FO $=$ FO^3 for Linear Orders with Monotone Binary Relations

Marie Fortin

University of Liverpool

YR-OWLS, June 16, 2020

The k-variable property

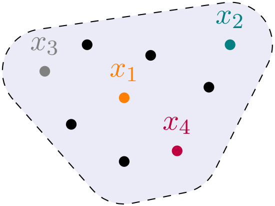
How many variables are needed in first-order logic ?

The k-variable property

How many variables are needed in first-order logic ?

- Some properties require unboundedly many variables

$$\exists x_1.\exists x_2.\exists x_3.\exists x_4.\ \bigwedge_{1\leq i<j\leq 4} x_i \neq x_j$$

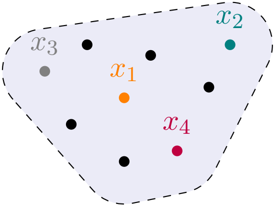


The k-variable property

How many variables are needed in first-order logic ?

- Some properties require unboundedly many variables

$$\exists x_1.\exists x_2.\exists x_3.\exists x_4.\ \bigwedge_{1\le i<j\le 4} x_i \neq x_j$$



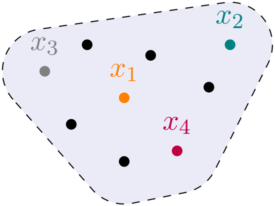
- ... but not in every class of models:

The k-variable property

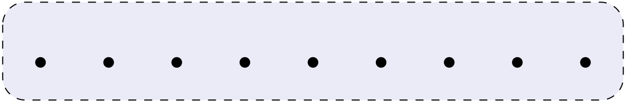
How many variables are needed in first-order logic ?

- Some properties require unboundedly many variables

$$\exists x_1.\exists x_2.\exists x_3.\exists x_4.\ \bigwedge_{1\le i<j\le 4} x_i \neq x_j$$



- ... but not in every class of models:
  $\exists x.\exists y.\big(x < y \wedge \exists x.\,(y < x \wedge \exists y.\, x < y)\big)$

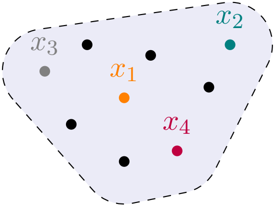


The k-variable property

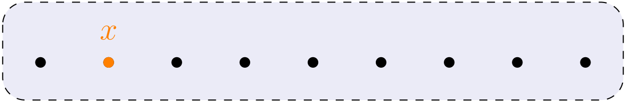
How many variables are needed in first-order logic ?

- Some properties require unboundedly many variables

$$\exists x_1.\exists x_2.\exists x_3.\exists x_4.\ \bigwedge_{1\leq i<j\leq 4} x_i \neq x_j$$



- ... but not in every class of models:
  $\exists x.\exists y.\big(x < y \wedge \exists x.\,(y < x \wedge \exists y.\, x < y)\big)$

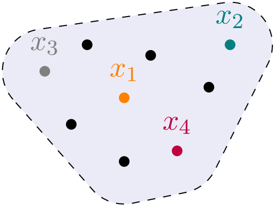


The k-variable property

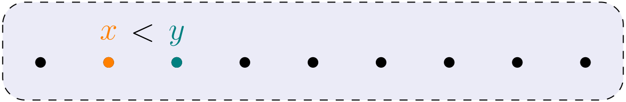
How many variables are needed in first-order logic ?

- Some properties require unboundedly many variables

$$\exists x_1.\exists x_2.\exists x_3.\exists x_4.\ \bigwedge_{1\leq i<j\leq 4} x_i \neq x_j$$



- ... but not in every class of models:
  $\exists x.\exists y.\big(x < y \wedge \exists x.\,(y < x \wedge \exists y.\, x < y)\big)$

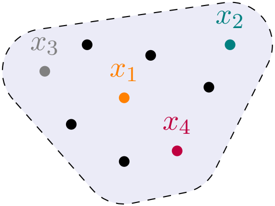


The k-variable property

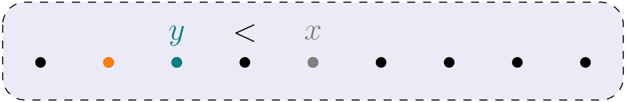
How many variables are needed in first-order logic ?

- Some properties require unboundedly many variables

$$\exists x_1.\exists x_2.\exists x_3.\exists x_4.\ \bigwedge_{1\leq i<j\leq 4} x_i \neq x_j$$



- ... but not in every class of models:
  $\exists x.\exists y.\big(x < y \wedge \exists x.\,(y < x \wedge \exists y.\, x < y)\big)$

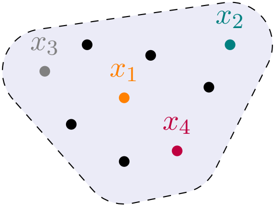


The k-variable property

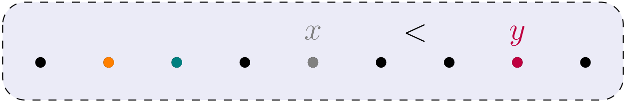
How many variables are needed in first-order logic ?

- Some properties require unboundedly many variables

$$\exists x_1.\exists x_2.\exists x_3.\exists x_4.\ \bigwedge_{1\leq i<j\leq 4} x_i \neq x_j$$



- ... but not in every class of models:
$\exists x.\exists y.\big(x < y \wedge \exists x.\,(y < x \wedge \exists y.\, x < y)\big)$

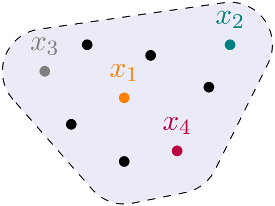


The k-variable property

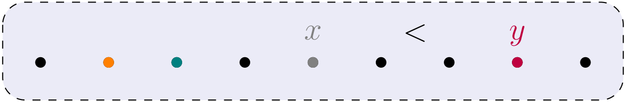
How many variables are needed in first-order logic ?

- Some properties require unboundedly many variables

$\exists x_1.\exists x_2.\exists x_3.\exists x_4.\ \bigwedge_{1\leq i<j\leq 4} x_i \neq x_j$



- ... but not in every class of models:
  $\exists x.\exists y.\big(x < y \wedge \exists x.\,(y < x \wedge \exists y.\,x < y)\big)$



Over linear orders, FO $=$ FO^3.

Bounded variable logics

Why do we care about the number of variables?

Bounded variable logics

Why do we care about the number of variables?

- (Descriptive) complexity

Bounded variable logics

Why do we care about the number of variables?

- (Descriptive) complexity
- Temporal logics

Bounded variable logics

Why do we care about the number of variables?

- (Descriptive) complexity
- Temporal logics

[Gabbay 1981] In any class of time flows, TFAE:

- There exists an expressively complete finite set of FO-definable (multi-dimensional) temporal connectives
- There exists k such that every first-order sentence is equivalent to one with at most k variables

Example

$\boxed{\text{Over } \textbf{linear orders}, \ \mathsf{FO} = \mathsf{FO}^3.}$

Example

$\boxed{\text{Over } \textbf{linear orders}\text{, } \mathsf{FO} = \mathsf{FO}^3.}$

Two classical techniques to prove $\mathsf{FO} = \mathsf{FO}^k$ (over a class $\mathcal{C}$)

Example

$\boxed{\text{Over } \textbf{linear orders}, \text{FO} = \text{FO}^3.}$

Two classical techniques to prove $\text{FO} = \text{FO}^k$ (over a class $\mathcal{C}$)

1. Corollary of expressive completeness of a temporal logic

Example

$\boxed{\text{Over } \textbf{linear orders}\text{, FO} = \text{FO}^3.}$

Two classical techniques to prove $\text{FO} = \text{FO}^k$ (over a class $\mathcal{C}$)

1. Corollary of expressive completeness of a temporal logic

Example: Over complete linear orders,
$\text{FO}^3 \subseteq \text{FO} = \text{LTL} \subseteq \text{FO}^3$ [Kamp 1968]

Example

$\boxed{\text{Over } \textbf{linear orders}, \text{ FO} = \text{FO}^3.}$

Two classical techniques to prove $\mathsf{FO} = \mathsf{FO}^k$ (over a class $\mathcal{C}$)

1. Corollary of expressive completeness of a temporal logic

Example: Over complete linear orders,
$\mathsf{FO}^3 \subseteq \mathsf{FO} = \mathsf{LTL} \subseteq \mathsf{FO}^3$ [Kamp 1968]

Over (arbitrary) linear orders,
$\mathsf{FO}^3 \subseteq \mathsf{FO} = \mathsf{LTL}$ with Stavi connectives $\subseteq \mathsf{FO}^3$
[Gabbay, Hodkinson, Reynolds 1993]

Example

$\boxed{\text{Over } \textbf{linear orders}, \text{ FO} = \text{FO}^3.}$

Two classical techniques to prove $\mathsf{FO} = \mathsf{FO}^k$ (over a class $\mathcal{C}$)

1. Corollary of expressive completeness of a temporal logic
2. Ehrenfeucht-Fraïssé games with k pebbles

Example

$\boxed{\text{Over } \textbf{linear orders}, \text{ FO} = \text{FO}^3.}$

Two classical techniques to prove $\mathsf{FO} = \mathsf{FO}^k$ (over a class $\mathcal{C}$)

1. Corollary of expressive completeness of a temporal logic
2. Ehrenfeucht-Fraïssé games with k pebbles

Example: Over complete linear orders,
$\mathsf{FO} = \mathsf{FO}^3$ [Immerman, Kozen 1989]

Example

$\boxed{\text{Over } \textbf{linear orders}, \text{ FO} = \text{FO}^3.}$

Two classical techniques to prove FO $=$ FO^k (over a class $\mathcal{C}$)

1. Corollary of expressive completeness of a temporal logic
   0 or 1 free variables
2. Ehrenfeucht-Fraïssé games with k pebbles
   up to k free variables

Known results (non-exhaustive)

$$\boxed{\begin{array}{c}\text{Over linear orders,}\\ \mathsf{FO} = \mathsf{FO}^3\\ \text{[Immerman-Kozen'89]}\end{array}} \checkmark$$

Known results (non-exhaustive)

$$\boxed{\begin{array}{c}\text{Over linear orders,}\\ \mathsf{FO} = \mathsf{FO}^3\\ \text{[Immerman-Kozen'89]}\end{array}}\ \checkmark$$

What happens if we have additional binary relations?

Known results (non-exhaustive)

Over linear orders,
$$\mathsf{FO} = \mathsf{FO}^3$$
[Immerman-Kozen'89]

✓

What happens if we have additional binary relations?

Over ordered graphs,
$$\forall k, \mathsf{FO} \neq \mathsf{FO}^k$$
[Rossman'08]

✗

Known results (non-exhaustive)

Over linear orders,
$\mathsf{FO} = \mathsf{FO}^3$
[Immerman-Kozen'89]
✓

What happens if we have additional binary relations?

Over ordered graphs,
$\forall k, \mathsf{FO} \neq \mathsf{FO}^k$
[Rossman'08]
✗

Over $(\mathbb{R}, <, +1)$,
$\mathsf{FO} = \mathsf{FO}^3$
[AHRW'15]
✓

Known results (non-exhaustive)

Over linear orders,
FO $=$ FO^3
[Immerman-Kozen'89]
✓

What happens if we have additional binary relations?

Over ordered graphs,
$\forall k,$ FO $\neq$ FO^k
[Rossman'08]
✗

Over $(\mathbb{R}, <, +1)$,
FO $=$ FO^3
[AHRW'15]
✓

Over Mazurkiewicz traces,
FO $=$ FO^3
[Gastin-Mukund'02]
✓

Over MSCs,
FO $=$ FO^3
[Bollig-F.-Gastin'18]
✓

Known results (non-exhaustive)

Over linear orders,
$\mathsf{FO} = \mathsf{FO}^3$
[Immerman-Kozen'89]

✓

What happens if we have additional binary relations?

Over ordered graphs,
$\forall k, \mathsf{FO} \neq \mathsf{FO}^k$
[Rossman'08]

✗

Over $(\mathbb{R}, <, +1)$,
$\mathsf{FO} = \mathsf{FO}^3$
[AHRW'15]

✓

Over Mazurkiewicz traces,
$\mathsf{FO} = \mathsf{FO}^3$
[Gastin-Mukund'02]

✓

Over MSCs,
$\mathsf{FO} = \mathsf{FO}^3$
[Bollig-F.-Gastin'18]

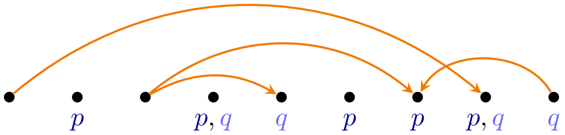
✓

What do these 4 positive results have in common?

Generalisation [F.'19]

$\mathsf{FO} = \mathsf{FO}^3$ over structures with

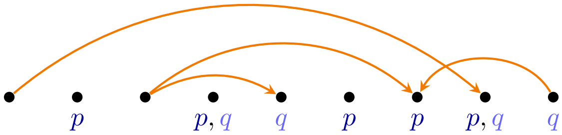
- one linear order $\leq$,
- "interval-preserving" binary relations $R_1, R_2, \ldots$,
- arbitrary unary predicates $p, q, \ldots$



Generalisation [F.'19]

FO = FO^3 over structures with

- one linear order $\leq$,
- "interval-preserving" binary relations $R_1, R_2, \ldots$,
- arbitrary unary predicates $p, q, \ldots$



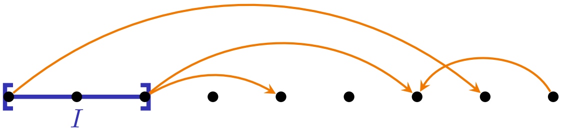
R is interval-preserving if for all intervals I,

- $R(I)$ is an interval of $(\mathsf{Im}(R), \leq)$
- $R^{-1}(I)$ is an interval of $(\mathsf{dom}(R), \leq)$

Generalisation [F.'19]

FO = FO^3 over structures with

- one linear order $\leq$,
- "interval-preserving" binary relations $R_1, R_2, \ldots$,
- arbitrary unary predicates $p, q, \ldots$



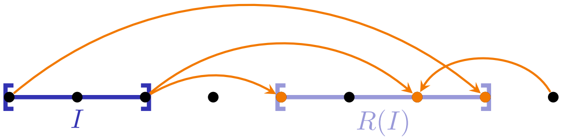
R is interval-preserving if for all intervals I,

- $R(I)$ is an interval of $(\mathrm{Im}(R), \leq)$
- $R^{-1}(I)$ is an interval of $(\mathrm{dom}(R), \leq)$

Generalisation [F.'19]

FO = FO^3 over structures with

- one linear order $\leq$,
- "interval-preserving" binary relations $R_1, R_2, \ldots$,
- arbitrary unary predicates $p, q, \ldots$



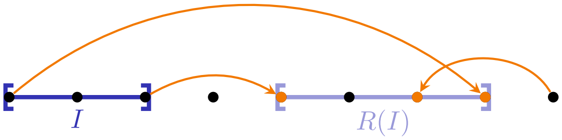
R is interval-preserving if for all intervals I,

- $R(I)$ is an interval of $(\mathrm{Im}(R), \leq)$
- $R^{-1}(I)$ is an interval of $(\mathrm{dom}(R), \leq)$

Generalisation [F.'19]

FO $=$ FO^3 over structures with

- one linear order $\leq$,
- "interval-preserving" binary relations $R_1, R_2, \ldots$,
- arbitrary unary predicates $p, q, \ldots$



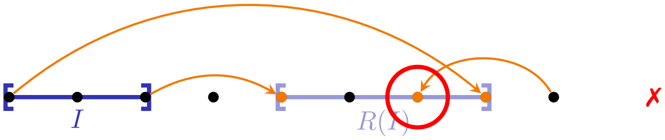
R is interval-preserving if for all intervals I,

- $R(I)$ is an interval of $(\mathsf{Im}(R), \leq)$
- $R^{-1}(I)$ is an interval of $(\mathsf{dom}(R), \leq)$

Generalisation [F.'19]

FO = FO^3 over structures with

- one linear order $\leq$,
- "interval-preserving" binary relations $R_1, R_2, \ldots$,
- arbitrary unary predicates $p, q, \ldots$



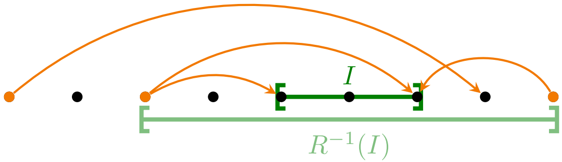
R is interval-preserving if for all intervals I,

- $R(I)$ is an interval of $(\mathsf{Im}(R), \leq)$
- $R^{-1}(I)$ is an interval of $(\mathsf{dom}(R), \leq)$

Generalisation [F.'19]

FO = FO^3 over structures with

- one linear order $\leq$,
- "interval-preserving" binary relations $R_1, R_2, \ldots$,
- arbitrary unary predicates $p, q, \ldots$



R is interval-preserving if for all intervals I,

- $R(I)$ is an interval of $(\mathsf{Im}(R), \leq)$
- $R^{-1}(I)$ is an interval of $(\mathsf{dom}(R), \leq)$

A special case: monotone partial functions

Any relation R corresponding to a monotone partial function is interval-preserving.

A special case: monotone partial functions

Any relation R corresponding to a monotone partial function is interval-preserving.

- $R(I)$ is an interval of $(\mathsf{Im}(R), \leq)$

A special case: monotone partial functions

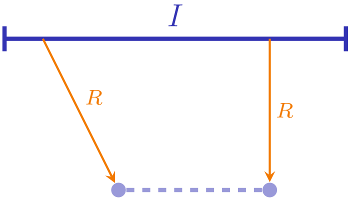
Any relation R corresponding to a monotone partial function is interval-preserving.

- $R(I)$ is an interval of $(\mathsf{Im}(R), \leq)$

A special case: monotone partial functions

Any relation R corresponding to a monotone partial function is interval-preserving.

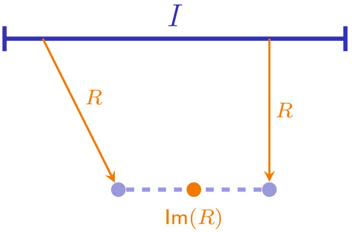
- $R(I)$ is an interval of $(\mathsf{Im}(R), \leq)$



A special case: monotone partial functions

Any relation R corresponding to a monotone partial function is interval-preserving.

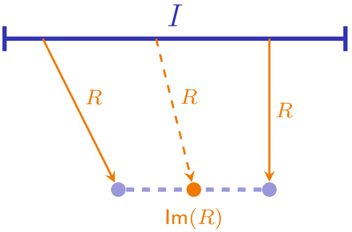
- $R(I)$ is an interval of $(\mathsf{Im}(R), \leq)$



A special case: monotone partial functions

Any relation R corresponding to a monotone partial function is interval-preserving.

- $R(I)$ is an interval of $(\mathsf{Im}(R), \leq)$



A special case: monotone partial functions

Any relation R corresponding to a monotone partial function is interval-preserving.

- $R(I)$ is an interval of $(\mathsf{Im}(R), \leq)$
- $R^{-1}(I)$ is an interval of $(\mathsf{dom}(R), \leq)$

A special case: monotone partial functions

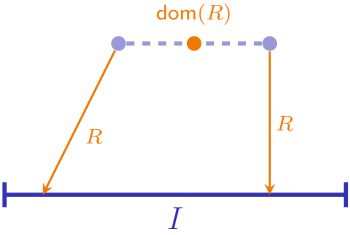
Any relation R corresponding to a monotone partial function is interval-preserving.

- $R(I)$ is an interval of $(\mathrm{Im}(R), \leq)$
- $R^{-1}(I)$ is an interval of $(\mathrm{dom}(R), \leq)$

A special case: monotone partial functions

Any relation R corresponding to a monotone partial function is interval-preserving.

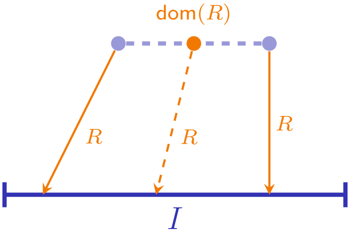
- $R(I)$ is an interval of $(\mathrm{Im}(R), \leq)$
- $R^{-1}(I)$ is an interval of $(\mathrm{dom}(R), \leq)$



A special case: monotone partial functions

Any relation R corresponding to a monotone partial function is interval-preserving.

- $R(I)$ is an interval of $(\mathsf{Im}(R), \leq)$
- $R^{-1}(I)$ is an interval of $(\mathsf{dom}(R), \leq)$



Applications

$\mathsf{FO} = \mathsf{FO}^3$ over

1. Linear orders with partial non-decreasing or non-increasing functions (new)

Applications

$\mathsf{FO} = \mathsf{FO}^3$ over

1. Linear orders with partial non-decreasing or non-increasing functions (new)
2. Linear orders: finite or infinite words, $\mathbb{R}$, $\mathbb{Q}$, ordinals...

Applications

$\mathsf{FO} = \mathsf{FO}^3$ over

1. Linear orders with partial non-decreasing or non-increasing functions (new)
2. Linear orders: finite or infinite words, $\mathbb{R}$, $\mathbb{Q}$, ordinals...
3. $(\mathbb{R}, \leq, +1)$, $(\mathbb{R}, \leq, (+q)_{q \in \mathbb{Q}})$ ...

Applications

$\mathsf{FO} = \mathsf{FO}^3$ over

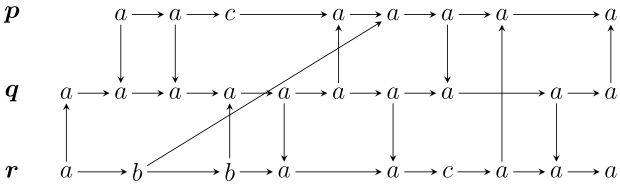
1. Linear orders with partial non-decreasing or non-increasing functions (new)
2. Linear orders: finite or infinite words, $\mathbb{R}$, $\mathbb{Q}$, ordinals...
3. $(\mathbb{R}, \leq, +1)$, $(\mathbb{R}, \leq, (+q)_{q \in \mathbb{Q}})$ $\ldots$
4. $(\mathbb{R}, \leq)$ + polynomial functions (new)

Applications

5. Message sequence charts (MSCs)

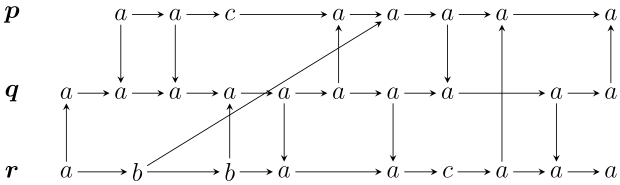
Applications

5. Message sequence charts (MSCs)



Applications

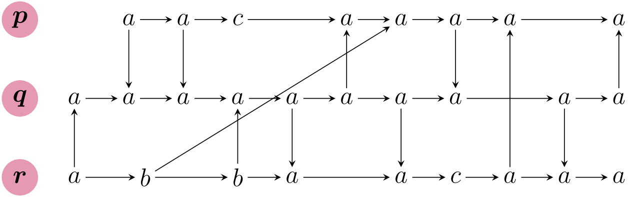
5. Message sequence charts (MSCs)



Executions of **message-passing systems**

Applications

5. Message sequence charts (MSCs)

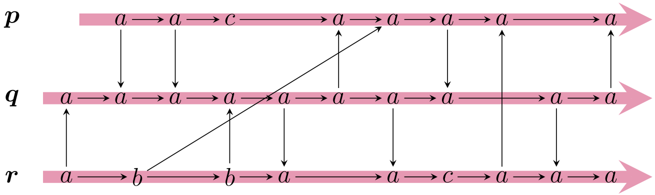


Executions of **message-passing systems**

- Fixed, finite set of processes

Applications

5. Message sequence charts (MSCs)

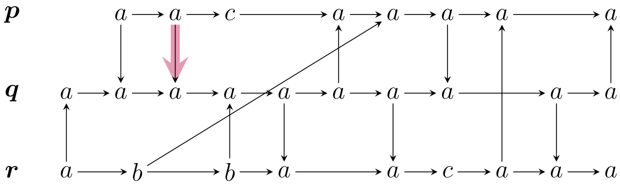


Executions of **message-passing systems**

- Fixed, finite set of processes
- Process order $\leq_{\mathsf{proc}}$

Applications

5. Message sequence charts (MSCs)

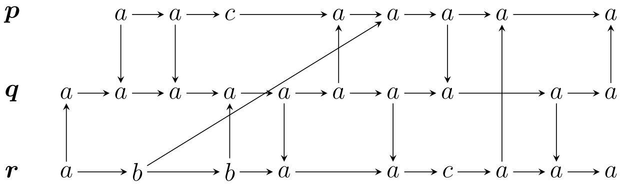


Executions of **message-passing systems**

- Fixed, finite set of processes
- Process order $\leq_{\mathsf{proc}}$
- Message relations $\lhd_{p,q}$

Applications

5. Message sequence charts (MSCs)

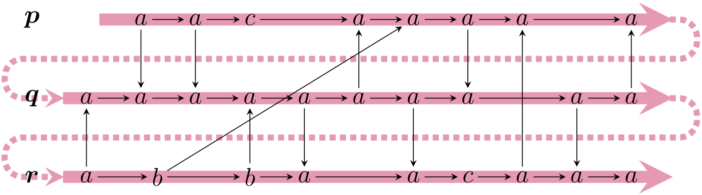


Executions of **message-passing systems**

- Fixed, finite set of processes
- Process order $\leq_{\text{proc}}$
- Message relations $\lhd_{p,q}$

Applications

5. Message sequence charts (MSCs)

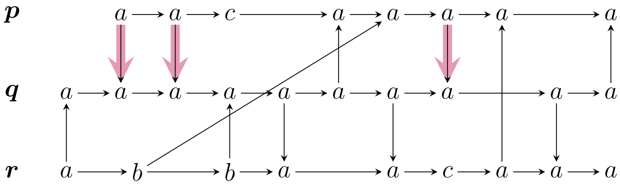


Executions of **message-passing systems**

- Fixed, finite set of processes
- Process order $\leq_{\mathsf{proc}}$ — Extended to a linear order
- Message relations $\lhd_{p,q}$

Applications

5. Message sequence charts (MSCs)

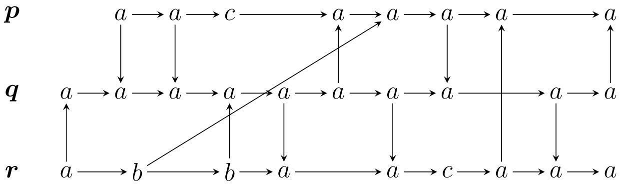


Executions of **message-passing systems**

- Fixed, finite set of processes
- Process order $\leq_{\mathsf{proc}}$ — Extended to a linear order
- Message relations $\lhd_{p,q}$ — FIFO $\rightarrow$ monotone

Applications

5. Message sequence charts (MSCs)



Executions of **message-passing systems**

- Fixed, finite set of processes
- Process order $\leq_{\text{proc}}$
- Message relations $\lhd_{p,q}$

Extended to a linear order

FIFO $\to$ monotone

$\to$ Interval-preserving structure

Applications

$\mathsf{FO} = \mathsf{FO}^3$ over structures with

- one linear order $\leq$,
- "interval-preserving" binary relations $R_1, R_2, \ldots$,
- arbitrary unary predicates $p, q, \ldots$

1. Linear orders with partial non-decreasing or non-increasing functions (new)
2. Linear orders: finite or infinite words, $\mathbb{R}$, $\mathbb{Q}$, ordinals...
3. $(\mathbb{R}, \leq, +1)$, $(\mathbb{R}, \leq, (+q)_{q\in\mathbb{Q}})$ $\ldots$
4. $(\mathbb{R}, \leq)$ + polynomial functions (new)
5. MSCs
6. Mazurkiewicz traces

How does the interval-preserving assumption help?

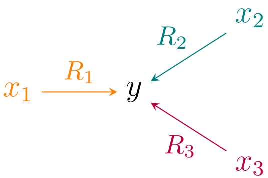
How does the interval-preserving assumption help?

$$\varphi(x_1, x_2, x_3) = \exists y.\, R_1(x_1, y) \land R_2(x_2, y) \land R_3(x_3, y)$$

$$\begin{tikzcd}
 & & x_2 \arrow[dl, "R_2"'] \\
x_1 \arrow[r, "R_1"] & y & \\
 & & x_3 \arrow[ul, "R_3"]
\end{tikzcd}$$

How does the interval-preserving assumption help?

$$\varphi(x_1, x_2, x_3) = \exists y.\, R_1(x_1, y) \wedge R_2(x_2, y) \wedge R_3(x_3, y)$$



$\boxed{\text{Equivalent FO}^3 \text{ formula?}}$

How does the interval-preserving assumption help?

$$\varphi(x_1, x_2, x_3) = \exists y. \, R_1(x_1, y) \wedge R_2(x_2, y) \wedge R_3(x_3, y)$$

Equivalent FO^3 formula?

How does the interval-preserving assumption help?

$$\varphi(x_1, x_2, x_3) = \exists y.\, R_1(x_1, y) \wedge R_2(x_2, y) \wedge R_3(x_3, y)$$

Equivalent FO^3 formula?

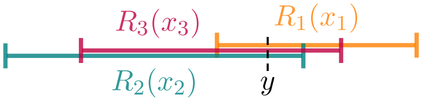
How does the interval-preserving assumption help?

$$\begin{aligned}\varphi(x_1, x_2, x_3) &= \exists y.\, R_1(x_1, y) \wedge R_2(x_2, y) \wedge R_3(x_3, y) \\ &\equiv \Big(\exists y.\, R_1(x_1, y) \wedge R_2(x_2, y) \wedge \qquad\qquad \Big) \wedge \\ & \Big(\exists y.\, R_1(x_1, y) \wedge R_3(x_3, y) \wedge \qquad\qquad \Big) \wedge \\ & \Big(\exists y.\, R_2(x_2, y) \wedge R_2(x_3, y) \wedge \qquad\qquad \Big)\end{aligned}$$

Equivalent FO^3 formula?

How does the interval-preserving assumption help?

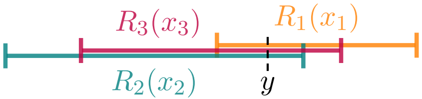
$$
\begin{aligned}
\varphi(x_1, x_2, x_3) &= \exists y.\, R_1(x_1, y) \wedge R_2(x_2, y) \wedge R_3(x_3, y) \\
&\equiv \Big(\exists y.\, R_1(x_1, y) \wedge R_2(x_2, y) \wedge \exists x.\, R_3(x, y)\Big) \wedge \\
&\quad\; \Big(\exists y.\, R_1(x_1, y) \wedge R_3(x_3, y) \wedge \exists x.\, R_2(x, y)\Big) \wedge \\
&\quad\; \Big(\exists y.\, R_2(x_2, y) \wedge R_2(x_3, y) \wedge \exists x.\, R_1(x, y)\Big)
\end{aligned}
$$



Equivalent FO^3 formula?

How does the interval-preserving assumption help?

$$\begin{aligned}\varphi(x_1,x_2,x_3) &= \exists y.\, R_1(x_1,y) \wedge R_2(x_2,y) \wedge R_3(x_3,y)\\ &\equiv \Big(\exists x_3.\, R_1(x_1,x_3) \wedge R_2(x_2,x_3) \wedge \exists x_1.\, R_3(x_1,x_3)\Big) \wedge\\ &\quad\ \Big(\exists x_2.\, R_1(x_1,x_2) \wedge R_3(x_3,x_2) \wedge \exists x_1.\, R_2(x_1,x_2)\Big) \wedge\\ &\quad\ \Big(\exists x_1.\, R_2(x_2,x_1) \wedge R_2(x_3,x_1) \wedge \exists x_2.\, R_1(x_2,x_1)\Big)\end{aligned}$$



Equivalent FO^3 formula?

The proof

$\mathsf{FO} = \mathsf{FO}^3$ over structures with

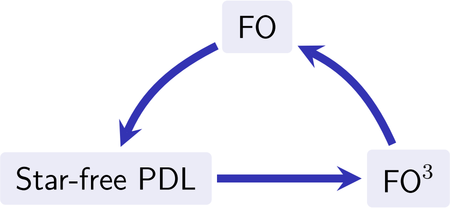
- one linear order $\leq$,
- "interval-preserving" binary relations $R_1, R_2, \ldots$,
- arbitrary unary predicates $p, q, \ldots$

The proof

$\mathsf{FO} = \mathsf{FO}^3$ over structures with

- one linear order $\leq$,
- "interval-preserving" binary relations $R_1, R_2, \ldots$,
- arbitrary unary predicates $p, q, \ldots$

Key idea: Go through an intermediate language:
Star-free Propositional Dynamic Logic.

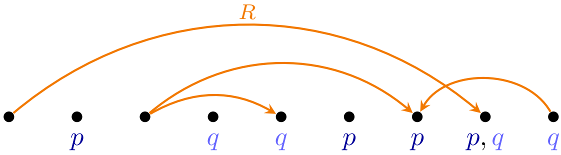


Star-free Propositional Dynamic Logic

Examples

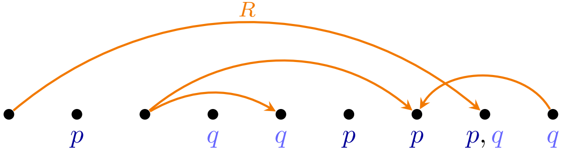
Star-free Propositional Dynamic Logic

Examples



Star-free Propositional Dynamic Logic

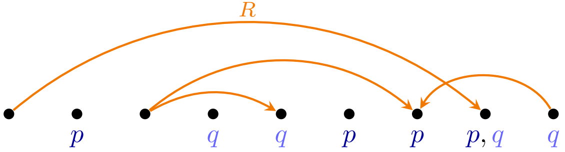
Examples



$(p \wedge \neg q) \vee (q \wedge \neg p)$	✗	✓	✗	✓	✓	✓	✓	✗	✓

Star-free Propositional Dynamic Logic

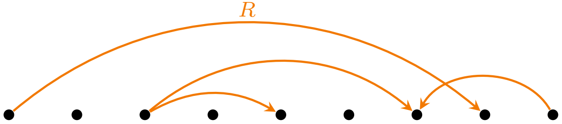
Examples



$(p \wedge \neg q) \vee (q \wedge \neg p)$	✗	✓	✗	✓	✓	✓	✓	✗	✓
$\langle R \rangle\, q$	✓	✗	✓	✗	✗	✗	✗	✗	✗

Star-free Propositional Dynamic Logic

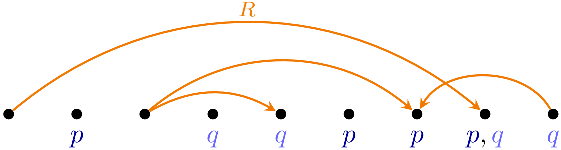
Examples



		p		q	q	p	p	p, q	q
$(p \wedge \neg q) \vee (q \wedge \neg p)$	✗	✓	✗	✓	✓	✓	✓	✗	✓
$\langle R \rangle\, q$	✓	✗	✓	✗	✗	✗	✗	✗	✗
$\langle \leq \cdot R^{-1} \rangle\, q$	✓	✓	✓	✓	✓	✓	✓	✗	✗

Star-free Propositional Dynamic Logic

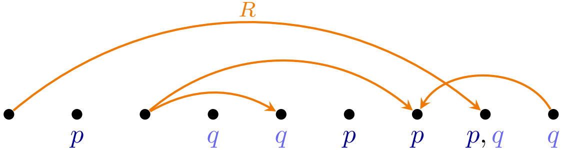
Examples



$(p \wedge \neg q) \vee (q \wedge \neg p)$	✗	✓	✗	✓	✓	✓	✓	✗	✓
$\langle R \rangle\, q$	✓	✗	✓	✗	✗	✗	✗	✗	✗
$\langle \leq \cdot R^{-1} \rangle\, q$	✓	✓	✓	✓	✓	✓	✓	✗	✗
$\langle \leq \cdot \{\langle R \rangle\, q\}? \cdot \leq \rangle\, p$	✓	✓	✓	✗	✗	✗	✗	✗	✗

Star-free Propositional Dynamic Logic

Examples

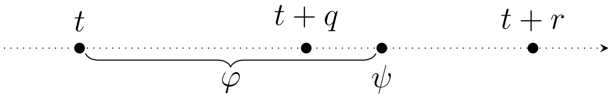


$(p \wedge \neg q) \vee (q \wedge \neg p)$	✗	✓	✗	✓	✓	✓	✓	✗	✓
$\langle R \rangle\, q$	✓	✗	✓	✗	✗	✗	✗	✗	✗
$\langle \leq \cdot R^{-1} \rangle\, q$	✓	✓	✓	✓	✓	✓	✓	✗	✗
$\langle \leq \cdot \{\langle R \rangle\, q\}? \cdot \leq \rangle\, p$	✓	✓	✓	✗	✗	✗	✗	✗	✗
$\langle R^c \cap \leq \rangle\, (p \wedge q)$	✗	✓	✓	✓	✓	✓	✓	✓	✗

Star-free Propositional Dynamic Logic
Examples

Over $(\mathbb{R}, <, \{+q \mid q \in \mathbb{Q}_+\})$,

$\varphi \, \mathsf{U}_{(q,r)} \, \psi \equiv$

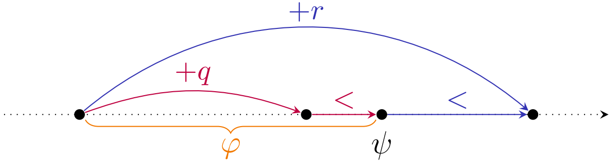


Star-free Propositional Dynamic Logic

Examples

Over $(\mathbb{R}, <, \{+q \mid q \in \mathbb{Q}_+\})$,

$$\varphi \,\mathsf{U}_{(q,r)}\, \psi \equiv \left\langle (+q \cdot <) \cap (+r \cdot <^{-1}) \cap (< \cdot \{\neg\varphi\}? \cdot <)^{\mathsf{c}} \right\rangle \psi$$



Star-free Propositional Dynamic Logic

Syntax

State formulas:

$$\varphi ::= P \mid \varphi \vee \varphi \mid \neg\varphi \mid \langle\pi\rangle\,\varphi$$

Path formulas:

$$\pi ::= \leq \mid R \mid \{\varphi\}? \mid \pi^{-1} \mid \pi \cdot \pi \mid \pi \cup \pi \mid \pi^{\mathsf{c}}$$

$\mathsf{PDL}_{\mathsf{sf}}$

Star-free Propositional Dynamic Logic

Syntax

State formulas:

$$\varphi ::= P \mid \varphi \vee \varphi \mid \neg\varphi \mid \langle \pi \rangle\, \varphi$$

Path formulas:

$$\pi ::= \leq \mid R \mid \{\varphi\}? \mid \pi^{-1} \mid \pi \cdot \pi \mid \pi \cup \pi \mid \pi^{\mathsf{c}}$$

$\mathsf{PDL}_{\mathsf{sf}}$

Combines features from

- Propositional Dynamic Logic [Fisher-Ladner 1979]
- Star-free regular expressions
- The calculus of relations

Star-free Propositional Dynamic Logic

Syntax

State formulas:

$$\varphi ::= P \mid \varphi \vee \varphi \mid \neg\varphi \mid \langle\pi\rangle\,\varphi$$

Path formulas:

$$\pi ::= \leq \mid R \mid \{\varphi\}? \mid \pi^{-1} \mid \pi \cdot \pi \mid \pi \cup \pi \mid \pi^{\mathsf{c}}$$

$\mathsf{PDL}_{\mathsf{sf}}$

Combines features from

- Propositional Dynamic Logic [Fisher-Ladner 1979]
- Star-free regular expressions
- The calculus of relations

Theorem: [Tarski-Givant 1987 (calculus of relations)]
$\mathsf{PDL}_{\mathsf{sf}}$ and FO^3 are expressively equivalent

A fragment of Star-free PDL

A fragment of Star-free PDL

State formulas:

$$\varphi ::= P \mid \varphi \vee \varphi \mid \neg\varphi \mid \langle\pi\rangle\,\varphi$$

Path formulas:

$$\pi ::= \leq \mid R \mid \{\varphi\}? \mid \pi^{-1} \mid \pi \cdot \pi \mid \pi \cup \pi \mid \pi^{\mathsf{c}}$$

$\mathsf{PDL_{sf}}$

$$\pi ::= \leq \mid R \mid \{\varphi\}? \mid \pi^{-1} \mid \pi \cdot \pi \mid \pi \cap \pi \mid (\leq \cdot \pi \cdot \leq)^{\mathsf{c}} \mid (\leq \cdot \pi \cdot \geq)^{\mathsf{c}} \mid (\geq \cdot \pi \cdot \leq)^{\mathsf{c}} \mid (\geq \cdot \pi \cdot \geq)^{\mathsf{c}}$$

$\mathsf{PDL_{sf}^{int}}$

A fragment of Star-free PDL

State formulas:

$$\varphi ::= P \mid \varphi \vee \varphi \mid \neg\varphi \mid \langle\pi\rangle\,\varphi$$

Path formulas:

$$\pi ::= \leq \mid R \mid \{\varphi\}? \mid \pi^{-1} \mid \pi \cdot \pi \mid \pi \cup \pi \mid \pi^{\mathsf{c}}$$

$\mathrm{PDL}_{\mathsf{sf}}$

$$\pi ::= \leq \mid R \mid \{\varphi\}? \mid \pi^{-1} \mid \pi \cdot \pi \mid \pi \cap \pi \mid (\leq \cdot \pi \cdot \leq)^{\mathsf{c}} \mid (\leq \cdot \pi \cdot \geq)^{\mathsf{c}} \mid (\geq \cdot \pi \cdot \leq)^{\mathsf{c}} \mid (\geq \cdot \pi \cdot \geq)^{\mathsf{c}}$$

$\mathrm{PDL}_{\mathsf{sf}}^{\mathsf{int}}$

Lemma: $\forall \pi \in \mathrm{PDL}_{\mathsf{sf}}^{\mathsf{int}}$, $[\![\pi]\!]$ is interval-preserving

Equivalences over interval-preserving structures

$$\begin{tikzcd}
\mathsf{FO} \arrow[r] & \mathsf{PDL}_{\mathsf{sf}}^{\mathsf{int}} \arrow[d] \\
\mathsf{FO}^3 \arrow[u] & \mathsf{PDL}_{\mathsf{sf}} \arrow[l]
\end{tikzcd}$$

Equivalences over interval-preserving structures

$$\begin{tikzcd}
\mathsf{FO} \arrow[r] & \mathsf{PDL}_{\mathsf{sf}}^{\mathsf{int}} \arrow[d, "\text{def.}"] \\
\mathsf{FO}^3 \arrow[u, "\text{def.}"] & \mathsf{PDL}_{\mathsf{sf}} \arrow[l]
\end{tikzcd}$$

Equivalences over interval-preserving structures

$$\begin{tikzcd}
\mathsf{FO} \arrow[r] & \mathsf{PDL}_{\mathsf{sf}}^{\mathsf{int}} \arrow[d, "\text{def.}"] \\
\mathsf{FO}^3 \arrow[u, "\text{def.}"] & \mathsf{PDL}_{\mathsf{sf}} \arrow[l, "\text{trivial induction}"']
\end{tikzcd}$$

Equivalences over interval-preserving structures

$$\begin{tikzcd}
\mathsf{FO} \arrow[r] & \mathsf{PDL}_{\mathsf{sf}}^{\mathsf{int}} \arrow[d, "\text{def.}"] \\
\mathsf{FO}^3 \arrow[u, "\text{def.}"] & \mathsf{PDL}_{\mathsf{sf}} \arrow[l, "\substack{\text{trivial}\\ \text{induction}}"]
\end{tikzcd}$$

- State formula $\varphi \in \mathsf{PDL}_{\mathsf{sf}} \quad \rightsquigarrow \quad \varphi^{\mathsf{FO}}(x) \in \mathsf{FO}$

- Path formula $\pi \in \mathsf{PDL}_{\mathsf{sf}} \quad \rightsquigarrow \quad \pi^{\mathsf{FO}}(x, y) \in \mathsf{FO}$

Equivalences over interval-preserving structures

$$\begin{tikzcd} \mathsf{FO} \arrow[r] & \mathsf{PDL}_{\mathsf{sf}}^{\mathsf{int}} \arrow[d, "\text{def.}"] \\ \mathsf{FO}^3 \arrow[u, "\text{def.}"] & \mathsf{PDL}_{\mathsf{sf}} \arrow[l, "\text{trivial induction}"] \end{tikzcd}$$

- State formula $\varphi \in \mathsf{PDL}_{\mathsf{sf}} \quad \rightsquigarrow \quad \varphi^{\mathsf{FO}}(x) \in \mathsf{FO}$

$$\langle \pi \rangle \varphi \quad \rightsquigarrow \quad \exists y . \pi^{\mathsf{FO}}(x, y) \wedge \varphi^{\mathsf{FO}}(y)$$

- Path formula $\pi \in \mathsf{PDL}_{\mathsf{sf}} \quad \rightsquigarrow \quad \pi^{\mathsf{FO}}(x, y) \in \mathsf{FO}$

$$\pi_1 \cdot \pi_2 \quad \rightsquigarrow \quad \exists z . \pi_1^{\mathsf{FO}}(x, z) \wedge \pi_2^{\mathsf{FO}}(z, y)$$

Equivalences over interval-preserving structures

$$\boxed{\mathsf{FO}} \xrightarrow{?} \boxed{\mathsf{PDL}^{\mathsf{int}}_{\mathsf{sf}}}$$

Any FO formula $\Phi(x_1, \ldots, x_n)$ is equivalent to a finite positive boolean combination of formulas of the form $\pi^{\mathsf{FO}}(x_i, x_j)$, where $\pi \in \mathsf{PDL}^{\mathsf{int}}_{\mathsf{sf}}$.

Equivalences over interval-preserving structures

$$\mathsf{FO} \xrightarrow{\;?\;} \mathsf{PDL}^{\mathsf{int}}_{\mathsf{sf}}$$

Any FO formula $\Phi(x_1, \ldots, x_n)$ is equivalent to a finite positive boolean combination of formulas of the form $\pi^{\mathsf{FO}}(x_i, x_j)$, where $\pi \in \mathsf{PDL}^{\mathsf{int}}_{\mathsf{sf}}$.

Proof: by induction on Φ.

Equivalences over interval-preserving structures

$$\boxed{\mathsf{FO}} \xrightarrow{?} \boxed{\mathsf{PDL}^{\mathsf{int}}_{\mathsf{sf}}}$$

Any FO formula $\Phi(x_1, \ldots, x_n)$ is equivalent to a finite positive boolean combination of formulas of the form $\pi^{\mathsf{FO}}(x_i, x_j)$, where $\pi \in \mathsf{PDL}^{\mathsf{int}}_{\mathsf{sf}}$.

Proof: by induction on Φ.

- Atomic formulas, disjunction: easy

Equivalences over interval-preserving structures

$$\mathsf{FO} \xrightarrow{?} \mathsf{PDL}^{\mathsf{int}}_{\mathsf{sf}}$$

Any FO formula $\Phi(x_1, \ldots, x_n)$ is equivalent to a finite positive boolean combination of formulas of the form $\pi^{\mathsf{FO}}(x_i, x_j)$, where $\pi \in \mathsf{PDL}^{\mathsf{int}}_{\mathsf{sf}}$.

Proof: by induction on Φ.

- Negation: Express π^{c} using $(\leq \cdot \pi \cdot \leq)^{\mathsf{c}}$, $(\leq \cdot \pi \cdot \geq)^{\mathsf{c}}$, $(\geq \cdot \pi \cdot \leq)^{\mathsf{c}}$, $(\geq \cdot \pi \cdot \geq)^{\mathsf{c}}$.

Equivalences over interval-preserving structures

$$\boxed{\mathsf{FO}} \xrightarrow{?} \boxed{\mathsf{PDL}^{\mathsf{int}}_{\mathsf{sf}}}$$

Any FO formula $\Phi(x_1, \ldots, x_n)$ is equivalent to a finite positive boolean combination of formulas of the form $\pi^{\mathsf{FO}}(x_i, x_j)$, where $\pi \in \mathsf{PDL}^{\mathsf{int}}_{\mathsf{sf}}$.

Proof: by induction on Φ.

- Existential quantification: Similar to the example before.

Equivalences over interval-preserving structures

$$\boxed{\mathsf{FO}} \xrightarrow{?} \boxed{\mathsf{PDL}^{\mathsf{int}}_{\mathsf{sf}}}$$

Any FO formula $\Phi(x_1, \ldots, x_n)$ is equivalent to a finite positive boolean combination of formulas of the form $\pi^{\mathsf{FO}}(x_i, x_j)$, where $\pi \in \mathsf{PDL}^{\mathsf{int}}_{\mathsf{sf}}$.

Proof: by induction on Φ.

- Existential quantification: Similar to the example before.

  $\exists x. \bigwedge_i \pi_i^{\mathsf{FO}}(x_i, x)$

Equivalences over interval-preserving structures

$$\boxed{\mathsf{FO}} \xrightarrow{?} \boxed{\mathsf{PDL}^{\mathsf{int}}_{\mathsf{sf}}}$$

Any FO formula $\Phi(x_1, \ldots, x_n)$ is equivalent to a finite positive boolean combination of formulas of the form $\pi^{\mathsf{FO}}(x_i, x_j)$, where $\pi \in \mathsf{PDL}^{\mathsf{int}}_{\mathsf{sf}}$.

Proof: by induction on Φ.

- Existential quantification: Similar to the example before.

$$\underbrace{\exists x. \bigwedge_i \pi_i^{\mathsf{FO}}(x_i, x)}_{\text{intersection of } n \text{ intervals}}$$

Equivalences over interval-preserving structures

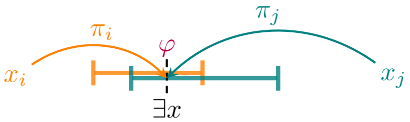
$$\boxed{\mathsf{FO}} \xrightarrow{\ ?\ } \boxed{\mathsf{PDL}^{\mathsf{int}}_{\mathsf{sf}}}$$

Any FO formula $\Phi(x_1, \ldots, x_n)$ is equivalent to a finite positive boolean combination of formulas of the form $\pi^{\mathsf{FO}}(x_i, x_j)$, where $\pi \in \mathsf{PDL}^{\mathsf{int}}_{\mathsf{sf}}$.

Proof: by induction on Φ.

- Existential quantification: Similar to the example before.

$$\underbrace{\exists x. \bigwedge_i \pi_i^{\mathsf{FO}}(x_i, x)}_{\text{intersection of } n \text{ intervals}}$$



Equivalences over interval-preserving structures

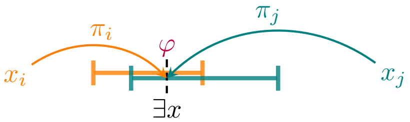
$$\boxed{\mathsf{FO}} \xrightarrow{?} \boxed{\mathsf{PDL}_{\mathsf{sf}}^{\mathsf{int}}}$$

Any FO formula $\Phi(x_1, \ldots, x_n)$ is equivalent to a finite positive boolean combination of formulas of the form $\pi^{\mathsf{FO}}(x_i, x_j)$, where $\pi \in \mathsf{PDL}_{\mathsf{sf}}^{\mathsf{int}}$.

Proof: by induction on Φ.

- Existential quantification: Similar to the example before.

$$\underbrace{\exists x. \bigwedge_i \pi_i^{\mathsf{FO}}(x_i, x)}_{\text{intersection of } n \text{ intervals}} \equiv \underbrace{\bigwedge_{i,j} (\pi_i \cdot \{\varphi\}? \cdot \pi_j^{-1})^{\mathsf{FO}}(x_i, x_j)}_{\text{pairwise intersections}}$$



Conclusion

- Over linearly ordered structures with interval-preserving binary relations,

$$\boxed{\mathsf{FO} = \mathsf{PDL}_{\mathsf{sf}} = \mathsf{FO}^3}$$

Conclusion

- Over linearly ordered structures with interval-preserving binary relations,

$$\boxed{\mathsf{FO} = \mathsf{PDL}_{\mathsf{sf}} = \mathsf{FO}^3}$$

- Covers many classical classes of structures: linear orders, real-time signals, MSCs, ...

Conclusion

- Over linearly ordered structures with interval-preserving binary relations,

$$\boxed{\mathsf{FO} = \mathsf{PDL}_{\mathsf{sf}} = \mathsf{FO}^3}$$

- Covers many classical classes of structures: linear orders, real-time signals, MSCs, ...
- Star-free PDL is a useful technical tool, but also an interesting logic on its own.

Conclusion

- Over linearly ordered structures with interval-preserving binary relations,

$$\boxed{\mathsf{FO} = \mathsf{PDL}_{\mathsf{sf}} = \mathsf{FO}^3}$$

- Covers many classical classes of structures: linear orders, real-time signals, MSCs, ...
- Star-free PDL is a useful technical tool, but also an interesting logic on its own.

Further directions:

- Generalizations to ther types of orders (trees...), relations of arity > 2?

Conclusion

- Over linearly ordered structures with interval-preserving binary relations,

$$\boxed{\mathsf{FO} = \mathsf{PDL}_{\mathsf{sf}} = \mathsf{FO}^3}$$

- Covers many classical classes of structures: linear orders, real-time signals, MSCs, ...
- Star-free PDL is a useful technical tool, but also an interesting logic on its own.

Further directions:

- Generalizations to ther types of orders (trees...), relations of arity > 2?
- Uniform approach for proving completeness of temporal logics?

Conclusion

- Over linearly ordered structures with interval-preserving binary relations,

$$\boxed{\mathsf{FO} = \mathsf{PDL}_{\mathsf{sf}} = \mathsf{FO}^3}$$

- Covers many classical classes of structures: linear orders, real-time signals, MSCs, ...
- Star-free PDL is a useful technical tool, but also an interesting logic on its own.

Further directions:

- Generalizations to ther types of orders (trees...), relations of arity > 2?
- Uniform approach for proving completeness of temporal logics?

Conclusion

- Over linearly ordered structures with interval-preserving binary relations,

$$\boxed{\mathsf{FO} = \mathsf{PDL}_{\mathsf{sf}} = \mathsf{FO}^3}$$

- Covers many classical classes of structures: linear orders, real-time signals, MSCs, ...
- Star-free PDL is a useful technical tool, but also an interesting logic on its own.

Further directions:

- Generalizations to ther types of orders (trees...), relations of arity > 2?
- Uniform approach for proving completeness of temporal logics?

Thank you!