Ontology Languages (COMP321)
Solution for Exercise 2

1. Let $\mathcal{T} = \{A \sqsubseteq \exists r.B, E \sqsubseteq A\}$. Show that $\mathcal{T} \not\models A \sqsubseteq E$ by giving an interpretation $\mathcal{I}$ such that $\mathcal{I} \models \mathcal{T}$ and $\mathcal{I} \not\models A \sqsubseteq E$.

   Solution: Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ with $\Delta^{\mathcal{I}} = \{a, b\}$, $A^{\mathcal{I}} = \{a\}$, $B^{\mathcal{I}} = \{b\}$, $E^{\mathcal{I}} = \emptyset$, $r^{\mathcal{I}} = \{(a, b)\}$. Then $\mathcal{I} \models \mathcal{T}$ and $\mathcal{I} \not\models A \sqsubseteq E$.

2. Let $\mathcal{T}$ be an $\mathcal{EL}$-TBox consisting of the following concept inclusions:

$$\begin{aligned} \mathsf{Bird} &\equiv \mathsf{Vertebrate} \sqcap \exists \mathsf{has.Wing} \\ \mathsf{Reptile} &\sqsubseteq \mathsf{Vertebrate} \sqcap \exists \mathsf{lays.Egg} \end{aligned}$$

   (a) Is $\mathcal{T}$ in normal form? Explain.

   Solution: No, it is not.

   (b) Given $\mathcal{T}$, compute an $\mathcal{EL}$-TBox $\mathcal{T}'$ in normal form using the pre-processing algorithm from the lecture.

   Solution: We start by removing $\equiv$. Namely, $\mathcal{T}$ is transformed to

$$\begin{aligned} \mathsf{Bird} &\sqsubseteq \mathsf{Vertebrate} \sqcap \exists \mathsf{has.Wing} \\ \mathsf{Vertebrate} \sqcap \exists \mathsf{has.Wing} &\sqsubseteq \mathsf{Bird} \\ \mathsf{Reptile} &\sqsubseteq \mathsf{Vertebrate} \sqcap \exists \mathsf{lays.Egg} \end{aligned}$$

   Now we remove conjunctions on the right-hand-side of inclusions and obtain

$$\begin{aligned} \mathsf{Bird} &\sqsubseteq \mathsf{Vertebrate} \\ \mathsf{Bird} &\sqsubseteq \exists \mathsf{has.Wing} \\ \mathsf{Vertebrate} \sqcap \exists \mathsf{has.Wing} &\sqsubseteq \mathsf{Bird} \\ \mathsf{Reptile} &\sqsubseteq \mathsf{Vertebrate} \\ \mathsf{Reptile} &\sqsubseteq \exists \mathsf{lays.Egg} \end{aligned}$$

It remains to simplify the third inclusion. Introduce a new concept name X and rewrite the TBox to $\mathcal{T}'$:

$$\begin{array}{rcl}
\mathsf{Bird} & \sqsubseteq & \mathsf{Vertebrate} \\
\mathsf{Bird} & \sqsubseteq & \exists \mathsf{has}.\mathsf{Wing} \\
\mathsf{Vertebrate} \sqcap X & \sqsubseteq & \mathsf{Bird} \\
\exists \mathsf{has}.\mathsf{Wing} & \sqsubseteq & X \\
\mathsf{Reptile} & \sqsubseteq & \mathsf{Vertebrate} \\
\mathsf{Reptile} & \sqsubseteq & \exists \mathsf{lays}.\mathsf{Egg}
\end{array}$$

$\mathcal{T}'$ is in normal form.

(c) Apply the algorithm deciding $A \sqsubseteq_{\mathcal{T}'} B$ (equivalently, $\mathcal{T}' \models A \sqsubseteq B$), where A, B are concept names. Use the normalised TBox $\mathcal{T}'$ as input.

Solution: We initialise

$$\begin{array}{rcl}
S(\mathsf{Bird}) & = & \{\mathsf{Bird}\} \\
S(\mathsf{Vertebrate}) & = & \{\mathsf{Vertebrate}\} \\
S(\mathsf{Wing}) & = & \{\mathsf{Wing}\} \\
S(\mathsf{Egg}) & = & \{\mathsf{Egg}\} \\
S(\mathsf{Reptile}) & = & \{\mathsf{Reptile}\} \\
R(\mathsf{has}) & = & \emptyset \\
R(\mathsf{lays}) & = & \emptyset
\end{array}$$

- Apply (simpleR) to inclusion (1) and obtain

$$S(\mathsf{Bird}) = \{\mathsf{Bird}, \mathsf{Vertebrate}\}.$$

- Apply (simpleR) to inclusion (5) and obtain

$$S(\mathsf{Reptile}) = \{\mathsf{Reptile}, \mathsf{Vertebrate}\}.$$

- Apply (rightR) to inclusion (2) and obtain

$$R(\mathsf{has}) = \{(\mathsf{Bird}, \mathsf{Wing})\}.$$

- Apply (rightR) to inclusion (6) and obtain

$$R(\mathsf{lays}) = \{(\mathsf{Reptile}, \mathsf{Egg})\}.$$

- Apply (leftR) to inclusion (4) and obtain

$$S(\mathsf{Bird}) = \{(\mathsf{Bird}, \mathsf{Vertebrate}, X\}.$$

- No rules are applicable. Notice, for example, that applying (conjR) to inclusion (3) does not give us anything new.

Thus, we end up with

$$\begin{aligned}
S(\mathsf{Bird}) &= \{\mathsf{Bird}, \mathsf{Vertebrate}, X\} \\
S(\mathsf{Vertebrate}) &= \{\mathsf{Vertebrate}\} \\
S(\mathsf{Wing}) &= \{\mathsf{Wing}\} \\
S(\mathsf{Egg}) &= \{\mathsf{Egg}\} \\
S(\mathsf{Reptile}) &= \{\mathsf{Reptile}, \mathsf{Vertebrate}\} \\
R(\mathsf{has}) &= \{(\mathsf{Bird}, \mathsf{Wing})\} \\
R(\mathsf{lays}) &= \{(\mathsf{Reptile}, \mathsf{Egg})\}
\end{aligned}$$

(d) Using the output of the algorithm, decide whether

- $\mathsf{Reptile} \sqsubseteq_{\mathcal{T}'} \mathsf{Vertebrate}$
- $\mathsf{Vertebrate} \sqsubseteq_{\mathcal{T}'} \mathsf{Bird}$

Solution: In the first case, the answer is yes because

$$\mathsf{Vertebrate} \in S(\mathsf{Reptile}).$$

In the second case the answer is no because

$$\mathsf{Bird} \notin S(\mathsf{Vertebrate}).$$

3. Let $\mathcal{T}$ be an $\mathcal{EL}$-TBox containing the following concept inclusions:

$$\begin{aligned} A &\sqsubseteq X && (1)\\ A &\sqsubseteq Y && (2)\\ B &\sqsubseteq B' && (3)\\ X \sqcap Y &\sqsubseteq Z && (4)\\ Z &\sqsubseteq \exists r.B && (5)\\ \exists r.B' &\sqsubseteq A' && (6) \end{aligned}$$

(a) Is $\mathcal{T}$ in normal form?

Solution: Yes, it is in normal form.

Apply the algorithm for deciding $E \sqsubseteq_{\mathcal{T}} F$ (equivalently, $\mathcal{T} \models E \sqsubseteq F$), where E, F are concept names. Use $\mathcal{T}$ as input.

Solution: Initialise $S(A) = \{A\}$, $S(B) = \{B\}$, $S(Y) = \{Y\}$, $S(X) = \{X\}$, $S(Z) = \{Z\}$, $S(A') = \{A'\}$, $S(B') = \{B'\}$, $R(r) = \emptyset$.

- Apply (simpleR) to inclusion (1) and obtain $S(A) = \{A, X\}$
- Apply (simpleR) to inclusion (2) and obtain $S(A) = \{A, X, Y\}$
- Apply (simpleR) to inclusion (3) and obtain $S(B) = \{B, B'\}$
- Apply (conjR) to inclusion (4) and obtain $S(A) = \{A, X, Y, Z\}$
- Apply (rightR) to inclusion (5) and obtain $R(r) = \{(A, B)\}$
- Apply (rightR) to inclusion (5) and obtain $R(r) = \{(Z, B), (A, B)\}$
- Apply (leftR) to inclusion (6) and obtain $S(A) = \{A, X, Y, Z, A'\}$ and $S(Z) = \{Z, A'\}$.

No more rule is applicable. Thus $S(A) = \{A, X, Y, Z, A'\}$, $S(B) = \{B, B'\}$, $S(Y) = \{Y\}$, $S(X) = \{X\}$, $S(Z) = \{Z, A'\}$, $S(A') = \{A'\}$, $S(B') = \{B'\}$, $R(r) = \{(Z, B), (A, B)\}$.

(c) Using the output of the algorithm, decide whether

- $A \sqsubseteq_{\mathcal{T}} Z$: Yes, because $Z \in S(A)$;
- $B \sqsubseteq_{\mathcal{T}} Z$: No, because $Z \notin S(B)$;
- $X \sqsubseteq_{\mathcal{T}} Y$: No, because $Y \notin S(X)$;
- $A \sqsubseteq_{\mathcal{T}} A'$: Yes, because $A' \in S(A)$;
- $B \sqsubseteq_{\mathcal{T}} B'$: Yes, because $B' \in S(B)$.