# How undecidable are HyperLTL and HyperCTL$^*$?

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e.g. "no secret information should leak to low-level users"

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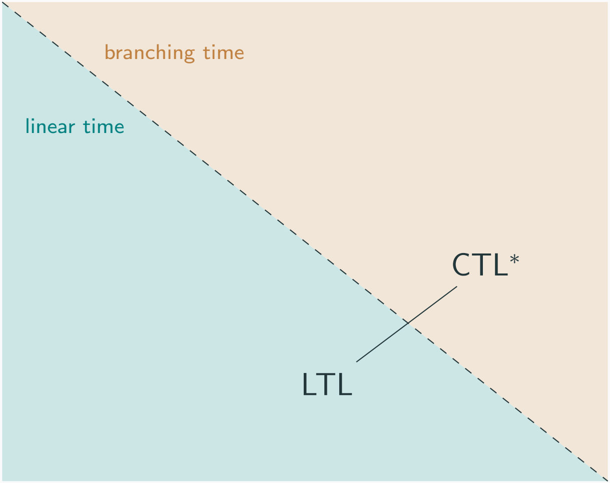
- Noninterference
- Observational determinism
- Declassification
- $\cdots$

# HyperLTL and HyperCTL*



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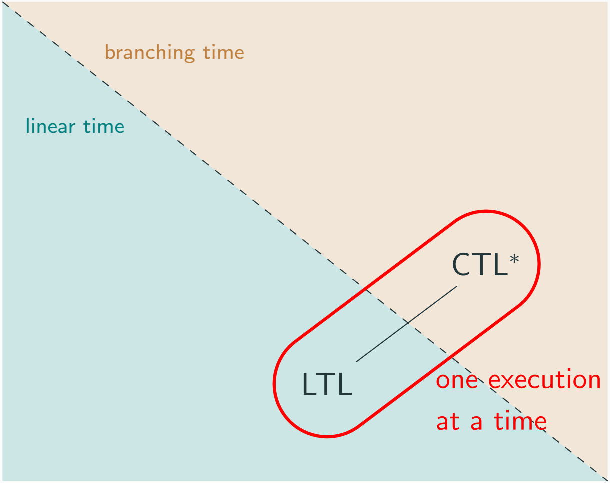

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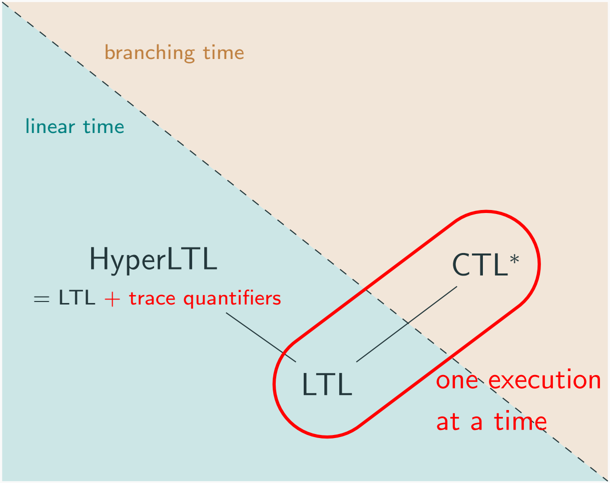

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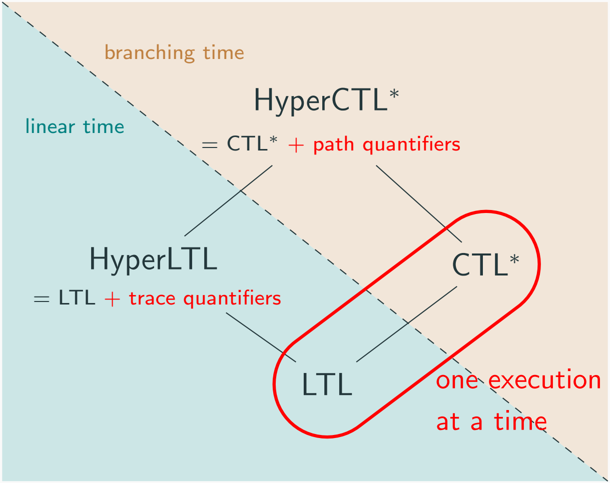

# Linear Temporal Logic (LTL)

**LTL**

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Properties of individual traces $\mathbb{N} \rightarrow 2^{\mathsf{AP}}$

Cannot compare executions

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Always in prenex normal form: $\underbrace{Q_1\pi_1.Q_n\pi_n\ldots}_{\text{trace quantifiers}}\ \underbrace{\varphi}_{\text{LTL formula}}$

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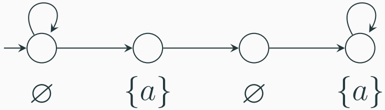

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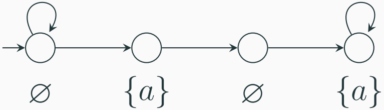


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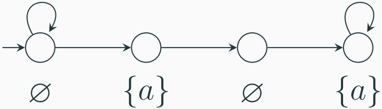


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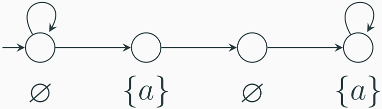


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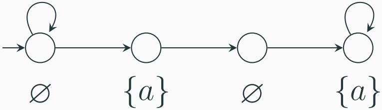


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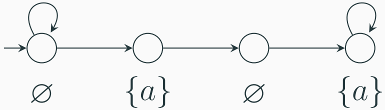


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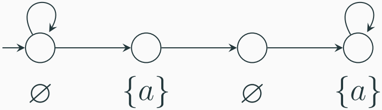

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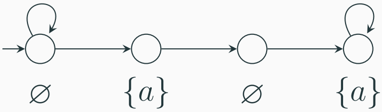


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- Strict generalization of both HyperLTL and CTL$^*$

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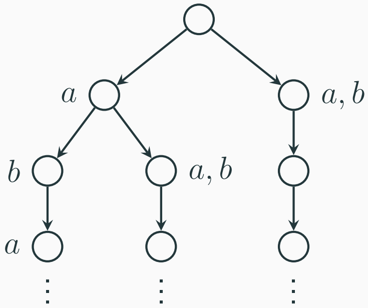

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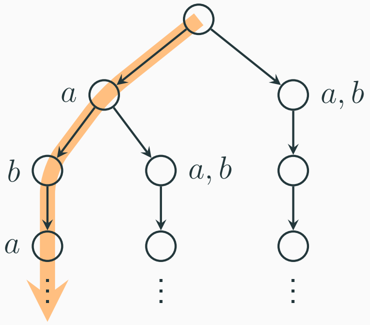

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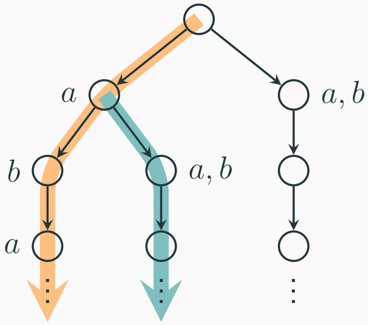

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**How** undecidable is HyperLTL or HyperCTL* satisfiability?

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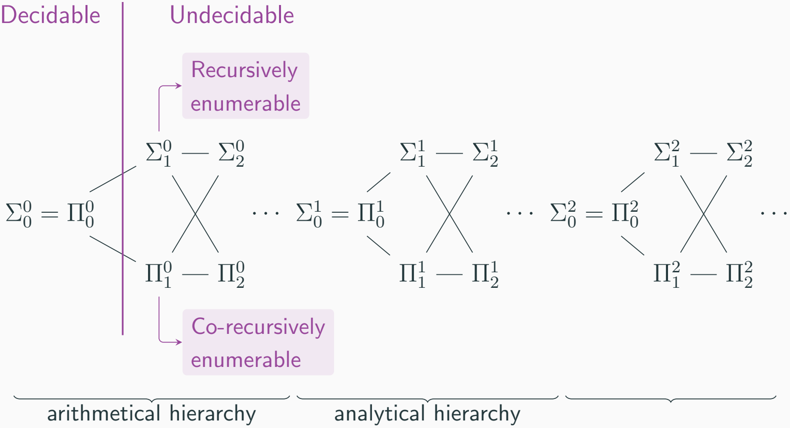

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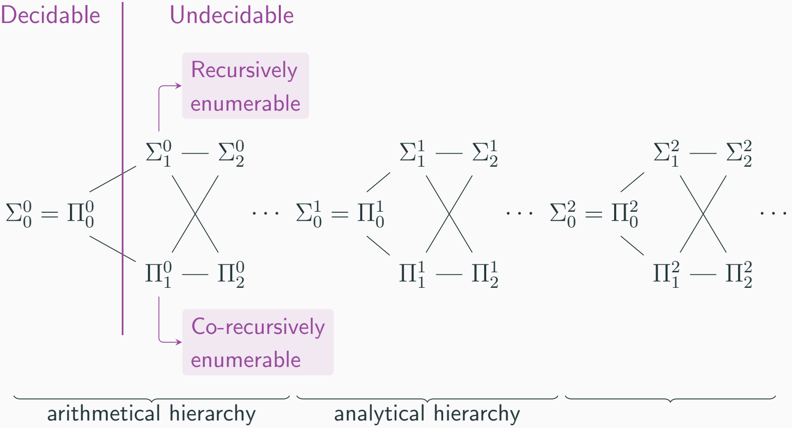


$\Sigma_i^j$: definable by a formula of the form $\underbrace{\exists^*\forall^*\exists^*\forall^*\cdots}_{\substack{\text{i blocks}\\ (j+1)\text{th order}\\ \text{variables}}}\ \underbrace{\varphi}_{\substack{j\text{-th order}\\ \text{arithmetic}}}$

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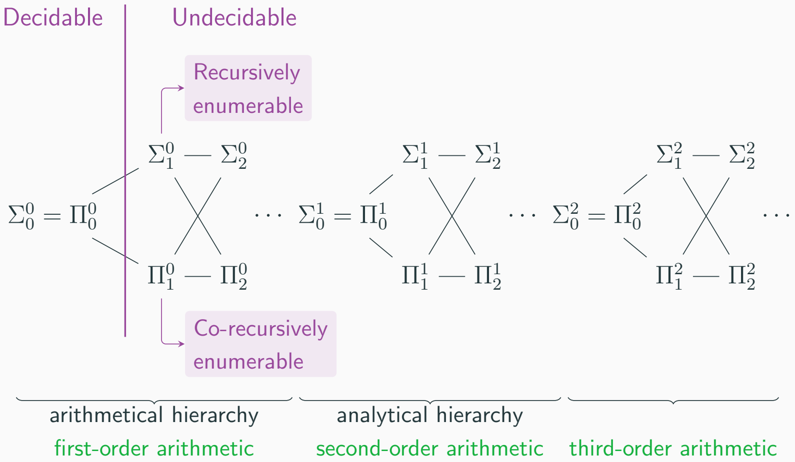


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  [Clarkson, Finkbeiner, Koleini, Micinski, Rabe, Sánchez 2014]

# What Was Known

- HyperLTL satisfiability is $\Sigma_1^0$-hard. [Finkbeiner, Hahn 2016]
- HyperLTL satisfiability **restricted to finite sets of traces** is $\Sigma_1^0$-complete (complete for the class of recursively enumerable problems).
  [Finkbeiner, Hahn, Hans 2018], [Mascle, Zimmermann, 2020]
- HyperCTL$^*$ satisfiability is $\Sigma_1^1$-hard.
  [Clarkson, Finkbeiner, Koleini, Micinski, Rabe, Sánchez 2014]
- **Finite-state** HyperCTL$^*$ satisfiability is $\Sigma_1^0$-complete.
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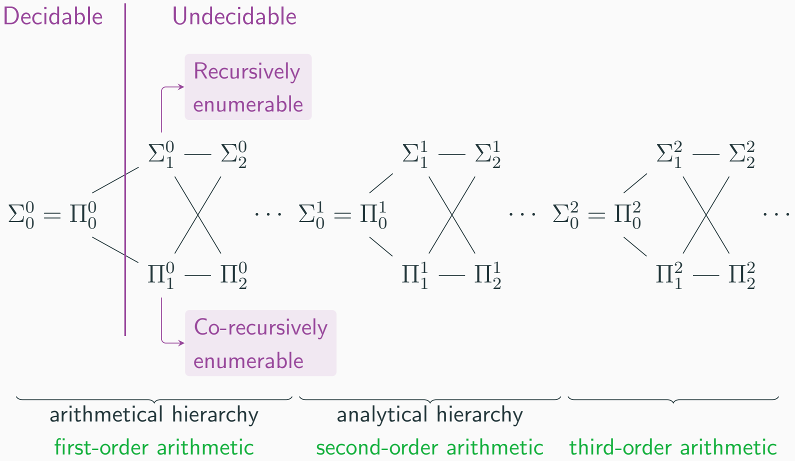
# Completeness Results



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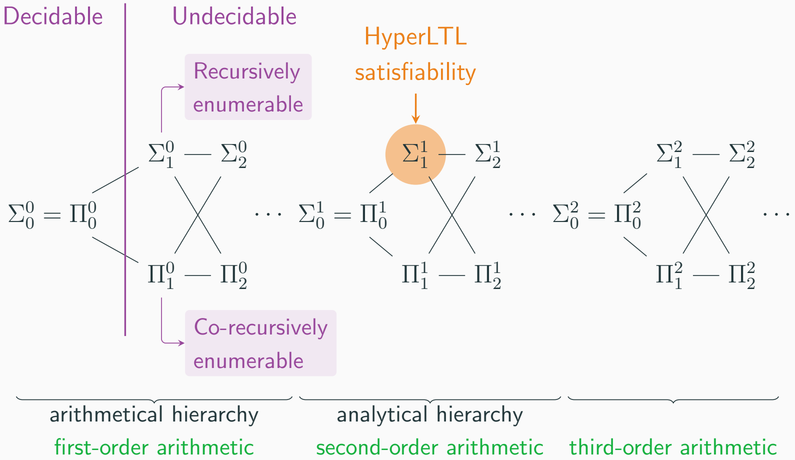

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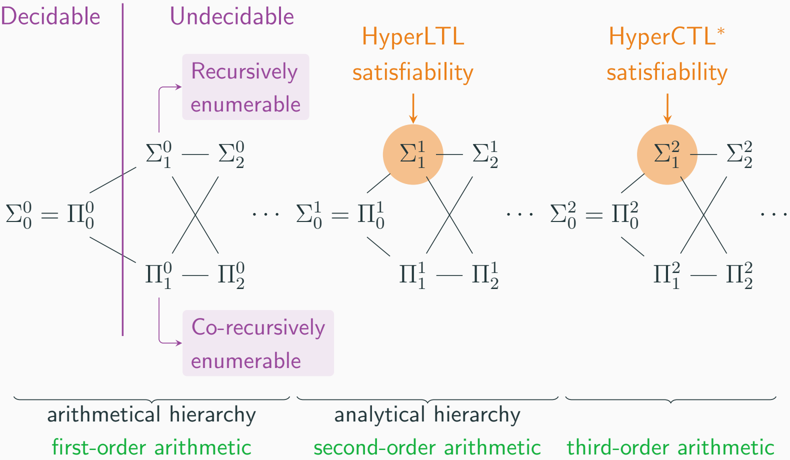

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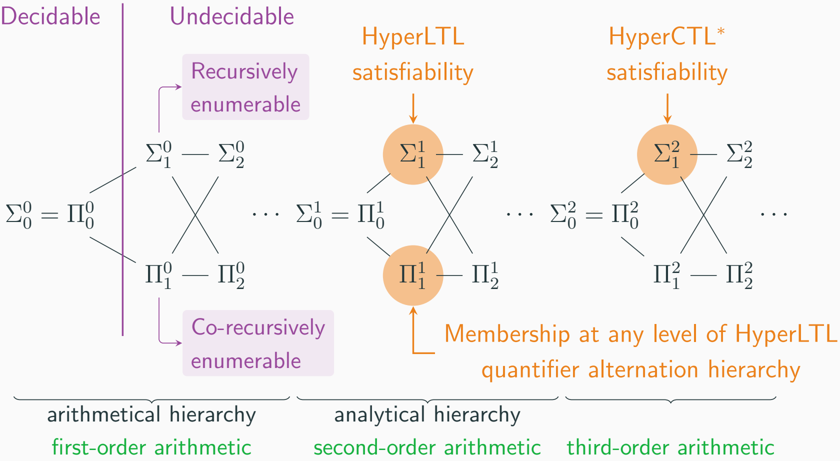

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- Each tile is encoded by an atomic proposition
- Each row is encoded by a trace
- Rows/traces are ordered vertically using a special atomic proposition $y$ true exactly once in each trace: $y$ is true at time $i$ on the trace representing row $\mathbb{N} \times i$

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$$\forall \pi.\, (\neg y_\pi \,\mathsf{U}\, (y_\pi \wedge \mathsf{X}\,\mathsf{G}\, \neg y_\pi))$$

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4. two traces with the same position for $y$ are identical

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5. tiles match horizontally
6. tiles match vertically
7. the specified tile occurs infinitely often on the trace where $y$ is true at $0$

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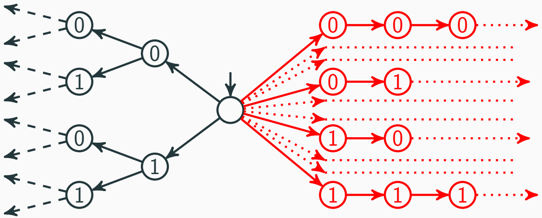
**Why?**

- Every satisfiable HyperLTL formula has a countable model
- Some formulas of HyperCTL$^*$ require models of cardinality $\mathfrak{c} = |2^{\mathbb{N}}|$ (and this bound is optimal)

# Models of Cardinality at Least $\mathfrak{c}$



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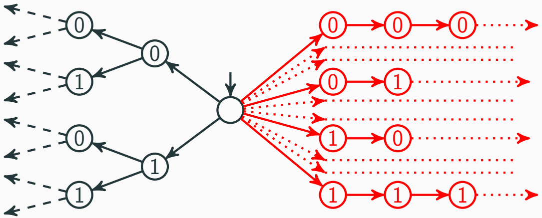


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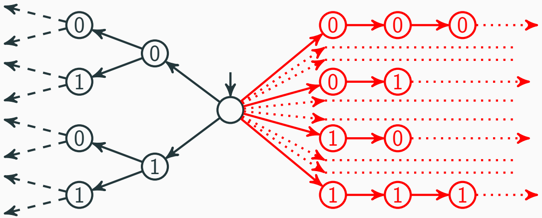


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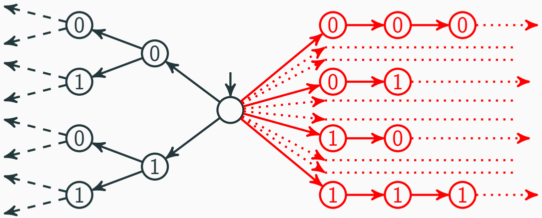


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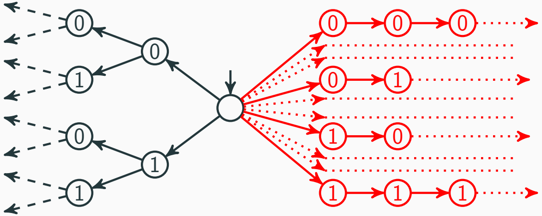


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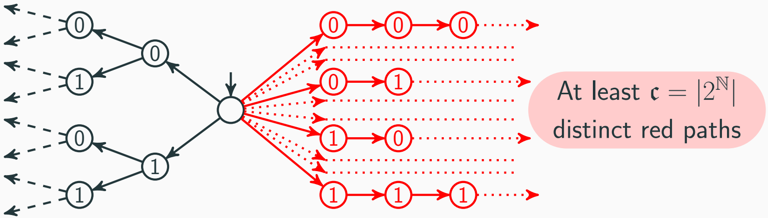


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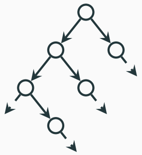

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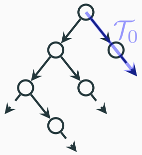

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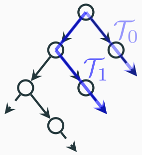

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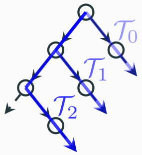

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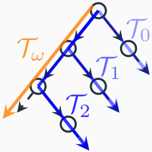

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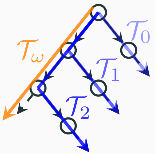


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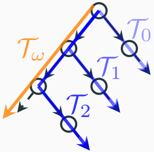


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# Size of Minimal Models

**Theorem**

- Every satisfiable HyperCTL$^*$ formula has a model of cardinality at most $\mathfrak{c} = |2^{\mathbb{N}}|$.
- There is a satisfiable HyperCTL$^*$ formula that does not have any model of cardinality less than $\mathfrak{c}$.

# HyperCTL$^*$ Satisfiability

**Theorem**

HyperCTL$^*$ satisfiability is $\Sigma_1^2$-complete.

$$\underbrace{\exists x_1, \ldots, \exists x_n.}_{\text{third-order variables}} \underbrace{\Phi_0(x, x_1, \ldots, x_n)}_{\text{second-order arithmetic}}$$

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From an existential third-order arithmetic formula $\Phi(x)$ and $n$, construct $\psi$ such that $\mathbb{N} \models \Phi(n)$ iff $\psi$ is satisfiable:

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- One atomic proposition $p_i$ for each third-order $x_i$.
  Existential third-order quantifiers $\rightsquigarrow$ satisfiability

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- HyperLTL satisfiability problem is $\Sigma_1^1$-complete, thus highly undecidable
- HyperCTL$^*$ satisfiability problem is $\Sigma_1^2$-complete, which is infinitely higher than $\Sigma_1^1$ in the hierarchy
- First (and optimal) bound on the minimal size of models for HyperCTL$^*$:
  - every satisfiable formula has a model with at most $\mathfrak{c}$ many states
  - there is satisfiable formula that requires $\mathfrak{c}$ many states

# Conclusion

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Other results:

- HyperLTL satisfiability is still $\Sigma_1^1$-complete when restricted to ultimately periodic traces.
- HyperCTL$^*$ satisfiability restricted to countable or finitely branching transition systems is equivalent to the problem of evaluating a second-order arithmetic formula.
- deciding if a HyperLTL formula is equivalent to one with $n$ quantifier alternations is exactly as hard unsatisfiability, i.e. $\Pi_1^1$-complete.

**Thank you!**