Interpolants and Explicit Definitions in Horn Description Logics

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Explicit Definitions

Fix a DL $\mathcal{L}$ and signature a Σ.

An explicit definition in $\mathcal{L}(\Sigma)$ of a concept name A under an ontology $\mathcal{O}$ is an $\mathcal{L}(\Sigma)$-concept C such that $\mathcal{O} \models A \equiv C$.

$\rightarrow$ **Existence? Size?**

(Projective) Beth Definability Property

Projective Beth Definibility Property

A DL $\mathcal{L}$ has the PBDP if for all $\mathcal{L}$-ontologies $\mathcal{O}$, concept names A, and signatures $\Sigma \subseteq \text{sig}(\mathcal{O})$,

A is implicitly definable from Σ under $\mathcal{O}$ $\iff$ A is explicitely $\mathcal{L}(\Sigma)$-definable under $\mathcal{O}$

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A is **implicitely definable** from Σ under $\mathcal{O}$ if $(\mathcal{I} \models \mathcal{O}$ and $\mathcal{J} \models \mathcal{O}$ and $\mathcal{I}|_\Sigma = \mathcal{J}|_\Sigma)$ implies $A^\mathcal{I} = A^\mathcal{J}$

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$\rightarrow$ If $\mathcal{L}$ has PBDP, then $\mathcal{L}$-explicit definition existence reduces to subsumption checking

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$$\boxed{\begin{array}{c}A \text{ is implicitly definable} \\ \text{from } \Sigma \text{ under } \mathcal{O}\end{array}} \iff \boxed{\begin{array}{c}A \text{ is explicitely } \mathcal{L}(\Sigma)\text{-} \\ \text{definable under } \mathcal{O}\end{array}}$$

- $\mathcal{ALC}(\mathcal{S})(\mathcal{I})(\mathcal{F})$ has the PBDP

[ten Cate, Franconi, Seylan 2013]

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  [ten Cate, Franconi, Seylan 2013]
- $\mathcal{EL}(\mathcal{H})$ has the PBDP [Lutz, Seylan, Wolter 2019]

  [Konev, Lutz, Ponomaryov, Wolter 2010]

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- $\mathcal{ALCO}$ does **not** have the PBDP
  [ten Cate, Conradie, Marx, Venema 2006]
- $\mathcal{ALCH}$ does **not** have PBDP
  [ten Cate, Franconi, Seylan 2013]
- $\mathcal{ELO}$ does **not** have the PBDP
  [Artale, Mazzullo, Ozaki, Wolter 2021]

Craig Interpolation Property

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An $\mathcal{L}$-interpolant for $C_1 \sqsubseteq C_2$ under ontologies $\mathcal{O}_1$ and $\mathcal{O}_2$ is a concept D such that

- $\text{sig}(D) \subseteq \text{sig}(\mathcal{O}_1, C_1) \cap \text{sig}(\mathcal{O}_2, C_2)$
- $\mathcal{O}_1 \cup \mathcal{O}_2 \models C_1 \sqsubseteq D$
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Remark: explicit definitions $\rightarrow$ interpolants

$\mathcal{O}_\Sigma = \mathcal{O}$ where $X \notin \Sigma$ is replaced with X'

Assume $\mathcal{O}, \mathcal{O}_\Sigma \models A \equiv A'$

Explicit definition for A in $\mathcal{L}(\Sigma)$ under $\mathcal{O}$ $\iff$ $\mathcal{L}$-interpolant for $A \sqsubseteq A'$ under $\mathcal{O}, \mathcal{O}_\Sigma$

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A DL $\mathcal{L}$ has the CI if for all $\mathcal{L}$-ontologies $\mathcal{O}_1, \mathcal{O}_2$ and $\mathcal{L}$-concepts C_1, C_2 such that $\mathcal{O}_1 \cup \mathcal{O}_2 \models C_1 \sqsubseteq C_2$, there exists an $\mathcal{L}$-interpolant for $C_1 \sqsubseteq C_2$ under $\mathcal{O}_1, \mathcal{O}_2$.

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Remark

CIP $\Rightarrow$ PBDP

- $\mathcal{ALC}(\mathcal{S})(\mathcal{I})(\mathcal{F})$ has the CIP [ten Cate, Franconi, Seylan 2013]
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$\rightarrow$ focus on extensions of $\mathcal{EL}$

Failure of Projective Beth Definability Property

Theorem

$\mathcal{ELO}$, $\mathcal{EL}_u$, $\mathcal{EL}^{++}$ with (a) a single role inclusion $r \circ s \sqsubseteq s$ or (b) a single transitivity inclusion $s \circ s$ and role hierarchies $r_1 \sqsubseteq r_2$, $\mathcal{ELI}$, $\mathcal{ELI}_u$, Horn-$\mathcal{ALC}$ and Horn-$\mathcal{ALCI}$ do not enjoy the CIP nor PBDP.

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Reminder: $\mathcal{EL}$, $\mathcal{ELH}$ enjoy the CIP and PBDP.

$\mathcal{EL}^{++}$ and $\mathcal{EL}_u^{++}$

$A \sqsubseteq \exists r.E$ $\quad$ $E \sqsubseteq \exists s.B$ $\quad$ $\exists s.B \sqsubseteq A$ $\quad$ $r \circ s \sqsubseteq s$

$\Sigma = \{s, E\}$

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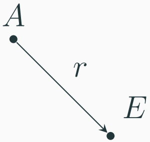

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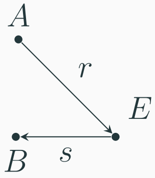

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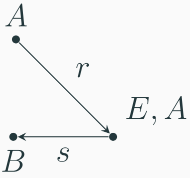

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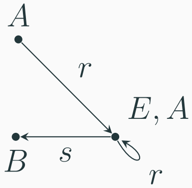

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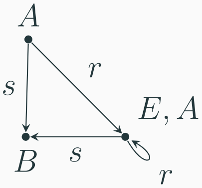

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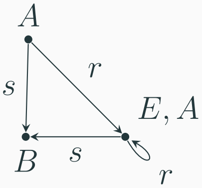

$$\mathcal{O} \models \forall x.(A(x) \leftrightarrow \exists y.E(y) \land \forall z.(s(y,z) \rightarrow s(x,z)))$$

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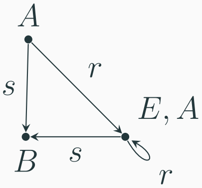

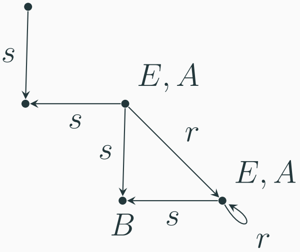

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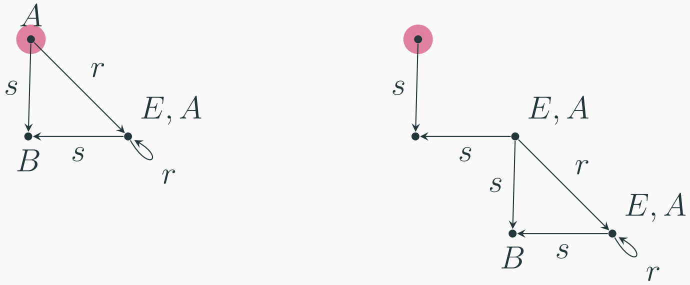

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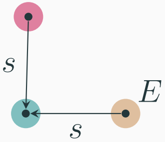

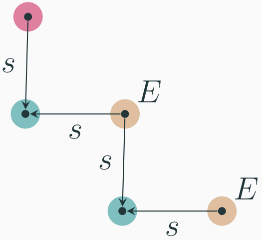

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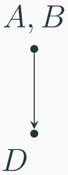

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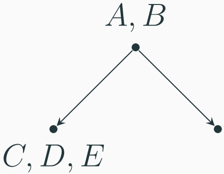

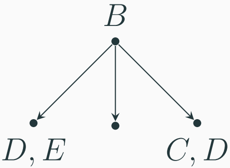

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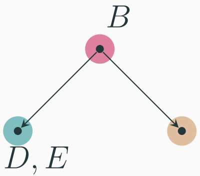

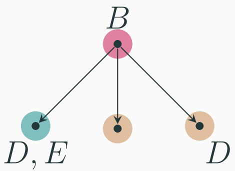

Decision Problem and Size of Interpolants (1/2)

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Theorem

Let $\mathcal{L} \in \{\mathcal{EL}_u, \mathcal{ELO}, \mathcal{ELO}_u, \mathcal{EL}^{++}, \mathcal{EL}^{++}_u\}$.

- $\mathcal{L}$-interpolant existence and $\mathcal{L}$-explicit definition existence are in PTIME.
- If an interpolant or explicit definition exists, then there is one of at most exponential size. This bound is optimal.

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Proof techniques.

- There is an interpolant for $\mathcal{O}_1 \cup \mathcal{O}_2 \models C_1 \sqsubseteq C_2$ iff C_2 is true at the root of the Σ-reduct of the canonical model for $\mathcal{O}_1 \cup \mathcal{O}_2$ and C_1.

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- Bound the size of a derivation tree for C_2.

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Let $\mathcal{L} \in \{\mathcal{ELI}, \mathcal{ELI}_u, \mathcal{ELIO}, \mathcal{ELIO}_u\}$.

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- Check intersection.

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Thank you!