Quantifying the Efficiency of Congestion Games

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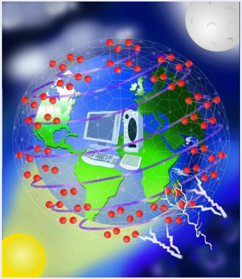
WINE Tutorial, December 2014

Motivation: Systems with Selfish Agents

Our Focus

- Problems in which multiple agents interact

Motivation: the Internet



- Billions of users
- Tens of thousands of autonomous systems
- By design, centralized control is impossible
  - Technical constraints – resources
  - Political constraints – ISP, countries
- Decentralized operation and ownership
- Distributed control by competing entities

Motivation: Systems with Selfish Agents

Selfish Agents

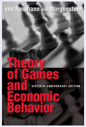
- Have their own private objectives
- Are rational and selfish
  - Make choices to maximize their profit
  - Profit depends on choices of all agents

Goal

- Algorithms that account for strategic behavior by selfish agents

Natural Tool: GAME THEORY

- Theory of rational behavior in competitive, collaborative settings
  - [von Neumann/Morgenstern 1944]



Objectives

This Talk

- Understand consequences of non-cooperative behavior
- What is the “cost” of selfish behavior?
  - the price of anarchy [Koutsoupias/Papadimitriou 99]
  - the price of stability [Anshelevich et al. 04]

Our Scenario

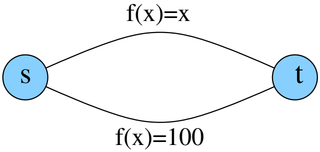
- General model for non-cooperative sharing of resources
- Congestion games

Example

Motivating Example

Example

100 cars need to go from s to t.



Question

What will selfish network users do?

Motivating Example

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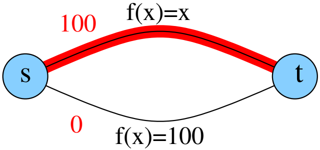


Question

What will selfish network users do?

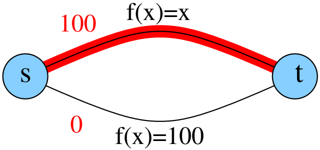
Claim

In Nash equilibrium all traffic will take the top link.

Can we do better?

Example

100 cars need to go from s to t.



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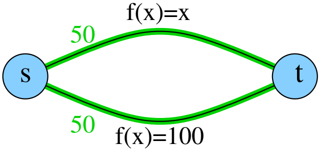
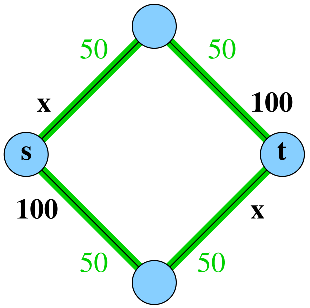


Consider instead

traffic split equally

- 50 cars have delay 100 (same as before)
- 50 cars have delay 50 (big improvement!)

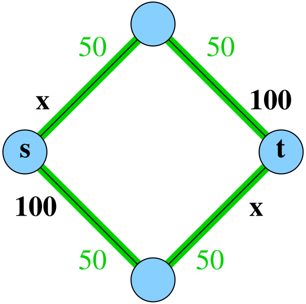
Initial Network



delay=150

Braess's Paradox [Braess, 68]

Initial Network



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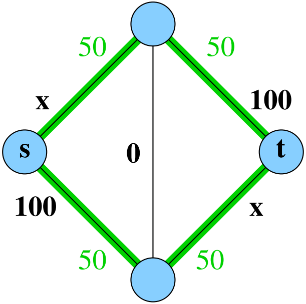
Augmented Network



What now?

Braess's Paradox [BRAESS, 68]

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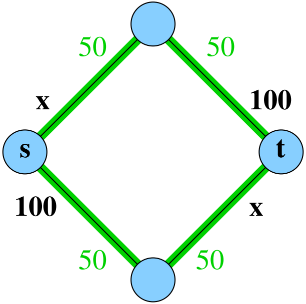

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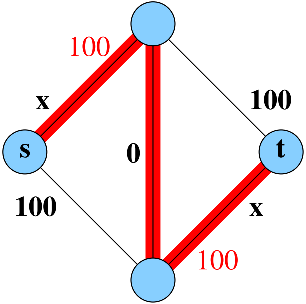

delay=200

(Weighted) congestion games

$$\Gamma = (\mathcal{N}, (w_i)_{i\in\mathcal{N}}, E, (S_i)_{i\in\mathcal{N}}, (c_e)_{e\in E})$$

- $\mathcal{N}$ ... set of k players
- w_i ... weight of player $i \in \mathcal{N}$

 vs. 

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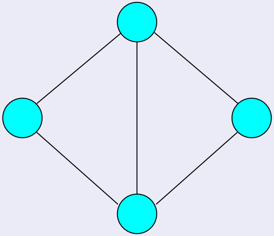
  e.g. edges in a graph



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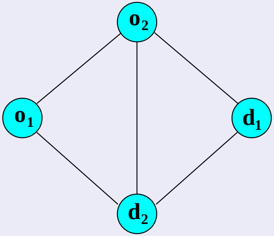

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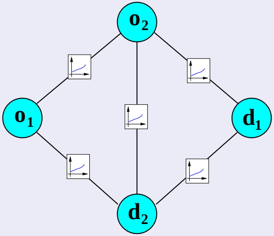
- c_e …latency function of resource e


vs.

Subclasses of (weighted) congestion games

- unweighted congestion games (or simply congestion games):

$$w_i = 1 \quad \text{for all player } i \in \mathcal{N}$$

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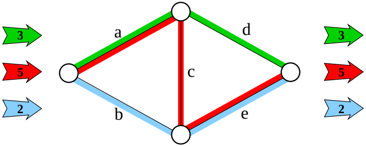
- network congestion games



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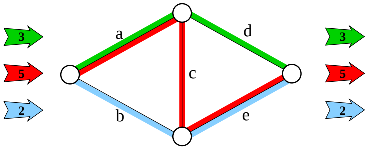
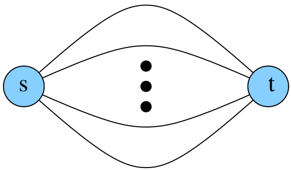


- singleton congestion games



Load and Private Cost

Strategy profile

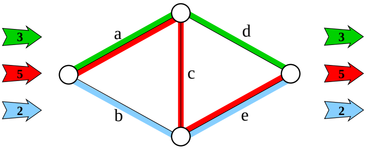
$$\mathsf{s} = (s_1, \ldots, s_n) \in S_1 \times \ldots \times S_n$$

Traffic on resource $e \in E$

$$x_e(\mathsf{s}) = \sum_{i \in \mathcal{N} : e \in s_i} w_i$$

Private cost of player $i \in \mathcal{N}$

$$\mathsf{C}_i(\mathsf{s}) = w_i \cdot \sum_{e \in s_i} c_e(x_e(\mathsf{s}))$$



$$\mathsf{C}_1(\mathsf{s}) = 3 \cdot (c_a(8) + c_d(3))$$

$$\mathsf{C}_2(\mathsf{s}) = 5 \cdot (c_a(8) + c_c(5) + c_e(7))$$

$$\mathsf{C}_3(\mathsf{s}) = 2 \cdot (c_b(2) + c_e(7))$$

Nash Equilibrium

Nash Equilibrium

A strategy profile s is a Nash equilibrium if and only if all players $i \in \mathcal{N}$ are satisfied, that is,

$$\mathrm{C}_i(\mathrm{s}) \leq \mathrm{C}_i(\mathrm{s}_{-i}, s_i') \qquad \text{for all } i \in \mathcal{N} \text{ and } s_i' \in S_i.$$

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Remarks

- For simplicity we restrict to pure Nash equilibria.
- Many results hold also for mixed Nash equilibria.
  - Players randomize over their pure strategies
  - Guaranteed to exist [NASH, 1951]

Existence of pure NE: positive result

Theorem [Rosenthal, 1973]

Every unweighted congestion game possesses a pure Nash equilibrium.

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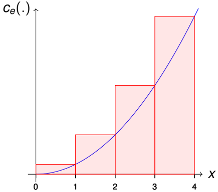
Every unweighted congestion game possesses a pure Nash equilibrium.

Define $\Phi : (S_1 \times ... \times S_n) \to \mathbb{N}$ by

$$\Phi(s) = \sum_{e \in E} \sum_{j=1}^{x_e(s)} c_e(j).$$



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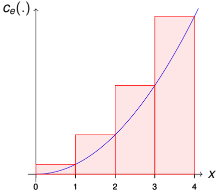


Consider two strategy profiles $s = (s_1, \ldots, s_k)$ and $s' = (s'_i, s_{-i})$:

$$\begin{aligned} \Phi(s) - \Phi(s') &= \sum_{e \in s_i - s'_i} c_e(x_e(s)) - \sum_{e \in s'_i - s_i} c_e(x_e(s')) \\ &= C_i(s) - C_i(s'). \end{aligned}$$

Therefore: $\Phi(s)$ minimal $\Rightarrow$ s is Nash equilibrium.

Existence of pure NE: negative result

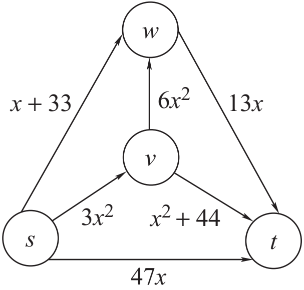
[LIBMAN & ORDA 2001, FOTAKIS ET AL. 2004, GOEMANS ET AL. 2005]

Theorem

There is a weighted network congestion game that does not admit a pure Nash equilibrium.

Consider the following instance:

- 2 players
- $w_1 = 1$
- $w_2 = 2$



Existence of pure NE in weighted games

Theorem [Fotakis, Kontogiannis, Spirakis, 2004]

Every weighted congestion game with linear latency functions possesses a pure Nash equilibrium.

Proof is based on the following potential function:

$$\begin{aligned}\widetilde{\Phi}(\mathrm{s}) &= \sum_{i\in\mathcal{N}} w_i \cdot \sum_{e\in s_i} (c_e(x_e(\mathrm{s})) + c_e(w_i)) \\ &= \sum_{e\in E} x_e(\mathrm{s}) \cdot c_e(x_e(\mathrm{s})) + \sum_{i\in\mathcal{N}} w_i \cdot \sum_{e\in s_i} c_e(w_i).\end{aligned}$$

If $\mathrm{s} = (s_1, \ldots, s_k)$ and $\mathrm{s}' = (s_j', \mathrm{s}_{-j})$ for some $j \in \mathcal{N}$ and $s_j' \in S_j$, then

$$\widetilde{\Phi}(\mathrm{s}) - \widetilde{\Phi}(\mathrm{s}') = 2 \cdot (C_j(\mathrm{s}) - C_j(\mathrm{s}')).$$

Existence and Complexity of Pure NE

- Do weighted congestion games always possess pure Nash Equilibria?
  - Yes, for unweighted players. [ROSENTHAL, '73]

Rosenthals Potential Function

$$\Phi(s) = \sum_{e \in E} \sum_{i=1}^{x_e(s)} c_e(i)$$

If a player decreases her cost by Δ then also the potential decreases by Δ.

Existence and Complexity of Pure NE

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  - No. [Libman, Orda, '01] [Fotakis, Kontogiannis, Spirakis, '04] [Goemans, Mirrokni, Vetta, '05]
  - Characterisation. [Harks, Klimm, '12]

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- Complexity of deciding for pure Nash equilibria?
  - NP-complete [Dunkel, Schulz, '06]

- Complexity of computing pure Nash equilibia (unweighted)?
  - PLS-complete [Fabrikant, Papadimitriou, Talwar, '04]

Price of Anarchy

http://thetyee.ca/News/2007/10/10/ChinaAutoMad/

Price of Anarchy

Social Cost

- Different definitions possible
- Here: Total Latency

$$\mathrm{SC}(s) = \sum_{i \in \mathcal{N}} C_i(s) = \sum_{e \in E} x_e(s) \cdot c_e(x_e(s))$$

- Let $\mathcal{G}$ be a class of games.

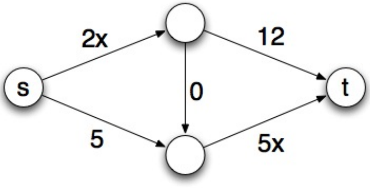
Price of Anarchy

[KOUTSOUPIAS, PAPADIMITRIOU, STACS'99]

$$\mathrm{PoA}(\mathcal{G}) = \sup_{\substack{\Gamma \in \mathcal{G}, \\ s \text{ is NE in } \Gamma}} \frac{\mathrm{SC}(s)}{\mathrm{OPT}}$$

Price of Anarchy: Example

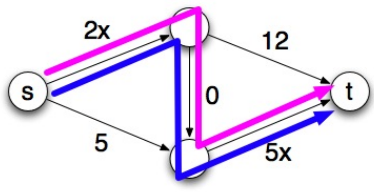
Network with 2 (unweighted) players



Symmetric

Price of Anarchy: Example

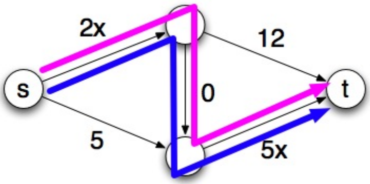
Nash Equilibrium



SC = 14 + 14 = 28

Price of Anarchy: Example

Nash Equilibrium



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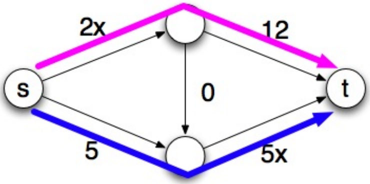
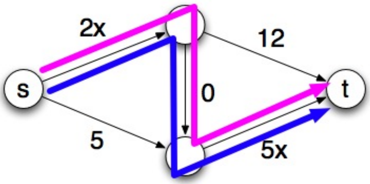
OPT



SC = 14 + 10 = 24

Price of Anarchy: Example

Nash Equilibrium



SC = 14 + 14 = 28

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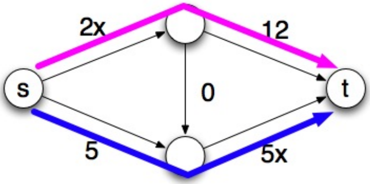

SC = 14 + 10 = 24

Price of Anarchy = 28/24 = 7/6

If multiple equilibria, look at worst one

Price of Anarchy: State of the Art

(1) Analytical simple classes of cost functions
$\Rightarrow$ exact formula for PoA.

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- bounded degree polynomials [Aland et al., STACS'06]

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- non-atomic (Wardrop model) [Roughgarden, Tardos, JACM'00]
- unweighted [Roughgarden, STOC'09]
- weighted [Bhawalkar, Gairing, Roughgarden, ESA'10]

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- weighted [BHAWALKAR, GAIRING, ROUGHGARDEN, ESA'10]

(3) Understanding of game complexity required for worst-case PoA to be realized.

- Ideally independent of cost functions.
- e.g. symmetric strategy sets, singleton strategy sets

Abstract Setup

- n players, each picks a strategy s_i
- player i incurs cost $\mathrm{C}_i(\mathrm{s})$

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- n players, each picks a strategy s_i
- player i incurs cost $\mathrm{C}_i(\mathrm{s})$
- Important Assumption:
  objective function is $SC(\mathrm{s}) = \sum_i \mathrm{C}_i(\mathrm{s})$

Definition: [Roughgarden, STOC'09]

A game is $(\lambda, \mu) -$ *smooth* if for every pair s, s^* of outcomes:

$$\sum_i \mathrm{C}_i(\mathrm{s}_{-i}, s_i^*) \leq \lambda \cdot SC(\mathrm{s}^*) + \mu \cdot SC(\mathrm{s}).$$

$(\lambda > 0, \mu < 1)$

Smoothness $\implies$ PoA bound

Theorem

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$$PoA(G) \leq \frac{\lambda}{1 - \mu}.$$

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Proof: s is a NE, s^* is optimum

$$\begin{aligned} SC(s) &= \sum_i C_i(s) \\ &\leq \sum_i C_i(s_{-i}, s_i^*) \\ &\leq \lambda \cdot SC(s^*) + \mu \cdot SC(s) \end{aligned}$$

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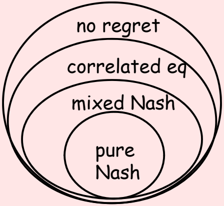
BUT: smoothness is stronger



Back to congestion games

- $\mathcal{C}$ ... arbitrary class of cost functions

Consider the set:

- $\mathcal{A}(\mathcal{C}) = \{(\lambda, \mu) : x^* \cdot c(x + x^*) \leq \lambda \cdot x^* \cdot c(x^*) + \mu \cdot x \cdot c(x)\}$

where

- $0 \leq \mu < 1$ and $\lambda > 0$
- constraints range over all $c \in \mathcal{C}$ and $x \geq 0$ and $x^* > 0$.

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Local smoothness implies global smoothness

For a class of functions $\mathcal{C}$, if $(\lambda, \mu) \in \mathcal{A}(\mathcal{C})$ then every weighted congestion game with cost functions in $\mathcal{C}$ is (λ, μ)-smooth.

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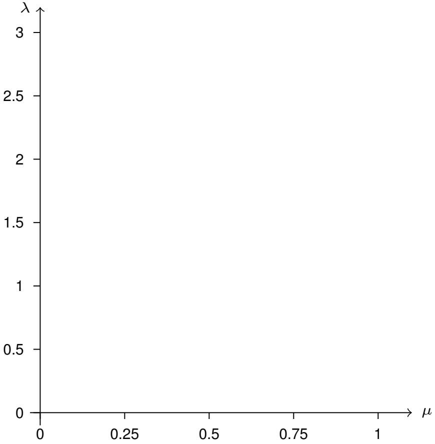
For unweighted congestion games: redefine $\mathcal{A}(\mathcal{C})$:

- $\mathcal{A}(\mathcal{C}) = \{(\lambda, \mu) : x^* \cdot c(x + 1) \leq \lambda \cdot x^* \cdot c(x^*) + \mu \cdot x \cdot c(x)\}$
- and restrict x, x^* to be integer

Example

- Unweighted congestion games with $\mathcal{C} = \{c_1\}$
  - $c_1(x) = x$
- $\mathcal{A}(\mathcal{C}) = \{(\lambda, \mu) : \lambda \geq \frac{c(x+1)}{c(x^*)} - \mu \cdot \frac{x \cdot c(x)}{x^* \cdot c(x^*)}\}$



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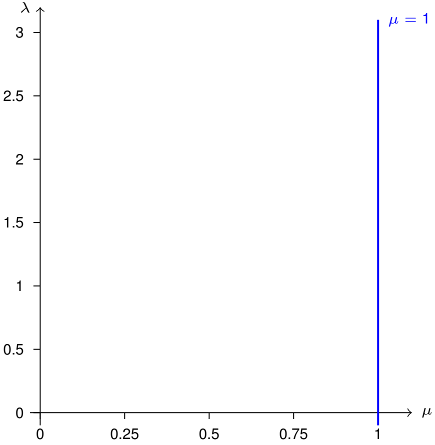

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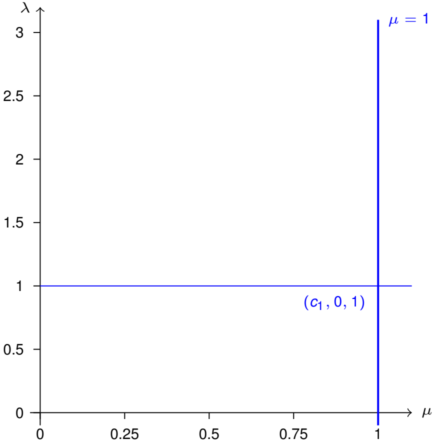

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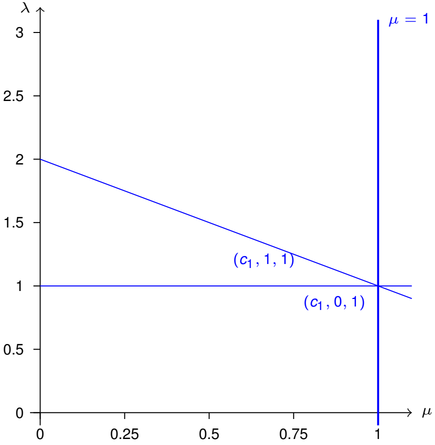

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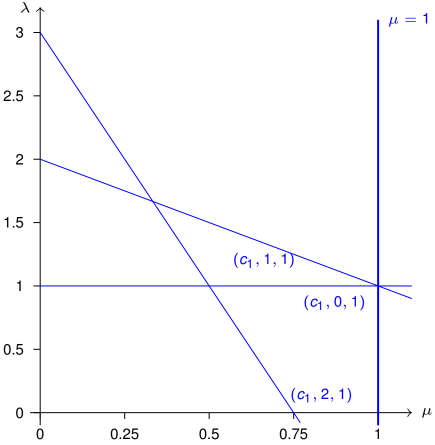

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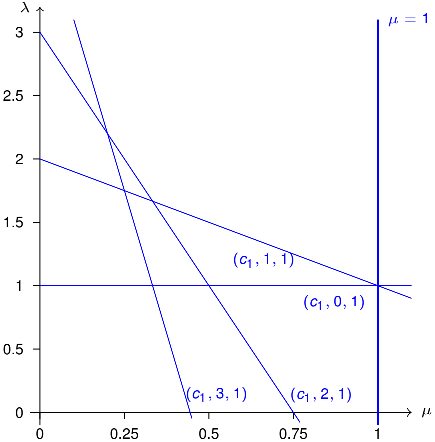

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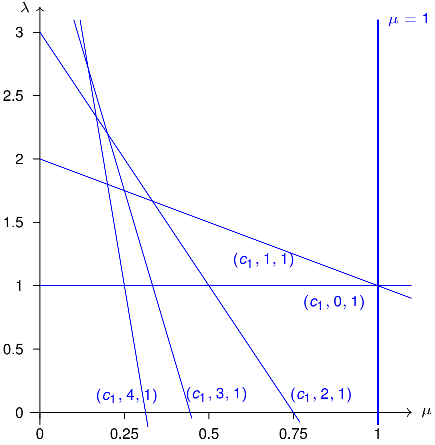

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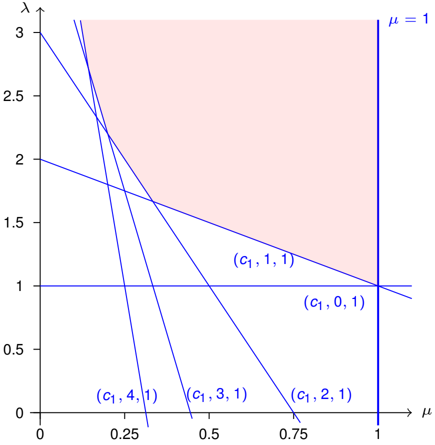

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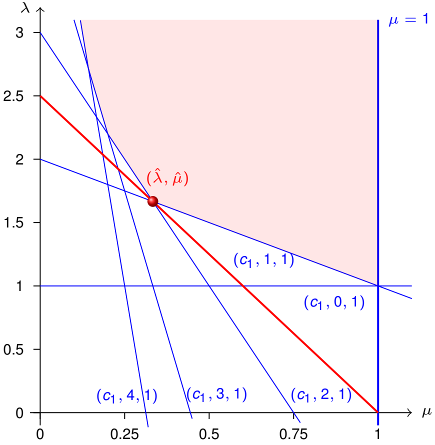

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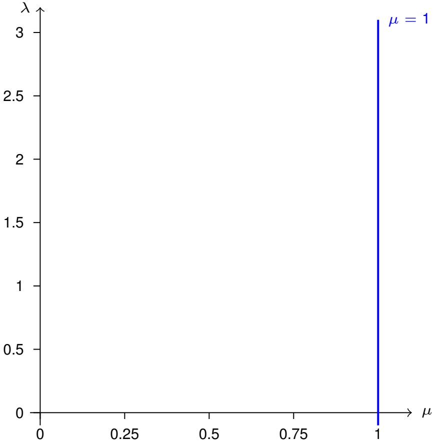

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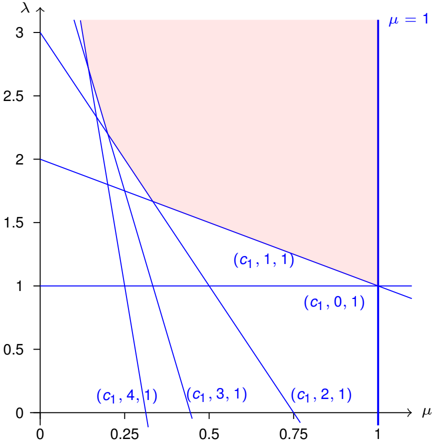

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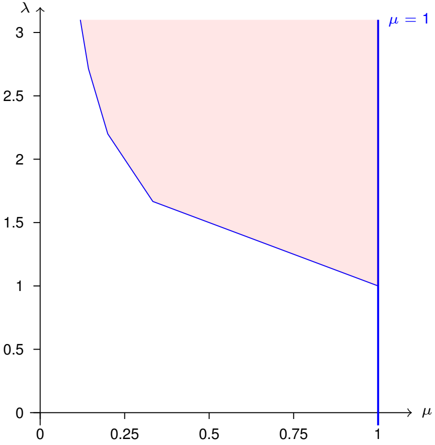

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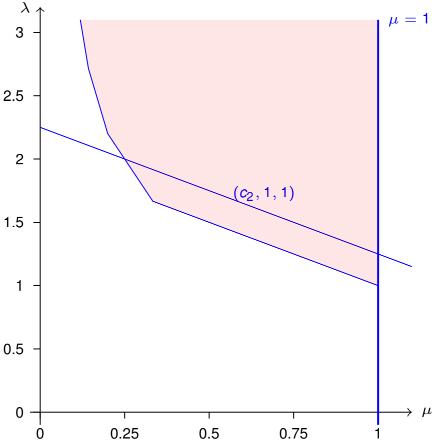

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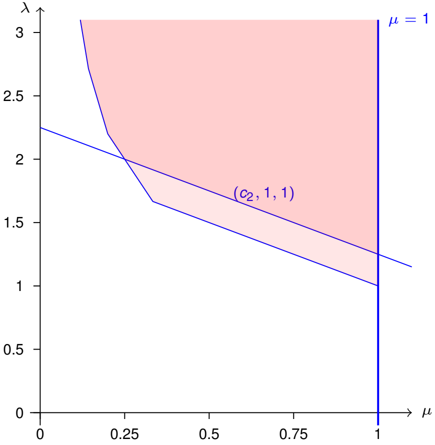

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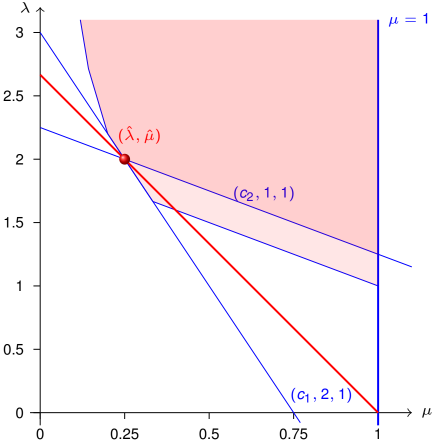

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- $\zeta(\mathcal{C})$ for linear/polynomial cost functions?

PoA for linear/polynomial

- Polynomial latency functions: $\mathcal{C}_d = \left\{ c \mid c(x) = \sum_{i=0}^{d} a_i \cdot x^i \right\}$
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Corollary

For the linear case ($d = 1$) we have:

(a) weighted congestion games: PoA $= \Phi^2 = \frac{3+\sqrt{5}}{2} \approx 2.618$

(b) unweighted congestion games: PoA $= 2.5$

Lower bound for unweighted games

Proof Sketch.

- $n \geq \lfloor \Phi_d \rfloor + 2$ player
- $E = \{g_1, \ldots, g_n\} \,\dot\cup\, \{h_1, \ldots, h_n\}$
- $c_{g_*}(x) = a \cdot x^d, \quad c_{h_*}(x) = x^d$
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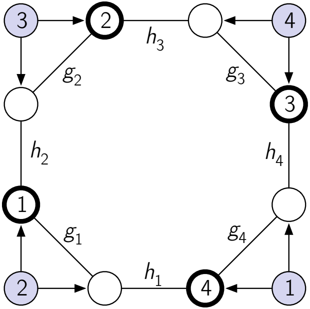

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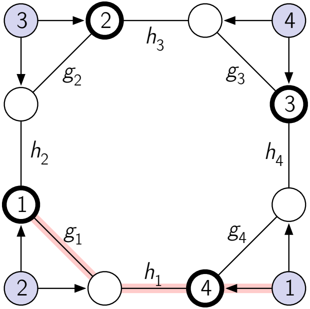

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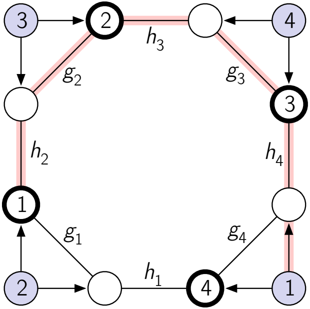

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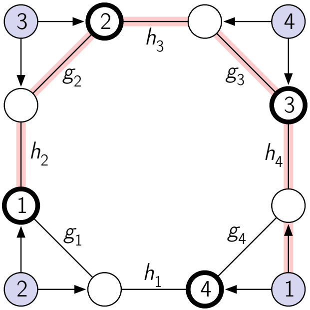
Choose $a > 0$ such that $P = (P_i)_{i \in [n]}$ NE with $\mathrm{C}_i(P) = \mathrm{C}_i(P_{-i}, Q_i)$.



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Definition: Price of Stability

For a game G:

$$\mathrm{PoS}(G) = \min_{\text{s is NE}} \frac{\mathrm{SC(s)}}{\mathrm{OPT}}$$

For a class of games $\mathcal{G}$:

$$\mathrm{PoS}(\mathcal{G}) = \sup_{G \in \mathcal{G}} \mathrm{PoS}(G)$$

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New challenges

- What is a good choice for R?
- How can we incorporate the description of R in the PoA methodology?

Potential Games and Price of Stability

Exact Potential Games:

- All games that admit a potential function Φ, s.t. for all outcomes s, all player i, and all alternative strategies s_i',

$$C_i(s_i', \mathrm{s}_{-i}) - C_i(\mathrm{s}) = \Phi(s_i', \mathrm{s}_{-i}) - \Phi(\mathrm{s}).$$

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- Every congestion game is an exact potential game.

  [ROSENTHAL, 1973]

- For every exact potential game there exists a congestion game having the same potential function. [MONDERER, SHAPLEY, 1996]

Potential Games and Price of Stability

Theorem

Suppose that we have a potential game with potential function Φ, and assume that for any outcome s, we have

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for some constants $A, B \geq 0$. Then the price of stability is at most $A \cdot B$.

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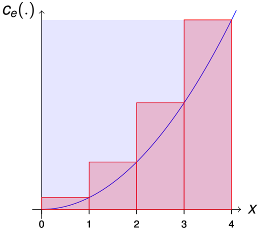

PoS for polynomial (unweighted) congestion games

Theorem [CHRISTODOULOU, GAIRING, 2013]

For polynomial congestion games with cost functions from $\mathcal{C}_d$ we have

$$\mathrm{PoS} = \max_{r>1} \frac{(2^d d + 2^d - 1) \cdot r^{d+1} - (d+1) \cdot r^d + 1}{(2^d + d - 1) \cdot r^{d+1} - (d+1) \cdot r^d + 2^d d - d + 1}.$$

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- $d = 1$:

$$\max_{r>1} \frac{3\,r^2 - 2\,r + 1}{2\,r^2 - 2\,r + 2} = 1 + \frac{\sqrt{3}}{3} \approx 1.577$$

- $d = 2$:

$$\max_{r>1} \frac{11\,r^3 - 3\,r^2 + 1}{5\,r^3 - 3\,r^2 + 7} \approx 2.361$$

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d	PoS
1	1.577
2	2.361
3	3.321
4	4.398
5	5.525
6	6.656
7	7.765
8	8.847

Upper bound high level proof idea:

- Consider NE s with $\Phi(s) \leq \Phi(s^*)$

$$\Rightarrow \qquad \mathrm{SC}(s) \leq \mathrm{SC}(s) + \Phi(s^*) - \Phi(s) \tag{1}$$

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- Use linear combination $(1 - \nu) \cdot (1) + \nu \cdot (2)$ of the above and apply smoothness techniques.

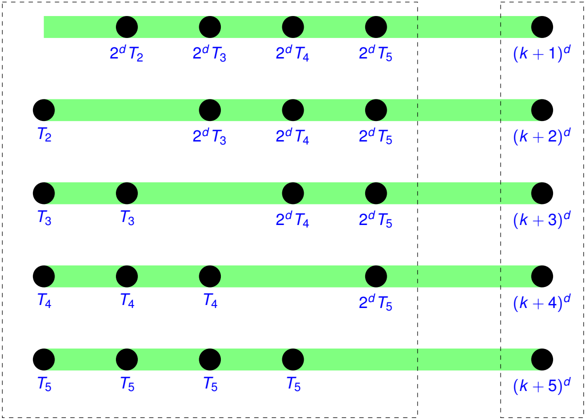
Upper Bound: Key Insights

- Suffices to show local smoothness; i.e. $\forall x, x^* \in \mathbb{N}$ and $c \in \mathcal{C}_d$:

$$f(x, x^*, c, \nu) \leq \mu \cdot x \cdot c(x) + \lambda \cdot x^* \cdot c(x^*)$$

- Sufficient to consider $c(x) = x^d$.
- Tight constraints (x, x^*) are $(0, 1), (1, 1)$ and $(k \cdot r, k)$ for $k \to \infty$
- λ, μ and ν can be determined (in terms of r) as the “solution” of those 3 constraints.
- The hard part is to show that all other constraints are satisfied.
  - Without determining roots of high order polynomials.

Price of Stability: Structure of Lower Bound



Here: $n = 5$

All cost functions of the form:

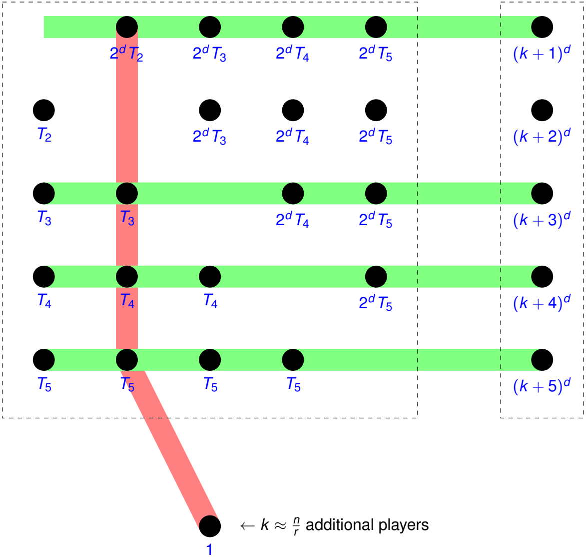
$$c_e(x) = \alpha_e \cdot x^d$$

$$T_i = \frac{(k+i)^d - (k+i-1)^d}{2^{2d} - 1}$$

$\leftarrow k \approx \frac{n}{r}$ additional players

1

Price of Stability: Structure of Lower Bound



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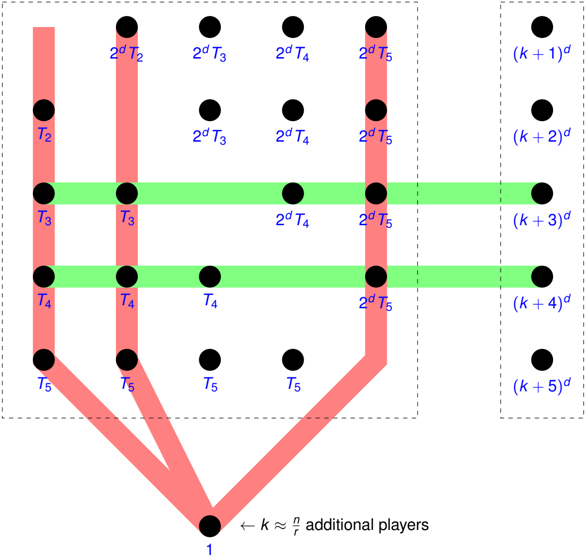

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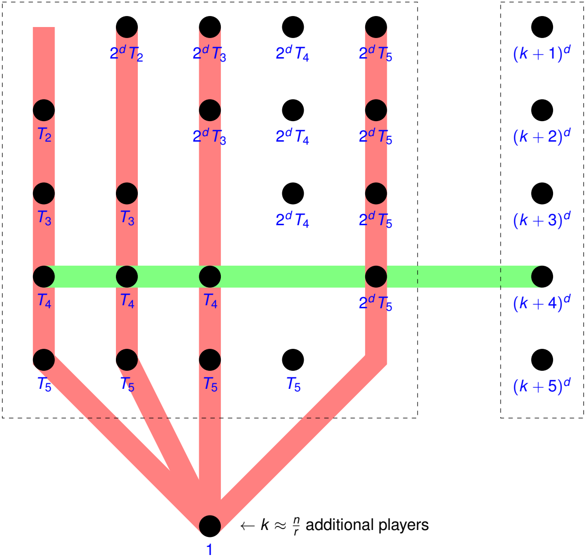

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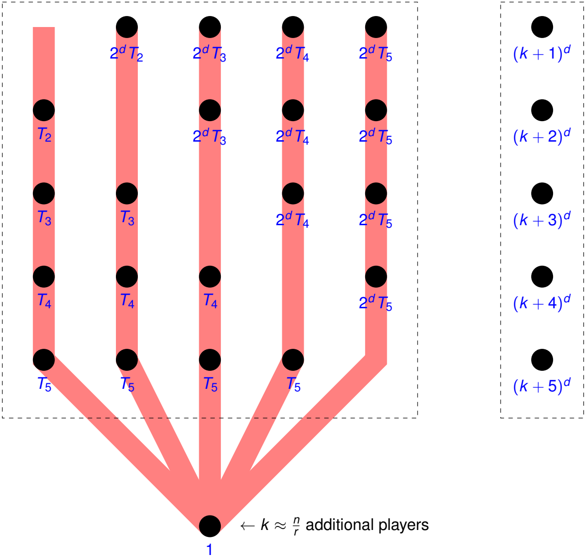

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Unique NE

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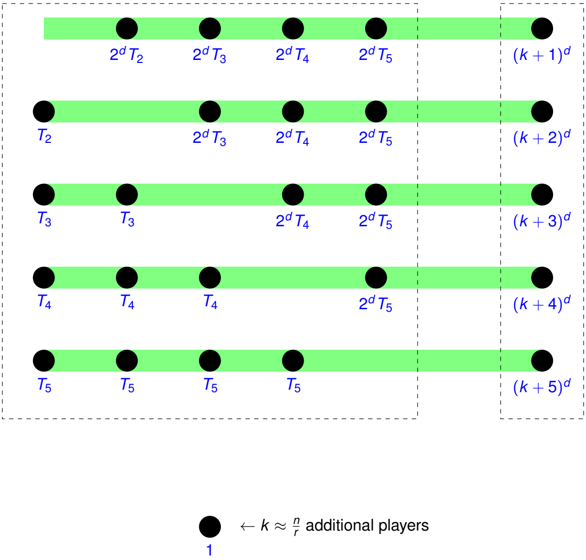

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Optimum

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  - $\Rightarrow SC(s) \leq \widetilde{\Phi}(s) \leq 2 \cdot SC(s)$
  - $\Rightarrow PoS(\mathcal{C}_1) \leq 2$

Conclusion and open problems

Take Home Points

- There is a strong theory on the PoA in congestion games
  - Exact values for polynomial cost functions.
  - Recipe for general functions.
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Thanks. Any questions?