COMP116 – Work Sheet Five

Associated Module Learning Outcomes

1. basic understanding of the range of techniques used to analyse and reason about computational settings.
2. Basic understanding of manipulating complex numbers and translating between different representations.

Question 1: Complex Number Manipulation

We have seen in the lectures (*vide* also Section 5.4, pages 202–207 of the module textbook) there are a number of equivalent approaches that can be adopted in specifying the structure of a complex number $z \in \mathbb{C}$. This question concerns not only translating between different formalisms but also commenting on the advantages and drawbacks of different schemes. We recall the following:

a. **Standard** form of $z \in \mathbb{C}$: $z = \alpha + \imath\beta$ ($\alpha = \Re(z) \in \mathbb{R}$; $\beta = \Im(z) \in \mathbb{R}$)

b. **Matrix** form of $\in \mathbb{C}$, (denoted $\mathbf{M}_z$)

$$\mathbf{M}_z = \begin{pmatrix} \Re(z) & -\Im(z) \\ \Im(z) & \Re(z) \end{pmatrix}$$

c. **Argand** form of $z \in \mathbb{C}$. A coordinate in the **Complex plane** having the form $(\Re(z), \Im(z))$ defining the **vector** $< \Re(z), \Im(z) >$ relative to the origin $(0, 0)$ in this plane. **Note:** Recall that vectors have size and direction but not position.

d. **Polar** form of $z \in \mathbb{C}$. Here z is specified by a pair of Real values $z = (r, \theta)$ in which r is the size of z and θ (called the *argument* of z, denoted $\arg z$) is the angle in **radians** formed by the i **vector** $< \Re(z), \Im(z) >$ taken counter-clockwise from the $\Re$-axis.

e. **Euler** form of $z \in \mathbb{C}$. The value z is described by $z = re^{\imath\theta}$ wherein r is the size of z and $\theta = \arg z$.

In Figure 1 the notion of polar and Argand form is expanded in more detail.

In the questions below the Complex numbers x and y are in Standard form with specification

$$\begin{aligned} x &= 3 + 4\imath \\ y &= 12 + 5\imath \end{aligned}$$

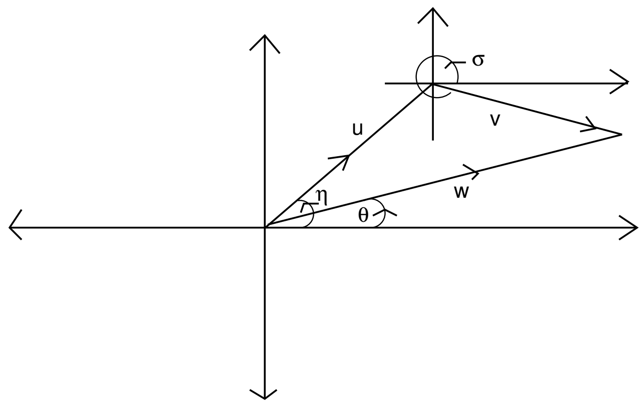


Figure 1: Polar Coordinates: $\arg u = \eta$; $\arg v = \sigma$; $\arg w = \arg(u+v) = \theta$

a. Give the matrix forms $\mathbf{M}_x$ and $\mathbf{M}_y$.

b State the values $|x|$, $|y|$ (the sizes of x and y).

c. Give the *polar* coordinate forms of x and y. (**Hint:** check the translation from Argand to Polar coordinates on p. 205 of the module textbook).

d. Which do you consider the "easiest" method to calculate $\arg(x \cdot y)$?

   1. Find $\arg(\gamma + \delta\imath)$ with $\gamma + \delta\imath$ the standard form of $(3+4\imath)\cdot(12+5\imath)$?
   2. Construct the Euler representations of x and y?

e. Suppose that x is replaced by $x' \in \mathbb{C}$ with $x' = 4 + 3\imath$ and, similarly y by y' having $y' = 5 + 12\imath$. What are the Complex numbers $x \cdot x'$ and $y \cdot y'$?

f. Having observed the outcomes from (e), what, if anything, can you deduce about $(\alpha + \beta\imath)\cdot(\beta + \alpha\imath)$?

Question 2 - Complex Numbers, Fractal Sets and Computer Graphics

One of the more colourful applications of Complex number properties in Computer Science arises in the behaviour of **Complex quadratic functions** when these are used to classify numbers in the Complex Plane.

The structures defined are known as *Julia Sets* (after the early 20th Century French scholar Gaston Julia[1]).

Every Complex Number $c \in \mathbb{C}$ gives rise to a Julia Set as follows. Letting $J(c)$ be the Julia set associated with c,

$$J(c) \ = \ \{z \in \mathbb{C} \ : \ \forall n \in \mathbb{W}, |z_{n+1}| \leq R, \ (z_{n+1} = z_n^2 + c \text{ and } z_0 = z)\}$$

The value $R \in \mathbb{R}$ is an *escape radius* for c and must satisfy $R(R-1) \geq |c|$, i.e.

$$R \geq \frac{1 \ + \ \sqrt{1 + 4|c|}}{2}$$

Informally $J(c)$ contains only those $z \in \mathbb{C}$ for which, no matter how often one iterates the computation $z_n^2 + c$ (starting from $z_0 = z$) the resulting value will always have size (modulus) at most R.

Julia Sets have been used in (at least) two ways within Computer Science:

1. As a methods for creating very intricate graphical images.
2. As an approach to computer music, promoted in work of Elaine Walker.[2]

The Java source code at

`http://www.csc.liv.ac.uk/~ped/COMP116/TUTORIALS/Julia.java`

Provides an (extremely basic) generator for graphical display of Julia sets. Some selected outputs are shown in Tables 1 and 2.

In each of these the colour assigned to a pixel is determined (rather crudely) by the "exit time", i.e. the number of tests made prior to the z iterating to a value outside the escape radius.

[1] G. Julia. Memoire sur l'iteration des fonctions rationnelles. *J. Math. Pures Appl.*, 8:47–245, 1918., see module textbook pages 250–2

[2] E. Walker. Chaos melody theory. *Music in Music Technology New York University, Master's thesis*, 2001.

a. Compile and try running the Java code referred to above. The program asks for **three** parameters (all **double**):

   i. The **Real** part of the Complex number, c, being used.

   ii. The **Imaginary** part of the Complex number, c, being used.

   iii. The **escape radius**, R, used both to determine the section of the Complex plane considered and the point at which $f_c^n(z)$ takes z out of the Julia set. This value, R, is required to satisfy $R(R-1) \geq |c|$ (the input will loop until an appropriate R is provided). A value of $R \sim 4$ suffices for the cases below.

b. What effect is noticeable keeping $\Im(c) = 0$ and allowing $\Re(c)$ to take the values in

$$\{-1.5, -1, -0.5\}?$$

c. What effect is noticeable keeping $\Im(c) = 0$ and allowing $\Re(c)$ to take the values in

$$\{1.5, 1, 0.5\}?$$

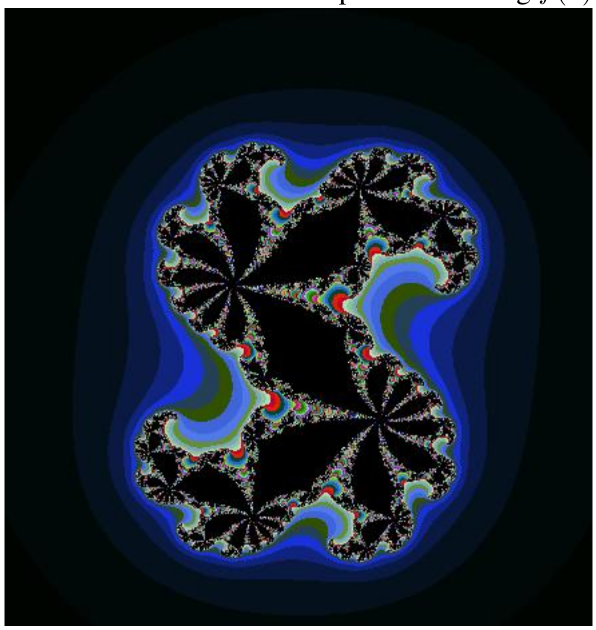
d. Is similar behaviour evident using $\Re(c) = 0$ and $\Im(c)$ in

$$\{-1.5, -1, -0.5\}$$

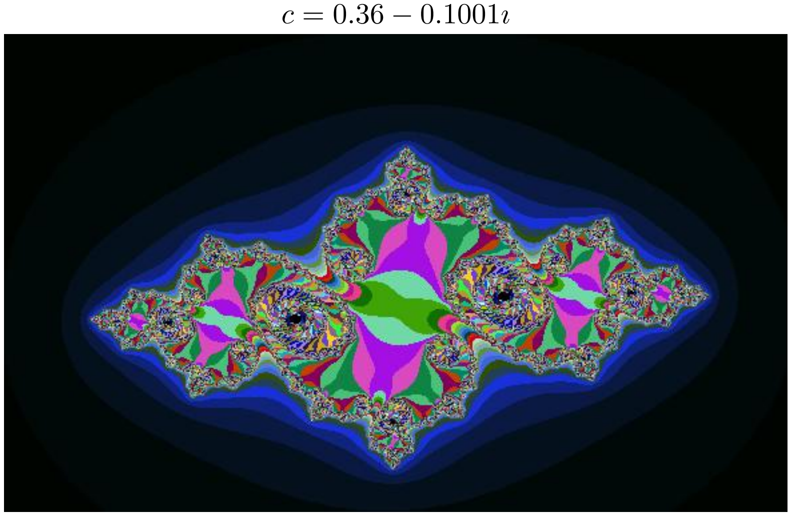
   Finally compare these with $\Re(c) = 0$ and $\Im(c)$ from

$$\{1.5, 1, 0.5\}$$

Table 1: Julia Sets in Complex Plane using $f(z) = z^2 + c$

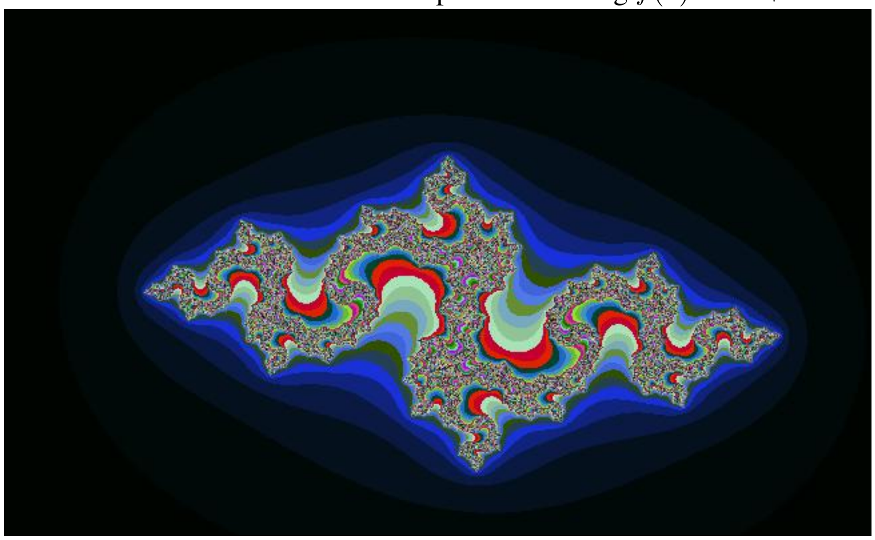


$c = 0.36 - 0.1001\imath$

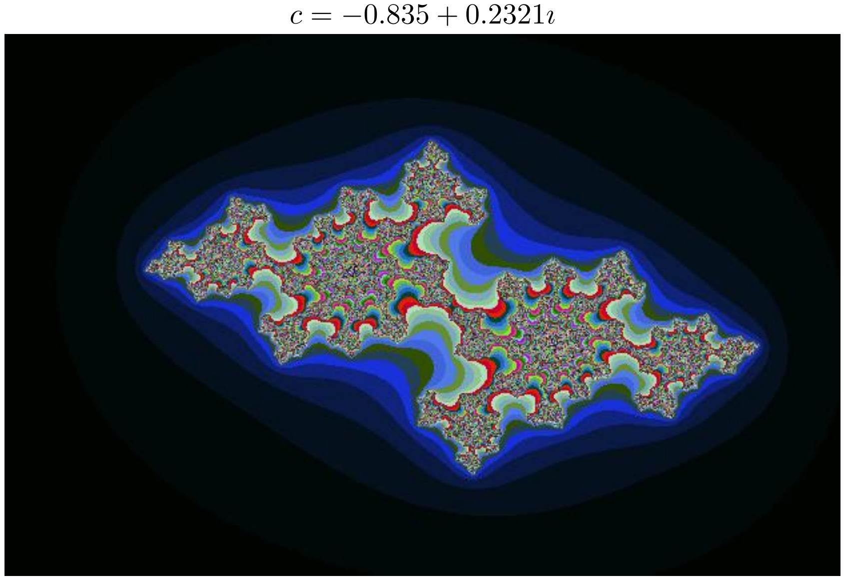


$c = -0.758 - 0.121\imath$

Table 2: Julia Sets in Complex Plane using $f(z) = z^2 + c$



$c = -0.835 + 0.2321\imath$



$c = -0.70176 + 0.38424\imath$