\frac{1}{3}-\epsilon)_{*\epsilon}$ p bins: $(\frac{2}{3}-\epsilon)_{*\epsilon}$	$(\frac{2}{3}-\epsilon)_{*\epsilon}$	$(\frac{2}{3}-\epsilon)_{*\epsilon}$	$(\frac{2}{3}-\epsilon)_{*\epsilon}$	- -	- -	- -	- -	$8k+21k\epsilon$
	$\frac{33}{36}_{*\epsilon}$	$\frac{1}{3}_{*\epsilon}$, $\mathbf{\frac{1}{3}}, \mathbf{\frac{1}{3}}$	$(\frac{1}{3}-\epsilon)_{*\epsilon}$, $\mathbf{\frac{1}{3}}, \mathbf{\frac{1}{3}}$	$(\frac{1}{2}-\epsilon)_{*\epsilon}$, $\mathbf{\frac{1}{3}}$	$(\frac{2}{3}-\epsilon)_{*\epsilon}$, $\mathbf{\frac{1}{3}}$	$(\frac{2}{3}-\epsilon)_{*\epsilon}$, $\mathbf{\frac{1}{3}}$	$\mathbf{\frac{1}{3}}, \mathbf{\frac{1}{3}}, \mathbf{\frac{1}{3}}$	$\mathbf{\frac{1}{3}}, \mathbf{\frac{1}{3}}, \mathbf{\frac{1}{3}}$	$\mathbf{\frac{1}{3}}, \mathbf{\frac{1}{3}}, \mathbf{\frac{1}{3}}$	$\mathbf{\frac{1}{3}}, \mathbf{\frac{1}{3}}, \mathbf{\frac{1}{3}}$	$18k+21k\epsilon-\frac{q}{6}$
Stage 3	$\frac{33}{36}_{*\epsilon}$	$\frac{1}{3}_{*\epsilon}$	$(\frac{1}{6}-\epsilon)_{*\epsilon}, \frac{1}{3}$	$(\frac{1}{6}-\epsilon)_{*\epsilon}, \frac{1}{3}$	$(\frac{1}{6}-\epsilon)_{*\epsilon}, \frac{1}{3}$	$(\frac{1}{2}-\epsilon)_{*\epsilon}$	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$	$\frac{1}{3}$	-	-	$7k+\frac{2k}{3}+21k\epsilon$
	$\frac{33}{36}_{*\epsilon}$	$\frac{1}{3}_{*\epsilon}$, $\mathbf{\frac{1}{2}-\frac{\epsilon}{4}}$	$(\frac{1}{6}-\epsilon)_{*\epsilon}, \frac{1}{3}$, $\mathbf{\frac{1}{2}-\frac{\epsilon}{4}}$	$(\frac{1}{6}-\epsilon)_{*\epsilon}, \frac{1}{3}$, $\mathbf{\frac{1}{2}-\frac{\epsilon}{4}}$	$(\frac{1}{6}-\epsilon)_{*\epsilon}, \frac{1}{3}$, $\mathbf{\frac{1}{2}-\frac{\epsilon}{4}}$	$(\frac{1}{2}-\epsilon)_{*\epsilon}$, $\mathbf{\frac{1}{2}-\frac{\epsilon}{4}}$	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$	$\frac{1}{3}$, $\mathbf{\frac{1}{2}-\frac{\epsilon}{4}}$	$\mathbf{\frac{1}{2}-\frac{\epsilon}{4}}$, $\mathbf{\frac{1}{2}-\frac{\epsilon}{4}}$	$\mathbf{\frac{1}{2}-\frac{\epsilon}{4}}$, $\mathbf{\frac{1}{2}-\frac{\epsilon}{4}}$	$17k+\frac{2k}{3}$ $+16k\epsilon$
Stage 4	$\frac{33}{36}_{*\epsilon}$	$\frac{1}{3}_{*\epsilon}$	$(\frac{1}{6}-\epsilon)_{*\epsilon}, \frac{1}{3}$	$(\frac{1}{6}-\epsilon)_{*\epsilon}, \frac{1}{3}$	$(\frac{1}{6}-\epsilon)_{*\epsilon}, \frac{1}{3}$	$(\frac{1}{2}-\epsilon)_{*\epsilon}$	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{2}-\frac{\epsilon}{4}$	-	$8k+\frac{2}{3}k+20.5k\epsilon$
	$\frac{33}{36}_{*\epsilon}$	$\frac{1}{3}_{*\epsilon}$, $\mathbf{\frac{1}{2}}$	$(\frac{1}{6}-\epsilon)_{*\epsilon}, \frac{1}{3}$, $\mathbf{\frac{1}{2}}$	$(\frac{1}{6}-\epsilon)_{*\epsilon}, \frac{1}{3}$, $\mathbf{\frac{1}{2}}$	$(\frac{1}{6}-\epsilon)_{*\epsilon}, \frac{1}{3}$, $\mathbf{\frac{1}{2}}$	$(\frac{1}{2}-\epsilon)_{*\epsilon}$, $\mathbf{\frac{1}{2}}$	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$	$\frac{1}{3}$, $\mathbf{\frac{1}{2}}$	$\frac{1}{2}-\frac{\epsilon}{4}$, $\mathbf{\frac{1}{2}}$	$\mathbf{\frac{1}{2}}, \mathbf{\frac{1}{2}}$	$17k+\frac{2k}{3}$ $+20.5k\epsilon$
Stage 5	$\frac{33}{36}_{*\epsilon}$	$\frac{1}{3}_{*\epsilon}$	$(\frac{1}{6}-\epsilon)_{*\epsilon}, \frac{1}{3}$	$(\frac{1}{6}-\epsilon)_{*\epsilon}, \frac{1}{3}$	$(\frac{1}{6}-\epsilon)_{*\epsilon}, \frac{1}{3}$	$(\frac{1}{2}-\epsilon)_{*\epsilon}$,	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{2}-\frac{\epsilon}{4}$	$\frac{1}{2}, \frac{1}{2}$	$9k+\frac{2k}{3}+20.5k\epsilon$
	$\frac{33}{36}_{*\epsilon}$	$\frac{1}{3}_{*\epsilon}$, $\mathbf{\frac{1}{2}+\frac{\epsilon}{4}}$	$(\frac{1}{6}-\epsilon)_{*\epsilon}, \frac{1}{3}$, $\mathbf{\frac{1}{2}+\frac{\epsilon}{4}}$	$(\frac{1}{6}-\epsilon)_{*\epsilon}, \frac{1}{3}$, $\mathbf{\frac{1}{2}+\frac{\epsilon}{4}}$	$(\frac{1}{6}-\epsilon)_{*\epsilon}, \frac{1}{3}$, $\mathbf{\frac{1}{2}+\frac{\epsilon}{4}}$	$(\frac{1}{2}-\epsilon)_{*\epsilon}$, $\mathbf{\frac{1}{2}+\frac{\epsilon}{4}}$	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$	$\frac{1}{3}$, $\mathbf{\frac{1}{2}+\frac{\epsilon}{4}}$	$\frac{1}{2}-\frac{\epsilon}{4}$, $\mathbf{\frac{1}{2}+\frac{\epsilon}{4}}$	$\frac{1}{2}, \frac{1}{2}$	$17k+\frac{2k}{3}$ $+24.5k\epsilon$
Stage 6	$\frac{21}{36}_{*\epsilon}$	$\frac{1}{3}_{*\epsilon}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}_{*\epsilon}$	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{2}-\frac{\epsilon}{4}$, $\frac{1}{2}+\frac{\epsilon}{4}$	$\frac{1}{2}, \frac{1}{2}$	$8k+\frac{2k}{3}$ $+21k\epsilon$
	$\frac{21}{36}_{*\epsilon}$	$\frac{1}{3}_{*\epsilon}$	$\frac{1}{3}$, $\mathbf{\frac{2}{3}}$	$\frac{1}{3}$, $\mathbf{\frac{2}{3}}$	$\frac{1}{3}$, $\mathbf{\frac{2}{3}}$	$\frac{1}{3}_{*\epsilon}$, $\mathbf{\frac{2}{3}}$	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$	$\frac{1}{3}$, $\mathbf{\frac{2}{3}}$	$\frac{1}{2}-\frac{\epsilon}{4}$, $\frac{1}{2}+\frac{\epsilon}{4}$	$\frac{1}{2}, \frac{1}{2}$	$17k+\frac{k}{3}$ $+21k\epsilon$
Stage 7	$\frac{21}{36}_{*\epsilon}$	-	-	-	-	-	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$	$\frac{1}{3}, \frac{2}{3}$	$\frac{1}{2}-\frac{\epsilon}{4}$, $\frac{1}{2}+\frac{\epsilon}{4}$	$\frac{1}{2}, \frac{1}{2}$	$5k$ $+21k\epsilon$
(Final)	$\frac{21}{36}_{*\epsilon}$	**1**	**1**	**1**	**1**	**1**	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$	$\frac{1}{3}, \frac{2}{3}$	$\frac{1}{2}-\frac{\epsilon}{4}$, $\frac{1}{2}+\frac{\epsilon}{4}$	$\frac{1}{2}, \frac{1}{2}$	$18k$ $+21k\epsilon$

Table 5: The optimal schedule for Case 2.2. For each stage, the first row is the configuration just before items arrival and the second row is the configuration at the end of the stage. The very last row is the final configuration. Bolded entries are new items arrived in the corresponding stage. The notation y_{*z} means packing a bin with a load y of z-items. * Note that in Stage 2, there are two phases: the first row shows the configuration right before the items arrival in Phase 1 and the second row the end of both phases (in Phase 2, ϵ-items of a size of $\frac{2p+q}{6}$ departed).

The remaining stages run as follows. To make it easier to follow, recall that after the departure of a size of $10k + 15k\epsilon$ of ϵ-items in Stage 2, the configuration of $\mathcal{A}$ was

# bins	$\boldsymbol{6k}$	$\boldsymbol{6k}$	$\boldsymbol{3k}$	$\boldsymbol{6k}$
$\mathcal{A}$	$(\frac{2}{3}+\epsilon)_{*\epsilon}$	$(\frac{1}{2}+\epsilon)_{*\epsilon}$	$(\frac{1}{3}+\epsilon)_{*\epsilon}$	ϵ
total size: $8k + 21k\epsilon$				
# bins: $21k$				

After this, $30k + p$ items of size $\frac{1}{3}$ arrived and a further size of $\frac{2p+q}{6}$ of ϵ-items departed. $\mathcal{A}$ uses at least $4k$ new bins in this stage. At the end of Stage 2, the number of bins used by $\mathcal{A}$ is at least $25k$.

3. In Stage 3,

   - We first let a total size of $2k + \frac{2p+q}{6}$ of ϵ-items and a total number of $25k - p - q$ of $\frac{1}{3}$- items depart until the configuration of $\mathcal{A}$ is as follows. Note that we can have

$\mathcal{A}$: # bins	$6k$	$2p+q$	$6k-2p-q$	$3k$	$6k$	k	$3k$	$2k$	$2k$	$2k$	k	$13k$	total
Stage 1	-	-	-	-	-	-	-	-	-	-	-	-	0
Stage 2	$(\frac{2}{3}+\epsilon)_{*\epsilon}$	$(\frac{1}{2}+\epsilon)_{*\epsilon}$	$(\frac{1}{2}+\epsilon)_{*\epsilon}$	$(\frac{1}{3}+\epsilon)_{*\epsilon}$	ϵ	-	-	-	-	-	-	-	$8k+21k\epsilon$
Stage 3	$(\frac{1}{2}+\epsilon)_{*\epsilon}$	$\frac{1}{3}, (\frac{1}{6}+\epsilon)_{*\epsilon}$	$(\frac{1}{2}+\epsilon)_{*\epsilon}$	ϵ	ϵ	$\frac{1}{3}, \frac{1}{3}$	$\frac{1}{3}$	-	-	-	-	-	$7k+\frac{2k}{3}+21k\epsilon$
Stage 4	$(\frac{1}{2}+\epsilon)_{*\epsilon}$	$\frac{1}{3}, (\frac{1}{6}+\epsilon)_{*\epsilon}$	$(\frac{1}{2}+\epsilon)_{*\epsilon}$	ϵ	ϵ	$\frac{1}{3}, \frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{2}-\frac{\epsilon}{4}$	-	-	-	-	$8k+\frac{2k}{3}+20.5k\epsilon$
Stage 5	$(\frac{1}{2}+\epsilon)_{*\epsilon}$	$\frac{1}{3}, (\frac{1}{6}+\epsilon)_{*\epsilon}$	$(\frac{1}{2}+\epsilon)_{*\epsilon}$	ϵ	ϵ	$\frac{1}{3}, \frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{2}-\frac{\epsilon}{4}$	$\frac{1}{2}$	-	-	-	$9k+\frac{2k}{3}+20.5k\epsilon$
Stage 6	$(\frac{1}{3}+\epsilon)_{*\epsilon}$	$\frac{1}{3}, \epsilon$	$(\frac{1}{3}+\epsilon)_{*\epsilon}$	ϵ	ϵ	$\frac{1}{3}, \frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{2}-\frac{\epsilon}{4}$	$\frac{1}{2}$	$\frac{1}{2}+\frac{\epsilon}{4}$	-	-	$8k+\frac{2k}{3}+21k\epsilon$
Stage 7	ϵ	ϵ	ϵ	ϵ	ϵ	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{2}-\frac{\epsilon}{4}$	$\frac{1}{2}$	$\frac{1}{2}+\frac{\epsilon}{4}$	$\frac{2}{3}$	-	$5k+21k\epsilon$
Final	ϵ	ϵ	ϵ	ϵ	ϵ	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{2}-\frac{\epsilon}{4}$	$\frac{1}{2}$	$\frac{1}{2}+\frac{\epsilon}{4}$	$\frac{2}{3}$	1	$18k+21k\epsilon$

Table 6: The schedule of $\mathcal{A}$ for Case 2.2 right before items arrival in the corresponding stages. The last row shows the final configuration. The notation y_{*z} means packing a bin with a load y of z-items. The last column shows the total load of all the bins.

such configuration for $\mathcal{A}$ because $a_2 > k$ (by Observation 4) and $a_1 + a_2 \geq 4k$ (by Observation 3). We also show the corresponding configuration of $\mathcal{O}$.

# bins	$6k$	$2p+q$	$6k-2p-q$
$\mathcal{A}$	$(\frac{1}{2}+\epsilon)_{*\epsilon}$	$\frac{1}{3}, (\frac{1}{6}+\epsilon)_{*\epsilon}$	$(\frac{1}{2}+\epsilon)_{*\epsilon}$
	$9k$	k	$3k$
	ϵ	$\frac{1}{3}, \frac{1}{3}$	$\frac{1}{3}$
total size: $7k + \frac{2k}{3} + 21k\epsilon$			
# bins: $25k$			

# bins	2	k	$k+2p+q$
$\mathcal{O}$	$\frac{33}{36}_{*\epsilon}$	$\frac{1}{3}_{*\epsilon}$	$(\frac{1}{6}-\epsilon)_{*\epsilon}, \frac{1}{3}$
	$11k-2p-q$	k	k
	$(\frac{1}{2}-\epsilon)_{*\epsilon}$	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$	$\frac{1}{3}$
total size: $7k + \frac{2k}{3} + 21k\epsilon$			
# bins: $15k + 2$			

- We then release $20k$ items of size $\frac{1}{2}-\frac{\epsilon}{4}$. For $\mathcal{A}$, only bins of load ϵ and $\frac{1}{3}$ can accommodate one such item, therefore, $n_3 \geq (20k - 12k)/2 = 4k > 2k$. Note that although $n_3 = 4k$, the adversary only relies on the fact that $n_3 > 2k$. The number of bins used by $\mathcal{A}$ is at least $27k$. The corresponding packing configuration of $\mathcal{O}$ is

# bins	2	k	$k+2p+q$	$11k-2p-q$
$\mathcal{O}$	$\frac{33}{36}_{*\epsilon}$	$\frac{1}{3}_{*\epsilon}, \frac{1}{2}-\frac{\epsilon}{4}$	$(\frac{1}{6}-\epsilon)_{*\epsilon}, \frac{1}{3}, \frac{1}{2}-\frac{\epsilon}{4}$	$(\frac{1}{2}-\epsilon)_{*\epsilon}, \frac{1}{2}-\frac{\epsilon}{4}$
	k	k	$3k$	
	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$	$\frac{1}{3}, \frac{1}{2}-\frac{\epsilon}{4}$	$\frac{1}{2}-\frac{\epsilon}{4}, \frac{1}{2}-\frac{\epsilon}{4}$	
total size: $17k + \frac{2k}{3} + 21k\epsilon - 5k\epsilon = 17k + \frac{2k}{3} + 16k\epsilon$				
# bins: $18k + 2$				

4. In Stage 4,

   - We let $18k$ items of size $(\frac{1}{2}-\frac{\epsilon}{4})$ depart until the configuration of $\mathcal{A}$ is as follows. The corresponding configuration of $\mathcal{O}$ is also shown.

# bins	$\boldsymbol{6k}$	$\boldsymbol{2p+q}$	$\boldsymbol{6k-2p-q}$
$\mathcal{A}$	$(\frac{1}{2}+\epsilon)_{*\epsilon}$	$\frac{1}{3}, (\frac{1}{6}+\epsilon)_{*\epsilon}$	$(\frac{1}{2}+\epsilon)_{*\epsilon}$
	$\boldsymbol{9k}$	$\boldsymbol{k}$	$\boldsymbol{3k}$
	ϵ	$\frac{1}{3}, \frac{1}{3}$	$\frac{1}{3}$
	$\boldsymbol{2k}$		
	$\frac{1}{2}-\frac{\epsilon}{4}$		
total size: $8k+\frac{2k}{3}+21k\epsilon-\frac{k\epsilon}{2}$			
# bins: $27k$			

# bins	$\boldsymbol{2}$	$\boldsymbol{k}$	$\boldsymbol{k+2p+q}$
$\mathcal{O}$	$\frac{33}{36}_{*\epsilon}$	$\frac{1}{3}_{*\epsilon}$	$(\frac{1}{6}-\epsilon)_{*\epsilon}, \frac{1}{3}$
	$\boldsymbol{11k-2p-q}$	$\boldsymbol{k}$	$\boldsymbol{k}$
	$(\frac{1}{2}-\epsilon)_{*\epsilon}$	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$	$\frac{1}{3}$
	$\boldsymbol{2k}$		
	$\frac{1}{2}-\frac{\epsilon}{4}$		
total size: $8k+\frac{2k}{3}+21k\epsilon-\frac{k\epsilon}{2}$			
# bins: $17k+2$			

- Next we release $18k$ items of size $\frac{1}{2}$. For $\mathcal{A}$, only bins of load ϵ, $\frac{1}{3}$ and $\frac{1}{2}-\frac{\epsilon}{4}$ can accommodate one such item, therefore, $n_4 \geq (18k-14k)/2 = 2k$. The number of bins used by $\mathcal{A}$ is at least $29k$. The corresponding packing configuration of $\mathcal{O}$ is

# bins	$\boldsymbol{2}$	$\boldsymbol{k}$	$\boldsymbol{k+2p+q}$	$\boldsymbol{11k-2p-q}$
$\mathcal{O}$	$\frac{33}{36}_{*\epsilon}$	$\frac{1}{3}_{*\epsilon}, \frac{1}{2}$	$(\frac{1}{6}-\epsilon)_{*\epsilon}, \frac{1}{3}, \frac{1}{2}$	$(\frac{1}{2}-\epsilon)_{*\epsilon}, \frac{1}{2}$
	$\boldsymbol{k}$	$\boldsymbol{k}$	$\boldsymbol{2k}$	$\boldsymbol{k}$
	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$	$\frac{1}{3}, \frac{1}{2}$	$\frac{1}{2}-\frac{\epsilon}{4}, \frac{1}{2}$	$\frac{1}{2}, \frac{1}{2}$
total size: $17k+\frac{2k}{3}+21k\epsilon-\frac{k\epsilon}{2}$				
# bins: $18k+2$				

5. In Stage 5,

   - We let $16k$ items of size $\frac{1}{2}$ depart until the packing configuration of $\mathcal{A}$ is as follows.

# bins	$\boldsymbol{6k}$	$\boldsymbol{2p+q}$	$\boldsymbol{6k-2p-q}$
$\mathcal{A}$	$(\frac{1}{2}+\epsilon)_{*\epsilon}$	$\frac{1}{3}, (\frac{1}{6}+\epsilon)_{*\epsilon}$	$(\frac{1}{2}+\epsilon)_{*\epsilon}$
	$\boldsymbol{9k}$	$\boldsymbol{k}$	$\boldsymbol{3k}$
	ϵ	$\frac{1}{3}, \frac{1}{3}$	$\frac{1}{3}$
	$\boldsymbol{2k}$	$\boldsymbol{2k}$	
	$\frac{1}{2}-\frac{\epsilon}{4}$	$\frac{1}{2}$	
total size: $9k+\frac{2k}{3}+21k\epsilon-\frac{k\epsilon}{2}$			
# bins: $29k$			

# bins	$\boldsymbol{2}$	$\boldsymbol{k}$	$\boldsymbol{k+2p+q}$
$\mathcal{O}$	$\frac{33}{36}_{*\epsilon}$	$\frac{1}{3}_{*\epsilon}$	$(\frac{1}{6}-\epsilon)_{*\epsilon}, \frac{1}{3}$
	$\boldsymbol{11k-2p-q}$	$\boldsymbol{k}$	$\boldsymbol{k}$
	$(\frac{1}{2}-\epsilon)_{*\epsilon}$	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$	$\frac{1}{3}$
	$\boldsymbol{2k}$	$\boldsymbol{k}$	
	$\frac{1}{2}-\frac{\epsilon}{4}$	$\frac{1}{2}, \frac{1}{2}$	
total size: $9k+\frac{2k}{3}+21k\epsilon-\frac{k\epsilon}{2}$			
# bins: $18k+2$			

   - We then release $16k$ items of size $\frac{1}{2}+\frac{\epsilon}{4}$. For $\mathcal{A}$, only bins of load ϵ, $\frac{1}{3}$ and $\frac{1}{2}-\frac{\epsilon}{4}$ can accommodate one such item, therefore, $n_5 \geq 16k-14k = 2k$. The number of bins used by $\mathcal{A}$ is at least $31k$. The corresponding configuration of $\mathcal{O}$ is

# bins	$\boldsymbol{2}$	$\boldsymbol{k}$	$\boldsymbol{k+2p+q}$	$\boldsymbol{11k-2p-q}$
$\mathcal{O}$	$\frac{33}{36}_{*\epsilon}$	$\frac{1}{3}_{*\epsilon}, \frac{1}{2}+\frac{\epsilon}{4}$	$(\frac{1}{6}-\epsilon)_{*\epsilon}, \frac{1}{3}, \frac{1}{2}+\frac{\epsilon}{4}$	$(\frac{1}{2}-\epsilon)_{*\epsilon}, \frac{1}{2}+\frac{\epsilon}{4}$
	$\boldsymbol{k}$	$\boldsymbol{k}$	$\boldsymbol{2k}$	$\boldsymbol{k}$
	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$	$\frac{1}{3}, \frac{1}{2}+\frac{\epsilon}{4}$	$\frac{1}{2}-\frac{\epsilon}{4}, \frac{1}{2}+\frac{\epsilon}{4}$	$\frac{1}{2}, \frac{1}{2}$
total size: $17k+\frac{2k}{3}+21k\epsilon+\frac{7k\epsilon}{2}$				
# bins: $18k+2$				

6. In Stage 6,

- We let a total size of $2k$ of ϵ-items and a total number of $14k$ of $(\frac{1}{2}+\frac{\epsilon}{4})$-item depart until the configuration of $\mathcal{A}$ is as follows.

# bins	$\boldsymbol{6k}$	$\boldsymbol{2p+q}$	$\boldsymbol{6k-2p-q}$
$\mathcal{A}$	$(\frac{1}{3}+\epsilon)_{*\epsilon}$	$\frac{1}{3}, \epsilon$	$(\frac{1}{3}+\epsilon)_{*\epsilon}$
	$\boldsymbol{9k}$	$\boldsymbol{k}$	$\boldsymbol{3k}$
	ϵ	$\frac{1}{3}, \frac{1}{3}$	$\frac{1}{3}$
	$\boldsymbol{2k}$	$\boldsymbol{2k}$	$\boldsymbol{2k}$
	$\frac{1}{2}-\frac{\epsilon}{4}$	$\frac{1}{2}$	$\frac{1}{2}+\frac{\epsilon}{4}$
total size: $8k+\frac{2k}{3}+21k\epsilon$			
# bins: $31k$			

# bins	$\boldsymbol{2}$	$\boldsymbol{k}$	$\boldsymbol{k+2p+q}$
$\mathcal{O}$	$\frac{21}{36}{}_{*\epsilon}$	$\frac{1}{3}{}_{*\epsilon}$	$\frac{1}{3}$
	$\boldsymbol{11k-2p-q}$	$\boldsymbol{k}$	$\boldsymbol{k}$
	$\frac{1}{3}{}_{*\epsilon}$	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$	$\frac{1}{3}$
	$\boldsymbol{2k}$	$\boldsymbol{k}$	
	$\frac{1}{2}-\frac{\epsilon}{4}, \frac{1}{2}+\frac{\epsilon}{4}$	$\frac{1}{2}, \frac{1}{2}$	
total size: $8k+\frac{2k}{3}+21k\epsilon$			
# bins: $18k+2$			

- We then release $13k$ items of size $\frac{2}{3}$. For $\mathcal{A}$, only bins of load ϵ and $\frac{1}{3}$ can accommodate one such item, therefore, $n_6 \geq 13k-12k = k$. The number of bins used by $\mathcal{A}$ is at least $32k$. The corresponding configuration of $\mathcal{O}$ is

# bins	$\boldsymbol{2}$	$\boldsymbol{k}$	$\boldsymbol{k+2p+q}$	$\boldsymbol{11k-2p-q}$
$\mathcal{O}$	$\frac{21}{36}{}_{*\epsilon}$	$\frac{1}{3}{}_{*\epsilon}$	$\frac{1}{3}, \frac{2}{3}$	$\frac{1}{3}{}_{*\epsilon}, \frac{2}{3}$
	$\boldsymbol{k}$	$\boldsymbol{k}$	$\boldsymbol{2k}$	$\boldsymbol{k}$
	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$	$\frac{1}{3}, \frac{2}{3}$	$\frac{1}{2}-\frac{\epsilon}{4}, \frac{1}{2}+\frac{\epsilon}{4}$	$\frac{1}{2}, \frac{1}{2}$
total size: $17k+\frac{k}{3}+21k\epsilon$				
# bins: $18k+2$				

7. In Stage 7,

- We let a total size of $4k-\frac{2p+q}{3}$ of ϵ-items, a total number of $k+2p+q$ of $\frac{1}{3}$-items, and a number of $12k$ of $\frac{2}{3}$-items depart until the configuration of $\mathcal{A}$ is as follows.

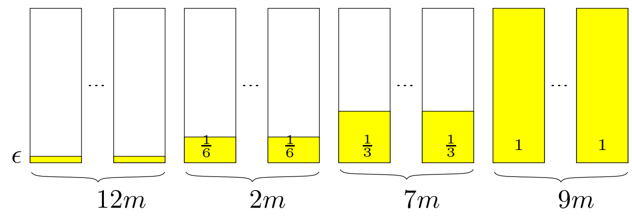
# bins	$\boldsymbol{21k}$	$\boldsymbol{4k}$	$\boldsymbol{2k}$
$\mathcal{A}$	ϵ	$\frac{1}{3}$	$\frac{1}{2}-\frac{\epsilon}{4}$
	$\boldsymbol{2k}$	$\boldsymbol{2k}$	$\boldsymbol{k}$
	$\frac{1}{2}$	$\frac{1}{2}+\frac{\epsilon}{4}$	$\frac{2}{3}$
total size: $5k+21k\epsilon$			
# bins: $32k$			

# bins	$\boldsymbol{2}$	$\boldsymbol{k}$	$\boldsymbol{k}$
$\mathcal{O}$	$\frac{21}{36}{}_{*\epsilon}$	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$	$\frac{1}{3}, \frac{2}{3}$
	$\boldsymbol{2k}$	$\boldsymbol{k}$	
	$\frac{1}{2}-\frac{\epsilon}{4}, \frac{1}{2}+\frac{\epsilon}{4}$	$\frac{1}{2}, \frac{1}{2}$	
total size: $5k+21k\epsilon$			
# bins: $5k+2$			

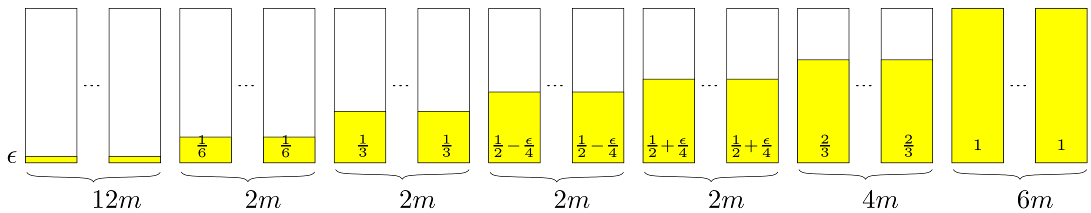
- A final of $13k$ items of size 1 are released, making $n_7 = 13k$. Totally, $\mathcal{A}$ uses $21k+4k+2k+2k+2k+k+13k = 45k$ bins.

# bins	$\boldsymbol{21k}$	$\boldsymbol{4k}$	$\boldsymbol{2k}$	$\boldsymbol{2k}$
$\mathcal{A}$	ϵ	$\frac{1}{3}$	$\frac{1}{2}-\frac{\epsilon}{4}$	$\frac{1}{2}$
	$\boldsymbol{2k}$	$\boldsymbol{k}$	$\boldsymbol{13k}$	
	$\frac{1}{2}+\frac{\epsilon}{4}$	$\frac{2}{3}$	1	
total size: $18k+21k\epsilon$				
# bins: $45k$				

# bins	$\boldsymbol{2}$	$\boldsymbol{13k}$	$\boldsymbol{k}$
$\mathcal{O}$	$\frac{21}{36}{}_{*\epsilon}$	1	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$
	$\boldsymbol{k}$	$\boldsymbol{2k}$	$\boldsymbol{k}$
	$\frac{1}{3}, \frac{2}{3}$	$\frac{1}{2}-\frac{\epsilon}{4}, \frac{1}{2}+\frac{\epsilon}{4}$	$\frac{1}{2}, \frac{1}{2}$
total size: $18k+21k\epsilon$			
# bins: $18k+2$			



(a) Case 3.1



(b) Case 3.2

Figure 3: The final configuration of $\mathcal{A}$ achieved by the adversary in Case 3.

In both Case 2.1 and Case 2.2, we show that there is an adversarial sequence such that the on-line algorithm $\mathcal{A}$ uses $45k$ bins at the end while the optimal off-line algorithm uses no more than $18k + 2$ bins at any time.

Case 3: $21k > n_1 \geq 18k$.

For the sake of simplicity, we let $m = 3k/2$. In other words, $14m > n_1 \geq 12m$. The following observation can be proved by contradiction, similar to Observation 2.

Observation 5. *If $14m > n_1 \geq 12m$, then $\mathcal{A}$ uses at least $8m$ bins with load at least $\frac{2}{3} + \epsilon$, $10m$ bins with at least $\frac{1}{2} + \epsilon$, and $11m$ bins with at least $\frac{1}{3} + \epsilon$ at the end of Stage 1.*

2. In Stage 2, we let a total size of $\frac{16m}{3} + 12m\epsilon$ of ϵ-items depart until the configuration of $\mathcal{A}$ becomes

# bins	$8m$	$2m$	m	m
$\mathcal{A}$	$(\frac{2}{3} + \epsilon)_{*\epsilon}$	$(\frac{1}{2} + \epsilon)_{*\epsilon}$	$(\frac{1}{3} + \epsilon)_{*\epsilon}$	ϵ
total size: $6m + \frac{2m}{3} + 12m\epsilon$				
# bins: $12m$				

We then release $32m$ items of size $\frac{1}{6}$. The total size of items becomes $12m + 12m\epsilon$. A $(\frac{2}{3}+\epsilon)$-bin can accommodate one item, $(\frac{1}{2}+\epsilon)$-bin two, $(\frac{1}{3}+\epsilon)$-bin three, and ϵ-bin five. Number of new bins required by $\mathcal{A}$ is at least $(32m - 8m - 4m - 3m - 5m)/6 = 2m$.

3. Stage 3 is divided into two phases.

   - In Phase 1 of Stage 3, we let $30m$ items of size $\frac{1}{6}$ depart until the configuration of $\mathcal{A}$ becomes

# bins	$8m$	$2m$	m	m	$2m$
$\mathcal{A}$	$(\frac{2}{3}+\epsilon)_{*\epsilon}$	$(\frac{1}{2}+\epsilon)_{*\epsilon}$	$(\frac{1}{3}+\epsilon)_{*\epsilon}$	ϵ	$\frac{1}{6}$

total size: $7m + 12m\epsilon$
 # bins: $14m$

We then release $15m$ items of size $\frac{1}{3}$.

- Phase 2 is similar to Stage 2 of Case 2. If $\mathcal{A}$ packs a $\frac{1}{3}$-item into a bin of load $\frac{1}{2}+\epsilon$, we depart a total size of $\frac{1}{6}$ of ϵ-items from that bin. For every two such bins, we further release one $\frac{1}{3}$-item. Repeat departing ϵ-items of size $\frac{1}{6}$ and releasing items of size $\frac{1}{3}$ as long as $\mathcal{A}$ packs $\frac{1}{3}$-item into bin of load $\frac{1}{2}+\epsilon$.

  Let x and y be the number of bins with load $\frac{1}{2}+\epsilon$ and $\frac{1}{6}$, respectively, that has packed at least one $\frac{1}{3}$-item at the end of Stage 3. Suppose $x = 2p + q$, for some integers p and q and $q = 0$ or 1. In Phase 2, a total size of $\frac{2p+q}{6}$ of ϵ-items departed while p items of size $\frac{1}{3}$ arrived.

At the end of Stage 3, the total size of items is $12m + 12m\epsilon - \frac{q}{6}$, among them, $6m + \frac{2m}{3} + 12m\epsilon - \frac{2p+q}{6}$ are ϵ-items, the number of $\frac{1}{6}$-items is $2m$, and the number of $\frac{1}{3}$-items is $15m + p$.

We now consider the new bins used by $\mathcal{A}$ at the end of Stage 3. Let a_1 and a_2 be the number of new bins that have packed exactly one $\frac{1}{3}$-item and at least two $\frac{1}{3}$-items, respectively, i.e., $n_3 = a_1 + a_2$. We observe the following property about $a_1 + a_2$.

Observation 6. *The number of new bins used by $\mathcal{A}$ in Stage 3, $n_3 = a_1 + a_2 \geq 2m$.*

Proof. Considering the load of the bins and the total size of all items, we have $(8m+x)(\frac{2}{3}+\epsilon)+(2m-x)(\frac{1}{2}+\epsilon)+m(\frac{2}{3}+\epsilon)+m(\frac{2}{3}+\epsilon)+y(\frac{2}{3}+\frac{1}{6})+(2m-y)(\frac{1}{6})+a_2+\frac{a_1}{3} \geq 12m+12m\epsilon-\frac{q}{6}$. Simplifying gives $\frac{x}{6} + \frac{2y}{3} + a_2 + \frac{a_1}{3} \geq 4m$. Using the properties $x \leq 2m - q$, and $y \leq 2m$, we can derive that $a_1 + a_2 \geq 2m$. □

We further consider two subcases: $a_1 + a_2 \geq 7m$ and $7m > a_1 + a_2 \geq 2m$.

Case 3.1: $a_1 + a_2 \geq 7m$

Before we move on to how the adversary continues, we first show how $\mathcal{O}$ packs items in Stages 2 and 3.

2. In Stage 2, a total size of $\frac{16m}{3} + 12m\epsilon$ of ϵ-items departed. After the arrival of $32m$ items of size $\frac{1}{6}$, the configuration of $\mathcal{O}$ becomes

# bins	1	m	$2m$	$4m$	$5m$
$\mathcal{O}$	$1_{*\epsilon}$	$\frac{2}{3}_{*\epsilon}, \frac{1}{6}, \frac{1}{6}$	$1_{*\epsilon}$	$1_{*\epsilon}$	$6 \times \frac{1}{6}$

total size: $12m + 12m\epsilon$
 # bins: $12m + 1$

3. In Stage 3, $30m$ items of size $\frac{1}{6}$ and a total size of $\frac{2p+q}{6}$ of ϵ-items departed. The number of $\frac{1}{3}$-items arrived is $15m + p$. The configuration of $\mathcal{O}$ becomes

$\mathcal{O}$: # bins	**1**	**1**	$\boldsymbol{m}$	$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{2m-p-q}$	$\boldsymbol{4m}$	$\boldsymbol{m}$	$\boldsymbol{m}$	$\boldsymbol{m}$	$\boldsymbol{2m}$	**total**
Stage 1	-	-	-	-	-	-	-	-	-	-	-	0
	$\mathbf{1}_{*\epsilon}$	$\mathbf{1}_{*\epsilon}$	$\mathbf{1}_{*\epsilon}$	$\mathbf{1}_{*\epsilon}$	$\mathbf{1}_{*\epsilon}$	$\mathbf{1}_{*\epsilon}$	$\mathbf{1}_{*\epsilon}$	$\mathbf{1}_{*\epsilon}$	$\mathbf{1}_{*\epsilon}$	$\mathbf{1}_{*\epsilon}$	$\mathbf{1}_{*\epsilon}$	$12m+2$
Stage 2	$1_{*\epsilon}$	-	$\frac{2}{3}_{*\epsilon}$	$1_{*\epsilon}$	$1_{*\epsilon}$	$1_{*\epsilon}$	$1_{*\epsilon}$	-	-	-	-	$6m+\frac{2m}{3}+12m\epsilon$
	$1_{*\epsilon}$	-	$\frac{2}{3}_{*\epsilon}, \mathbf{\frac{1}{6}}, \mathbf{\frac{1}{6}}$	$1_{*\epsilon}$	$1_{*\epsilon}$	$1_{*\epsilon}$	$1_{*\epsilon}$	$\mathbf{6 \times \frac{1}{6}}$	$\mathbf{6 \times \frac{1}{6}}$	$\mathbf{6 \times \frac{1}{6}}$	$\mathbf{6 \times \frac{1}{6}}$	$12m+12m\epsilon$
Stage 3*	$1_{*\epsilon}$	-	$\frac{2}{3}_{*\epsilon}$	$1_{*\epsilon}$	$1_{*\epsilon}$	$1_{*\epsilon}$	$1_{*\epsilon}$	$\frac{1}{6}, \frac{1}{6}$	-	-	-	$7m+12m\epsilon$
	$1_{*\epsilon}$	-	$\frac{2}{3}_{*\epsilon}, \mathbf{\frac{1}{3}}$	$\frac{2}{3}_{*\epsilon}, \mathbf{\frac{1}{3}}$	$\frac{5}{6}_{*\epsilon}$	$1_{*\epsilon}$	$1_{*\epsilon}$	$\frac{1}{6}, \frac{1}{6}, \mathbf{\frac{1}{3}}, \mathbf{\frac{1}{3}}$	$\mathbf{\frac{1}{3}}, \mathbf{\frac{1}{3}}, \mathbf{\frac{1}{3}}$	$\mathbf{\frac{1}{3}}, \mathbf{\frac{1}{3}}, \mathbf{\frac{1}{3}}$	$\mathbf{\frac{1}{3}}, \mathbf{\frac{1}{3}}, \mathbf{\frac{1}{3}}$	$12m+12m\epsilon-\frac{q}{6}$
Stage 4	$1_{*\epsilon}$	-	-	-	-	-	-	$\frac{1}{6}, \frac{1}{6}, \frac{1}{3}, \frac{1}{3}$	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$	$\frac{1}{3}, \frac{1}{3}$	-	$2m+\frac{2m}{3}+12m\epsilon$
(Final)	$1_{*\epsilon}$	-	**1**	**1**	**1**	**1**	**1**	$\frac{1}{6}, \frac{1}{6}, \frac{1}{3}, \frac{1}{3}$	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$	$\frac{1}{3}, \frac{1}{3}$	**1**	$11m+\frac{2m}{3}+12m\epsilon$

Table 7: The optimal schedule for Case 3.1. For each stage, the first row is the configuration just before items arrival and the second row is the configuration at the end of the stage. The very last row is the final configuration. Bolded entries are new items arrived in the corresponding stage. The notation y_{*z} means packing a bin with a load y of z-items. * Note that in Stage 3, there are two phases: the first row shows the configuration right before the items arrival in Phase 1 and the second row the end of both phases (in Phase 2, ϵ-items of a size of $\frac{2p+q}{6}$ departed).

# bins	**1**	$\boldsymbol{m+p}$	$\boldsymbol{q}$	$\boldsymbol{2m-p-q}$	$\boldsymbol{4m}$	$\boldsymbol{m}$	$\boldsymbol{4m}$
$\mathcal{O}$	$1_{*\epsilon}$	$\frac{2}{3}_{*\epsilon}, \frac{1}{3}$	$\frac{5}{6}_{*\epsilon}$	$1_{*\epsilon}$	$1_{*\epsilon}$	$\frac{1}{6}, \frac{1}{6}, \frac{1}{3}, \frac{1}{3}$	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$
total size: $12m+12m\epsilon-\frac{q}{6}$							
# bins: $12m+1$							

We then proceed to Stage 4. Table 7 gives a summary of the packing of $\mathcal{O}$ while Table 8 shows the configuration of $\mathcal{A}$ right before items arrival in each stage with the final target also shown in Figure 3(a).

- In Stage 4, we let $8m+p$ items of size $\frac{1}{3}$ and a total size of $6m+\frac{2m}{3}-\frac{2p+q}{6}$ of ϵ-items depart until the configuration of $\mathcal{A}$ is as follows. The number of $\frac{1}{3}$-items remained is $7m$, $\frac{1}{6}$-items is $2m$, and ϵ-items is $12m$. Total size departed equals $(9+\frac{1}{3})m-\frac{q}{6}>9m$. This configuration is possible because $a_1+a_2 \geq 7m$.

# bins	$\boldsymbol{12m}$	$\boldsymbol{2m}$	$\boldsymbol{7m}$
$\mathcal{A}$	ϵ	$\frac{1}{6}$	$\frac{1}{3}$
total size: $2m+\frac{2m}{3}+12m\epsilon$			
# bins: $21m$			

# bins	**1**	$\boldsymbol{m}$	$\boldsymbol{m}$	$\boldsymbol{m}$
$\mathcal{O}$	$1_{*\epsilon}$	$\frac{1}{6}, \frac{1}{6}, \frac{1}{3}, \frac{1}{3}$	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$	$\frac{1}{3}, \frac{1}{3}$
total size: $2m+\frac{2m}{3}+12m\epsilon$				
# bins: $3m+1$				

- Then release $9m$ items of size 1, requiring $9m$ new bins. Therefore, the total number of bins used by $\mathcal{A}$ equals $12m+2m+7m+9m=30m=45k$.

# bins	$\boldsymbol{12m}$	$\boldsymbol{2m}$	$\boldsymbol{7m}$	$\boldsymbol{9m}$
$\mathcal{A}$	ϵ	$\frac{1}{6}$	$\frac{1}{3}$	1
total size: $11m+\frac{2m}{3}+12m\epsilon$				
# bins: $30m$				

# bins	**1**	$\boldsymbol{m}$	$\boldsymbol{m}$	$\boldsymbol{m}$	$\boldsymbol{9m}$
$\mathcal{O}$	$1_{*\epsilon}$	$\frac{1}{6}, \frac{1}{6}, \frac{1}{3}, \frac{1}{3}$	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$	$\frac{1}{3}, \frac{1}{3}$	1
total size: $11m+\frac{2m}{3}+12m\epsilon$					
# bins: $12m+1$					

$\mathcal{A}$: # bins	$8m$	$2m$	m	m	$2m$	$7m$	$9m$	total
Stage 1	-	-	-	-	-	-	-	0
Stage 2	$(\frac{2}{3}+\epsilon)_{*\epsilon}$	$(\frac{1}{2}+\epsilon)_{*\epsilon}$	$(\frac{1}{3}+\epsilon)_{*\epsilon}$	ϵ	-	-	-	$6m+\frac{2m}{3}+12m\epsilon$
Stage 3	$(\frac{2}{3}+\epsilon)_{*\epsilon}$	$(\frac{1}{2}+\epsilon)_{*\epsilon}$	$(\frac{1}{3}+\epsilon)_{*\epsilon}$	ϵ	$\frac{1}{6}$	-	-	$7m+12m\epsilon$
Stage 4	ϵ	ϵ	ϵ	ϵ	$\frac{1}{6}$	$\frac{1}{3}$	-	$2m+\frac{2m}{3}+12m\epsilon$
Final	ϵ	ϵ	ϵ	ϵ	$\frac{1}{6}$	$\frac{1}{3}$	1	$11m+\frac{2m}{3}+12m\epsilon$

Table 8: The schedule of $\mathcal{A}$ for Case 3.1 right before items arrival in the corresponding stages. The last row shows the final configuration. The notation y_{*z} means packing a bin with a load y of z-items. The last column shows the total load of all the bins.

Case 3.2: $7m > a_1 + a_2 \geq 2m$

Before we move on to how the adversary continues, we first show how $\mathcal{O}$ packs items in Stages 2 and 3.

2. In Stage 2, a total size of $\frac{16m}{3} + 12m\epsilon$ of ϵ-items departed with a total size of $6m + \frac{2m}{3} + 12m\epsilon$ of ϵ-items remained from Stage 1. After the arrival of $32m$ items of size $\frac{1}{6}$-items, the configuration of $\mathcal{O}$ becomes

# bins	2	m	$4m$	m	m	$3m$	$2m$
$\mathcal{O}$	$\frac{18}{24}{}_{*\epsilon}$	$\frac{2}{3}{}_{*\epsilon}, \frac{1}{6}, \frac{1}{6}$	$(\frac{1}{3}-\epsilon)_{*\epsilon}, 4 \times \frac{1}{6}$	$(\frac{2}{3}-\epsilon)_{*\epsilon}, \frac{1}{6}, \frac{1}{6}$	$(1-\epsilon)_{*\epsilon}$	$1_{*\epsilon}$	$6 \times \frac{1}{6}$
total size: $12m + 12m\epsilon$							
# bins: $12m + 2$							

3. In Stage 3, $30m$ items of size $\frac{1}{6}$ and a total size of $\frac{2p+q}{6}$ of ϵ-items departed. The number of $\frac{1}{3}$-items arrived is $15m + p$. The configuration of $\mathcal{O}$ becomes

# bins	2	m	$4m$	$m+p$	q	$m-p-q$	$3m$	$2m$
$\mathcal{O}$	$\frac{18}{24}{}_{*\epsilon}$	$\frac{2}{3}{}_{*\epsilon}, \frac{1}{6}, \frac{1}{6}$	$(\frac{1}{3}-\epsilon)_{*\epsilon}, \frac{1}{3}, \frac{1}{3}$	$(\frac{2}{3}-\epsilon)_{*\epsilon}, \frac{1}{3}$	$(\frac{5}{6}-\epsilon)_{*\epsilon}$	$(1-\epsilon)_{*\epsilon}$	$1_{*\epsilon}$	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$
total size: $12m + 12m\epsilon - \frac{q}{6}$								
# bins: $12m + 2$								

We then proceed with the adversary. Table 9 gives a summary of the packing of $\mathcal{O}$ while Table 10 shows the configuration of $\mathcal{A}$ right before items arrival in each stage with the final target also shown in Figure 3(b). We have one more observation.

Observation 7. *For Case 3.2, we have $a_2 + y \geq 2m$, i.e., $a_2 \geq 2m - y$.*

Proof. The observation can be proved by using the inequalities $\frac{x}{6} + \frac{2y}{3} + a_2 + \frac{a_1}{3} \geq 4m$ (shown in the proof of Observation 6), $x \leq 2m$ and $a_1 + a_2 < 7m$. $\square$

The remaining stages run as follows. Recall that in Stage 3, the configuration of $\mathcal{A}$ after $30m$ items of size $\frac{1}{6}$ depart.

$\mathcal{O}$ **: # bins**	$\mathbf{2}$	$\boldsymbol{m}$	$\mathbf{4}\boldsymbol{m}$	$\boldsymbol{m}$	$\boldsymbol{p}$	$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{m-2p-q}$	$\mathbf{3}\boldsymbol{m}$	$\mathbf{2}\boldsymbol{m}$	**total**
Stage 1	-	-	-	-	-	-	-	-	-	-	0
	$\mathbf{1}_{*\epsilon}$	$\mathbf{1}_{*\epsilon}$	$\mathbf{1}_{*\epsilon}$	$\mathbf{1}_{*\epsilon}$	$\mathbf{1}_{*\epsilon}$	$\mathbf{1}_{*\epsilon}$	$\mathbf{1}_{*\epsilon}$	$\mathbf{1}_{*\epsilon}$	$\mathbf{1}_{*\epsilon}$	$\mathbf{1}_{*\epsilon}$	$12m+2$
Stage 2	$\frac{18}{24}{}_{*\epsilon}$	$\frac{2}{3}{}_{*\epsilon}$	$(\frac{1}{3}-\epsilon)_{*\epsilon}$	$(\frac{2}{3}-\epsilon)_{*\epsilon}$	$(1-\epsilon)_{*\epsilon}$	$(1-\epsilon)_{*\epsilon}$	$(1-\epsilon)_{*\epsilon}$	$(1-\epsilon)_{*\epsilon}$	$1_{*\epsilon}$	-	$6m+\frac{2m}{3}+12m\epsilon$
	$\frac{18}{24}{}_{*\epsilon}$	$\frac{2}{3}{}_{*\epsilon}$, $\mathbf{\frac{1}{6}}$, $\mathbf{\frac{1}{6}}$	$(\frac{1}{3}-\epsilon)_{*\epsilon}$, $\mathbf{4\times\frac{1}{6}}$	$(\frac{2}{3}-\epsilon)_{*\epsilon}$, $\mathbf{\frac{1}{6}}$, $\mathbf{\frac{1}{6}}$	$(1-\epsilon)_{*\epsilon}$	$(1-\epsilon)_{*\epsilon}$	$(1-\epsilon)_{*\epsilon}$	$(1-\epsilon)_{*\epsilon}$	$1_{*\epsilon}$	$\mathbf{6\times\frac{1}{6}}$	$12m$ $+12m\epsilon$
Stage 3*	$\frac{18}{24}{}_{*\epsilon}$	$\frac{2}{3}{}_{*\epsilon}$, $\frac{1}{6}$, $\frac{1}{6}$	$(\frac{1}{3}-\epsilon)_{*\epsilon}$	$(\frac{2}{3}-\epsilon)_{*\epsilon}$	$(1-\epsilon)_{*\epsilon}$	$(1-\epsilon)_{*\epsilon}$	$(1-\epsilon)_{*\epsilon}$	$(1-\epsilon)_{*\epsilon}$	$1_{*\epsilon}$	-	$7m+12m\epsilon$
	$\frac{18}{24}{}_{*\epsilon}$	$\frac{2}{3}{}_{*\epsilon}$, $\frac{1}{6}$, $\frac{1}{6}$	$(\frac{1}{3}-\epsilon)_{*\epsilon}$, $\mathbf{\frac{1}{3}}$, $\mathbf{\frac{1}{3}}$	$(\frac{2}{3}-\epsilon)_{*\epsilon}$, $\mathbf{\frac{1}{3}}$	$(\frac{2}{3}-\epsilon)_{*\epsilon}$, $\mathbf{\frac{1}{3}}$	$(1-\epsilon)_{*\epsilon}$	$(\frac{5}{6}-\epsilon)_{*\epsilon}$	$(1-\epsilon)_{*\epsilon}$	$1_{*\epsilon}$	$\mathbf{\frac{1}{3}}$, $\mathbf{\frac{1}{3}}$, $\mathbf{\frac{1}{3}}$	$12m$ $+12m\epsilon-\frac{q}{6}$

$\mathcal{O}$ **: # bins**$^+$	$\mathbf{2}$	$\boldsymbol{m}$	$\mathbf{2}\boldsymbol{m}$	$\boldsymbol{m}$	$\boldsymbol{m}$	$\mathbf{2}\boldsymbol{p+q}$	$\mathbf{2}\boldsymbol{m-2p-q}$	$\mathbf{3}\boldsymbol{m}$	$\mathbf{2}\boldsymbol{m}$	**total**
Stage 4	$\frac{18}{24}{}_{*\epsilon}$	$\frac{2}{3}{}_{*\epsilon}$, $\frac{1}{6}$, $\frac{1}{6}$	$(\frac{1}{6}-\epsilon)_{*\epsilon}$, $\frac{1}{3}$	$(\frac{1}{6}-\epsilon)_{*\epsilon}$, $\frac{1}{3}$	$(\frac{1}{6}-\epsilon)_{*\epsilon}$, $\frac{1}{3}$	$(\frac{1}{6}-\epsilon)_{*\epsilon}$, $\frac{1}{3}$	$(\frac{1}{2}-\epsilon)_{*\epsilon}$	$1_{*\epsilon}$	-	$7m+12m\epsilon$
	$\frac{18}{24}{}_{*\epsilon}$	$\frac{2}{3}{}_{*\epsilon}$, $\frac{1}{6}$, $\frac{1}{6}$	$(\frac{1}{6}-\epsilon)_{*\epsilon}$, $\frac{1}{3}$, $\mathbf{\frac{1}{2}-\frac{\epsilon}{4}}$	$(\frac{1}{6}-\epsilon)_{*\epsilon}$, $\frac{1}{3}$, $\mathbf{\frac{1}{2}-\frac{\epsilon}{4}}$	$(\frac{1}{6}-\epsilon)_{*\epsilon}$, $\frac{1}{3}$, $\mathbf{\frac{1}{2}-\frac{\epsilon}{4}}$	$(\frac{1}{6}-\epsilon)_{*\epsilon}$, $\frac{1}{3}$, $\mathbf{\frac{1}{2}-\frac{\epsilon}{4}}$	$(\frac{1}{2}-\epsilon)_{*\epsilon}$, $\mathbf{\frac{1}{2}-\frac{\epsilon}{4}}$	$1_{*\epsilon}$	$\mathbf{\frac{1}{2}-\frac{\epsilon}{4}}$, $\mathbf{\frac{1}{2}-\frac{\epsilon}{4}}$	$12m$ $+9.5m\epsilon$
Stage 5	$\frac{18}{24}{}_{*\epsilon}$	$\frac{2}{3}{}_{*\epsilon}$, $\frac{1}{6}$, $\frac{1}{6}$	$(\frac{1}{6}-\epsilon)_{*\epsilon}$, $\frac{1}{3}$	$(\frac{1}{6}-\epsilon)_{*\epsilon}$, $\frac{1}{3}$	$(\frac{1}{6}-\epsilon)_{*\epsilon}$, $\frac{1}{3}$	$(\frac{1}{6}-\epsilon)_{*\epsilon}$, $\frac{1}{3}$	$(\frac{1}{2}-\epsilon)_{*\epsilon}$	$1_{*\epsilon}$	$\frac{1}{2}-\frac{\epsilon}{4}$	$8m+11.5m\epsilon$
	$\frac{18}{24}{}_{*\epsilon}$	$\frac{2}{3}{}_{*\epsilon}$, $\frac{1}{6}$, $\frac{1}{6}$	$(\frac{1}{6}-\epsilon)_{*\epsilon}$, $\frac{1}{3}$, $\mathbf{\frac{1}{2}+\frac{\epsilon}{4}}$	$(\frac{1}{6}-\epsilon)_{*\epsilon}$, $\frac{1}{3}$, $\mathbf{\frac{1}{2}+\frac{\epsilon}{4}}$	$(\frac{1}{6}-\epsilon)_{*\epsilon}$, $\frac{1}{3}$, $\mathbf{\frac{1}{2}+\frac{\epsilon}{4}}$	$(\frac{1}{6}-\epsilon)_{*\epsilon}$, $\frac{1}{3}$, $\mathbf{\frac{1}{2}+\frac{\epsilon}{4}}$	$(\frac{1}{2}-\epsilon)_{*\epsilon}$, $\mathbf{\frac{1}{2}+\frac{\epsilon}{4}}$	$1_{*\epsilon}$	$\frac{1}{2}-\frac{\epsilon}{4}$, $\mathbf{\frac{1}{2}+\frac{\epsilon}{4}}$	$12m$ $+13.5m\epsilon$
Stage 6	$\frac{12}{24}{}_{*\epsilon}$	$\frac{1}{6}$, $\frac{1}{6}$,	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}{}_{*\epsilon}$	$1_{*\epsilon}$	$\frac{1}{2}-\frac{\epsilon}{4}$, $\frac{1}{2}+\frac{\epsilon}{4}$	$7m+\frac{m}{3}+12m\epsilon$
	$\frac{12}{24}{}_{*\epsilon}$	$\frac{1}{6}$, $\frac{1}{6}$, $\mathbf{\frac{2}{3}}$	$\frac{1}{3}$, $\mathbf{\frac{2}{3}}$	$\frac{1}{3}$, $\mathbf{\frac{2}{3}}$	$\frac{1}{3}$, $\mathbf{\frac{2}{3}}$	$\frac{1}{3}$, $\mathbf{\frac{2}{3}}$	$\frac{1}{3}{}_{*\epsilon}$, $\mathbf{\frac{2}{3}}$	$1_{*\epsilon}$	$\frac{1}{2}-\frac{\epsilon}{4}$, $\frac{1}{2}+\frac{\epsilon}{4}$	$12m+12m\epsilon$
Stage 7	$\frac{12}{24}{}_{*\epsilon}$	$\frac{1}{6}$, $\frac{1}{6}$, $\frac{2}{3}$	$\frac{1}{3}$, $\frac{2}{3}$	$\frac{2}{3}$	-	-	-	-	$\frac{1}{2}-\frac{\epsilon}{4}$, $\frac{1}{2}+\frac{\epsilon}{4}$	$5m+\frac{2m}{3}+12m\epsilon$
(Final)	$\frac{12}{24}{}_{*\epsilon}$	$\frac{1}{6}$, $\frac{1}{6}$, $\frac{2}{3}$	$\frac{1}{3}$, $\frac{2}{3}$	$\frac{2}{3}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\frac{1}{2}-\frac{\epsilon}{4}$, $\frac{1}{2}+\frac{\epsilon}{4}$	$11m+\frac{2m}{3}+12m\epsilon$

Table 9: The optimal schedule for Case 3.2. For each stage, the first row is the configuration just before items arrival and the second row is the configuration at the end of the stage. The very last row is the final configuration. Bolded entries are new items arrived in the corresponding stage. The notation y_{*z} means packing a bin with a load y of z-items. * Note that in Stage 3, there are two phases: the first row shows the configuration right before the items arrival in Phase 1 and the second row the end of both phases (in Phase 2, ϵ-items of a size of $\frac{2p+q}{6}$ departed). $^+$ Note that the 4-6th columns with bins $2m, m, m$ are the $4m$ bins in the 3rd column above while the column $2m{-}2p{-}q$ are the m and $m{-}2p{-}q$ bins in the 5th and 9th columns above.

$\mathcal{A}$: # bins	$8m$	$2p{+}q$	$2m{-}2p{-}q$	m	m	y	$2m{-}y$	$2m{-}y$	y	$2m$	$2m$	$4m$	$6m$	total
Stage 1	-	-	-	-	-	-	-	-	-	-	-	-	-	0
Stage 2	$(\frac{2}{3}{+}\epsilon)_{*\epsilon}$	$(\frac{1}{2}{+}\epsilon)_{*\epsilon}$	$(\frac{1}{2}{+}\epsilon)_{*\epsilon}$	$(\frac{1}{3}{+}\epsilon)_{*\epsilon}$	ϵ	-	-	-	-	-	-	-	-	$6m{+}\frac{2m}{3}{+}12m\epsilon$
Stage 3	$(\frac{2}{3}{+}\epsilon)_{*\epsilon}$	$(\frac{1}{2}{+}\epsilon)_{*\epsilon}$	$(\frac{1}{2}{+}\epsilon)_{*\epsilon}$	$(\frac{1}{3}{+}\epsilon)_{*\epsilon}$	ϵ	$\frac{1}{6}$	$\frac{1}{6}$	-	-	-	-	-	-	$7m{+}12m\epsilon$
Stage 4	$(\frac{1}{2}{+}\epsilon)_{*\epsilon}$	$(\frac{1}{6}{+}\epsilon)_{*\epsilon}, \frac{1}{3}$	$(\frac{1}{2}{+}\epsilon)_{*\epsilon}$	$(\frac{1}{3}{+}\epsilon)_{*\epsilon}$	ϵ	$\frac{1}{6}, \frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{3}, \frac{1}{3}$	$\frac{1}{3}$	-	-	-	-	$7m{+}12m\epsilon$
Stage 5	$(\frac{1}{2}{+}\epsilon)_{*\epsilon}$	$(\frac{1}{6}{+}\epsilon)_{*\epsilon}, \frac{1}{3}$	$(\frac{1}{2}{+}\epsilon)_{*\epsilon}$	$(\frac{1}{3}{+}\epsilon)_{*\epsilon}$	ϵ	$\frac{1}{6}, \frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{3}, \frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{2}{-}\frac{\epsilon}{4}$	-	-	-	$8m{+}11.5m\epsilon$
Stage 6	$(\frac{1}{3}{+}\epsilon)_{*\epsilon}$	$\epsilon, \frac{1}{3}$	$(\frac{1}{3}{+}\epsilon)_{*\epsilon}$	$(\frac{1}{3}{+}\epsilon)_{*\epsilon}$	ϵ	$\frac{1}{6}, \frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{3}, \frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{2}{-}\frac{\epsilon}{4}$	$\frac{1}{2}{+}\frac{\epsilon}{4}$	-	-	$7m{+}\frac{m}{3}{+}12m\epsilon$
Stage 7	ϵ	ϵ	ϵ	ϵ	ϵ	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{2}{-}\frac{\epsilon}{4}$	$\frac{1}{2}{+}\frac{\epsilon}{4}$	$\frac{2}{3}$	-	$5m{+}\frac{2m}{3}{+}12m\epsilon$
Final	ϵ	ϵ	ϵ	ϵ	ϵ	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{2}{-}\frac{\epsilon}{4}$	$\frac{1}{2}{+}\frac{\epsilon}{4}$	$\frac{2}{3}$	1	$11m{+}\frac{2m}{3}{+}12m\epsilon$

Table 10: The schedule of $\mathcal{A}$ for Case 3.2 right before items arrival in the corresponding stages. The last row shows the final configuration. The notation y_{*z} means packing a bin with a load y of z-items. The last column shows the total load of all the bins.

# bins	$8m$	$2m$	m	m	$2m$
$\mathcal{A}$	$(\frac{2}{3}+\epsilon)_{*\epsilon}$	$(\frac{1}{2}+\epsilon)_{*\epsilon}$	$(\frac{1}{3}+\epsilon)_{*\epsilon}$	ϵ	$\frac{1}{6}$
total size: $7m+12m\epsilon$					
# bins: $14m$					

There were then $15m{+}p$ items of size $\frac{1}{3}$ arrived and a further size of $\frac{2p+q}{6}$ of ϵ-items departed. $\mathcal{A}$ uses $2m$ more bins in this stage. At the end of Stage 3, $\mathcal{A}$ uses at least $16m$ bins.

4. In Stage 4, we keep a total size of $5m+\frac{m}{3}+12m\epsilon-\frac{2p+q}{3}$ of ϵ-items, a total number of $2m$ of $\frac{1}{6}$-items, and a total number of $4m+2p+q$ of $\frac{1}{3}$-items; and let a total size of $5m-\frac{q}{6}$ of items depart until the configuration of $\mathcal{A}$ is as follows.

# bins	$8m$	$2p{+}q$	$2m{-}2p{-}q$
$\mathcal{A}$	$(\frac{1}{2}+\epsilon)_{*\epsilon}$	$(\frac{1}{6}+\epsilon)_{*\epsilon}, \frac{1}{3}$	$(\frac{1}{2}+\epsilon)_{*\epsilon}$
	m	m	y
	$(\frac{1}{3}+\epsilon)_{*\epsilon}$	ϵ	$\frac{1}{6}, \frac{1}{3}$
	$2m{-}y$	$2m{-}y$	y
	$\frac{1}{6}$	$\frac{1}{3}, \frac{1}{3}$	$\frac{1}{3}$
total size: $7m+12m\epsilon$			
# bins: $16m$			

# bins	2	m
$\mathcal{O}$	$\frac{18}{24}_{*\epsilon}$	$\frac{2}{3}_{*\epsilon}, \frac{1}{6}, \frac{1}{6}$
	$4m$	$2p{+}q$
	$(\frac{1}{6}{-}\epsilon)_{*\epsilon}, \frac{1}{3}$	$(\frac{1}{6}{-}\epsilon)_{*\epsilon}, \frac{1}{3}$
	$2m{-}2p{-}q$	$3m$
	$(\frac{1}{2}{-}\epsilon)_{*\epsilon}$	$1{*}\epsilon$
total size: $7m+12m\epsilon$		
# bins: $10m+2$		

Next release $10m$ items of size $\frac{1}{2}{-}\frac{\epsilon}{4}$. For $\mathcal{A}$, only bins of load $\frac{1}{3}+\epsilon$, ϵ, $\frac{1}{6}$, $\frac{1}{3}$ and $\frac{1}{2}$ can accommodate one such item, therefore, $n_4 \geq (10m-4m-y)/2 \geq 2m$ because $y \leq 2m$. The number of bins used by $\mathcal{A}$ is at least $18m$. The corresponding packing configuration of $\mathcal{O}$ is

# bins	2	m	$4m$	$2p{+}q$	$2m{-}2p{-}q$	$3m$	$2m$
$\mathcal{O}$	$\frac{18}{24}_{*\epsilon}$	$\frac{2}{3}_{*\epsilon}, \frac{1}{6}, \frac{1}{6}$	$(\frac{1}{6}{-}\epsilon)_{*\epsilon}, \frac{1}{3}, \frac{1}{2}{-}\frac{\epsilon}{4}$	$(\frac{1}{6}{-}\epsilon)_{*\epsilon}, \frac{1}{3}, \frac{1}{2}{-}\frac{\epsilon}{4}$	$(\frac{1}{2}{-}\epsilon)_{*\epsilon}, \frac{1}{2}{-}\frac{\epsilon}{4}$	$1_{*\epsilon}$	$\frac{1}{2}{-}\frac{\epsilon}{4}, \frac{1}{2}{-}\frac{\epsilon}{4}$
total size: $12m+12m\epsilon-2.5m\epsilon=12m+9.5m\epsilon$							
# bins: $12m+2$							

5. In Stage 5, we let $8m$ items of size $\frac{1}{2}-\frac{\epsilon}{4}$ depart until the configuration of $\mathcal{A}$ is as follows.

# bins	$\boldsymbol{8m}$	$\boldsymbol{2p+q}$	$\boldsymbol{2m-2p-q}$
$\mathcal{A}$	$(\frac{1}{2}+\epsilon)_{*\epsilon}$	$(\frac{1}{6}+\epsilon)_{*\epsilon}, \frac{1}{3}$	$(\frac{1}{2}+\epsilon)_{*\epsilon}$
	$\boldsymbol{m}$	$\boldsymbol{m}$	$\boldsymbol{y}$
	$(\frac{1}{3}+\epsilon)_{*\epsilon}$	ϵ	$\frac{1}{6}, \frac{1}{3}$
	$\boldsymbol{2m-y}$	$\boldsymbol{2m-y}$	$\boldsymbol{y}$
	$\frac{1}{6}$	$\frac{1}{3}, \frac{1}{3}$	$\frac{1}{3}$
	$\boldsymbol{2m}$		
	$\frac{1}{2}-\frac{\epsilon}{4}$		
total size: $8m+11.5m\epsilon$			
# bins: $18m$			

# bins	$\boldsymbol{2}$	$\boldsymbol{m}$
$\mathcal{O}$	${\frac{18}{24}}_{*\epsilon}$	${\frac{2}{3}}_{*\epsilon}, \frac{1}{6}, \frac{1}{6}$
	$\boldsymbol{4m}$	$\boldsymbol{2p+q}$
	$(\frac{1}{6}-\epsilon)_{*\epsilon}, \frac{1}{3}$	$(\frac{1}{6}-\epsilon)_{*\epsilon}, \frac{1}{3}$
	$\boldsymbol{2m-2p-q}$	$\boldsymbol{3m}$
	$(\frac{1}{2}-\epsilon)_{*\epsilon}$	$1{*}\epsilon$
	$\boldsymbol{2m}$	
	$\frac{1}{2}-\frac{\epsilon}{4}$	
total size: $8m+11.5m\epsilon$		
# bins: $12m+2$		

Release $8m$ items of size $\frac{1}{2}+\frac{\epsilon}{4}$. For $\mathcal{A}$, only bins of load $\frac{1}{3}+\epsilon$, ϵ, $\frac{1}{6}$, $\frac{1}{3}$, and $\frac{1}{2}-\frac{\epsilon}{4}$ can accommodate one such item, therefore, $n_5 \geq 8m-m-m-(2m-y)-y-2m = 2m$. The number of bins used by $\mathcal{A}$ is at least $20m$. The corresponding packing configuration of $\mathcal{O}$ is

# bins	$\boldsymbol{2}$	$\boldsymbol{m}$	$\boldsymbol{4m}$	$\boldsymbol{2p+q}$	$\boldsymbol{2m-2p-q}$	$\boldsymbol{3m}$	$\boldsymbol{2m}$
$\mathcal{O}$	${\frac{18}{24}}_{*\epsilon}$	${\frac{2}{3}}_{*\epsilon}, \frac{1}{6}, \frac{1}{6}$	$(\frac{1}{6}-\epsilon)_{*\epsilon}, \frac{1}{3}, \frac{1}{2}+\frac{\epsilon}{4}$	$(\frac{1}{6}-\epsilon)_{*\epsilon}, \frac{1}{3}, \frac{1}{2}+\frac{\epsilon}{4}$	$(\frac{1}{2}-\epsilon)_{*\epsilon}, \frac{1}{2}+\frac{\epsilon}{4}$	$1_{*\epsilon}$	$\frac{1}{2}-\frac{\epsilon}{4}, \frac{1}{2}+\frac{\epsilon}{4}$
total size: $12m+12m\epsilon+1.5m\epsilon = 12m+13.5m\epsilon$							
# bins: $12m+2$							

6. In Stage 6, we keep a total size of $3m+\frac{2m}{3}+12m\epsilon-\frac{2p+q}{3}$ of ϵ-items, a total number of $2m$ of $\frac{1}{6}$-items, a total number of $4m+2p+q$ of $\frac{1}{3}$-items, a total number of $2m$ of $\frac{1}{2}-\frac{\epsilon}{4}$-items, and a total number of $2m$ of $\frac{1}{2}+\frac{\epsilon}{4}$-items. We let $6m$ items of size $\frac{1}{2}+\frac{\epsilon}{4}$ and a total size of $m+\frac{2m}{3}$ of ϵ-items depart until the configuration of $\mathcal{A}$ is as follows.

# bins	$\boldsymbol{8m}$	$\boldsymbol{2p+q}$	$\boldsymbol{2m-2p-q}$
$\mathcal{A}$	$(\frac{1}{3}+\epsilon)_{*\epsilon}$	$\epsilon, \frac{1}{3}$	$(\frac{1}{3}+\epsilon)_{*\epsilon}$
	$\boldsymbol{m}$	$\boldsymbol{m}$	$\boldsymbol{y}$
	$(\frac{1}{3}+\epsilon)_{*\epsilon}$	ϵ	$\frac{1}{6}, \frac{1}{3}$
	$\boldsymbol{2m-y}$	$\boldsymbol{2m-y}$	$\boldsymbol{y}$
	$\frac{1}{6}$	$\frac{1}{3}, \frac{1}{3}$	$\frac{1}{3}$
	$\boldsymbol{2m}$	$\boldsymbol{2m}$	
	$\frac{1}{2}-\frac{\epsilon}{4}$	$\frac{1}{2}+\frac{\epsilon}{4}$	
total size: $7m+\frac{m}{3}+12m\epsilon$			
# bins: $20m$			

# bins	$\boldsymbol{2}$	$\boldsymbol{m}$
$\mathcal{O}$	${\frac{12}{24}}_{*\epsilon}$	$\frac{1}{6}, \frac{1}{6}$
	$\boldsymbol{4m}$	$\boldsymbol{2p+q}$
	$\frac{1}{3}$	$\frac{1}{3}$
	$\boldsymbol{2m-2p-q}$	$\boldsymbol{3m}$
	${\frac{1}{3}}_{*\epsilon}$	$1{*}\epsilon$
	$\boldsymbol{2m}$	
	$\frac{1}{2}-\frac{\epsilon}{4}, \frac{1}{2}+\frac{\epsilon}{4}$	
total size: $7m+\frac{m}{3}+12m\epsilon$		
# bins: $12m+2$		

We release $7m$ items of size $\frac{2}{3}$. For $\mathcal{A}$, only bins of load ϵ, $\frac{1}{6}$ and $\frac{1}{3}$ can accommodate such item, therefore, $n_6 \geq 7m-m-(2m-y)-y = 4m$. The number of bins used by $\mathcal{A}$ is at least $24m$. The corresponding configuration of $\mathcal{O}$ is

# bins	2	m	$4m$	$2p+q$	$2m-2p-q$	$3m$	$2m$
$\mathcal{O}$	$\frac{12}{24*\epsilon}$	$\frac{1}{6},\frac{1}{6},\frac{2}{3}$	$\frac{1}{3},\frac{2}{3}$	$\frac{1}{3},\frac{2}{3}$	$\frac{1}{3*\epsilon},\frac{2}{3}$	$1_{*\epsilon}$	$\frac{1}{2}-\frac{\epsilon}{4},\frac{1}{2}+\frac{\epsilon}{4}$

total size: $12m + 12m\epsilon$

 # bins: $12m + 2$

7. In Stage 7, we keep $12m$ items of size ϵ, $2m$ items of size $\frac{1}{6}$, $2m$ items of size $\frac{1}{3}$, $2m$ items of size $\frac{1}{2}-\frac{\epsilon}{4}$, $2m$ items of size $\frac{1}{2}+\frac{\epsilon}{4}$, and $4m$ items of size $\frac{2}{3}$. We let a total size of $6m + \frac{m}{3}$ of items depart until the configuration of $\mathcal{A}$ is as follows.

# bins	$12m$	$2m$	$2m$	$2m$	$2m$	$4m$
$\mathcal{A}$	ϵ	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}-\frac{\epsilon}{4}$	$\frac{1}{2}+\frac{\epsilon}{4}$	$\frac{2}{3}$

total size: $5m + \frac{2m}{3} + 12m\epsilon$

 # bins: $24m$

# bins	2	m	$2m$	m	$2m$
$\mathcal{O}$	$\frac{12}{24*\epsilon}$	$\frac{1}{6},\frac{1}{6},\frac{2}{3}$	$\frac{1}{3},\frac{2}{3}$	$\frac{2}{3}$	$\frac{1}{2}-\frac{\epsilon}{4},\frac{1}{2}+\frac{\epsilon}{4}$

total size: $5m + \frac{2m}{3} + 12m\epsilon$

 # bins: $6m + 2$

We then release another $6m$ items of size 1, therefore, $n_7 = 6m$. The total number of bins used by $\mathcal{A}$ is $30m = 45k$.

# bins	$12m$	$2m$	$2m$	$2m$	$2m$	$4m$	$6m$
$\mathcal{A}$	ϵ	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}-\frac{\epsilon}{4}$	$\frac{1}{2}+\frac{\epsilon}{4}$	$\frac{2}{3}$	1

total size: $11m + \frac{2m}{3} + 12m\epsilon$

 # bins: $30m$

# bins	2	m	$2m$	m	$2m$	$6m$
$\mathcal{O}$	$\frac{12}{24*\epsilon}$	$\frac{1}{6},\frac{1}{6},\frac{2}{3}$	$\frac{1}{3},\frac{2}{3}$	$\frac{2}{3}$	$\frac{1}{2}-\frac{\epsilon}{4},\frac{1}{2}+\frac{\epsilon}{4}$	1

total size: $11m + \frac{2m}{3} + 12m\epsilon$

 # bins: $12m + 2$

In summary, for all three cases, $\mathcal{A}$ uses $30m = 45k$ bins while there is a schedule that uses at most $12m + 2 = 18k + 2$ bins and thus Theorem 1 follows.

3 1-competitive if and only if size-2 bins are used

In this section, we show that using size-2 bins is both necessary (Theorem 9) and sufficient (Theorem 8) to achieve 1-competitiveness. Any-fit (AF) is an algorithm that always packs a new item into a non-empty bin arbitrarily as long as the bin can accommodate the item.

Theorem 8. *Any fit algorithm with size-2 bins is 1-competitive.*

Proof. Suppose AF uses n size-2 bins for a sequence of items. When AF first uses n bins due to the arrival of a new item X (size ≤ 1), all the existing $n-1$ bins must have a load greater than 1, otherwise, X can be packed into one of these bins and AF does not need to open a new bin. In other words, the total load of items is at least $n - 1 + s$ where s is the size of X. Any algorithm using unit-size bins needs at least n bins to pack all these items. Therefore, the maximum number of size-2 bins used by AF is at most that used by the optimal off-line algorithm. $\square$

Theorem 9. *No on-line algorithm can be* 1*-competitive by using size-x bins, for any $x < 2$.*

Proof. Suppose $x = 2 - \epsilon$, for some small $\epsilon > 0$. Let k be a positive integer such that $\frac{2}{k} \geq \epsilon > \frac{1}{k}$. Notice the size satisfies the property $2 - \frac{2}{k} \leq 2 - \epsilon = x < 2 - \frac{1}{k}$. The adversary works in two phases.

In the first phase, release k^3 items of size $\frac{1}{k}$. The total load of the items is k^2 and all items can be packed into k^2 unit-size bins. If the on-line algorithm uses more than k^2 bins, we are done. So we only need to consider the case in which the on-line algorithm uses at most k^2 bins. We are going to prove that the on-line algorithm uses at least $2k$ bins with load at least $1-\frac{1}{k}$. Let Y be the number of such bins. Note that the maximum possible load of items of size $\frac{1}{k}$ in a size-$(2-\epsilon)$ bin is $2 - \frac{2}{k}$ and the maximum possible load of items of size $\frac{1}{k}$ such that the load is strictly less than $1 - \frac{1}{k}$ is $1 - \frac{2}{k}$. Then the total load accommodated by the on-line algorithm is at most $Y(2 - \frac{2}{k}) + (k^2 - Y)(1 - \frac{2}{k}) = k^2 - 2k + Y$. This load cannot be smaller than the total load of items, i.e, $k^2 - 2k + Y \geq k^2$. In other words, $Y \geq 2k$.

In the second phase, we retain a load of $1-\frac{1}{k}$ in k bins and let all other items depart. Then release $k^2 - k + 1$ items of size 1. Notice that none of these items can be packed into an existing bin because $1 + 1-\frac{1}{k}$ is greater than x, the size of the bin. Therefore, the total number of bins used by the on-line algorithm is $k^2 + 1$.

On the other hand, the optimal off-line algorithm can pack the items released in the first phase in a way that those retained in the second phase are packed into $k - 1$ bins and the departing items into the other $k^2 - k + 1$ bins. The latter bins can be reused for the size-1 items released in the second phase. Hence, the maximum number of unit-size bins used by the optimal off-line algorithm is k^2, which is strictly smaller than the maximum number of size-x bins used by the on-line algorithm, i.e., the on-line algorithm is not 1-competitive. □

4 Trade-off between bin size and competitive ratio

In this section, we discuss results where the on-line algorithm uses bins of size $1 < b < 2$ while the optimal off-line algorithm uses bins of unit size. We first give a general lower bound for any on-line algorithm. Then we analyze the performance of first-fit (packs to the first bin that can fit), best-fit (heaviest loaded bin) and worst-fit (lightest loaded bin) giving their upper bounds.

4.1 General lower bound for $1 < b < 2$

In this section, we describe two adversaries, one gives better lower bound for $1 < b < 1.5$ and the other for $1.5 \leq b < 2$. The first adversary attempts to obtain the following lemma.

Lemma 10. *No on-line algorithm using size-b bins can be better than $\frac{2}{b}$-competitive.*

Proof. Consider any on-line algorithm $\mathcal{A}$. Let ϵ be a small constant such that $\frac{1}{\epsilon}$ is an integer and $k = \frac{1}{\epsilon} - 2$. The adversary runs in 3 stages, using items of 3 sizes: ϵ, $\frac{b}{2} + \epsilon$ and 1, each type arriving in different stage. Roughly speaking, ϵ-items are released in Stage 1; items depart in Stage 2 so that a $(\frac{b}{2}+\epsilon)$-item can be packed into existing bins but not a 1-item; finally in

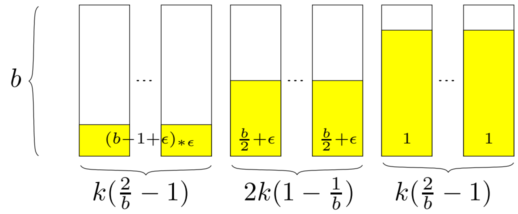


Figure 4: The final configuration of the on-line algorithm achieved by the adversary in Lemma 10.

Stage 3, more items depart so that all 1-items arriving have to be packed in separate new bins. The exact number of items arriving and departing are as follows.

1. Release ϵ-items of total size k. If $\mathcal{A}$ uses more than $\frac{2k}{b}$ bins, we are done. If $\mathcal{A}$ uses at most $\frac{2k}{b}$ bins, we claim that there must be at least $(\frac{2}{b}-1)k$ bins with load $\geq b-1+\epsilon$, otherwise, the total possible load accommodated by $\mathcal{A}$ is less than $((\frac{2}{b}-1)k-1)b+(k+1)(b-1+\epsilon) = k-1+(k+1)\epsilon < k$, contradiction.

2. Let items depart until the configuration of $\mathcal{A}$ becomes $\{\, k(\frac{2}{b}-1){:}(b-1+\epsilon)_{*\epsilon} \,\}$.

   Then release k items of size $\frac{b}{2}+\epsilon$. At most one such item can be packed into an existing bin or an empty bin. So, at least $k-(\frac{2}{b}-1)k = 2k(1-\frac{1}{b})$ new bins are opened.

3. Let items depart until the configuration of $\mathcal{A}$ is $\{\, k(\frac{2}{b}-1){:}(b-1+\epsilon)_{*\epsilon}, 2k(1-\frac{1}{b}){:}(\frac{b}{2}+\epsilon) \,\}$. Finally, release $k(\frac{2}{b}-1)$ items of size 1. None of the items can be packed into existing bins, thus, another $(\frac{2}{b}-1)k$ new bins are opened. Number of bins used becomes $\frac{2k}{b}$. (See Figure 4.)

Note that $k(\frac{2}{b}-1)+2k(1-\frac{1}{b}) = k$. Now, we show that there is a schedule that uses at most $k+O(1)$ unit-size bins at any time. To achieve this, at the end of Stage 3, we require that the ϵ-items and $(\frac{b}{2}+\epsilon)$-items to be packed into at most $k-(\frac{2}{b}-1)k+O(1)$ bins, i.e., $2k(1-\frac{1}{b})+O(1)$ bins. Since ϵ is a small constant, it is possible to pack ϵ-items with a $(\frac{b}{2}+\epsilon)$-item to make a load of very close to 1, precisely, at least $1-\epsilon$. It can be verified that the total size of ϵ-items and $(\frac{b}{2}+\epsilon)$-items, i.e., $k(\frac{2}{b}-1)(b-1+\epsilon)+2k(1-\frac{1}{b})(\frac{b}{2}+\epsilon)$, equals $2k(1-\frac{1}{b})+O(1)$, implying that they can be packed into $2k(1-\frac{1}{b})+O(1)$ bins. This implies a packing for Stage 2 such that the rest of the $k-2k(1-\frac{1}{b})$ items of size $\frac{b}{2}+\epsilon$ are each packed into a separate unit-sized bin and the total number of bins used is $k+O(1)$. In Stage 1, the ϵ-items can be packed into the $k+O(1)$ bins since the total size is only k. $\square$

The second adversary makes use of unit fraction items, i.e., in the form $\frac{1}{w}$, for some integer w. Let m be the largest integer such that $b-\frac{1}{m}<1$ and let $k = m!(m-1)!$. We define the functions $\alpha(i)$ and $\beta(i)$ for any positive integer $1 \leq i \leq m$ as follows. Let $\alpha(1) = k, \beta(i) = \sum_{j=1}^{i} \alpha(j)$,

$$\alpha(i) = \beta(i-1)\left(\tfrac{m+1-i}{m+2-i} - \tfrac{m-i}{m+1-i}\right).$$

In other words, $\alpha(i) = \frac{\beta(i-1)}{(m+2-i)(m+1-i)}$. E.g., $\alpha(2) = \frac{k}{m(m-1)}$, $\alpha(3) = \frac{k+\frac{k}{m(m-1)}}{(m-1)(m-2)}$.

Before we show the lemma for the second adversary, we state the following property to be used in the analysis.

Fact 11. *Both $\alpha(i)$ and $\beta(i)$ are integer multiples of $(m-i+1)!(m-i)!$*

Proof. By definition, $\alpha(1) = \beta(1) = m!(m-1)!$, the fact is true for $i = 1$. Suppose the fact is true for all $j \leq p$ for some p. By definition, $\alpha(p+1) = \frac{\beta(p)}{(m+1-p)(m-p)}$ which is a multiple of $(m-p)!(m-p-1)!$, by the induction hypothesis. Also by induction, $\beta(p)$ is a multiple of $(m-p+1)!(m-p)!$, i.e., $\beta(p)$ is also a multiple of $(m-p-1)!(m-p)!$. Notice that $\beta(p+1) = \beta(p) + \alpha(p+1)$ and since both $\beta(p)$ and $\alpha(p+1)$ are multiples of $(m-p)!(m-p-1)!$, $\beta(p+1)$ is also a multiple of $(m-p)!(m-p-1)!$. □

Lemma 12. *No on-line algorithm using size-b bins, for $1 < b < 2$, can be better than $\frac{\beta(m)}{m!(m-1)!}$-competitive, where m is the largest integer such that $b - \frac{1}{m} < 1$.*

Proof. Consider any on-line algorithm $\mathcal{A}$. The adversary runs in m stages. In Stage 1, we release $\frac{1}{m}$-items up to a total size of k, i.e., km such items. For each stage $2 \leq i \leq m$, we let some items released in previous stage depart and then release some $\frac{1}{m+1-i}$-items, such that in Stage i, the following invariants are maintained: (1) items of a total size $\beta(i-1)\frac{m+1-i}{m+2-i}$ depart and the same size of $\frac{1}{m+1-i}$-items are released, keeping the total size of items being k; and (2) $\mathcal{A}$ uses at least $\alpha(i)$ new bins at the end of the stage. Stage i proceeds as follows.

a. Retain one $\frac{1}{m+2-i}$-item from each of the $\alpha(i-1)$ new bins used in Stage $i-1$ and let all other $\frac{1}{m+2-i}$-items depart, i.e., only retain $\alpha(i-1)$ such items.

b. Release items of size $\frac{1}{m+1-i}$ until the total size of all items is k.

We are going to prove by induction that the invariants are maintained. Notice that the total size of items that we have considered is always an integer, which can be showed to be a consequence of Fact 11.

We first show that the invariants hold for Stage 2. The total size of $\frac{1}{m}$-items departed equals $\beta(1) - \alpha(1)\frac{1}{m} = \beta(1)\frac{m-1}{m}$. A total size of $\beta(1)\frac{m-1}{m}$ items of size $\frac{1}{m-1}$ are then released. Notice that at most $m-2$ such items can be packed into an existing bins, otherwise, the load of a bin is at least $\frac{1}{m} + 1 > b$, by the definition of m. In other words, a total size of at least $\beta(1)(\frac{m-1}{m} - \frac{m-2}{m-1})$ items cannot be packed into existing bins, implying that $\mathcal{A}$ needs at least $\alpha(2)$ new bins. Therefore, the invariants hold for Stage 2.

Suppose the invariants hold for all stages up to Stage p for some $p \geq 2$. Consider Stage $p+1$. The total size of items departed equals $\beta(p-1)\frac{m+1-p}{m+2-p} - \alpha(p-1)\frac{1}{m+1-p}$. By some arithmetic, we can show that this equals $\beta(p)\frac{m-p}{m+1-p}$. After the departure, every existing bin has a load of at least $\frac{1}{m}$. Using a similar argument as the base case, at most $m-p-1$ items of size $\frac{1}{m-p}$ released in this stage can be packed into existing bins, leaving a total size of $\beta(p)\frac{m-p}{m+1-p} - \beta(p)\frac{m-p-1}{m-p}$ to be packed into at least $\alpha(p+1)$ new bins. Therefore, the invariants also hold for Stage $p+1$. At the end of Stage m, $\mathcal{A}$ uses a total of at least $\beta(m)$ bins.

Consider the optimal off-line algorithm. Note that the total size of items at any time is kept at k. Furthermore, the total size of items of the same type of items departing and arriving is always an integer because of Fact 11, and since the item size is unit fraction, we can always pack the same type of item fully into the same unit-size bin. Therefore, the optimal off-line algorithm only needs k bins. □

The corollary below follows directly from Lemmas 10 and 12 (see Figure 5 for the trend).

Corollary 13. *No on-line algorithm using size-b bins is better than* $\max\{\frac{\beta(m)}{m!(m-1)!}, \frac{2}{b}\}$*-competitive, where m is the largest integer such that $b - \frac{1}{m} < 1$.*

4.2 Performance of First-Fit for $1 < b < 2$

We first analyze the upper bound of the first-fit algorithm (FF) using size-b bins. To simplify the discussion, we refer to two properties pointed out by Coffman et al. [7]. (1) We can focus on the input sequences such that the maximum number of bins used by FF when the last item is packed and not before. (2) No non-empty bin ever becomes empty during the execution of FF on input sequences satisfying the first property. It can be shown [7] that the two properties are satisfied because FF will work out the same packing for the modified input sequence, e.g., a modified sequence in which the items packed to that non-empty bin are removed. By the second property, we can label the non-empty bins by the order they became non-empty, i.e., bin i refers to the i-th bin used by FF, and the labels never change.

Theorem 14. *The competitive ratio of FF using size-b bins is at most* $\min\{\frac{2b+1}{2b-1}, \frac{5-b}{(2b-1)}, \frac{b^2+3}{b(2b-1)}\}$.

Proof. Let k denote the maximum number of bins used by the optimal off-line algorithm using unit-size bins and n be the maximum number of size-b bins used by FF. Suppose x is the largest labelled among the bins (i.e., the last bin) that FF ever packs an item of size $\leq 1/2$. Let B be the last bin that FF opens with an item of size $\leq b/2$. When FF opens B, let y be the number of bins (including B) with label $> x$ whose smallest item has a size in the range $(1/2, b/2]$. Let $z = n - x - y$. We claim that the following inequalities hold.

$$\begin{aligned}
(x-1)(b-\tfrac{1}{2}) &\leq k &(1)\\
\tfrac{xb}{2} + (y-1) &\leq k &(2)\\
(x+y)(b-1) + \tfrac{zb}{2} &\leq k &(3)\\
x(b-1) + \tfrac{y}{2} + \tfrac{zb}{2} &\leq k &(4)\\
y + z &\leq k &(5)
\end{aligned}$$

(1) When FF packs an item of size at most $\frac{1}{2}$ into the x-th bin, all existing bins must have a load at least $b - \frac{1}{2}$, otherwise, FF would pack the item into those bins instead. The total item size is $\geq (x-1)(b-\frac{1}{2})$, which must be $\leq k$, the total size that can be accommodated by the optimal off-line algorithm using unit-size bins; Inequality (1) follows. (2) When FF opens the bin B with an item of size $\leq \frac{b}{2}$, the first x bins must have load at least $\frac{b}{2}$; and there must be at least $y-1$ bins with label $> x$ containing two items of size at least $\frac{1}{2}$, otherwise, there will be a bin with load $\leq \frac{b}{2}$ and FF does not need to open B. Thus, Inequality (2) follows. (3) When FF packs the first item into bin-n, all existing bins must have a load at least $b-1$ (otherwise, FF can pack the item in those bins instead of opening a new bin) and there must be z bins containing an item of size $\geq \frac{b}{2}$. (4) A similar argument gives

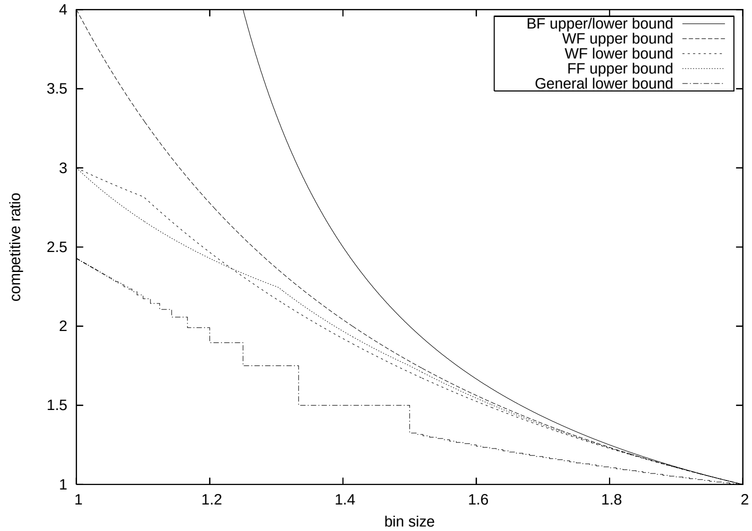


Figure 5: Trade-off between bin size b and competitive ratio.

Algorithm	upper bound	lower bound
BF	$\frac{1}{b-1}$	$\frac{1}{b-1}$
WF	$\frac{4}{b^2}$	$\min\{\frac{2+b}{b}, \frac{b^2-8b+20}{4b}\}$
FF	$\min\{\frac{2b+1}{2b-1}, \frac{5-b}{(2b-1)}, \frac{b^2+3}{b(2b-1)}\}$	$\max\{\frac{\beta(m)}{m!(m-1)!}, \frac{2}{b}\}$

Table 11: Summary of results for bin size $1 \leq b < 2$.

Inequality (4). (5) is due to the fact that items packed to the last $y+z$ bins must have size greater than $\frac{1}{2}$, each of them must be packed into a different bin in the optimal off-line algorithm that uses unit-size bins.

Using Inequalities (1) and (5), we have (i) $x+y+z \leq \frac{2k}{2b-1} + 1 + k = \frac{2b+1}{2b-1}k + 1$. Using Inequalities (1), (2) and (3), we can show that (ii) $x+y+z \leq \frac{5-b}{(2b-1)}k + O(1)$. Inequalities (1), (2) and (4) give (iii) $x+y+z \leq \frac{b^2+3}{b(2b-1)}k + O(1)$. Together the theorem follows. □

Figure 5 shows how the competitive ratio of FF varies with b. Notice that our formula in Theorem 14 reaches the value 1 when $b = 2$ matching Theorem 8; yet when $b = 1$, the value is 3, not matching the existing best upper bound of 2.788 [7]. We leave it as an open question to close the gap between the upper and lower bounds.

4.3 Performance of Best-Fit and Worst-Fit for $1 < b < 2$

To have a more complete picture about the performance of the class of any-fit algorithms, we also study the performance of best-fit (BF) and worst-fit (WF) (see the upper three curves in Figure 5 and Table 11).

Theorem 15. *(i) BF using size-b bins is $\frac{1}{b-1}$-competitive, this bound is tight. (ii) WF using size-b bins is $\frac{4}{b^2}$-competitive; on the other hand, its competitive ratio is no better than $\min\{\frac{2+b}{b}, \frac{b^2-8b+20}{4b}\}$.*

Proof. **Upper bound of BF.** Suppose BF uses a maximum of n bins. When BF first packs an item into bin n, the load of each of the other $n-1$ bins for $i < n$ is at least $b-1$, otherwise, BF can pack the item into those bins instead of opening a new bin. Therefore, the optimal off-line algorithm needs $k \geq (n-1)(b-1)$ bins, and hence the competitive ratio of BF is at most $\frac{1}{b-1}$ since $n \leq \frac{k}{b-1} + 1$.

Upper bound of WF. Let k denote the maximum number of unit-size bins used by the optimal off-line algorithm. Suppose WF uses a maximum of n size-b bins. Let x be the number of bins that do not contain items of size $> \frac{b}{2}$ at the time instance t_1 when WF packs the first item into bin n. For each of these x bins, say B_i, let $t_{B_i} \leq t_1$ be the latest time instance such that B_i changes from empty to non-empty. Let B be the bin such that t_B is the largest. Let $y = n - x$. We claim the following inequalities hold.

$$\frac{yb}{2} + x(b-1) \leq k \qquad (6)$$

$$(x-1)\frac{b}{2} \leq k \qquad (7)$$

The inequalities can be proved in a similar way as in the analysis for FF.

(6) At t_1, WF has packed an item of size greater than $\frac{b}{2}$ into $n-x$ bins (by the definition of x). Furthermore, at t_1, WF packs an item into bin n, thus, each of the first $n-1$ bins must have at least a load of $b-1$. Therefore, the total load of items at t_1 is at least $y\frac{b}{2} + x(b-1)$.

(7) Consider the time instance t_B. There are two cases for the item that WF packs into B: the size of the item is less than or equal to $\frac{b}{2}$, and the size is greater than $\frac{b}{2}$. In the former case, the total load of items at t_B is at least $(x-1)(b-\frac{b}{2}) = (x-1)\frac{b}{2}$. In the latter case, bin B has an item of size greater than $\frac{b}{2}$ at t_B but no such item at t_1. Therefore, there must be a time instance that WF packs an item to bin B while its load is at least $\frac{b}{2}$. By the property of WF, the load of all the other bins is also at least $\frac{b}{2}$ at that time instance, i.e., the total load is at least $(x-1)\frac{b}{2}$; otherwise, WF should have packed the item into those bins instead. In both cases, the total load at t_B is at least $(x-1)\frac{b}{2}$.

By the two inequalities and simple arithmetic, we can show that $n = x + y \leq \frac{4}{b^2} + 1$. The lower bounds for both BF and WF will be given below. It is worth mentioning that the lower bound of BF can be extended to the case where $b = 1$ so that the competitive ratio of BF is unbounded.

Lower bound of BF. Recall that BF packs a newly arrived item to the heaviest loaded bin that can fit the item. Roughly speaking, the adversary runs in stages and attempts to make BF use one more bin in each stage each with load just less than $\frac{b}{2}$ (all being small items) at the end of the stage. To force BF to do this, an (large) item of size just greater than $\frac{b}{2}$ arrives in each stage, followed by some small items of total size just less than $\frac{b}{2}$; then the large item departs. The sizes of the items are set in a way such that BF has to use a new bin for the large item, making this bin of higher priority for packing the small items arriving in the same stage. Precisely, the adversary works as follows.

Let k be an arbitrarily large integer. We define a sequence of small positive constants ϵ_i, for $1 \leq i \leq k$, such that (i) $\epsilon_1 < \min\{1-\frac{b}{2}, b-1\}$, and (ii) $\epsilon_{i+1} = \frac{\epsilon_i}{2}$. To simplify the proof, we assume that we can pick an ϵ_k satisfying the above properties such that $b-1$ is divisible by ϵ_k (the other case can be proved similarly). The adversary runs in k stages.

In Stage 1, $\frac{b-1+\epsilon_1}{\epsilon_k}$ items of size ϵ_k are released, i.e., a load of $b-1+\epsilon_1$ is released. In each subsequent stage, Stage i for $2 \leq i \leq k$, an item of size $1-\epsilon_{i-1}+\epsilon_i$ is first released, followed by $\frac{b-1+\epsilon_i}{\epsilon_k}$ items of size ϵ_k, and then the $(1-\epsilon_{i-1}+\epsilon_i)$-item departs. Notice the following properties of the items released:

1. The load of ϵ_k items released in each stage is less than $\frac{b}{2}$ because $b-1+\epsilon_1 < b-1+(1-\frac{b}{2}) = \frac{b}{2}$ and the number of ϵ_k items released in each stage is decreasing as ϵ_i is decreasing.

2. The size of the $(1-\epsilon_{i-1}+\epsilon_i)$-item released in Stage i, for $i \geq 2$ is larger than $\frac{b}{2}$ since $1-\epsilon_{i-1}+\epsilon_i = 1-\frac{\epsilon_{i-1}}{2} > 1-\epsilon_1 \geq 1-(1-\frac{b}{2}) = \frac{b}{2}$.

3. All the items released in the same stage can be packed into a single bin of size b because $(1-\epsilon_{i-1}+\epsilon_i)+\epsilon_k\frac{b-1+\epsilon_i}{\epsilon_k} = b-\epsilon_{i-1}+2\epsilon_i = b$.

With these properties, we will show that BF uses $\geq k$ bins while the optimal off-line algorithm uses $\leq k(b-1)+2$ bins, implying that BF is no better than $\frac{1}{b-1}$-competitive.

We claim that in each stage, BF packs the newly arrived items into a new bin making its load become $b-1+\epsilon_i$ at the end of the stage. The claim holds for Stage 1 because BF packs all those items into a single bin whose load becomes $b-1+\epsilon_1$. Assume the claim holds up to Stage i.

In Stage $i+1$, BF must pack the new $(1-\epsilon_{i-1}+\epsilon_i)$-item into a new bin, otherwise, if it is packed into an existing bin $1 \leq j \leq i$, the load of this bin becomes $(b-1+\epsilon_j)+(1-\epsilon_i+\epsilon_{i+1}) = b+\epsilon_j-\epsilon_{i+1} > b$ because $\epsilon_{i+1} = \frac{\epsilon_i}{2} < \epsilon_j$, contradiction. Note that the load of any existing i bins is at most $b-1+\epsilon_1$ and the load of the new bin is $1-\epsilon_i+\epsilon_{i+1}$. By Property (2) above, the load of the new bin is the highest among all bins. Therefore, BF packs the further arriving items of size ϵ_k into the new bin instead of the existing bins (this can be done because of Property (3)). The departure of the $1-\epsilon_i+\epsilon_{i+1}$ item leaves the new bin to have a load of $b-1+\epsilon_{i+1}$ and the claim holds for Stage $i+1$. Thus, BF uses k bins.

On the other hand, the optimal off-line algorithm can use a dedicate bin to store the $1-\epsilon_i+\epsilon_{i+1}$ items (at any time, there is at most one such item), and $k(b-1)+1$ bins to store the ϵ_k items because the total load of such items is $k(b-1)+\sum \epsilon_i \leq k(b-1)+2\epsilon_1$. Altogether the optimal off-line algorithm only uses $k(b-1)+2$ bins. Therefore, BF using size-b bins, $b > 1$, is no better than $\frac{1}{b-1}$-competitive.

Lower bound of WF. Recall that WF packs a newly arrived item to the lightest loaded bin that can fit the item. Let k be an arbitrarily large integer constant such that kb is an integer. The adversary attempts to first force WF to use $2k$ bins each with a load just more than $\frac{b}{2}$. Then some items depart from the first $2k$ bins and items with increasing size are released, this repeats until the load of each of the first $2k$ bins is just more than $b-1$. Finally, more items of size 1 are released, which have to be packed into some new bins. We are going to show that WF uses $\min\{(2+b)k, \frac{(b^2-8b+20)k}{4}\} + O(1)$ bins while the optimal off-line algorithm uses $\leq kb+3$ bins.

Let $x = (b-1)k+1$, and $y = \frac{b}{2}k+1$. Intuitively, x and y are the minimum numbers of items of size $\frac{1}{k}$ required to make the load of a bin greater than $b-1$ and $\frac{b}{2}$, respectively. The adversary works in 3 stages. In Stage 1, the following steps are repeated for $2k$ times: (i) an item of size $\frac{b}{2}$ is released, (ii) followed by an item with size $\frac{1}{k}$, (iii) the item with size $\frac{b}{2}$ departs, and (iv) finally $y-1$ items of size $\frac{1}{k}$ are released. Consider how WF packs the above item into bins of size b. We observe that after every round of the 4 steps, WF uses one more bin and each of the existing bins contains y items of size $\frac{1}{k}$, i.e., has a load of more than $\frac{b}{2}$. The base case is easy. Assume the observation holds after some round. Then, the newly released item of size $\frac{b}{2}$ cannot be packed into any existing bin, requiring a new bin. The $\frac{1}{k}$-item followed is also packed into a new bin because WF packs to the lightest load bin (all existing bins have load $> \frac{b}{2}$). After the departure of the $\frac{b}{2}$ item, the new bin used has the lightest load among the others, and so all the remaining $y-1$ items are packed into this bin as well, making its load $\frac{y}{k} > \frac{b}{2}$.

Stage 2 is divided into $y-x$ rounds. In round j, for $1 \le j \le y-x-1$, an $\frac{1}{k}$-item from each of the bins used in Stage 1 departs, then 2 items of size $\frac{y+j}{k}$ are released. We claim that at the end of round j, for any $1 < j < y-x-1$, WF uses $2k+2j$ bins and each of them has a load $\ge \frac{y-j}{k}$. In round 1, the newly arrived $\frac{y+1}{k}$-items have to be packed into a new bin because each of the first $2k$ bins have a load of $\frac{y-1}{k}$ after items departed and $\frac{y+1}{k} + \frac{y-1}{k} > b$. The 2 newly arrived $\frac{y+1}{k}$-items are also packed into 2 different bins because their size is $> \frac{b}{2}$. Two new bins are used and thus the claim holds for round 1. Assume that the claim holds for round j. In round $j+1$, all of the existing bins have a load of $\ge \frac{y-j-1}{k}$ after items departed. The newly arrived $\frac{y+j+1}{k}$-items have to be packed into a new bin because $\frac{y-j-1}{k} + \frac{y+j+1}{k} > b$. The two newly arrived $\frac{y+j+1}{k}$-items also have to be packed into 2 different bins because their size $> \frac{b}{2}$. WF uses $2k+2j+2$ bins, each has a load $\ge \frac{y-j-1}{k}$, and the claim holds. At the end of Stage 2, WF uses $2k+2(y-x-1) = 2k+k(2-b)$ bins and each of them has a load $\ge \frac{x}{k}$.

In Stage 3, items of size 1 are released. The amount of 1-items are set in a way such that the optimal off-line algorithm can pack all items into $kb+3$ bins as we claimed. In other words, we require that (i) the number of 1-items plus the number of items of size larger than $\frac{b}{2} > \frac{1}{2}$ (those released in Stage 2) does not exceed $kb+3$ since each of these items have to be packed in a separate bin of size 1; and (ii) the number of 1-items plus the total item size left at the end of Stage 2 does not exceed $kb+3$. By arithmetic, this means $\min\{2(b-1)k, \frac{(b^2-4b+4)k}{4}\}$ items with size 1 are released (note that we can pick k in such a way that it is divisible by 4). In this stage, all the $\min\{2(b-1)k, \frac{(b^2-4b+4)k}{4}\}$ items of size 1 have to be packed into different new bins because $\frac{x}{k}+1 > b$ and $b < 2$. Combining the 3 stages, WF uses $\min\{(2+b)k, \frac{(b^2-8b+20)k}{4}\} + O(1)$ bins at the end of Stage 3.

On the other hand, during Stage 1, the optimal off-line algorithm can use 1 bin to hold the item with size $\frac{b}{2}$, and pack the other items with $\frac{1}{k}$ into $kb+2$ bins. The $\frac{1}{k}$ items are arranged such that the departing items in Stage 2 comes from 2 groups, the group $G1$ consists of $2(\frac{(2-b)k}{2})$ bins, and the group $G2$ consists of $\min\{2(b-1)k, \frac{(b^2-4b+4)k}{4}\}$ bins. Note that in each round of Stage 2, total size of items depart is 2, and total size of items release ≤ 2. The items are arranged such that in round j of Stage 2, $(y+j)$ items depart from 2 of the bins in $G1$, and remaining items depart from bins in $G2$. The space in the 2 bins of $G1$ can be reused to pack the 2 newly released size $\frac{y+j}{k}$ items, and the space in $G2$ can be later reused to pack the size 1 items in Stage 3. Hence, the number of bins used by the optimal off-line algorithm is at most $kb + 1$. As a result, WF using size-b bins is no better than $\min\{\frac{2+b}{b}, \frac{(b^2-8b+20)}{4b}\}$-competitive. $\square$

5 Concluding remarks

In this paper, we have shown a 2.5 lower bound for dynamic bin packing, revealing that dynamic bin packing of general items is more difficult than unit fraction items. An open question is to close the gap between this 2.5 lower bound and the 2.788 upper bound [7]. We believe it is possible to push down the upper bound by analyzing some modified version of FF. One can also analyze other algorithms, like the class of Harmonic algorithms [18], yet our preliminary study showed that some versions of Harmonic algorithm have a non-constant lower bound for DBP; further investigation on other variants of Harmonic algorithms is desirable. We also give the first resource augmentation analysis for dynamic bin packing, showing that doubling bin size is both necessary and sufficient to achieve 1-competitiveness. Trade-off between bin size and competitive ratio is also studied. Note that the formula derived for the upper bound of FF does not yet match the general lower bound. We are attempting to give tighter bounds to close the gap.

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