Forgetting and uniform interpolation in large-scale description logic terminologies

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Abstract. We develop a framework for forgetting concepts and roles (aka uniform interpolation) in terminologies in the lightweight description logic $\mathcal{EL}$ extended with role inclusions and domain and range restrictions. Three different notions of forgetting, preserving, respectively, concept inclusions, concept instances, and answers to conjunctive queries, with corresponding languages for uniform interpolants are investigated. Experiments based on SNOMED CT (Systematised Nomenclature of Medicine Clinical Terms) and NCI (National Cancer Institute Ontology) demonstrate that forgetting is often feasible in practice for large-scale terminologies.

1 Introduction

The main application of ontologies in computer science is to fix the vocabulary of an application domain and to provide a formal theory that defines the meaning of terms built from the vocabulary and their relationships. Current applications lead to the development of very large and comprehensive ontologies such as the medical ontology SNOMED CT (Systematised Nomenclature of Medicine Clinical Terms) (Spackman 2000) containing about $380\,000$ concept definitions and the National Cancer Institute ontology (NCI) (Sioutos *et al.* 2006) containing more than $60\,000$ axioms. For ontologies $\mathcal{T}$ of this size, it is often of interest to *forget* a subvocabulary Σ of the vocabulary of $\mathcal{T}$; i.e., to transform $\mathcal{T}$ into a new ontology $\mathcal{T}_\Sigma$ (called a Σ-*interpolant of* $\mathcal{T}$) that contains no symbols from Σ and that is indistinguishable from $\mathcal{T}$ regarding its consequences that do not use Σ. In AI, this problem has been studied under a variety of names such as *forgetting* and *variable elimination* (Reiter and Lin 1994; Eiter and Wang 2008; Lang *et al.* 2003). In mathematical logic, this problem has been investigated as the *uniform interpolation problem* (Visser 1996). Computing Σ-interpolants of ontologies has a number of potential applications, e.g.,

Re-use of ontologies: when using ontologies such as SNOMED CT in an application, often only a very small fraction of its vocabulary is of interest. In this case, one could use a Σ-interpolant instead of the whole ontology, where Σ is the vocabulary not of interest for the application.

Predicate hiding: an ontology developer might not want to publish an ontology completely because a certain part of its vocabulary is not intended for public use. Again, publishing Σ-interpolants, where Σ is the vocabulary to be hidden, appears to be a solution to this problem.

Exhibiting hidden relations between terms: large ontologies are difficult to maintain as small changes to its axioms can have drastic and damaging effects. To analyze possibly unwanted consequences over a certain part Γ of the vocabulary, an ontology developer can automatically generate a complete axiomatization of the relations between terms over Γ by computing a Σ-interpolant, where Σ is the complement of Γ.
Ontology versioning: to check whether two versions of an ontology have the same consequences over their common vocabulary (or a subset thereof), one can first compute their interpolants by forgetting the vocabulary not shared by the two versions and then check whether the two interpolants are logically equivalent (i.e., have the same models).

In the description of Σ-interpolants given above, we have neither specified a language in which they are axiomatized nor did we specify the language wrt. which Σ-interpolants should be indistinguishable from the original ontology. The choice of the latter language depends on the application: for example, if one is interested in *inclusions between concepts*, then a Σ-interpolant should imply the same concept inclusions using no symbols from Σ as the original ontology. On the other hand, if the ontology is used to query instance data using *conjunctive queries*, then a Σ-interpolant together with any instance data using no symbols from Σ should imply the same certain answers to conjunctive queries using no symbols from Σ as the original ontology.

Regarding the language $\mathcal{L}$ in which Σ-interpolants should be axiomatized, one has to find a compromise between the following three conflicting goals:
(R) Standard reasoning problems (e.g., logical equivalence) in $\mathcal{L}$ should not be more complex than reasoning in the language underlying the ontology.
(I) Σ-interpolants in $\mathcal{L}$ should be uniquely determined up to logical equivalence: if $\mathcal{T}_1'$ and $\mathcal{T}_2'$ are Σ-interpolants in $\mathcal{L}$ of ontologies $\mathcal{T}_1$ and $\mathcal{T}_2$ that have the same consequences not using Σ, then $\mathcal{T}_1'$ and $\mathcal{T}_2'$ should be logically equivalent.
(E) The language $\mathcal{L}$ should be powerful enough to admit *finite* and *succinct* (ideally, polynomial size) axiomatizations of Σ-interpolants, and it should be possible to compute Σ-interpolants efficiently (ideally, in polynomial time).

For ontologies given in standard description logics (DLs) such as $\mathcal{EL}$ and any language between $\mathcal{ALC}$ and $\mathcal{SHIQO}$, there do not exist languages $\mathcal{L}$ achieving all these goals simultaneously.[1] To illustrate this point, let $\mathcal{L}$ be second-order logic. Then $\mathcal{L}$ trivially satisfies (E) but fails to satisfy (R) and (I), for ontologies in any standard DL.

In this paper, we consider forgetting in the lightweight description logic $\mathcal{EL}$ underlying the designated OWL2-EL profile of the upcoming OWL Version 2 extended with role inclusions and domain and range restrictions (Baader *et al.* 2008). This choice it motivated by the fact that forgetting appears to be of particular interest for large-scale and comprehensive ontologies and that many such ontologies are given in this language. We introduce three DLs for axiomatizing Σ-interpolants satisfying criteria (R) and (I) and preserving, respectively, inclusions between concepts, concept instances, and answers to conjunctive queries. These DLs do not satisfy (E), as Σ-interpolants sometimes to not exist or are of exponential size. We demonstrate that, nevertheless, Σ-interpolants typically exist and can be computed in practice for large-scale termi-

[1] This follows from the fact that deciding whether TBoxes in these DLs imply the same concept inclusions over a signature is by at least one exponential harder than deciding logical equivalence (Lutz and Wolter 2007; Lutz *et al.* 2007).

Concept C	Translation $C^\sharp$	Concept C	Translation $C^\sharp$
$\top$	$x = x$	$\mathsf{dom}(r)$	$\exists y\,(r(x,y))$
A	$A(x)$	$\mathsf{ran}(r)$	$\exists y\,(r(y,x))$
$C \sqcap D$	$C^\sharp(x) \wedge D^\sharp(x)$	$\exists u.C$	$(x = x) \wedge \exists y\, C^\sharp(y)$
$\exists r.C$	$\exists y\,(r(x,y) \wedge C^\sharp(y))$	$\exists r_1 \sqcap \cdots \sqcap r_n.C$	$\exists y(r_1(x,y) \wedge \cdots \wedge r_n(x,y) \wedge C^\sharp(y))$

Fig. 1. Standard translation $\cdot^\sharp$

nologies such as SNOMED CT and appropriate versions of NCI. Detailed proofs and additional experiments are available in the appendix.

2 Preliminaries

Let $\mathsf{N_C}$, $\mathsf{N_R}$, and $\mathsf{N_I}$ be countably infinite and mutually disjoint sets of concept names, role names, and individual names. $\mathcal{EL}$-concepts C are built according to the rule

$$C := \quad A \quad | \quad \top \quad | \quad C \sqcap D \quad | \quad \exists r.C,$$

where $A \in \mathsf{N_C}$, $r \in \mathsf{N_R}$, and C, D range over $\mathcal{EL}$-concepts. The set of $\mathcal{ELH}^r$-inclusions consists of *concept inclusions* $C \sqsubseteq D$ and *concept equations* $C \equiv D$, *domain restrictions* $\mathsf{dom}(r) \sqsubseteq C$, *range restrictions* $\mathsf{ran}(r) \sqsubseteq C$ and *role inclusions* $r \sqsubseteq s$, where C, D are $\mathcal{EL}$-concepts and $r, s \in \mathsf{N_R}$. An $\mathcal{ELH}^r$-TBox $\mathcal{T}$ is a finite set of $\mathcal{ELH}^r$-inclusions. An $\mathcal{ELH}^r$-TBox is called $\mathcal{ELH}^r$*-terminology* if all its concept inclusions and equations are of the form $A \sqsubseteq C$ and $A \equiv C$ and no concept name occurs more than once on the left hand side. In what follows we use $A \bowtie C$ to denote expressions of the form $A \sqsubseteq C$ and $A \equiv C$.

Assertions of the form $A(a)$ and $r(a, b)$, where $a, b \in \mathsf{N_I}$, $A \in \mathsf{N_C}$, and $r \in \mathsf{N_R}$, are called ABox-assertions. An ABox is a finite set of ABox-assertions. By $\mathsf{obj}(\mathcal{A})$ we denote the set of individual names in $\mathcal{A}$. A knowledge base (KB) is a pair $(\mathcal{T}, \mathcal{A})$ consisting of a TBox $\mathcal{T}$ and an ABox $\mathcal{A}$. Assertions of the form $C(a)$ and $r(a, b)$, where $a, b \in \mathsf{N_I}$, C a $\mathcal{EL}$-concept, and $r \in \mathsf{N_R}$, are called instance assertions. To define the semantics of DLs considered in this paper we make use of the fact that DL-expressions can be regarded as formulas in FO, where FO denotes the set of first-order predicate logic formulas with equality using unary predicates in $\mathsf{N_C}$, binary predicates in $\mathsf{N_R}$, and constants from $\mathsf{N_I}$; see Figure 1 (in which the DL-constructors not considered so far are defined later). In what follows, we will not distinguish between DL-expressions and their translation into FO and regard TBoxes, ABoxes and KBs as finite subsets of FO. Thus, we use $\mathcal{T} \models \varphi$ to denote that φ follows from $\mathcal{T}$ in first-order logic even if φ is an $\mathcal{ELH}^r$-inclusion and $\mathcal{T}$ a subset of FO and similar conventions apply to DLs introduced later in this paper. FO (and, therefore, $\mathcal{ELH}^r$) is interpreted in models $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$, where the *domain* $\Delta^\mathcal{I}$ is a non-empty set, and $\cdot^\mathcal{I}$ is a function mapping each concept name A to a subset $A^\mathcal{I}$ of $\Delta^\mathcal{I}$, each role name r to a binary relation $r^\mathcal{I} \subseteq \Delta^\mathcal{I} \times \Delta^\mathcal{I}$, and each individual name a to an element $a^\mathcal{I} \in \Delta^\mathcal{I}$.

The most important ways of querying $\mathcal{ELH}^r$-TBoxes and KBs are subsumption (check whether $\mathcal{T} \models \alpha$ for an $\mathcal{ELH}^r$-inclusion α), instance checking (check whether

$(\mathcal{T}, \mathcal{A}) \models \alpha$ for an instance assertion α), and conjunctive query answering. To define the latter, call a first-order formula $q(\boldsymbol{x})$ a *conjunctive query* if it is of the form $\exists \boldsymbol{y} \psi(\boldsymbol{x}, \boldsymbol{y})$, where ψ is a conjunction of expressions $A(t)$ and $r(t_1, t_2)$ with t, t_1, t_2 drawn from $\mathsf{N_I}$ and sequences of variables $\boldsymbol{x}$ and $\boldsymbol{y}$. If $\boldsymbol{x}$ has length k, then a sequence $\boldsymbol{a}$ of elements of $\mathsf{obj}(\mathcal{A})$ of length k is called a *certain answer* to $q(\boldsymbol{x})$ of a KB $(\mathcal{T}, \mathcal{A})$ if $(\mathcal{T}, \mathcal{A}) \models q(\boldsymbol{a})$.

3 Forgetting

A signature Σ is a subset of $\mathsf{N_C} \cup \mathsf{N_R}$[2]. Given a signature Σ, we set $\overline{\Sigma} = (\mathsf{N_C} \cup \mathsf{N_R}) \setminus \Sigma$. Given a concept, role, concept inclusion, TBox, ABox, FO-sentence, set of FO-sentences E, we denote by $\mathsf{sig}(E)$ the signature of E, that is, the set of concept and role names occurring in it. We use the term $\mathcal{ELH}^r_\Sigma$-inclusion (Σ-ABox, Σ-query, $\mathcal{L}_\Sigma$-sentence, etc.) to denote $\mathcal{ELH}^r$-inclusions (ABoxes, queries, $\mathcal{L}$-sentences, etc.) whose signature is contained in Σ.

To define forgetting, we first formalize the notion of inseparability between TBoxes wrt. a signature. Intuitively, two TBoxes $\mathcal{T}_1$ and $\mathcal{T}_2$ are inseparable wrt. a signature Σ if they have the same Σ-consequences, where the set of Σ-consequences considered can either reflect subsumption queries, instance queries, or conjunctive queries, depending on the application. We give the definitions for sets of FO-sentences because we later require these notions for a variety of DLs.

Definition 1. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be sets of FO-sentences and Σ a signature.

- $\mathcal{T}_1$ and $\mathcal{T}_2$ are concept Σ-inseparable, in symbols $\mathcal{T}_1 \equiv^C_\Sigma \mathcal{T}_2$, if for all $\mathcal{ELH}^r_\Sigma$-inclusions α: $\mathcal{T}_1 \models \alpha \Leftrightarrow \mathcal{T}_2 \models \alpha$.
- $\mathcal{T}_1$ and $\mathcal{T}_2$ are instance Σ-inseparable, in symbols $\mathcal{T}_1 \equiv^i_\Sigma \mathcal{T}_2$, if for all Σ-ABoxes $\mathcal{A}$ and Σ-instance assertions α using individual names from $\mathsf{obj}(\mathcal{A})$: $(\mathcal{T}_1, \mathcal{A}) \models \alpha \Leftrightarrow (\mathcal{T}_2, \mathcal{A}) \models \alpha$.
- $\mathcal{T}_1$ and $\mathcal{T}_2$ are query Σ-inseparable, in symbols $\mathcal{T}_1 \equiv^q_\Sigma \mathcal{T}_2$, if for all Σ-ABoxes $\mathcal{A}$, conjunctive Σ-queries $q(\boldsymbol{x})$, and vectors $\boldsymbol{a}$ of the same length as $\boldsymbol{x}$ of individual names in $\mathsf{obj}(\mathcal{A})$: $(\mathcal{T}_1, \mathcal{A}) \models q(\boldsymbol{a}) \Leftrightarrow (\mathcal{T}_2, \mathcal{A}) \models q(\boldsymbol{a})$.

The definition of forgetting (Σ-interpolants) is now straightforward.

Definition 2 (Σ-interpolant). Let $\mathcal{T}$ be an $\mathcal{ELH}^r$-TBox, Σ a finite signature, and $\mathcal{L}$ a set of FO-sentences. If $\mathcal{T}_\Sigma$ is a finite set of $\mathcal{L}_{\overline{\Sigma}}$-sentences such that $\mathcal{T} \models \varphi$ for all $\varphi \in \mathcal{T}_\Sigma$, then $\mathcal{T}_\Sigma$ is

- a concept Σ-interpolant of $\mathcal{T}$ in $\mathcal{L}$ if $\mathcal{T} \equiv^C_{\overline{\Sigma}} \mathcal{T}_\Sigma$;
- an instance Σ-interpolant of $\mathcal{T}$ in $\mathcal{L}$ if $\mathcal{T} \equiv^i_{\overline{\Sigma}} \mathcal{T}_\Sigma$;
- a query Σ-interpolant of $\mathcal{T}$ in $\mathcal{L}$ if $\mathcal{T} \equiv^q_{\overline{\Sigma}} \mathcal{T}_\Sigma$.

[2] We investigate forgetting for TBoxes for DLs without nominals; thus we do not include individual names into the signature.

One can show that every query Σ-interpolant is an instance Σ-interpolant and every instance Σ-interpolant is a concept Σ-interpolant. The converse implications do not hold, even for $\mathcal{ELH}^r$-terminologies:

Example 1. Let $\mathcal{T} = \{\mathsf{ran}(r) \sqsubseteq A_1, \mathsf{ran}(s) \sqsubseteq A_2, B \equiv A_1 \sqcap A_2\}$ and $\Sigma = \{A_1, A_2\}$. One can show that the empty TBox is a concept Σ-interpolant of $\mathcal{T}$. However, the empty TBox is not an instance Σ-interpolant of $\mathcal{T}$. To show this, consider the $\overline{\Sigma}$-ABox $\mathcal{A} = \{r(a_0, b), s(a_1, b)\}$. Then $(\mathcal{T}, \mathcal{A}) \models B(b)$ but $(\emptyset, \mathcal{A}) \not\models B(b)$. Observe that *no* $\mathcal{ELH}^r$-TBox (and even no $\mathcal{SHQ}$-TBox) is an instance Σ-interpolant of $\mathcal{T}$ because it is impossible to capture the ABox $\mathcal{A}$ in a DL in which one cannot refer to the range of distinct roles in one concept. On the other hand, the TBox $\mathcal{T}' = \{\mathsf{ran}(r) \sqcap \mathsf{ran}(s) \sqsubseteq B\}$ given in an extension of $\mathcal{ELH}^r$ is an instance Σ-interpolant of $\mathcal{T}$.

Example 2. Let $\mathcal{T} = \{A \sqsubseteq \exists s.\top, s \sqsubseteq r_1, s \sqsubseteq r_2\}$ and $\Sigma = \{s\}$. Then $\mathcal{T}' = \{A \sqsubseteq \exists r_1.\top \sqcap \exists r_2.\top\}$ is an instance Σ-interpolant of $\mathcal{T}$, but $\mathcal{T}'$ is not a query Σ-interpolant of $\mathcal{T}$. To show the latter, let $\mathcal{A} = \{A(a)\}$ and let $q = \exists x(r_1(a, x) \wedge r_2(a, x))$. Then $(\mathcal{T}, \mathcal{A}) \models q$ but $(\mathcal{T}', \mathcal{A}) \not\models q$. Again, no $\mathcal{ELH}^r$-TBox (and even no TBox in $\mathcal{SHIQ}$) is a query Σ-interpolant of $\mathcal{T}$. On the other hand, the TBox $\mathcal{T}'' = \{A \sqsubseteq \exists r_1 \sqcap r_2.\top\}$ given in an extension of $\mathcal{ELH}^r$ with conjunctions of roles names is a query Σ-interpolant of $\mathcal{T}$.

Besides of exhibiting examples where concept-, instance-, and query Σ-interpolants are distinct, Example 1 and 2 also show that even in extremely simple cases $\mathcal{ELH}^r$ and a variety of more expressive DLs are not sufficiently powerful to express instance and query Σ-interpolants of $\mathcal{ELH}^r$-terminologies. Rather surprisingly, there also exist simple examples in which $\mathcal{ELH}^r$-TBoxes are not sufficiently expressive to axiomatize concept Σ-interpolants of $\mathcal{ELH}^r$-terminologies.

Example 3. Let $\Sigma = \{\mathsf{Research\_Inst}, \mathsf{Education\_Inst}\}$ and $\mathcal{T}$ be

$$\begin{aligned} \mathsf{University} &\equiv \mathsf{Research\_Inst} \sqcap \mathsf{Education\_Inst} \\ \mathsf{School} &\sqsubseteq \mathsf{Education\_Inst} \\ \mathsf{ran}(\mathsf{PhD\_from}) &\sqsubseteq \mathsf{Research\_Inst} \end{aligned}$$

Then there does not exist an $\mathcal{ELH}^r$-TBox that is a concept Σ-interpolant of $\mathcal{T}$. Intuitively, the reason is that there is no $\mathcal{ELH}^r_{\overline{\Sigma}}$-TBox which follows from $\mathcal{T}$ and has the following infinite set of $\overline{\Sigma}$-consequences (which are consequences of $\mathcal{T}$):

$$\exists \mathsf{PhD\_from}.(\mathsf{School} \sqcap A) \sqsubseteq \exists \mathsf{PhD\_from}.(\mathsf{University} \sqcap A),$$

where $A \in \overline{\Sigma}$. On the other hand, the TBox $\mathcal{T}' = \{\mathsf{ran}(\mathsf{PhD\_from}) \sqcap \mathsf{School} \sqsubseteq \mathsf{University}\}$ given in an extension of $\mathcal{ELH}^r$ is a concept Σ-interpolant of $\mathcal{T}$.

We now introduce three extensions of $\mathcal{ELH}^r$ which we propose to axiomatize concept-, instance-, and query Σ-interpolants.

Definition 3 **($\mathcal{EL}^{\mathsf{ran},0}$, $\mathcal{EL}^{\mathsf{ran}}$, $\mathcal{EL}^{\mathsf{ran},\sqcap,u}$).** $\mathcal{C}^{\mathsf{ran},0}$-concepts are constructed using the following syntax rule

$$C := \quad D \quad | \quad \mathsf{ran}(r) \quad | \quad \mathsf{ran}(r) \sqcap D,$$

where D ranges over $\mathcal{EL}$-concepts and $r \in \mathsf{N_R}$. The set of $\mathcal{EL}^{\mathsf{ran},0}$-inclusions consists of concept inclusions $C \sqsubseteq D$ and role inclusions $r \sqsubseteq s$, where C is a $\mathcal{C}^{\mathsf{ran},0}$-concept, D an $\mathcal{EL}$-concept, and $r, s \in \mathsf{N_R}$.

$\mathcal{C}^{\mathsf{ran}}$-concepts are constructed using the following syntax rule

$$C := \quad A \quad | \quad \mathsf{ran}(r) \quad | \quad C \sqcap D \quad | \quad \exists r.C,$$

where $A \in \mathsf{N_C}$, C, D range over $\mathcal{C}^{\mathsf{ran}}$-concepts and $r \in \mathsf{N_R}$. The set of $\mathcal{EL}^{\mathsf{ran}}$-inclusions consists of all concept inclusions $C \sqsubseteq D$ and role inclusions $r \sqsubseteq s$, where C is a $\mathcal{C}^{\mathsf{ran}}$-concept, D an $\mathcal{EL}$-concept, and $r, s \in \mathsf{N_R}$.

Let u (the universal role) be a fresh logical symbol. $\mathcal{C}^{\sqcap,u}$-concepts are constructed using the following syntax rule

$$C := \quad A \quad | \quad C \sqcap D \quad | \quad \exists R.C \quad | \quad \exists u.C,$$

where $A \in \mathsf{N_C}$, C, D range over $\mathcal{C}^{\sqcap,u}$-concepts and $R = r_1 \sqcap \cdots \sqcap r_n$ with $r_1, \ldots, r_n \in \mathsf{N_R}$. The set of $\mathcal{EL}^{\mathsf{ran},\sqcap,u}$-inclusions consists of concept inclusions $C \sqsubseteq D$ and role inclusions $r \sqsubseteq s$, where C is a $\mathcal{C}^{\mathsf{ran}}$-concept, D a $\mathcal{C}^{\sqcap,u}$-concept, and $r, s \in \mathsf{N_R}$.

An X-TBox is a finite set of X-inclusions, where X ranges over $\mathcal{EL}^{\mathsf{ran}}$, $\mathcal{EL}^{\mathsf{ran},0}$, and $\mathcal{EL}^{\mathsf{ran},\sqcap,u}$.

We have the following inclusions:

$$\mathcal{ELH}^r \quad \lhd \quad \mathcal{EL}^{\mathsf{ran},0} \quad \lhd \quad \mathcal{EL}^{\mathsf{ran}} \quad \lhd \quad \mathcal{EL}^{\mathsf{ran},\sqcap,u}$$

where $\mathcal{L}_1 \lhd \mathcal{L}_2$ means that every $\mathcal{L}_1$-TBox is logically equivalent to some $\mathcal{L}_2$-TBox. The semantics of the additional constructors is straightforward and given in Figure 1. We regard the universal role u as a logical symbol (i.e., $u \notin \mathsf{N_R}$). This interpretation reflects the fact that the signature of the first-order translation of $\exists u.C$ coincides with the signature of C. Observe that the TBox given as a concept Σ-interpolant in Example 3 is an $\mathcal{EL}^{\mathsf{ran},0}$-TBox; the instance Σ-interpolant given in Example 1 is an $\mathcal{EL}^{\mathsf{ran}}$-TBox, and the query Σ-interpolant in Example 2 is an $\mathcal{EL}^{\mathsf{ran},\sqcap,u}$-TBox. The need for the universal role is illustrated by considering $\mathcal{T} = \{A \sqsubseteq \exists r.B\}$ and $\Sigma = \{r\}$. The empty TBox is a concept and an instance Σ-interpolant of $\mathcal{T}$. A query Σ-interpolant is given by $\mathcal{T}' = \{A \sqsubseteq \exists u.B\}$ and reflects the fact that $(\mathcal{T}, \mathcal{A}) \models \exists x B(x)$ for $\mathcal{A} = \{A(a)\}$.

We show that the languages introduced in Definition 3 satisfy criteria (R) and (I) from the introduction. (R) is a consequence of the following result.

Theorem 1. *The following problems are PTIME-complete for $\mathcal{EL}^{\mathsf{ran},\sqcap,u}$-TBoxes $\mathcal{T}$ and ABoxes $\mathcal{A}$: decide whether*

- *$\mathcal{T} \models C \sqsubseteq D$, for $C \sqsubseteq D$ an $\mathcal{EL}^{\mathsf{ran},\sqcap,u}$-inclusion;*
- *$(\mathcal{T}, \mathcal{A}) \models C(a)$, where C is an $\mathcal{EL}$-concept.*

Deciding whether $(\mathcal{T}, \mathcal{A}) \models q(\boldsymbol{a})$, where q is a conjunctive query, is NP-complete, and deciding this problem for fixed $q(\boldsymbol{a})$ (knowledge base complexity) is PTIME-complete.

It follows, in particular, that logical equivalence of $\mathcal{EL}^{\mathsf{ran},\sqcap,u}$-TBoxes is decidable in PTime. These complexity results are exactly the same as for $\mathcal{ELH}$-TBoxes (Rosati 2007). For (I) and the computation of Σ-interpolants below, we first investigate the relationship between the distinct inseparability notions introduced in Definition 1 and inseparability wrt. the new languages. (For $\mathcal{EL}$ this relationship is characterized in (Lutz and Wolter).) Let X range over the superscripts $\mathsf{ran}, 0$ and ran and $\mathsf{ran}, \sqcap, u$. Say that two finite sets of FO-sentences $\mathcal{T}_1$ and $\mathcal{T}_2$ are X-inseparable wrt. Σ, in symbols $\mathcal{T}_1 \equiv_\Sigma^X \mathcal{T}_2$, if $\mathcal{T}_1 \models \alpha \Leftrightarrow \mathcal{T}_2 \models \alpha$, for all $\mathcal{EL}_\Sigma^X$-inclusions α.

Theorem 2. *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be $\mathcal{EL}^{\mathsf{ran},\sqcap,u}$-TBoxes and Σ an infinite signature. Then the following holds:*

- $\mathcal{T}_1 \equiv_\Sigma^C \mathcal{T}_2$ *iff* $\mathcal{T}_1 \equiv_\Sigma^{\mathsf{ran},0} \mathcal{T}_2$*;*
- $\mathcal{T}_1 \equiv_\Sigma^i \mathcal{T}_2$ *iff* $\mathcal{T}_1 \equiv_\Sigma^{\mathsf{ran}} \mathcal{T}_2$*;*
- $\mathcal{T}_1 \equiv_\Sigma^q \mathcal{T}_2$ *iff* $\mathcal{T}_1 \equiv_\Sigma^{\mathsf{ran},\sqcap,u} \mathcal{T}_2$.

The condition that Σ is infinite is required only for the implication from right to left in Point 1. As we forget *finite* signatures, their complement is always infinite.

From Theorem 2 we immediately obtain that (I) is met for the three notions of Σ-interpolants. For example, assume that $\mathcal{T}_1$ and $\mathcal{T}_2$ are $\mathcal{ELH}^r$-TBoxes such that $\mathcal{T}_1 \equiv_{\overline{\Sigma}}^q \mathcal{T}_2$ and let $\mathcal{T}_1'$ and $\mathcal{T}_2'$ be query Σ-interpolants in $\mathcal{EL}^{\mathsf{ran},\sqcap,u}$ of $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively. By Theorem 2, $\mathcal{T}_1' \equiv_{\overline{\Sigma}}^{\mathsf{ran},\sqcap,u} \mathcal{T}_2'$. But then $\mathcal{T}_1'$ and $\mathcal{T}_2'$ are logically equivalent: we have $\mathcal{T}_1' \models \alpha$ for all $\alpha \in \mathcal{T}_2'$ because all such α are $\mathcal{EL}_{\overline{\Sigma}}^{\mathsf{ran},\sqcap,u}$-inclusions and $\mathcal{T}_2' \models \alpha$. The converse direction holds for the same reason.

4 Computing Σ-interpolants

We present algorithms computing Σ-interpolants for $\mathcal{ELH}^r$-terminologies satisfying certain acyclicity conditions. In this section, we assume wlog. that terminologies are *normalized* $\mathcal{ELH}^r$-terminologies; i.e., $\mathcal{ELH}^r$-terminologies $\mathcal{T}$ consisting of role inclusions and axioms of the form (here, and in what follows, we write $r \sqsubseteq_\mathcal{T} s$ if $\mathcal{T} \models r \sqsubseteq s$)

- $A \bowtie \exists r.B$, where $B \in \mathsf{N_C} \cup \{\top\}$;
- $A \bowtie B_1 \sqcap \cdots \sqcap B_n$, where $B_1, \ldots, B_n \in \mathsf{N_C}$;
- $\mathsf{dom}(s) \sqsubseteq A$, where $A \in \mathsf{N_C}$;
- $\mathsf{ran}(s) \sqsubseteq A$, where $A \in \mathsf{N_C}$

such that $\mathsf{dom}(s) \sqsubseteq A \in \mathcal{T}$ and $r \sqsubseteq_\mathcal{T} s$ imply $\mathsf{dom}(r) \sqsubseteq A \in \mathcal{T}$; $\mathsf{ran}(s) \sqsubseteq A \in \mathcal{T}$ and $r \sqsubseteq_\mathcal{T} s$ imply $\mathsf{ran}(r) \sqsubseteq A \in \mathcal{T}$; and $r \sqsubseteq_\mathcal{T} s$ and $s \sqsubseteq_\mathcal{T} r$ implies $r = s$. It is easy to see that every $\mathcal{ELH}^r$-terminology $\mathcal{T}$ can be transformed (in polynomial time) into a normalized terminology $\mathcal{T}'$ such that Σ-interpolants of $\mathcal{T}$ coincide with Σ' interpolants of $\mathcal{T}'$, where Σ' contains additional fresh concept names.

We give the acyclicity conditions required for the algorithms to terminate. The $\overline{\Sigma}$-cover $\mathcal{C}_\mathcal{T}^{\overline{\Sigma}}(r)$ of a role r wrt. a terminology $\mathcal{T}$ consists of all $s \in \overline{\Sigma}$ such that $r \sqsubseteq_\mathcal{T} s$ and there does not exist $r' \in \overline{\Sigma}$ with $r' \neq s$ and $r \sqsubseteq_\mathcal{T} r' \sqsubseteq_\mathcal{T} s$.

Definition 4 **(Σ-loop).** Let $\mathcal{T}$ be a normalized $\mathcal{ELH}^r$-terminology and Σ a signature. Define a relation $\prec_\Sigma \subseteq (\mathsf{N_C} \cap \Sigma) \times (\mathsf{N_C} \cap \Sigma)$ as follows: $A \prec_\Sigma B$ if $A, B \in \Sigma$ and
(a) $A \bowtie C \in \mathcal{T}$ for some C such that B occurs in C, or
(b) $A \bowtie \exists r.A' \in \mathcal{T}$ for some $A' \in \mathsf{N_C} \cup \{\top\}$ and $r \in \Sigma$ such that $\mathsf{dom}(r) \sqsubseteq B \in \mathcal{T}$, or

(c) $A \bowtie \exists r.A' \in \mathcal{T}$ for some $A' \in \mathsf{N_C} \cup \{\top\}$ and $r \in \Sigma$ such that there exists $s \in \mathcal{C}^{\Sigma}_{\mathcal{T}}(r)$ with $\mathsf{ran}(r) \sqsubseteq B \in \mathcal{T}$, $\mathsf{ran}(s) \sqsubseteq B \notin \mathcal{T}$.
We say that $\mathcal{T}$ contains a Σ-loop if $\prec_\Sigma$ contains a cycle.

The following example illustrates this definition and shows that the existence of Σ-loops typically entails the non-existence of Σ-interpolants, even in FO.

Example 4. Consider the set of inclusions

$$\mathsf{Elephant} \sqsubseteq \mathsf{Mammal} \tag{1}$$

$$\mathsf{Mammal} \sqsubseteq \exists \mathsf{has\_mother}.\mathsf{Mammal} \tag{2}$$

$$\mathsf{Mammal} \sqsubseteq \exists \mathsf{has\_mam'l\_father}.\top \tag{3}$$

$$\mathsf{dom}(\mathsf{has\_mam'l\_father}) \sqsubseteq \exists \mathsf{has\_mother}.\mathsf{Mammal} \tag{4}$$

$$\mathsf{ran}(\mathsf{has\_mam'l\_father}) \sqsubseteq \mathsf{Mammal} \tag{5}$$

$$\mathsf{has\_mam'l\_father} \sqsubseteq \mathsf{has\_mother} \tag{6}$$

and define $\mathcal{ELH}^r$-terminologies $\mathcal{T}_1 = \{(1),(2)\}$, $\mathcal{T}_2 = \{(1),(3),(4)\}$, and $\mathcal{T}_3 = \{(1),(3),(5),(6)\}$, and let $\Sigma_i = \mathsf{sig}(\mathcal{T}_i) \setminus \{\mathsf{Elephant}, \mathsf{has\_mother}\}$, for $i = 1,2,3$. Even in FO, there exists no concept/instance/query Σ_i-interpolant of $\mathcal{T}_i$. To see this observe that in all three cases an *infinite* axiomatization of such a Σ-interpolant is given by the inclusions

$$\{\mathsf{Elephant} \sqsubseteq \overbrace{\exists \mathsf{has\_mother}. \cdots \exists \mathsf{has\_mother}}^{n}.\top \mid n \geq 1\}.$$

This theory cannot be finitely axiomatized in FO without additional predicates. Observe that $\mathcal{T}_1$ contains a Σ_1-loop as axiom (2) implies $\mathsf{Mammal} \prec_{\Sigma_1} \mathsf{Mammal}$ by clause (a) of Definition 4 for Σ_1-loops; $\mathcal{T}_2$ contains a Σ_2'-loop as axioms (3) and (4) imply $\mathsf{Mammal} \prec_{\Sigma_2'} A \prec_{\Sigma_2'} \mathsf{Mammal}$ by clauses (a) and (b), where the fresh concept name A is due to normalization replacing (4) with the inclusions

$$\mathsf{dom}(\mathsf{has\_mam'l\_father}) \sqsubseteq A,\ A \sqsubseteq \exists \mathsf{has\_mother}.\mathsf{Mammal};$$

and $\mathcal{T}_3$ contains a Σ_3-loop as axioms (3), (5), and (6) imply $\mathsf{Mammal} \prec_{\Sigma_3} \mathsf{Mammal}$ by clause (c).

Call a concept name A primitive (pseudo-primitive) in a terminology $\mathcal{T}$ if A does not occur on the left hand side of any axiom in $\mathcal{T}$ (does not occur in the form $A \equiv C$ in $\mathcal{T}$).

The intuition behind the following algorithms for concept-, instance- and query Σ-interpolants is as follows: first, one can show using Theorem 2 and a sequent-style proof system for $\mathcal{ELH}^r$ that under the conditions of Theorems 3, 4 and 5 there exists a concept-, instance- and query Σ-interpolant, respectively, consisting of (in addition to role inclusions and domain and range restrictions) concept inclusions of the form

- $C \sqsubseteq A$ and $A \sqsubseteq C$,

where A is a concept name. In Figures 2 and 3 below, we compute the set $P_\Sigma(A)$ of concepts C such that the inclusion $C \sqsubseteq A$ is in the interpolant. The algorithm in Figure 2 is used to compute $P_\Sigma(A)$ for concept Σ-interpolants, and the algorithm in Figure 3 is used for instance- and query Σ-interpolants. In both cases, $P_\Sigma(A)$ is computed by making a case distinction:

For $A \in \mathsf{sig}(\mathcal{T})$, let $\mathsf{Pre}_\Sigma(A)$ consist of all $D = \mathsf{ran}(r)$, $D = \exists r.\top$, and $D \in \mathsf{N_C}$ such that $\mathcal{T} \models D \sqsubseteq A$ and $\mathsf{sig}(D) \subseteq \mathsf{sig}(\mathcal{T}) \cap \overline{\Sigma}$; construct $P_\Sigma(A)$ as follows:

- for A pseudo-primitive in $\mathcal{T}$, $P_\Sigma(A) = \mathsf{Pre}_\Sigma(A)$;
- if $A \equiv B_1 \sqcap \cdots \sqcap B_n \in \mathcal{T}$, then $P_\Sigma(A)$ is the set of $C_{B_1} \sqcap \cdots \sqcap C_{B_n}$, where
  - if $B_i \in \Sigma$: $C_{B_i} \in P_\Sigma(B_i)$;
  - if $B_i \notin \Sigma$: $C_{B_i} = B_i$;

  such that there are no two top-level conjuncts $\mathsf{ran}(s)$ and $\mathsf{ran}(s')$ with $s \neq s'$;
- if $A \equiv \exists r.A' \in \mathcal{T}$, then $P_\Sigma(A)$ is the union of $\mathsf{Pre}_\Sigma(A)$ and
  - if $A' \in \Sigma$: $\{\exists s.A' \mid s \sqsubseteq_\mathcal{T} r, s \in \overline{\Sigma}\}$;
  - if $A' \notin \Sigma$: the set of all concepts $\exists s.D$ such that $s \sqsubseteq_\mathcal{T} r$, $s \in \overline{\Sigma}$, and there exists $D' \in P_\Sigma(A')$ such that D does not contain a top-level conjunct of the form $\mathsf{ran}(r')$ and is the resulting concept when all top-level conjuncts of the form $\mathsf{ran}(s')$ with $s \sqsubseteq_\mathcal{T} s'$ are removed from D'.

Fig. 2. Computing $P_\Sigma(A)$ (for concept Σ-interpolants)

For $A \in \mathsf{sig}(\mathcal{T})$, let $\mathsf{Pre}_\Sigma(A)$ consist of all $D = \mathsf{ran}(r)$, $D = \exists r.\top$, and $D \in \mathsf{N_C}$ such that $\mathcal{T} \models D \sqsubseteq A$ and $\mathsf{sig}(D) \subseteq \mathsf{sig}(\mathcal{T}) \cap \overline{\Sigma}$; construct $P_\Sigma(A)$ as follows:

- for A pseudo-primitive in $\mathcal{T}$, $P_\Sigma(A) = \mathsf{Pre}_\Sigma(A)$;
- if $A \equiv B_1 \sqcap \cdots \sqcap B_n \in \mathcal{T}$, then
$$P_\Sigma(A) = \{C_{B_1} \sqcap \cdots \sqcap C_{B_n} \mid (B_i \in \overline{\Sigma} \text{ and } C_{B_i} = B_i) \text{ or } (B_i \in \Sigma \text{ and } C_{B_i} \in P_\Sigma(B_i))\};$$
- if $A \equiv \exists r.A' \in \mathcal{T}$, then $P_\Sigma(A)$ is the union of $\mathsf{Pre}_\Sigma(A)$ and
  - if $A' \in \Sigma$: $\{\exists s.A' \mid s \sqsubseteq_\mathcal{T} r, s \in \overline{\Sigma}\}$;
  - if $A' \notin \Sigma$: $\{\exists s.D \mid s \sqsubseteq_\mathcal{T} r, s \in \overline{\Sigma}, D' \in P_\Sigma(A')\}$.

Fig. 3. Computing $P_\Sigma(A)$ (for intance and query Σ-interpolants)

- in Point 1, A is pseudo-primitive;
- in Point 2, it is defined by a conjunction;
- in Point 3, it is defined as $\exists r.A'$.

Points 2 and 3 are recursive as they require the sets $P_\Sigma(B)$ when B is used in the definition of A. Σ-loops (or Σ-u-loops, defined below, in the case of query Σ-interpolants) describe exactly the situation in which the recursion does not terminate.

In Figures 4 and 5 below, we compute, in a similar way, the set $Q_\Sigma(A)$ of concepts C such that the inclusion $A \sqsubseteq C$ is in the interpolant. The algorithm in Figure 4 is used to compute $Q_\Sigma(A)$ for concept- and instance Σ-interpolants, and the algorithm in Figure 5 is used for query Σ-interpolants.

Having computed the sets $P_\Sigma(A)$ and $Q_\Sigma(A)$, for all concept names $A \in \mathsf{sig}(\mathcal{T})$, the corresponding concept-, instance-, or query Σ-interpolant $\mathcal{T}_\Sigma^X$ (where X stands for C, i, or q, respectively) consists of the following axioms, where A, r, and s range over $\mathsf{sig}(\mathcal{T}) \cap \overline{\Sigma}$:

- $r \sqsubseteq s$, for $r \sqsubseteq_\mathcal{T} s$;

- $D \sqsubseteq A$, for all $D \in P_\Sigma(A)$;
- $A \sqsubseteq D$, for all $D \in Q_\Sigma(A)$;
- $\mathsf{ran}(r) \sqsubseteq D$, for all $D \in Q_\Sigma(B)$ such that $\mathsf{ran}(r) \sqsubseteq B \in \mathcal{T}$ and $B \in \Sigma$;
- $\mathsf{dom}(r) \sqsubseteq D$, for all $D \in Q_\Sigma(B)$ such that $\mathsf{dom}(r) \sqsubseteq B \in \mathcal{T}$ and $B \in \Sigma$.

We now discuss a variety of runs of these algorithms in order to illustrate the difference between concept-, instance-, and query Σ-interpolants. We start by considering the computation of $P_\Sigma(A)$ and $Q_\Sigma(A)$ for concept and query Σ-interpolants.

Recall that concept Σ-interpolants are formulated using $\mathcal{EL}^{\mathsf{ran},0}$-inclusions, whereas instance Σ-interpolants use $\mathcal{EL}^{\mathsf{ran}}$-inclusions. For the former, the set $P_\Sigma(A)$ must consist of $\mathcal{C}^{\mathsf{ran},0}$-concepts only, whereas $\mathcal{C}^{\mathsf{ran}}$-concepts are allowed for the latter. This is reflected in the two algorithms computing $P_\Sigma(A)$ in Figures 2 and 3: in the recursion in Figure 2 we require a more complicated definition than in Figure 3 because we have to ensure that the concepts in $P_\Sigma(A)$ are $\mathcal{C}^{\mathsf{ran},0}$-concepts. The next example illustrates this difference by showing a run of these algorithms on Example 1.

Example 5. Consider the TBox and signature of Example 1. That is $\mathcal{T} = \{\mathsf{ran}(r) \sqsubseteq A_1, \mathsf{ran}(s) \sqsubseteq A_2, B \equiv A_1 \sqcap A_2\}$ and $\Sigma = \{A_1, A_2\}$. Then $Q_\Sigma(E) = \emptyset$ for all $E \in \{A_1, A_2\}$ and $Q_\Sigma(B) = \{B\}$. For the instance case, we compute P_Σ according to Figure 3:

- $P_\Sigma(A_1) = \{\mathsf{ran}(r_1), B\}$;
- $P_\Sigma(A_2) = \{\mathsf{ran}(r_2), B\}$;
- $P_\Sigma(B)$ consists of $\mathsf{ran}(r_1) \sqcap \mathsf{ran}(r_2)$, $B \sqcap B$, $\mathsf{ran}(r_1) \sqcap B$, $B \sqcap \mathsf{ran}(r_2)$.

Thus, we obtain the following finite $\mathcal{EL}^{\mathsf{ran}}$-TBox as axiomatization of an instance Σ-interpolant:

$$\begin{aligned}
\mathsf{ran}(r_1) \sqcap \mathsf{ran}(r_2) &\sqsubseteq B \\
B \sqcap B &\sqsubseteq B \\
\mathsf{ran}(r_1) \sqcap B &\sqsubseteq B \\
B \sqcap \mathsf{ran}(r_2) &\sqsubseteq B \\
B &\sqsubseteq B
\end{aligned}$$

From these axioms, only the first one is non-tautological, and so we obtain the axiomatization given in Example 1. Notice that this is not an axiomatization of a concept Σ-interpolant, as the first axiom is not an $\mathcal{EL}^{\mathsf{ran},0}$-inclusion. We, therefore, compute P_Σ according to Figure 2 which yields $P_\Sigma(B) = \{B \sqcap B, \mathsf{ran}(r_1) \sqcap B, B \sqcap \mathsf{ran}(r_2)\}$. The resulting axiomatization consists of tautologies only, and thus the empty TBox is a concept Σ-interpolant; cf. Example 1.

The following example shows a run of the algorithms for concept- and instance Σ-interpolants and illustrates the normalization of terminologies.

For $A \in \mathsf{sig}(\mathcal{T})$, let $\mathsf{Post}_\Sigma(A) = \{B \in \overline{\Sigma} \cap \mathsf{sig}(\mathcal{T}) \mid \mathcal{T} \models A \sqsubseteq B\}$ and construct $Q_\Sigma(A)$ as follows:

- for A primitive in $\mathcal{T}$, $Q_\Sigma(A) = \mathsf{Post}_\Sigma(A)$;
- if $A \bowtie B_1 \sqcap \cdots \sqcap B_n \in \mathcal{T}$, then

$$Q_\Sigma(A) = \mathsf{Post}_\Sigma(A) \ \cup \bigcup_{1 \leq i \leq n, B_i \in \Sigma} Q_\Sigma(B_i);$$

- if $A \bowtie \exists r.A' \in \mathcal{T}$, then $Q_\Sigma(A)$ is the union of $\mathsf{Post}_\Sigma(A)$,

$$\bigcup_{r \sqsubseteq_\mathcal{T} s} \{Q_\Sigma(B) \mid \mathsf{dom}(s) \sqsubseteq B \in \mathcal{T}, s \in \Sigma, B \in \Sigma\},$$

and

$$\{\exists s.E_s \mid s \in \mathcal{C}_\mathcal{T}^\Sigma(r)\},$$

where

$$E_s = \bigsqcap_{\substack{B \in \Sigma,\ \mathsf{ran}(r) \sqsubseteq B \in \mathcal{T} \\ \mathsf{ran}(s) \sqsubseteq B \notin \mathcal{T},\ D \in Q_\Sigma(B)}} D \ \sqcap \bigsqcap_{\substack{D \in Q_\Sigma(A') \\ A' \in \Sigma}} D \ \sqcap \bigsqcap_{\substack{B \in \overline{\Sigma} \\ \mathcal{T} \models \mathsf{ran}(r) \sqcap A' \sqsubseteq B}} B.$$

Fig. 4. Computing $Q_\Sigma(A)$ (for concept and instance Σ-interpolants)

Example 6. Let $\mathcal{T}$ be given as

$$\begin{aligned} B &\equiv \exists t.(A_1 \sqcap A_2) \\ t &\sqsubseteq s \\ s &\sqsubseteq r_1 \\ s &\sqsubseteq r_2 \\ \mathsf{ran}(r_1) &\sqsubseteq A_1 \\ \mathsf{ran}(r_2) &\sqsubseteq A_2 \end{aligned}$$

and $\Sigma = \{A_1, A_2\}$. Note that $\mathcal{T}$ is not a normalized $\mathcal{ELH}^r$-terminology. We transform $\mathcal{T}$ into a normalized terminology $\mathcal{T}'$:

$$\begin{aligned} B &\equiv \exists t.E \\ E &\equiv A_1 \sqcap A_2 \\ t &\sqsubseteq s \\ s &\sqsubseteq r_1 \\ s &\sqsubseteq r_2 \\ \mathsf{ran}(\ell) &\sqsubseteq A_1 \text{ for } \ell = r_1, t, s \\ \mathsf{ran}(\ell) &\sqsubseteq A_2 \text{ for } \ell = r_2, t, s \\ \mathsf{ran}(\ell) &\sqsubseteq E \text{ for } \ell = t, s \end{aligned}$$

For $A \in \mathsf{sig}(\mathcal{T})$, let $\mathsf{Post}_\Sigma(A) = \{B \in \overline{\Sigma} \cap \mathsf{sig}(\mathcal{T}) \mid \mathcal{T} \models A \sqsubseteq B\}$ and construct $Q_\Sigma(A)$ as follows:

- for A primitive in $\mathcal{T}$, $Q_\Sigma(A) = \mathsf{Post}_\Sigma(A)$;
- if $A \bowtie B_1 \sqcap \cdots \sqcap B_n \in \mathcal{T}$, then

$$Q_\Sigma(A) = \mathsf{Post}_\Sigma(A) \;\cup \bigcup_{1 \leq i \leq n, B_i \in \Sigma} Q_\Sigma(B_i);$$

- if $A \bowtie \exists r.A' \in \mathcal{T}$, then $Q_\Sigma(A)$ is the union of $\mathsf{Post}_\Sigma(A)$ and

$$\{\exists R.E\} \;\cup \bigcup_{r \sqsubseteq_\mathcal{T} s} \{Q_\Sigma(B) \mid \mathsf{dom}(s) \sqsubseteq B \in \mathcal{T}, s \in \Sigma, B \in \Sigma\},$$

  where
  - if $\mathcal{C}^\Sigma_\mathcal{T}(r) = \emptyset$: $R = u$;
  - if $\mathcal{C}^\Sigma_\mathcal{T}(r) \neq \emptyset$: $R = \bigsqcap_{s \in \mathcal{C}^\Sigma_\mathcal{T}(r)} s$;

  and

$$E = \bigsqcap_{\substack{B \in \Sigma,\, \mathsf{ran}(r) \sqsubseteq B \in \mathcal{T} \\ \forall s \in \mathcal{C}^\Sigma_\mathcal{T}(r) (\mathsf{ran}(s) \sqsubseteq B \notin \mathcal{T}) \\ D \in Q_\Sigma(B)}} D \;\sqcap \bigsqcap_{\substack{D \in Q_\Sigma(A') \\ A' \in \Sigma}} D \;\sqcap \bigsqcap_{\substack{B \in \overline{\Sigma} \\ \mathcal{T} \models \mathsf{ran}(r) \sqcap A' \sqsubseteq B}} B.$$

Fig. 5. Computing $Q_\Sigma(A)$ (for query Σ-interpolants)

where E is a fresh concept name not contained in $\mathsf{sig}(\mathcal{T})$. Let $\Sigma' = \Sigma \cup \{E\}$. Then a TBox $\mathcal{T}''$ is a concept/instance/query Σ-interpolant of $\mathcal{T}$ iff it is a concept/instance/query Σ'-interpolant of $\mathcal{T}'$.

For both the concept and instance case, the algorithms yields $P_{\Sigma'}(B) = \{B, \exists t.\top\}$, $Q_{\Sigma'}(B) = \{B, \exists t.\top\}$. Then the concept and instance Σ-interpolant coincide and consist of the obvious role inclusions and

$$\exists t.\top \sqsubseteq B, B \sqsubseteq \exists t.\top,$$

where redundent axioms such as $B \sqsubseteq B$ have been removed already.

For $\Sigma = \{A_1, A_2, t\}$ we obtain, besides of role inclusions,

$$B \sqsubseteq \exists s.\top$$

as the only axiom of the concept and instance Σ-interpolant.

The following theorems state that the algorithms terminate and that our construction indeed yields concept- and instance Σ-interpolants.

Theorem 3 (Concept Σ-interpolant). *Let Σ be a finite signature and $\mathcal{T}$ a normalized $\mathcal{ELH}^r$-terminology without Σ-loops. Then the algorithms computing $P_\Sigma(A)$ and $Q_\Sigma(A)$ in Figures 2 and 4, respectively, terminate, for all $A \in \mathsf{sig}(\mathcal{T})$.*

Let $\mathcal{T}^C_\Sigma$ be the TBox as defined above wrt. $P_\Sigma(A)$ and $Q_\Sigma(A)$. Then $\mathcal{T}^C_\Sigma$ is an concept Σ-interpolant of $\mathcal{T}$.

Theorem 4 (Instance Σ-interpolant). *Let Σ be a finite signature and $\mathcal{T}$ a normalized $\mathcal{ELH}^r$-terminology without Σ-loops. Then the algorithms computing $P_\Sigma(A)$ and $Q_\Sigma(A)$ in Figures 3 and 4, respectively, terminate, for all $A \in \mathsf{sig}(\mathcal{T})$.*

Let $\mathcal{T}^i_\Sigma$ be the TBox as defined above wrt. $P_\Sigma(A)$ and $Q_\Sigma(A)$. Then $\mathcal{T}^i_\Sigma$ is an instance Σ-interpolant of $\mathcal{T}$.

Query Σ-interpolants are axiomatized using $\mathcal{EL}^{\mathsf{ran},\sqcap,u}$-inclusions. We use a different algorithm in Figure 5 for computing the set $Q_\Sigma(A)$ consisting of $\mathcal{C}^{\sqcap,u}$-concepts instead of the algorithm in Figure 4 which yields $\mathcal{EL}$-concepts only. The following example illustrates this by showing a run of the algorithms on Example 2.

Example 7. Consider the TBox and signature of Example 2. That is, $\mathcal{T} = \{A \sqsubseteq \exists s.\top, s \sqsubseteq r_1, s \sqsubseteq r_2\}$ and $\Sigma = \{s\}$. We obtain $P_\Sigma(A) = \{A\}$. Observe that $\mathcal{C}^\Sigma_\mathcal{T}(s) = \{r_1, r_2\}$. In the instance case, we obtain

$$Q_\Sigma(A) = \{A, \exists r_1.\top, \exists r_2.\top\}$$

As $A \sqsubseteq A$ is tautological, we obtain the following axiomatization of an instance Σ-interpolant:

$$A \sqsubseteq \exists r_1.\top \sqcap \exists r_2.\top$$

For the query case, we have $Q_\Sigma(A) = \{A, \exists r_1 \sqcap r_2.\top\}$ and thus obtain

$$A \sqsubseteq \exists r_1 \sqcap r_2.\top$$

as an axiomatization of a query Σ-interpolant.

We now state that our construction of query Σ-interpolants terminates and is correct. Extend the relation $\prec_\Sigma$ to $\prec^u_\Sigma$ by adding (A, B) to $\prec_\Sigma$ if there are $A \bowtie \exists r.A' \in \mathcal{T}$ such that $\mathcal{C}^\Sigma_\mathcal{T}(r) = \emptyset$ and $\mathsf{ran}(r) \sqsubseteq B \in \mathcal{T}$. $\mathcal{T}$ contains Σ-u-loops if $\prec^u_\Sigma$ contains a cycle.

Theorem 5 (Query Σ-interpolant). *Let Σ be a finite signature and $\mathcal{T}$ a normalized $\mathcal{ELH}^r$-terminology without Σ-u-loops. Then the algorithms computing $P_\Sigma(A)$ and $Q_\Sigma(A)$ in Figures 3 and 5, respectively, terminate, for all $A \in \mathsf{sig}(\mathcal{T})$.*

Let $\mathcal{T}^q_\Sigma$ be the TBox as defined above wrt. $P_\Sigma(A)$ and $Q_\Sigma(A)$. Then $\mathcal{T}^q_\Sigma$ is a query Σ-interpolant of $\mathcal{T}$.

Notice that both, $P_\Sigma(A)$ and $Q_\Sigma(A)$, are of exponential size, in the worst case. For $P_\Sigma(A)$, this is clear from Point 2 of the construction: let $\mathcal{T}$ consist of $A \equiv B_1 \sqcap \cdots \sqcap B_n$ and $A^j_i \sqsubseteq B_i$ ($1 \leq i, j \leq n$) and let $\Sigma = \{B_i \mid 1 \leq i \leq n\}$. Then $P_\Sigma(A)$ is of size n^n, and one can show that there does not exist a shorter Σ-interpolant in $\mathcal{EL}^{\mathsf{ran},\sqcap,u}$. For $Q_\Sigma(A)$ this follows from the fact that one might have to construct a complete unfolding of the terminology.

If we admit disjunctions in C in axioms $C \sqsubseteq D$ of Σ-interpolants, then we can replace, in Point 2, $P_\Sigma(A)$ for $A \equiv B_1 \sqcap \cdots \sqcap B_n \in \mathcal{T}$ by the singleton set consisting of

$$\bigsqcap_{1 \leq i \leq n, B_i \in \overline{\Sigma}} B_i \sqcap \bigsqcap_{1 \leq i \leq n, B_i \in \Sigma} \bigsqcup_{C_{B_i} \in P_\Sigma(B_i)} C_{B_i}.$$

We will see below that in practice this construction leads to much smaller Σ-interpolants. However, this improvement does not come for free. Consider the language $\mathcal{EL}^{\mathsf{ran},\sqcup}$,

$\lvert\overline{\Sigma}\cap \mathsf{sig}(\cdot)\rvert$	SNOMED CT	$\lvert\overline{\Sigma}\cap \mathsf{sig}(\cdot)\rvert$	NCI
2 000	93.0%	5 000	97.0%
3 000	84.5%	10 000	81.1%
4 000	67.0%	15 000	72.0%
5 000	59.5%	20 000	59.2%

Table 1. Success rate of NUI

where the only difference to $\mathcal{EL}^{\mathsf{ran}}$ is that $\mathcal{C}^{\mathsf{ran}}$-concepts now admit '$\sqcup$' as a binary concept constructor. Every $\mathcal{EL}^{\mathsf{ran},\sqcup}$-TBox is logically equivalent to an (exponentially larger) $\mathcal{EL}^{\mathsf{ran}}$-TBox, and so $\mathcal{EL}^{\mathsf{ran},\sqcup}$ inherits many desirable properties from $\mathcal{EL}^{\mathsf{ran}}$. However, one can show that, in contrast to $\mathcal{EL}^{\mathsf{ran}}$, logical equivalence between $\mathcal{EL}^{\mathsf{ran},\sqcup}$-TBoxes is coNP-hard.

5 Experiments

We have implemented a prototype called NUI that computes instance Σ-interpolants as presented in Theorem 4. We have applied NUI to a version of SNOMED CT dated 09 February 2005 (without two left-identities) and the $\mathcal{ELH}^r$-fragment of the release 08.08d of NCI. The first terminology has approx. 380K axioms, almost the same number of concept names, and 56 role names. The $\mathcal{ELH}^r$-fragment of NCI has approx. 63K axioms, approx. 65K concept names, and 123 role names. We note that the algorithms given above compute (for ease of exposition) a large number of redundant axioms and NUI implements a variety of straightforward optimizations.

First observe that neither SNOMED CT nor NCI contain any Σ-loops, for any signature Σ. Thus, Σ-interpolants always exist and can, in principle, be computed using our algorithm.

In our experiments, we focus on the case of forgetting a large signature Σ (and keeping a "small" signature $\overline{\Sigma}\cap\mathsf{sig}(\cdot)$), as this corresponds to many application scenarios. The experiments have been performed on a standard PC with 2.13 GHz and 3 GB of RAM.

Success rate: Table 1 shows the rate at which NUI succeeds to compute instance Σ-interpolants of SNOMED CT and NCI wrt. various signatures. All failed cases are due to memory overflow after several hours. For each table entry, 100 samples have been used. The signatures contain concept and role names randomly selected from the full signature of SNOMED CT (we never forget the role 'roleGroup' as this would make forgetting trivial) and NCI, respectively. $\overline{\Sigma}\cap\mathsf{sig}(\cdot)$ always contains 20 role names. For NCI and signatures of size $\leq 4\,000$, NUI had a 100% success rate.

Size: We compare the size of instance Σ-interpolants of SNOMED CT and NCI computed by NUI with the size of extracted $\overline{\Sigma}\cap\mathsf{sig}(\cdot)$-modules; i.e., minimal *subsets* of the respective terminologies which preserve, e.g., inclusions between $\overline{\Sigma}\cap\mathsf{sig}(\cdot)$-concepts. We use MEX-modules (Konev *et al.* 2008a) of SNOMED CT and, respectively, $\top$-local modules (Cuenca Grau *et al.* 2007) of NCI. The size of Σ-interpolants, terminologies,

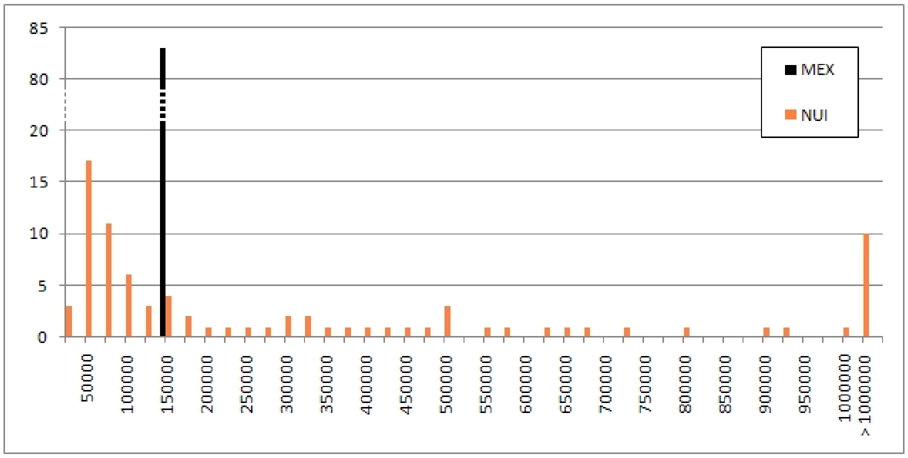

Fig. 6. Size distribution of MEX-modules and instance Σ-interpolants of SNOMED CT

and modules is measured as number of symbols rather than number of axioms as Σ-interpolants can contain large axioms.

For SNOMED CT, we generated 100 random signatures with $3\,000$ concept names and 20 role names in $\overline{\Sigma} \cap \mathsf{sig}(\cdot)$. For 17 of those signatures, NUI failed to compute an instance Σ-interpolant. For the remaining 83 signatures, Figure 6 shows the number interpolants and $\overline{\Sigma} \cap \mathsf{sig}(\cdot)$-modules (vertical axis) of a given size (horizontal axis). The size of the $\overline{\Sigma} \cap \mathsf{sig}(\cdot)$-modules lies between 125K and 150K symbols, i.e., it lies between 2.94% and 3.21% of SNOMED CT. 48.19% of instance Σ-interpolants are smaller than the corresponding modules. However, 10 instance Σ-interpolants contain more than one million symbols and the largest instance Σ-interpolant is more than 11 times larger than SNOMED CT.

For NCI, we computed instance Σ-interpolants and $\overline{\Sigma} \cap \mathsf{sig}(\cdot)$-modules wrt. random signatures with $7\,000$ concept names and 20 role names in $\overline{\Sigma} \cap \mathsf{sig}(\cdot)$. NUI succeeded to compute interpolants for 97 of 100 signatures, each within 25 min. Figure 7 shows the size distribution of the successfully computed interpolants and the corresponding modules. The size of the $\overline{\Sigma} \cap \mathsf{sig}(\cdot)$-modules lies between 140K and 160K symbols, i.e., it lies between 21.62% and 23.17% of NCI. 74.47% of the instance Σ-interpolants are smaller than the corresponding modules. On the other hand, 18 instance Σ-interpolants consist of more than 400K symbols and the largest instance Σ-interpolant is more than 12 times larger than NCI.

Forgetting with disjunction: All failures in Table 1 are due to the fact that $P_\Sigma(A)$ is too large. Indeed, if we admit disjunction and consider $\mathcal{EL}^{\mathsf{ran},\sqcup}$, then NUI succeeds to compute *all* Σ-interpolants from Table 1, each within 15 min. Moreover, for NCI, no signature for which NUI fails has been detected. For SNOMED CT, however, NUI still typically fails for $|\overline{\Sigma} \cap \mathsf{sig}(\cdot)| \geq 30\,000$.

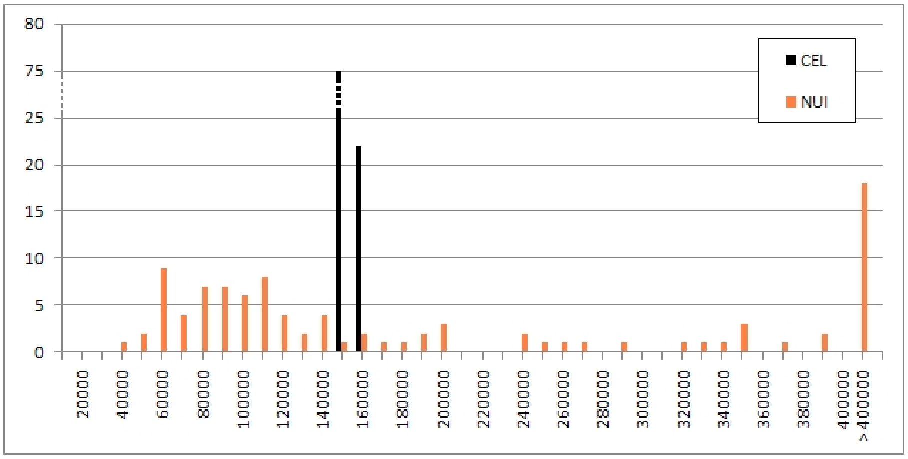

Fig. 7. Size distribution of CEL-modules and instance Σ-interpolants of NCI

6 Discussion

The notion of forgetting in DL ontologies has recently been investigated in a number of research papers. (Kontchakov *et al.* 2008; Wang *et al.* 2008) consider forgetting in DL-Lite and (Eiter *et al.* 2006) investigate in how far forgetting in DLs can be reduced to forgetting in logic programs. (Konev *et al.* 2008b) proposes forgetting for acyclic $\mathcal{EL}$-terminologies restricted to inclusions between concepts.

The main novel contributions of this paper are (i) the first algorithms with experimental results indicating the practical feasibility of forgetting in DL-terminologies and (ii) the first systematic analysis of the distinct languages required to axiomatize Σ-interpolants for distinct queries languages. Many open problems remain; e.g., we conjuncture that Σ-interpolants of $\mathcal{ELH}^r$-terminologies (and possibly even TBoxes) exist in the languages introduced whenever they exist in FO. Such a result would provide further justification for those languages. Secondly, it would be of interest to prove decidability (and complexity) of the decision problem whether there exists a Σ-interpolant for a given $\mathcal{ELH}^r$-terminology (TBox). Note that our acyclicity conditions are sufficient but not necessary for the existence of Σ-interpolants.

APPENDIX

The appendix is organized as follows. In the first three sections, we present basic results which are required for the proofs of the results presented in the paper. Subsequent sections then provide detailed proofs of the claims made in the paper. The first section presents basic semantic properties of $\mathcal{EL}^{\mathsf{ran},\sqcap,u}$ and its sublanguages. In the second section we briefly discuss basic material required about query answering. The third section presents a sequent style proof systems for $\mathcal{ELH}^r$-terminologies. Finally, we apply these results to present detailed proofs of the claims made in the paper. Throughout the appendix we will make use of two additional sets of concepts:

- $\mathcal{C}^u$ denotes the set of $\mathcal{C}^{\sqcap,u}$-concepts not using conjunctions of roles.
- $\mathcal{C}^\sqcap$ denotes the set of $\mathcal{C}^{\sqcap,u}$-concepts not using u.

We also sometimes make use of the fact that every $\mathcal{C}^{\sqcap,u}$-concept is equivalent to a concept $C_1 \sqcap \exists u.C_2 \sqcap \cdots \sqcap \exists u.C_n$, where $C_1, \ldots, C_n$ are $\mathcal{C}^\sqcap$-concepts.

A Basic properties of $\mathcal{EL}^{\mathsf{ran},\sqcap,u}$

In this section, we first provide a projective reduction of $\mathcal{EL}^{\mathsf{ran},\sqcap,u}$-TBoxes to $\mathcal{ELH}^r$-TBoxes. Many results for $\mathcal{EL}^{\mathsf{ran},\sqcap,u}$-TBoxes can thus be obtained by considering $\mathcal{ELH}^r$. Then we introduce different types of canonical models for $\mathcal{ELH}^r$-KBs and TBoxes (similar to (Lutz and Wolter 2007))

Proposition 1. *Let $\mathcal{T}$ be an $\mathcal{EL}^{\mathsf{ran},\sqcap,u}$-TBox. Then there exists an $\mathcal{ELH}^r$-TBox $\mathcal{T}'$ with* $\mathsf{sig}(\mathcal{T}) \subseteq \mathsf{sig}(\mathcal{T}')$ *such that*

- *Every model $\mathcal{I}'$ of $\mathcal{T}'$ is a model of $\mathcal{T}$;*
- *If $\mathcal{I}$ is a model of $\mathcal{T}$, then there exists model $\mathcal{I}'$ of $\mathcal{T}'$ which coincides with $\mathcal{I}$ regarding the interpretation of symbols in* $\mathsf{sig}(\mathcal{T})$ *and such that $\mathcal{I}' \models \mathcal{T}'$.*

Proof. Suppose $\mathcal{T}$ is given. Take

- for every concept $\mathsf{ran}(r)$ in $\mathcal{T}$ a fresh concept name X_r, replace all occurences of $\mathsf{ran}(r)$ in $\mathcal{T}$ by X_r and add the inclusion $\mathsf{ran}(r) \sqsubseteq X_r$ to $\mathcal{T}$;
- for every proper conjunction $R = r_1 \sqcap \cdots \sqcap r_m$ of role names in $\mathcal{T}$, introduce a fresh role name s_R, replace each occurrence of $r_1 \sqcap \cdots \sqcap r_m$ in $\mathcal{T}$ by s_R, and add the inclusions $s_R \sqsubseteq r_i$ to $\mathcal{T}$, for $1 \leq i \leq m$.
- for every occurrence of u in $\mathcal{T}$, take a fresh role name s and replace u in $\mathcal{T}$ by s.

It is easily seen that the resulting TBox $\mathcal{T}'$ is as required.

We now define canonical models for $\mathcal{ELH}^r$-knowledge bases and TBoxes. In the construction of canonical models and throughout this section (but not in later sections), we assume that

(i) $\mathcal{T}$ does not contain any domain restrictions;
(ii) $\mathcal{T}$ contains exactly one range restriction per role name;
(iii) if $r \sqsubseteq s, \mathsf{ran}(r) \sqsubseteq C, \mathsf{ran}(s) \sqsubseteq D$ are in $\mathcal{T}$, then $\mathcal{T} \models C \sqsubseteq D$; and

$$
\begin{aligned}
\Delta^{\mathcal{I}_\mathcal{K}} &:= \mathsf{obj}(\mathcal{A}) \cup \mathsf{NI}_{\mathsf{aux}} \\
A^{\mathcal{I}_\mathcal{K}} &:= \{a \in \mathsf{obj}(\mathcal{A}) \mid \mathcal{K} \models A(a)\} \cup \{x_{C,D} \in \mathsf{NI}_{\mathsf{aux}} \mid \mathcal{K} \models C \sqcap D \sqsubseteq A\} \\
r^{\mathcal{I}_\mathcal{K}} &:= \{(a,b) \in \mathsf{obj}(\mathcal{A}) \times \mathsf{obj}(\mathcal{A}) \mid s(a,b) \in \mathcal{A} \text{ and } \mathcal{K} \models s \sqsubseteq r\} \cup \\
&\quad\ \{(a, x_{C,D}) \in \mathsf{obj}(\mathcal{A}) \times \mathsf{NI}_{\mathsf{aux}} \mid \mathcal{K} \models \exists s.D(a), \mathsf{ran}_\mathcal{T}(s) = C, \text{ and } \mathcal{K} \models s \sqsubseteq r\} \cup \\
&\quad\ \{(x_{C,D}, x_{C',D'}) \in \mathsf{NI}_{\mathsf{aux}} \times \mathsf{NI}_{\mathsf{aux}} \mid \mathcal{K} \models C \sqcap D \sqsubseteq \exists s.D', \mathsf{ran}_\mathcal{T}(s) = C', \text{ and } \mathcal{K} \models s \sqsubseteq r\} \\
a^{\mathcal{I}_\mathcal{K}} &:= a, \text{ for all } a \in \mathsf{obj}(\mathcal{K})
\end{aligned}
$$

Fig. 8. Canonical model $\mathcal{I}_\mathcal{K}$ of a KB $\mathcal{K} = (\mathcal{T}, \mathcal{A})$

(iv) there are no $r, s \in \mathsf{N_R}$ with $r \neq s$, $\mathcal{T} \models r \sqsubseteq s$, and $\mathcal{T} \models s \sqsubseteq r$.

All these assumptions can be made wlog. This is true for Assumption (i) because $\mathsf{dom}(r) \sqsubseteq C$ is equivalent to $\exists r.\top \sqsubseteq C$; for Assumption (ii) because two range restrictions $\mathsf{ran}(r) \sqsubseteq C$ and $\mathsf{ran}(r) \sqsubseteq C'$ are equivalent to $\mathsf{ran}(r) \sqsubseteq C \sqcap C'$ and we can always introduce a range restriction $\mathsf{ran}(r) \sqsubseteq \top$ for each role name r; for Assumption (iii) because $\{r \sqsubseteq s, \mathsf{ran}(r) \sqsubseteq C, \mathsf{ran}(s) \sqsubseteq D\}$ is equivalent to $\{r \sqsubseteq s, \mathsf{ran}(r) \sqsubseteq C \sqcap D, \mathsf{ran}(s) \sqsubseteq D\}$; and for Assumption (iv) because, if $\mathcal{T} \models r \sqsubseteq s$ and $\mathcal{T} \models s \sqsubseteq r$ with $s \neq r$, we can simply substitute r with s in $\mathcal{T}$ and q.

Let $\mathsf{sub}(\mathcal{T})$ denote the set of all subconcepts of concepts used in $\mathcal{T}$, $\mathsf{rol}(\mathcal{T})$ the set of all role names occurring in $\mathcal{T}$, and $\mathsf{obj}(\mathcal{A})$ the set of individual names that occur in ABox $\mathcal{A}$ (we assume that $\mathsf{obj}(\mathcal{A})$ is not empty). In this section, we use $\mathsf{ran}_\mathcal{T}(r)$ to denote the (unique) concept C with $\mathsf{ran}(r) \sqsubseteq C \in \mathcal{T}$, and set

$$
\begin{aligned}
\mathsf{ran}(\mathcal{T}) &:= \{\mathsf{ran}_\mathcal{T}(r) \mid r \in \mathsf{rol}(\mathcal{T})\} \\
\mathsf{NI}_{\mathsf{aux}} &:= \{x_{C,D} \mid C \in \mathsf{ran}(\mathcal{T}) \text{ and } D \in \mathsf{sub}(\mathcal{T})\}.
\end{aligned}
$$

Now, the canonical model $\mathcal{I}_\mathcal{K}$ of a KB $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ is defined; see Figure 8.

$\mathcal{I}_\mathcal{K}$ contains many irrelevant points. We use $\mathsf{obj}(\mathcal{A})^\mathcal{I}$ to denote the set $\{a^\mathcal{I} \mid a \in \mathsf{obj}(\mathcal{A})\}$. A *path* in $\mathcal{I}$ is a finite sequence $d_0 r_1 d_1 \cdots r_n d_n$, $n \geq 0$, where $d_0 \in \mathsf{obj}(\mathcal{A})^\mathcal{I}$ and, for all $i < n$, $(d_i, d_{i+1}) \in r_{i+1}^\mathcal{I}$. We use $\mathsf{paths}(\mathcal{I})$ to denote the set of all paths in $\mathcal{I}$. If $p \in \mathsf{paths}(\mathcal{I})$, then $\mathsf{tail}(p)$ denotes the last element d_n in p. The *relevant* part of $\mathcal{I}_\mathcal{K}^r$ of $\mathcal{I}_\mathcal{K}$ is the restriction of $\mathcal{I}_\mathcal{K}$ to all d such that there is a path in $\mathcal{I}_\mathcal{K}$ with tail d.

The following result can be proved similarly to (Lutz and Wolter 2007).

Theorem 6. *Let $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ be an $\mathcal{ELH}^r$-KB. Then $\mathcal{I}_\mathcal{K}^r$ is a model of $\mathcal{K}$, it can be computed in polynomial time, and for all $x_{C,D} \in \Delta^{\mathcal{I}_\mathcal{K}^r}$, all $a \in \mathsf{obj}(\mathcal{A})$, and all $\mathcal{C}^u$-concepts C_0:*

- *$x_{C,D} \in C_0^{\mathcal{I}_\mathcal{K}^r}$ iff $\mathcal{K} \models C \sqcap D \sqsubseteq C_0$ iff $\mathcal{T} \models C \sqcap D \sqsubseteq C_0$;*
- *$\mathcal{K} \models C_0(a)$ iff $a^{\mathcal{I}_\mathcal{K}^r} \in C_0^{\mathcal{I}_\mathcal{K}^r}$.*

Observe that we can conclude that $\mathcal{T} \models C \sqsubseteq D$ is decidable in polynomial time, for $\mathcal{EL}^{\mathsf{ran},\sqcap,u}$-TBoxes $\mathcal{T}$, $\mathcal{EL}$-concepts C, and $\mathcal{C}^u$-concepts D. This follows from Theorem 6, Proposition 1, and by considering ABoxes of the form $\mathcal{A} = \{A(a)\}$ and adding an axiom $A \equiv C$ to $\mathcal{T}$: then $\mathcal{T} \models C \sqsubseteq D$ iff $(\mathcal{T}', \mathcal{A}) \models D(a)$, where $\mathcal{T}'$ is the resulting TBox when $A \equiv C$ is added to $\mathcal{T}$.

To deal with $\mathcal{EL}^{\mathsf{ran},\sqcap,u}$-inclusions instead of just inclusions of the form $C \sqsubseteq D$, where C is an $\mathcal{EL}$-concept and D an $\mathcal{C}^u$-concept, we consider unravelings of $\mathcal{I}^r_\mathcal{K}$. The *unraveling* $\mathcal{J}$ of a model $\mathcal{I}$ is defined as follows:

$$\begin{aligned}
\Delta^\mathcal{J} &:= \mathsf{paths}(\mathcal{I})\\
a^\mathcal{J} &:= a^\mathcal{I}\\
A^\mathcal{J} &:= \{p \mid \mathsf{tail}(p) \in A^\mathcal{I}\}\\
r^\mathcal{J} &:= \{(d,e) \mid d,e \in \mathsf{obj}(\mathcal{A})^\mathcal{I} \wedge (d,e) \in r^\mathcal{I}\} \cup\\
&\quad\ \{(p, p \cdot se) \mid p, p \cdot se \in \Delta^\mathcal{J} \text{ and } \mathcal{K} \models s \sqsubseteq r\}
\end{aligned}$$

where "$\cdot$" denotes concatenation. The following result can be proved similarly to results in (Lutz and Wolter 2007).

Theorem 7. *Let $\mathcal{K}$ be an $\mathcal{ELH}^r$ KB and $\mathcal{U}_\mathcal{K}$ the unraveling of $\mathcal{I}^r_\mathcal{K}$. Then $\mathcal{U}_\mathcal{K}$ is a model of $\mathcal{K}$ and*

(i) for all $p \in \Delta^{\mathcal{U}_\mathcal{K}}$ with $\mathsf{tail}(p) = x_{C,D}$, all $a \in \mathsf{obj}(\mathcal{A})$, and all $\mathcal{C}^{\mathsf{ran}} \cup \mathcal{C}^{\sqcap,u}$-concepts C_0:

- *$p \in C_0^{\mathcal{U}_\mathcal{K}}$ iff $\mathcal{K} \models C \sqcap D \sqsubseteq C_0$ iff $\mathcal{T} \models C \sqcap D \sqsubseteq C_0$;*
- *$\mathcal{K} \models C_0(a)$ iff $a^{\mathcal{U}_\mathcal{K}} \in C_0^{\mathcal{U}_\mathcal{K}}$.*

(ii) For all k-ary conjunctive queries q and individual names $a_1, \ldots, a_k$, we have $\mathcal{K} \models q[a_1, \ldots, a_k]$ iff $\mathcal{U}_\mathcal{K} \models q[a_1, \ldots, a_k]$.

We will frequently make use of the following conversion of $\mathcal{C}^{\mathsf{ran}}$-concepts C into ABoxes $\mathcal{A}_C$. Consider the following completion rules operating on constraint systems $\mathcal{C}$ consisting of expressions of the form $a : C$ and $(a, b) : r$.

- if $a : C_1 \sqcap C_2 \in \mathcal{C}$, then set $\mathcal{C} := \mathcal{C} \cup \{a : C_1, a : C_2\}$;
- if $a : \exists r.C \in \mathcal{C}$ and there does not exists b with $(a, b) : r \in \mathcal{C}$, then take a fresh b' and set $\mathcal{C} = \mathcal{C} \cup \{(a, b') : r, b' : C\}$;
- if $a : \mathsf{ran}(r) \in \mathcal{C}$ and there does not exist b with $(b, a) : r$ then take a new b and set $\mathcal{C} = \mathcal{C} \cup \{(b', a) : r\}$.

Denote by $\mathcal{C}_C$ the closure of $\mathcal{C} = \{a_C : C\}$ under the three rules above. Denote by $\mathcal{A}_C$ the ABox consisting of all $A(a)$ such that $a : A \in \mathcal{C}_C$, $\top(a)$ such that $a : D \in \mathcal{C}_C$ for some D, and $r(a, b)$ such that $(a, b) : r \in \mathcal{C}_C$. Observe that $\mathcal{A}_C$ can be computed in polynomial time.

Lemma 1. *For all $\mathcal{C}^{\mathsf{ran}}$-concepts C and $\mathcal{C}^{\mathsf{ran}} \cup \mathcal{C}^{\sqcap,u}$-concepts D:*

$$(\mathcal{T}, \mathcal{A}_C) \models D(a_C) \quad \Leftrightarrow \quad \mathcal{T} \models C \sqsubseteq D.$$

The model $\mathcal{U}_\mathcal{K}$ can be regarded as a minimal model of a KB $\mathcal{K}$. Let Γ be a set of $\mathcal{C}^{\mathsf{ran}}$-concepts and $\mathcal{T}$ an $\mathcal{ELH}^r$-TBox. We write

$$\mathcal{T} \cup \Gamma \models D$$

if, for every model $\mathcal{I}$ of $\mathcal{T}$ and all $d \in \Delta^\mathcal{I}$ the following holds: $d \in D^\mathcal{I}$ whenever for all $C \in \Gamma$, $d \in C^\mathcal{I}$ (in other words, $\mathcal{T} \models \bigsqcap_{C \in \Gamma} C \sqsubseteq D$, if we admit infinite conjunctions). Using compactness, Lemma 1, and Theorem 7 one can easily prove:

Theorem 8. *For all $\mathcal{ELH}^r$-TBoxes $\mathcal{T}$ and sets of $\mathcal{C}^{\mathsf{ran}}$-concepts Γ there exists a model $\mathcal{I}$ of $\mathcal{T}$ and $d \in \Delta^{\mathcal{I}}$ such that the following are equivalent, for all $\mathcal{C}^{\sqcap,u} \cup \mathcal{C}^{\mathsf{ran}}$-concepts D:*

- *$\mathcal{T} \cup \Gamma \models D$;*
- *$d \in D^{\mathcal{I}}$.*

Proof. Suppose such a model does not exist. By compactness, there exists a finite subset Γ' of Γ and a finite set Ξ of $\mathcal{C}^{\mathsf{ran}} \cup \mathcal{C}^{\sqcap,u}$-concepts such that

$$\mathcal{T} \models \bigsqcap_{D \in \Gamma'} D \sqsubseteq \bigsqcup_{F \in \Xi} F,$$

but

$$\mathcal{T} \not\models \bigsqcap_{D \in \Gamma'} D \sqsubseteq F,$$

for any $F \in \Xi$. Consider the ABox $\mathcal{A}_C$ for $C = \bigsqcap_{D \in \Gamma'} D$. By Theorem 7 and Lemma 1,

$$\mathcal{U}_{(\mathcal{T},\mathcal{A}_C)} \not\models F(a_C)$$

for any $F \in \Xi$. But then

$$\mathcal{U}_{(\mathcal{T},\mathcal{A}_C)} \not\models \bigsqcup_{F \in \Xi} F(a),$$

which implies that $(\mathcal{T}, \mathcal{A}_C) \not\models \bigsqcup_{F \in \Xi} F(a)$. We obtain

$$\mathcal{T} \not\models C \sqsubseteq \bigsqcup_{F \in \Xi} F,$$

and have obtained a contradiction.

We also require the following consequence of Theorem 7:

Theorem 9. *Let $\mathcal{T}$ be an $\mathcal{ELH}^r$-TBox and C a $\mathcal{C}^{\mathsf{ran}}$-concept.*
(i) If $\exists u.D$ is an $\mathcal{C}^{u,\sqcap}$-concept, then $\mathcal{T} \models C \sqsubseteq \exists u.D$ iff

- *there exists a subconcept $(\mathsf{ran}(r) \sqcap C')$ of C such that there exists a sequence $r'_1, \ldots, r'_n$ such that $\mathcal{T} \models \exists r.C' \sqsubseteq \exists r'_1. \cdots \exists r'_n.D$ or*
- *there exists a sequence $r'_1, \ldots, r'_n$ such that $\mathcal{T} \models C \sqsubseteq \exists r'_1. \cdots \exists r'_n.D$.*

(ii) If $\exists S.D$ is an $\mathcal{C}^{u,\sqcap}$-concept with $S = r_1 \sqcap \cdots \sqcap r_n$, then $\mathcal{T} \models C \sqsubseteq \exists S.D$ iff there exists s such that $s \sqsubseteq_{\mathcal{T}} r_i$ for $i \leq n$ and $\mathcal{T} \models C \sqsubseteq \exists s.D$.

B Query answering

We introduce some notation for query answering and discuss the relation to partial homomorphisms.

Let $\mathcal{I}$ be a model of $(\mathcal{T}, \mathcal{A})$ and $q(\boldsymbol{x}) = \exists \boldsymbol{y}.q'(\boldsymbol{x}, \boldsymbol{y})$ a conjunctive query for $\boldsymbol{x} = x_1, \ldots, x_k$. Wlog., we assume that $q(\boldsymbol{x})$ contains individual names from $\mathcal{A}$ only. We say that a vector $\boldsymbol{a} = a_1, \ldots, a_k$ of members of $\mathsf{obj}(\mathcal{A})$ is a π-match of $\mathcal{I}$ and $q(\boldsymbol{x})$ if π is a mapping assigning to every variable v in $\boldsymbol{x} \cup \boldsymbol{y}$ an element of $\Delta^{\mathcal{I}}$ such that

– $\mathcal{I} \models_\pi q'(\boldsymbol{x}, \boldsymbol{y})$;
– $\pi(x_i) = a_i^{\mathcal{I}}$ for $1 \leq i \leq k$.

Clearly, $\mathcal{I} \models q[\boldsymbol{a}]$ iff there exists a π such that $\boldsymbol{a}$ is a π-match for $\mathcal{I}$ and $q(\boldsymbol{x})$. Query answering is closely related to the existence of certain homomorphisms between models. Let Σ be a signature, O a set of individual names, and $\mathcal{I}_1, \mathcal{I}_2$ interpretations. A function $f : \Delta^{\mathcal{I}_1} \rightarrow \Delta^{\mathcal{I}_2}$ is called a (O, Σ)-homomorphism if

– $f(a^{\mathcal{I}_1}) = f(a^{\mathcal{I}_2})$ for all $a \in O$;
– $d \in A^{\mathcal{I}_1}$ implies $f(d) \in A^{\mathcal{I}_2}$ for all A in Σ;
– $(d_1, d_2) \in r^{\mathcal{I}_1}$ implies $(f(d_1), f(d_2) \in r^{\mathcal{I}_2}$ for all $r \in \Sigma$.

Now it is well-known (and straightforward to prove) that if there exists a (O, Σ)-homomorphism from $\mathcal{I}_1$ to $\mathcal{I}_2$ and $\mathcal{I}_1 \models q[\boldsymbol{a}]$ for a conjunctive Σ-query q using only individual names from O and $\boldsymbol{a} = a_1, \ldots, a_k$ from O, then $\mathcal{I}_2 \models q[\boldsymbol{a}]$. For the proof below we slightly refine this by considering partial homomorphisms. We consider such partial homomorphisms on certain models only, which we introduce first.

Let O be a finite set of individual names and $\mathcal{I}$ a model. $d \in \Delta^{\mathcal{I}}$ is called O-named if there exists $a \in O$ with $d = a^{\mathcal{I}}$. A model $\mathcal{I}$ is called an O-forest if

(F1) for every $d \in \Delta^{\mathcal{I}}$ which is not O-named, there exists at most one $d' \in \Delta^{\mathcal{I}}$ such that $(d', d) \in \bigcup_{r \in \mathsf{N_R}} r^{\mathcal{I}}$;
(F2) there are no infinite sequences $d_0, d_1, \ldots$ with $(d_{i+1}, d_i) \in \bigcup_{r \in \mathsf{N_R}} r^{\mathcal{I}}$ for all $i \geq 0$ such that all d_i are not O-named.
(F3) if $(d, d') \in \bigcup_{r \in \mathsf{N_R}} r^{\mathcal{I}}$ and d' is O-named, then d is O-named.

Observe that the model $\mathcal{U}_{\mathcal{K}}$ is an O-forest for $O = \mathsf{obj}(\mathcal{A})$. Clearly, every model can be turned into an O-forest by unraveling.

A partial function f from an O-forest $\mathcal{I}$ to a model $\mathcal{I}'$ is called an (O, n, Σ)-homomorphism if

(H1) for all $a \in O$: $a^{\mathcal{I}}$ is in the domain of f and $f(a^{\mathcal{I}}) = a^{\mathcal{I}'}$;
(H2) for all d, d' in the domain of f and $r \in \Sigma$: $(d, d') \in r^{\mathcal{I}}$ implies $(f(d), f(d')) \in r^{\mathcal{I}'}$;
(H3) for all d in the domain of f and $A \in \Sigma$: $d \in A^{\mathcal{I}}$ implies $f(d) \in A^{\mathcal{I}'}$;
(H4) if there does not exist a chain $d_1, \ldots, d_m$ with $(d_i, d_{i+1}) \in \bigcup_{r \in \Sigma} r^{\mathcal{I}}$ of length $m \geq n$ of not O-named d_i, then d_m is in the domain of f.

Now one can prove the following

Lemma 2. *Suppose $\mathcal{I}$ is an O-forest, $\mathcal{I}'$ a model and for every $m > 0$ there exists a (O, m, Σ)-homomorphism from $\mathcal{I}$ to $\mathcal{I}'$. Assume as well that $\mathcal{I} \models q[\boldsymbol{a}]$ with q a conjunctive Σ-query using only individual names from O and $\boldsymbol{a} = a_1, \ldots, a_k$ from O. Then $\mathcal{I}' \models q[\boldsymbol{a}]$.*

Proof. Assume that $\boldsymbol{a}$ is a π-match of $\mathcal{I}$ and $q(\boldsymbol{x}) = \exists \boldsymbol{y}. q'(\boldsymbol{x}, \boldsymbol{y})$ such that $\boldsymbol{a}$ consists of elements of O. By (F2) and (F3) in the definition of O-forests and (H1) and (H4) in the definition of partial homomorphisms, there exists $m > 0$ such that all $\pi(v)$, v from $\boldsymbol{x} \cup \boldsymbol{y}$, are in the domain of any (O, m, Σ)-homomorphism f. Take a (O, m, Σ)-homomorphism f. Then $\boldsymbol{a}$ is a π'-match of $\mathcal{I}'$ and $q(\boldsymbol{x})$ for $\pi'(v) = f(\pi(v))$.

Finally, we also need some way of constructing (O, m, Σ)-homomorphisms. Let $\mathcal{I}$ be a model. For each $d \in \Delta^{\mathcal{I}}$ and $m > 0$, let

$$t_{\mathcal{I}}^{m,\Sigma,\sqcap}(d) = \{C \in \mathcal{C}_{\Sigma}^{\sqcap} \mid \mathsf{rd}(C) \leq m, d \in C^{\mathcal{I}}\},$$

where rd is the role-depth of C.

Lemma 3. *Let Σ be a finite signature. Suppose $\mathcal{I}$ is an O-forest and $\mathcal{I}'$ a model such that*

(in1) $t_{\mathcal{I}}^{m,\Sigma,\sqcap}(a^{\mathcal{I}}) \subseteq t_{\mathcal{I}'}^{m,\Sigma,\sqcap}(a^{\mathcal{I}'})$*, for all* $a \in O$*;*
(in2) for all $d \in \Delta^{\mathcal{I}}$ *there exists* $d' \in \Delta^{\mathcal{I}'}$ *such that* $t_{\mathcal{I}}^{m,\Sigma,\sqcap}(d) \subseteq t_{\mathcal{I}'}^{m,\Sigma,\sqcap}(d')$.

Then there exists a (O, m, Σ)*-homomorphism* g *from* $\mathcal{I}$ *to* $\mathcal{I}'$.

Proof. We construct g by constructing a sequence $f_0, \ldots, f_m$ as follows: the domain $\mathsf{dom}(f_0)$ of f_0 consists of all $a^{\mathcal{I}}$ with $a \in O$ and all $d \in \Delta^{\mathcal{I}}$ such that there does not exist a d' with $(d', d) \in \bigcup_{r \in \Sigma} r^{\mathcal{I}}$. For $a^{\mathcal{I}}$ with $a \in O$ we set $f_0(a^{\mathcal{I}}) = a^{\mathcal{I}'}$. For every remaining $d \in \mathsf{dom}(f_0)$ choose a d' according to (in2) and set $f_0(d) = d'$. Observe that $t_{\mathcal{I}}^{m,\Sigma,\sqcap}(d) \subseteq t_{\mathcal{I}'}^{m,\Sigma,\sqcap}(f_0(d))$ for all $d \in \mathsf{dom}(f_0)$.

Now suppose that f_n has be constructed and

(in3) $t_{\mathcal{I}}^{m-n,\Sigma,\sqcap}(d) \subseteq t_{\mathcal{I}'}^{m-n,\Sigma,\sqcap}(f_n(d))$ for all $d \in \mathsf{dom}(f_n)$;
(in4) for $n > 0$: $d \in \mathsf{dom}(f_n)$ iff d is not O-named and there exists a sequence $d_0 r_1^{\mathcal{I}} d_1 r_2^{\mathcal{I}} \cdots r_n^{\mathcal{I}} d_n = d$ of which at most d_0 is O-named such that $r_i \in \Sigma$ and $d_0 \in \mathsf{dom}(f_0)$.

To construct f_{n+1} consider a $d \in \mathsf{dom}(f_n)$ and a not O-named d' such that $(d, d') \in \bigcup_{r \in \Sigma} r^{\mathcal{I}}$. The domain of f_{n+1} consists of all such d'. Let $R = \{r \in \Sigma \mid (d, d') \in r^{\mathcal{I}}\}$. Then

$$\exists R. \bigsqcap_{D \in t_{\mathcal{I}}^{m-n-1,\Sigma,\sqcap}(d')} D \ \in t_{\mathcal{I}}^{m-n,\Sigma,\sqcap}(d)$$

By (in3),

$$\exists R. \bigsqcap_{D \in t_{\mathcal{I}}^{m-n-1,\Sigma,\sqcap}(d')} D \ \in t_{\mathcal{I}}^{m-n,\Sigma,\sqcap}(f_n(d))$$

Thus, we can choose a d'' with $(f(d'), d'') \in r^{\mathcal{I}'}$ for all $r \in R$ and $t_{\mathcal{I}}^{m-n-1,\Sigma,\sqcap}(d') \subseteq t_{\mathcal{I}}^{m-n-1,\Sigma,\sqcap}(d'')$ and set $f_{n+1}(d') = d''$. This defines f_{n+1}. Observe that f_{n+1} is well-defined by (F1). Observe that f_{n+1} has the properties (in3) and (in4), by (F3).

Now we set $g = \bigcup_{0 \leq n \leq m} f_m$. It is readily checked that g is as required.

C Proof theory

We prove now two theorems on when a concept inclusion is entailed by an $\mathcal{ELH}^r$-terminology (Theorems 10 and 11) as a direct consequence of the consideration of possible proofs in a Gentzen-style proof system for $\mathcal{ELH}^r$ given in Figure 9. This proof system is a straightforward generalization of the proof system in (Hofmann 2005). It

$$\frac{}{C \sqsubseteq C}\ (\textsc{Ax}) \qquad \frac{}{C \sqsubseteq \top}\ (\textsc{AxTop})$$

$$\frac{C \sqsubseteq E}{C \sqcap D \sqsubseteq E}\ (\textsc{AndL1}) \qquad \frac{D \sqsubseteq E}{C \sqcap D \sqsubseteq E}\ (\textsc{AndL2})$$

$$\frac{C \sqsubseteq E \quad C \sqsubseteq D}{C \sqsubseteq D \sqcap E}\ (\textsc{AndR})$$

$$\frac{\mathsf{ran}(r) \sqcap C \sqsubseteq D}{\exists r.C \sqsubseteq \exists r.D}\ (\textsc{Ex})$$

$$\frac{C_A \sqsubseteq D}{A \sqsubseteq D}\ (\textsc{DefL}) \quad \frac{D \sqsubseteq C_A}{D \sqsubseteq A}\ (\textsc{DefR}) \quad \text{where } A \equiv C_A \in \mathcal{T}$$

$$\frac{C_A \sqsubseteq D}{A \sqsubseteq D}\ (\textsc{PDefL}) \quad \text{where } A \sqsubseteq C_A \in \mathcal{T}$$

$$\frac{\exists r.(C \sqcap A) \sqsubseteq D}{\exists r.C \sqsubseteq D}\ (\textsc{Ran}) \quad \text{where } \mathsf{ran}(r) \sqsubseteq A \in \mathcal{T}$$

$$\frac{B \sqcap \exists r.C \sqsubseteq D}{\exists r.C \sqsubseteq D}\ (\textsc{Dom}) \quad \text{where } \mathsf{dom}(r) \sqsubseteq B \in \mathcal{T}$$

$$\frac{A \sqcap \mathsf{ran}(r) \sqsubseteq C}{\mathsf{ran}(r) \sqsubseteq C}\ (\textsc{Inv}) \quad \text{where } \mathsf{ran}(r) \sqsubseteq A \in \mathcal{T}$$

$$\frac{\exists r.C \sqsubseteq D}{\exists s.C \sqsubseteq D}\ (\textsc{Sub}) \quad \frac{\mathsf{ran}(r) \sqsubseteq D}{\mathsf{ran}(s) \sqsubseteq D}\ (\textsc{InvSub}) \quad \text{where } s \sqsubseteq r \in \mathcal{T}$$

Fig. 9. Gentzen-style proof system for $\mathcal{ELH}^r$-terminologies

operates on *sequents* of the form $C \sqsubseteq D$, where C, D are $\mathcal{C}^{\mathsf{ran}}$-concepts; here the symbol $\sqsubseteq$ is treated as a syntactic separator. A *proof* (or a *derivation*) of a sequent $C \sqsubseteq D$ is a finite tree whose nodes are labelled with sequents, whose root is labelled with $C \sqsubseteq D$, whose terminal nodes (leaves) are labelled with axioms (instances of AX or AXTOP) and whose internal nodes are always labelled with the conclusion of a proof rule whose antecedent(s) are the labellings of the children. We use the notation $\mathcal{T} \vdash C \sqsubseteq D$ to indicate that that $C \sqsubseteq D$ is derivable from $\mathcal{T}$.

Cut elimination, correctness, and completeness can be proved similarly to (Hofmann 2005).

Lemma 4 (Cut elimination). *For all $\mathcal{ELH}^r$-terminologies $\mathcal{T}$ and $\mathcal{C}^{\mathsf{ran}}$-concepts C, D, and E the following holds: If $\mathcal{T} \vdash C \sqsubseteq D$ and $\mathcal{T} \vdash D \sqsubseteq E$ then $\mathcal{T} \vdash C \sqsubseteq E$.*

Lemma 5 (Correctness and completeness). *For all $\mathcal{ELH}^r$-terminologies $\mathcal{T}$ and $\mathcal{C}^{\mathsf{ran}}$-concepts C and D: $\mathcal{T} \models C \sqsubseteq D$ iff $\mathcal{T} \vdash C \sqsubseteq D$.*

Theorem 10. *Let $\mathcal{T}$ be an $\mathcal{ELH}^r$-terminology, A a concept name and $\exists r.D$ an $\mathcal{EL}$-concept. Assume*

$$\mathcal{T} \models \bigsqcap_{1 \le i \le l} \mathsf{ran}(s_i) \sqcap \bigsqcap_{1 \le j \le n} A_j \sqcap \bigsqcap_{1 \le k \le m} \exists r_k.D_k \sqsubseteq \exists r.D.$$

Then at least one of the following conditions holds:

(e1) *there exists r_i such that $r_i \sqsubseteq_{\mathcal{T}} r$ and $\mathcal{T} \models \mathsf{ran}(r_i) \sqcap D_i \sqsubseteq D$;*
(e2) *there exists A_i such that $\mathcal{T} \models A_i \sqsubseteq \exists r.D$;*
(e3) *there exists r_i such that $\mathcal{T} \models \exists r_i.\top \sqsubseteq \exists r.D$;*
(e4) *there exists s_i such that $\mathcal{T} \models \mathsf{ran}(s_i) \sqsubseteq \exists r.D$.*

Now assume that A is pseudo-primitive and

$$\mathcal{T} \models \bigsqcap_{1 \le i \le l} \mathsf{ran}(s_i) \sqcap \bigsqcap_{1 \le j \le n} A_j \sqcap \bigsqcap_{1 \le k \le m} \exists r_k.D_k \sqsubseteq A.$$

Then at least one of the following conditions holds:

(a1) *there exists A_i such that $\mathcal{T} \models A_i \sqsubseteq \exists A$;*
(a2) *there exists r_i such that $\mathcal{T} \models \exists r_i.\top \sqsubseteq A$;*
(a3) *there exists s_i such that $\mathcal{T} \models \mathsf{ran}(s_i) \sqsubseteq A$.*

Proof. We prove the theorem by induction on the length of proof $\mathcal{D}$ of the sequent

$$\bigsqcap_{1 \le i \le l} \mathsf{ran}(s_i) \sqcap \bigsqcap_{1 \le j \le n} A_j \sqcap \bigsqcap_{1 \le k \le m} \exists r_k.D_k \sqsubseteq \exists r.D.$$

Notice that if the left-hand side of this sequent is a conjunction (i.e., the $l+n+m > 1$), the last rule of $\mathcal{D}$ can only be ANDL1 or ANDL2. Then the theorem follows from the induction hypothesis.

Assume now that the left-hand side of the sequent is not a conjunction. Then, it is of one of the following forms.

1. $\mathsf{ran}(s_i) \sqsubseteq \exists r.D$,
2. $A_j \sqsubseteq \exists r.D$, or
3. $\exists r_k.D_k \sqsubseteq \exists r.D$.

Item 1 corresponds to **(e4)** and 2 to **(e2)**. It remains to consider sequents of the form $\exists r_k.D_k \sqsubseteq \exists r.D$. The last sequent of the proof of $\exists r_k.D_k \sqsubseteq \exists r.D$ can only be one of AX, EX, RAN, SUB or DOM. By induction on the proof length it can be shown that one of the following takes place:

(i) $\mathcal{T} \vdash B \sqsubseteq \exists r.D$, where $\exists s.\top \sqsubseteq B \in \mathcal{T}$ and $r_i \sqsubseteq_{\mathcal{T}} s$. It can be seen that $\mathcal{T} \vdash \exists s.\top \sqsubseteq \exists r.D$ and, by SUB, $\mathcal{T} \vdash \exists r_k.\top \sqsubseteq \exists r.D$, i.e., **(e3)**.

(ii) $\mathcal{T} \vdash \exists r.(C \sqcap A_1 \sqcap \cdots \sqcap A_p) \sqsubseteq \exists r.D$ where $r_i \sqsubseteq_{\mathcal{T}} r$ and $\mathsf{ran}(s_j) \sqsubseteq A_j \in \mathcal{T}$ for $1 < j < p$, and $r_i \sqsubseteq_{\mathcal{T}} s_1 \sqsubseteq_{\mathcal{T}} \cdots \sqsubseteq_{\mathcal{T}} s_p \sqsubseteq r$ and the last rule of the proof of $\exists r.(C \sqcap A_1 \sqcap \cdots \sqcap A_p) \sqsubseteq \exists r.D$ is EX, that is, the last sequent of its immediate subderivation is $(\mathsf{ran}(r) \sqcap C \sqcap A_1 \sqcap \cdots \sqcap A_p) \sqsubseteq D$. Notice that $\mathcal{T} \vdash (\mathsf{ran}(r) \sqcap C) \sqsubseteq (\mathsf{ran}(r) \sqcap C \sqcap A_1 \sqcap \cdots \sqcap A_p)$ and hence, by Lemma 4, $\mathcal{T} \vdash (\mathsf{ran}(r) \sqcap C) \sqsubseteq \exists r.D$, i.e., **(e1)**.

The second part of the theorem can be proved similarly.

Theorem 11. *Let $\mathcal{T}$ be an $\mathcal{ELH}^r$-terminology, A a concept name and $\exists s.C$ an $\mathcal{EL}$-concept. Assume*

$$\mathcal{T} \models A \sqsubseteq \exists s.C.$$

Then

1. *One of the following holds.*
   - *$A \bowtie \exists r.B \in \mathcal{T}$ and one of the following holds*
     - *$\mathcal{T} \models s \sqsubseteq r$ and $\mathcal{T} \models \mathsf{ran}(s) \sqcap B \sqsubseteq C$. In this case we say that the* proof depth *of A wrt. $\exists s.C$ is zero (in symbols, $\mathsf{pd}_{\exists s.C}(A) = 0$).*
     - *$\mathsf{dom}(s) \sqsubseteq B \in \mathcal{T}$ and $\mathcal{T} \models B \sqsubseteq \exists r.C$. We set $\mathsf{pd}_{\exists s.C}(A) = \mathsf{pd}_{\exists s.C}(B) + 1$.*
   - *$A \bowtie B_1 \sqcap \cdots \sqcap B_n \in \mathcal{T}$ and for some $i : 1 \leq i \leq n$ we have $\mathcal{T} \models B_i \sqsubseteq \exists r.C$. We set $\mathsf{pd}_{\exists s.C}(A) = \min\{\mathsf{pd}_{\exists s.C}(B_j) \mid \mathcal{T} \models B_j \sqsubseteq \exists r.C\} + 1$.*
2. *Proof depth wrt. $\exists s.C$ introduced above is well-defined for all $B \in \mathsf{N_C}$ such that $\mathcal{T} \models B \sqsubseteq \exists s.C$.*

Proof. The proof of Item 1 is similar to the proof of Theorem 10 and is left to the reader. Notice that if $\mathcal{T} \models B \sqsubseteq \exists s.C$, $B \sqsubseteq \exists s.C$ has a finite-size proof and so the proof depth wrt. $\exists s.C$ is well-defined.

D Relation between Σ-interpolants

In this section, we prove the claim that every concept Σ-interpolant is an instance Σ-interpolant, and every instance Σ-interpolant is a query Σ-interpolant.

Proposition 2. *Let $\mathcal{T}$ be an $\mathcal{ELH}^r$-TBox. Every query Σ-interpolant is a instance Σ-interpolant and every instance Σ-interpolant is a concept Σ-interpolant.*

Proof. Every instance Σ-interpolant is a concept Σ-interpolant: for suppose that $\mathcal{T}'$ is a instance Σ-interpolant of $\mathcal{T}$ and let α be an $\mathcal{ELH}^r_\Sigma$-inclusion. If α is of the form $r_1 \sqsubseteq r_2$ then $\mathcal{T} \models r_1 \sqsubseteq r_2$ iff $\mathcal{T}' \models r_1 \sqsubseteq r_2$ because $(\mathcal{T}, \mathcal{A}) \models r_2(a,b)$ iff $(\mathcal{T}', \mathcal{A}) \models r_2(a,b)$ for $\mathcal{A} = \{r_1(a,b)\}$. If α is of the form $C_1 \sqsubseteq C_2$, then consider the ABox $\mathcal{A}_{C_1}$ from Lemma 1. Since $\mathcal{T}'$ is an instance Σ-interpolant, we have

$$(\mathcal{T}, \mathcal{A}_{C_1}) \models C_2(a_{C_1}) \Leftrightarrow (\mathcal{T}', \mathcal{A}_{C_1}) \models C_2(a_{C_1}).$$

By Lemma 1, we have

$$\mathcal{T} \models C_1 \sqsubseteq C_2 \Leftrightarrow (\mathcal{T}, \mathcal{A}_{C_1}) \models C_2(a_{C_1})$$

and

$$\mathcal{T}' \models C_1 \sqsubseteq C_2 \Leftrightarrow (\mathcal{T}', \mathcal{A}_{C_1}) \models C_2(a_{C_1}),$$

Hence $\mathcal{T} \models C_1 \sqsubseteq C_2$ iff $\mathcal{T}' \models C_1 \sqsubseteq C_2$, as required.

It is trivial that every query Σ-interpolant is an instance Σ-interpolant.

E Complexity

In this section, we provide proofs of the complexity results stated in the paper. The only result which does not follow in a straighforward way from existing results (in (Baader *et al.* 2008; Rosati 2007)) is the first part of Proposition 1 stating that subsumption in $\mathcal{EL}^{\mathsf{ran},\sqcap,u}$ (and, therefore, logical equivalence of $\mathcal{EL}^{\mathsf{ran},\sqcap,u}$-TBoxes) is in PTIME. Finally, we prove coNP-hardness of logical equivalence for $\mathcal{EL}^{\mathsf{ran},\sqcup}$.

Theorem 1. *The following problems are PTIME-complete for $\mathcal{EL}^{\mathsf{ran},\sqcap,u}$-TBoxes $\mathcal{T}$:*

- *decide whether $\mathcal{T} \models C \sqsubseteq D$, where $C \sqsubseteq D$ is an $\mathcal{EL}^{\mathsf{ran},\sqcap,u}$-inclusion;*
- *decide whether $(\mathcal{T}, \mathcal{A}) \models C(a)$, where C is an $\mathcal{EL}$-concept.*

Deciding whether $(\mathcal{T}, \mathcal{A}) \models q(\boldsymbol{a})$, where q is a conjunctive query, is NP-complete, and knowledge base complexity of this problem is PTIME-complete.

Proof. (i) Let $\mathcal{T}$ be a $\mathcal{EL}^{\mathsf{ran},\sqcap,u}$-TBox and $C \sqsubseteq D$ a $\mathcal{EL}^{\mathsf{ran},\sqcap,u}$-inclusion. By Proposition 1, we may assume that $\mathcal{T}$ is a $\mathcal{ELH}^r$-TBox. Consider the ABox $\mathcal{A}_C$ and a_C from Lemma 1 with

$$\mathcal{T} \models C \sqsubseteq D \Leftrightarrow (\mathcal{T}, \mathcal{A}_C) \models D(a_C).$$

Clearly, it is sufficient to show that deciding $(\mathcal{T}, \mathcal{A}_C) \models D(a_C)$ is in PTIME. First we compute a new TBox $\mathcal{T}'$, a new ABox $\mathcal{A}'_C$, and a new $\mathcal{C}^u$-concept D' such that

$$(\mathcal{T}, \mathcal{A}_C) \models D(a_C) \Leftrightarrow (\mathcal{T}', \mathcal{A}'_C) \models D'(a_C).$$

To this end, take for every proper conjunction $R = r_1 \sqcap \cdots \sqcap r_m$ of role names in D a fresh role name s_R. Then define $\mathcal{T}'$, $\mathcal{A}'_C$ and D' recursively as follows:

- D' is obtained from D by replacing every occurrence of R in D by s_R;
- $\mathcal{T}'$ is obtained from $\mathcal{T}$ by adding the role inclusion $s \sqsubseteq s_R$ to $\mathcal{T}$ whenever $\mathcal{T} \models s \sqsubseteq r_i$ for all i with $1 \leq i \leq m$;
- $\mathcal{A}'_C$ is obtained from $\mathcal{A}_C$ by adding $s_R(a, b)$ to $\mathcal{A}_C$ whenever $r_i(a, b) \in \mathcal{A}$ for all i with $1 \leq i \leq m$.

Clearly we have $(\mathcal{T}, \mathcal{A}_C) \models D(a_C)$ whenever $(\mathcal{T}', \mathcal{A}'_C) \models D'(a_C)$. Conversely, suppose $(\mathcal{T}', \mathcal{A}'_C) \not\models D'(a_C)$. Then $\mathcal{U}_{\mathcal{K}'} \not\models D'(a_C)$, where $\mathcal{K}' = (\mathcal{T}', \mathcal{A}'_C)$, by Theorem 7. Clearly, $\mathcal{U}_{\mathcal{K}'}$ is a model of $(\mathcal{T}, \mathcal{A})$. Moreover, $(d_1, d_2) \in R^{\mathcal{U}_{\mathcal{K}'}}$ iff $(d_1, d_2) \in s_R^{\mathcal{U}_{\mathcal{K}'}}$, by the definition of $\mathcal{K}'$ and $\mathcal{U}_{\mathcal{K}'}$. Thus, $\mathcal{U}_{\mathcal{K}'} \not\models D(a_C)$ and so $(\mathcal{T}, \mathcal{A}) \not\models D(a_C)$.

Finally, $(\mathcal{T}', \mathcal{A}'_C) \models D'(a_C)$ iff $a_C^{\mathcal{I}^r_{\mathcal{K}'}} \in D'^{\mathcal{I}^r_{\mathcal{K}'}}$, by Theorem 6, and the latter can be checked in PTIME.

Instance checking: PTIME-completeness follows from Proposition 1 and the fact that instance checking in $\mathcal{ELH}^r$ is in PTIME (Baader *et al.* 2008).

Conjunctive queries: Again, using Proposition 1, PTIME completeness of knowledge-base complexity can be proved using the corresponding result for $\mathcal{ELH}$ (the extension of $\mathcal{EL}$ with role inclusions) in (Rosati 2007) and a by extending, in a straighforward way, the polynomial reduction of instance checking in $\mathcal{ELH}^r$ to instance checking in $\mathcal{ELH}$ from (Baader *et al.* 2008) to the case of conjunctive queries. NP-completeness of combined complexity can be proved in the same way using the corresponding result in (Rosati 2007) and the reduction of (Baader *et al.* 2008).

Theorem 12. *Deciding whether a $\mathcal{EL}^{\mathsf{ran},\sqcup}$-inclusion follows from a $\mathcal{EL}^{\mathsf{ran},\sqcup}$-TBox is coNP-hard. In particular, deciding logical equivalence of $\mathcal{EL}^{\mathsf{ran},\sqcup}$-TBoxes is coNP-hard.*

Proof. Let φ be a propositional formula in conjunctive normal form (CNF). Introduce, for every propositional variable p of φ concept names A_p and $A_{\neg p}$ and denote by φ^* the resulting concept when every occuurence of $\neg p$ in φ is replaced by $A_{\neg p}$ and every positive occurrence of p is replaced by A_p.

Take a fresh concept name A. Now set $\mathcal{T} = \{C \sqsubseteq A\}$, where

$$C = \bigsqcup_{p \in \mathsf{var}(\varphi)} (A_p \sqcap A_{\neg p}).$$

Claim. φ is is not satisfiable iff $\mathcal{T} \models \varphi^* \sqsubseteq A$.

Suppose φ is satisfiable. Let v be a variable assignment such that $v(\varphi) = 1$. Consider the interpretation $\mathcal{I}$ with $\Delta^{\mathcal{I}} = \{d\}$ and $d \in A_p^{\mathcal{I}}$ iff $v(p) = 1$ and $d \in A_{\neg p}^{\mathcal{I}}$ iff $v(p) = 0$. Let $A^{\mathcal{I}} = \emptyset$. Clearly $C^{\mathcal{I}} = \emptyset$ and so $\mathcal{I}$ is a model of $\mathcal{T}$. However, $d \in (\varphi^*)^{\mathcal{I}}$ and so $\mathcal{I} \not\models \varphi^* \sqsubseteq A$.

Conversely, suppose $\mathcal{T} \not\models \varphi^* \sqsubseteq A$. Take a model $\mathcal{I}$ of $\mathcal{T}$ such that $d \in (\varphi^*)^{\mathcal{I}}$ and $d \notin A^{\mathcal{I}}$. Define a propositional interpretation v by setting $v(p) = 1$ iff $d \in A_p^{\mathcal{I}}$. Then $v(\varphi) = 1$: note that $d \notin C^{\mathcal{I}}$ because $\mathcal{I}$ is a model of $\mathcal{T}$. Thus $d \in A_{\neg p}^{\mathcal{I}}$ implies $d \notin A_p^{\mathcal{I}}$. So $v(\varphi) = 1$ because $d \in (\varphi^*)^{\mathcal{I}}$. Hence φ is satisfiable.

F Relation between inseparability relations

In this section, we present a proof of the first main result of this paper, namely:

Theorem 2 Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be $\mathcal{EL}^{\mathsf{ran},\sqcap,u}$-TBoxes and Σ an infinite signature. Then the following holds:

- $\mathcal{T}_1 \equiv_\Sigma^C \mathcal{T}_2$ iff $\mathcal{T}_1 \equiv_\Sigma^{\mathsf{ran},0} \mathcal{T}_2$;
- $\mathcal{T}_1 \equiv_\Sigma^i \mathcal{T}_2$ iff $\mathcal{T}_1 \equiv_\Sigma^{\mathsf{ran}} \mathcal{T}_2$;
- $\mathcal{T}_1 \equiv_\Sigma^q \mathcal{T}_2$ iff $\mathcal{T}_1 \equiv_\Sigma^{\mathsf{ran},\sqcap,u} \mathcal{T}_2$.

Proof. By Proposition 1, we may assume that $\mathcal{T}_1$ and $\mathcal{T}_2$ are $\mathcal{ELH}^r$-TBoxes. We first prove the directions from left to right.

(1) Suppose $\mathcal{T}_1 \equiv_\Sigma^C \mathcal{T}_2$ but $\mathcal{T}_1 \not\equiv_\Sigma^{\mathsf{ran},0} \mathcal{T}_2$. Without loss of generality, there exists an $\mathcal{EL}_\Sigma^{\mathsf{ran},0}$-inclusion

$$\alpha = \mathsf{ran}(r) \sqcap C \sqsubseteq D,$$

such that $\mathcal{T}_1 \models \alpha$ but $\mathcal{T}_2 \not\models \alpha$. Take a model $\mathcal{I}$ of $\mathcal{T}_2$ with $d \in (\mathsf{ran}(r) \sqcap C)^{\mathcal{I}}$ but $d \notin D^{\mathcal{I}}$. Take a $d' \in \Delta^{\mathcal{I}}$ with $(d', d) \in r^{\mathcal{I}}$. Assume first that the exists a concept name A in $\Sigma \setminus \mathsf{sig}(\mathcal{T}_1 \cup \mathcal{T}_2)$. Define a new interpretation $\mathcal{I}'$ by modifying the interpretation of A to $A^{\mathcal{I}'} = \{d\}$. Then $\mathcal{I}'$ is a model of $\mathcal{T}_2$ since A does not occur in $\mathcal{T}_2$. However, $d' \in (\exists r.(C \sqcap A))^{\mathcal{I}'}$ and $d' \notin (\exists r.(D \sqcap A))^{\mathcal{I}'}$. Hence

$$\mathcal{T}_2 \not\models \exists r.(C \sqcap A) \sqsubseteq \exists r.(D \sqcap A).$$

This implies $\mathcal{T}_1 \not\models \exists r.(C \sqcap A) \sqsubseteq \exists r.(D \sqcap A)$, since $\mathcal{T}_1 \equiv^C_\Sigma \mathcal{T}_2$. But this is a contradiction since $\mathcal{T}_1 \models \alpha$ and $\{\alpha\} \models \exists r.(C \sqcap A) \sqsubseteq \exists r.(D \sqcap A)$.

If there does not exist an $A \in \Sigma \setminus \mathsf{sig}(\mathcal{T}_1 \cup \mathcal{T}_2)$, then take $s \in \Sigma \setminus \mathsf{sig}(\mathcal{T}_1 \cup \mathcal{T}_2)$ and define a new interpretation by modifying the interpretation of s to $s^{\mathcal{I}'} = \{(d, d)\}$. The argument is now similar to the one above and left to the reader.

(2) Suppose $\mathcal{T}_1 \equiv^i_\Sigma \mathcal{T}_2$ but $\mathcal{T}_1 \not\equiv^{\mathsf{ran}}_\Sigma \mathcal{T}_2$. As role inclusions are straighforward to handle, we assume that there exists an $\mathcal{EL}^{\mathsf{ran}}_\Sigma$-inclusion

$$\alpha = C \sqsubseteq D,$$

such that $\mathcal{T}_1 \models \alpha$ but $\mathcal{T}_2 \not\models \alpha$. Take a finite model $\mathcal{I}$ of $\mathcal{T}_2$ with $d \in C^{\mathcal{I}}$ and $d \notin D^{\mathcal{I}}$. Construct an ABox $\mathcal{A}$ by taking a individual name $a_{d'}$ for each $d' \in \Delta^{\mathcal{I}}$ and including into $\mathcal{A}$ the assertion $A(a_{d'})$ whenever A occurs in C and $d' \in A^{\mathcal{I}}$ and $r(a_{d'}, a_{d''})$ whenever $(d', d'') \in r^{\mathcal{I}}$ and r occurs in C. Then $(\mathcal{T}_2, \mathcal{A}) \not\models D(a_d)$. We get $(\mathcal{T}_1, \mathcal{A}) \not\models D(a_d)$. But from this we obtain a contradiction, as we clearly have $\mathcal{A} \models C(a_d)$ and $\mathcal{T}_1 \models C \sqsubseteq D$, and so $(\mathcal{T}_1, \mathcal{A}) \models D(a_d)$.

(3) Suppose $\mathcal{T}_1 \equiv^q_\Sigma \mathcal{T}_2$ but $\mathcal{T}_1 \not\equiv^{\mathsf{ran},\sqcap,u}_\Sigma \mathcal{T}_2$. Without loss of generality, there exists an $\mathcal{EL}^{\mathsf{ran},\sqcap,u}_\Sigma$-inclusion

$$\alpha = C \sqsubseteq D,$$

such that $\mathcal{T}_1 \models \alpha$ but $\mathcal{T}_2 \not\models \alpha$. We may assume that D is an $\mathcal{C}^\sqcap$-concept or of the form $D = \exists u.D'$, where D' is an $\mathcal{C}^\sqcap$-concept. We consider the latter case, leaving the first case to the reader. Take a finite model $\mathcal{I}$ of $\mathcal{T}_2$ with $d \in C^{\mathcal{I}}$ and $d \notin D^{\mathcal{I}}$. Construct an ABox $\mathcal{A}$ by taking an individual name $a_{d'}$ for each $d' \in \Delta^{\mathcal{I}}$ and including into $\mathcal{A}$ the assertion $A(a_{d'})$ whenever A occurs in C and $d' \in A^{\mathcal{I}}$ and the assertion $r(a_{d'}, a_{d''})$ whenever $(d', d'') \in r^{\mathcal{I}}$ and r occurs in C.

We associate a conjunctive query q_D with D in the well-known manner. To be precise, we sketch a definition of q_D by recursion. Assume

$$D' = A_1 \sqcap \cdots \sqcap A_n \sqcap \exists R_1.C_1 \sqcap \cdots \exists R_m.C_m,$$

where $R_1, \ldots, R_m$ are conjunctions of role names. Take a variable $x_{D'}$ and mutually distinct variable $x_{C_1}, \ldots, x_{C_m}$. q_D contains

- the conjuncts $A_i(x_{D'})$ for $1 \leq i \leq n$;
- $r(x_{D'}, x_{C_j})$ for every conjunct r of R_j and $1 \leq j \leq m$.

Add more conjuncts to q_D by dealing, recursively, in the same way with $C_1, \ldots, C_m$. At each step fresh variables are used. Let q_0 be the resulting conjunction. Now let $q_D = \exists \boldsymbol{x} q_0$, where $\boldsymbol{x}$ is the list of all variables in q_0.

Clearly $\mathcal{I} \not\models q_D$. Hence $(\mathcal{T}_2, \mathcal{A}) \not\models q_D$. So we obtain $(\mathcal{T}_1, \mathcal{A}) \not\models q_D$. But from this we obtain a contradiction, as we clearly have $\mathcal{A} \models C(a_d)$ and $\{D(a_d)\} \models q_D$. So, from $\mathcal{T}_1 \models C \sqsubseteq D$, we obtain $(\mathcal{T}_1, \mathcal{A}) \models q_D$.

We now come to the directions from right to left.

(1) $\mathcal{T}_1 \equiv_{\Sigma}^{\mathsf{ran},0} \mathcal{T}_2$ implies $\mathcal{T}_1 \equiv_{\Sigma}^{C} \mathcal{T}_2$ is trivial because every $\mathcal{ELH}^r$-inclusion is an $\mathcal{EL}^{\mathsf{ran},0}$-inclusion.

(2) Assume that $\mathcal{T}_1 \equiv_{\Sigma}^{\mathsf{ran}} \mathcal{T}_2$ but $\mathcal{T}_1 \not\equiv_{\Sigma}^{i} \mathcal{T}_2$. We may wlog. assume that there exists a Σ-ABox $\mathcal{A}$ such that at least one of the following holds:

- there exists $a, b \in \mathsf{obj}(\mathcal{A})$ and $r \in \Sigma$ such that $(\mathcal{T}_1, \mathcal{A}) \models r(a, b)$ and $(\mathcal{T}_2, \mathcal{A}) \not\models r(a, b)$.
- there exists an individual name $a \in \mathsf{obj}(\mathcal{A})$, and an $\mathcal{EL}_{\Sigma}$-concept C such that $(\mathcal{T}_1, \mathcal{A}) \models C(a)$ but $(\mathcal{T}_2, \mathcal{A}) \not\models C(a)$.

In the first case, there exists a role $s \in \Sigma$ such that $s(a, b) \in \mathcal{A}$ and $\mathcal{T}_1 \models s \sqsubseteq r$ and $\mathcal{T}_2 \not\models s \sqsubseteq r$. But this contradicts $\mathcal{T}_1 \equiv_{\Sigma}^{\mathsf{ran}} \mathcal{T}_2$.

In the second case, we first prove the following

Lemma 6. *Let $\mathcal{T}$ be an $\mathcal{ELH}^r$-TBox, Σ a signature, and $\mathcal{A}$ a Σ-ABox.*

For every $\mathcal{EL}_{\Sigma}$-concept D and $a \in \mathsf{N_I}$, $(\mathcal{T}, \mathcal{A}) \models D(a)$ iff there exists an $\mathcal{C}_{\Sigma}^{\mathsf{ran}}$-concept C such that $\mathcal{T} \models C \sqsubseteq D$ and $\mathcal{A} \models C(a)$.

Proof. The direction from right to left is straightforward, so we concentrate on the other direction.

Let D_0 be an $\mathcal{EL}_{\Sigma}$-concept, $a_0 \in \mathsf{N_I}$, and assume that $(\mathcal{T}, \mathcal{A}) \models D_0(a_0)$. Set, for every $a \in \mathsf{ob} = \mathsf{obj}(\mathcal{A}) \cup \{a_0\}$,

$$t_{\mathcal{A}}(a) = \{C \mid \mathcal{A} \models C(a),\ C \text{ an } \mathcal{C}_{\Sigma}^{\mathsf{ran}}\text{-concept}\}.$$

We show that $\mathcal{T} \cup t_{\mathcal{A}}(a_0) \models D_0$. Then, using compactness, we find an $\mathcal{C}_{\Sigma}^{\mathsf{ran}}$-concept C such that $\mathcal{T} \models C \sqsubseteq D_0$ and $\mathcal{A} \models C(a_0)$, as required.

Assume $\mathcal{T} \cup t_{\mathcal{A}}(a_0) \not\models D_0$. Take, for every $a \in \mathsf{ob}$, a model $\mathcal{I}_a$ of $\mathcal{T}$ with a point d_a such that for all $\mathcal{C}^{\mathsf{ran}}$-concepts C: $d^{\mathcal{I}_a} \in C^{\mathcal{I}_a}$ iff $\mathcal{T} \cup t_{\mathcal{A}}(a) \models C$. Such models exist by Theorem 8. We may assume that they are mutually disjoint. Take the following union $\mathcal{I}$ of the models $\mathcal{I}_a$:

- $\Delta^{\mathcal{I}} = \bigcup_{a \in \mathsf{ob}} \Delta^{\mathcal{I}_a}$;
- $A^{\mathcal{I}} = \bigcup_{a \in \mathsf{ob}} A^{\mathcal{I}_a}$, for $A \in \mathsf{N_C}$;
- $r^{\mathcal{I}} = \bigcup_{a \in \mathsf{ob}} r^{\mathcal{I}_a} \cup \{(d_a, d_b) \mid r'(a, b) \in \mathcal{A}, r' \sqsubseteq_{\mathcal{T}} r\}$, for $r \in \mathsf{N_R}$;
- $a^{\mathcal{I}} = d_a$, for $a \in \mathsf{ob}$.

For all $\mathcal{C}^{\mathsf{ran}}$-concepts C and all $a \in \mathsf{ob}$ the following holds for all $d \in \Delta^{\mathcal{I}_a}$:

$$d \in C^{\mathcal{I}_a} \text{ iff } d \in C^{\mathcal{I}}.$$

The proof is by induction on the construction of C. The interesting cases are $C = \mathsf{ran}(r)$ and $C = \exists r.D$ and the direction from right to left.

Assume first that $C = \mathsf{ran}(r)$. Let $d \in C^{\mathcal{I}} \cap \Delta^{\mathcal{I}_a}$. For $d \neq d_a$, $d \in C^{\mathcal{I}_a}$ follows immediately by IH. Assume $d = d_a$. Take d' with $(d', d) \in r^{\mathcal{I}}$. Again, if $d' \in \Delta^{\mathcal{I}_a}$, then the claim follows immediately from the IH. Now assume $d' \notin \Delta^{\mathcal{I}_a}$. Then $d' = b$ for some b with $r'(b, a) \in \mathcal{A}$ and $r' \sqsubseteq_{\mathcal{T}} r$. Then $\mathsf{ran}(r') \in t_{\mathcal{A}}(a)$. Hence $\mathcal{T} \cup t_{\mathcal{A}}(a) \models \mathsf{ran}(r)$ and we obtain $d \in C^{\mathcal{I}_a}$.

Assume now that $C = \exists r.D$ and $d \in C^{\mathcal{I}} \cap \Delta^{\mathcal{I}_a}$. For $d \neq d_a$, $d \in C^{\mathcal{I}_a}$ follows immediately by IH. Assume $d = d_a$. Take d' with $(d, d') \in r^{\mathcal{I}}$ and $d' \in D^{\mathcal{I}}$. Again, if $d' \in \Delta^{\mathcal{I}_a}$, then the claim follows immediately from the IH. Now assume $d' \notin \Delta^{\mathcal{I}_a}$. Then $d' = b$ for some b with $r'(a, b) \in \mathcal{A}$ and $r' \sqsubseteq_{\mathcal{T}} r$. By IH, $d' \in D^{\mathcal{I}_b}$. Hence $\mathcal{T} \cup t_{\mathcal{A}}(b) \models D$. By compactness, there exists a concept $E \in t_{\mathcal{A}}(b)$ such that $\mathcal{T} \models E \sqsubseteq D$. From $\mathcal{A} \models E(b)$ and $r'(a, b) \in \mathcal{A}$ we obtain $\mathcal{A} \models \exists r'.E(a)$. Therefore, $\exists r'.E \in t_{\mathcal{A}}(a)$. But then $\mathcal{T} \cup t_{\mathcal{A}}(a) \models \exists r'.D$ and we obtain $d_a \in C^{\mathcal{I}_a}$ using $r' \sqsubseteq_{\mathcal{T}} r$.

It follows that $\mathcal{I}$ is a model of $(\mathcal{T}, \mathcal{A})$ and $\mathcal{I} \not\models D_0(a_0)$. Hence $(\mathcal{T}, \mathcal{A}) \not\models D_0(a_0)$, and we have derived a contradiction.

We use this lemma to derive a contradiction: Suppose there exists an $a \in \mathsf{obj}(\mathcal{A})$ and an $\mathcal{EL}_{\Sigma}$-concept C such that $(\mathcal{T}_1, \mathcal{A}) \models C(a)$ and $(\mathcal{T}_2, \mathcal{A}) \not\models C(a)$. Then there exists an $\mathcal{C}^{\mathsf{ran}}_{\Sigma}$-concept C' such that $\mathcal{A} \models C'(a)$ and $\mathcal{T}_1 \models C' \sqsubseteq C$. Clearly $\mathcal{T}_2 \not\models C' \sqsubseteq C$, because $(\mathcal{T}_2, \mathcal{A}) \models C(a)$ otherwise. But this implies $\mathcal{T}_1 \not\equiv^{\mathsf{ran}}_{\Sigma} \mathcal{T}_2$.

(3) Assume that $\mathcal{T}_1 \equiv^{\mathsf{ran},\sqcap,u}_{\Sigma} \mathcal{T}_2$ but $\mathcal{T}_1 \not\equiv^{q}_{\Sigma} \mathcal{T}_2$. We may wlog. assume that there exists a Σ-ABox $\mathcal{A}$, a Σ-query $q(\boldsymbol{x})$, and a sequence of individual names $\boldsymbol{a}$ in $\mathsf{obj}(\mathcal{A})$ such that $(\mathcal{T}_1, \mathcal{A}) \models q(\boldsymbol{a})$ but $(\mathcal{T}_2, \mathcal{A}) \not\models q(\boldsymbol{a})$.

Let $\Sigma' = \Sigma \cap (\mathsf{sig}(\mathcal{T}_1 \cup \mathcal{T}_2) \cup \mathsf{sig}(q))$ and consider a model $\mathcal{I}'$ of $(\mathcal{T}_2, \mathcal{A})$ with $(\mathcal{T}_2, \mathcal{A}) \not\models q[\boldsymbol{a}]$. By Lemma 2, we obtain a contraction from the following

Lemma 7. *There exists an* $\mathsf{obj}(\mathcal{A})$*-forest* $\mathcal{I}$ *which is a model of* $(\mathcal{T}_1, \mathcal{A})$ *and such that for every* $n > 0$ *there exists an* $(\mathsf{obj}(\mathcal{A}), n, \Sigma')$*-homomorphism* f_n *from* $\mathcal{I}'$ *to* $\mathcal{I}$*.*

Proof. Assume $\mathcal{I}'$ is given. By Theorem 8, for every set Γ of $\mathcal{C}^{\mathsf{ran}}_{\Sigma}$-concepts there exists a model $\mathcal{I}_{\Gamma}$ of $\mathcal{T}_1$ and $d \in \Delta^{\mathcal{I}}$ such that $d \in C^{\mathcal{I}_{\Gamma}}$ iff $\mathcal{T}_1 \cup \Gamma \models C$, for all $\mathcal{C}^{\mathsf{ran}} \cup \mathcal{C}^{\sqcap,u}$-concepts C. Clearly, from this we obtain for each Γ a $\emptyset$-forest model $\mathcal{I}_{\Gamma}$ of $\mathcal{T}_1$ and $d \in \Delta^{\mathcal{I}_{\Gamma}}$ such that

$$d \in C^{\mathcal{I}} \Leftrightarrow \mathcal{T} \cup \Gamma \models C$$

for all $\mathcal{C}^{\sqcap,u}$-concepts C.

Take, for every $a \in \mathsf{obj}(\mathcal{A})$ such a model $\mathcal{I}'_a$ (with node d_a) for

$$\Gamma = t_{\mathcal{I}'}(a) = \{C \in \mathcal{C}^{\mathsf{ran}}_{\Sigma} \mid a^{\mathcal{I}'} \in C^{\mathcal{I}'}\}$$

Remove from $\mathcal{I}'_a$ all nodes d' distinct from d_a from which d_a is reachable along $\bigcup_{r \in \mathsf{N_C}} r^{\mathcal{I}'_a}$ and call the resulting model $\mathcal{I}_a$. Clearly, $\mathcal{I}_a$ is still a model of $\mathcal{T}_1$.

Take the following union $\mathcal{I}$ of the models $\mathcal{I}_a$:

- $\Delta^{\mathcal{I}} = \bigcup_{a \in \mathsf{obj}(\mathcal{A})} \Delta^{\mathcal{I}_a}$;
- $A^{\mathcal{I}} = \bigcup_{a \in \mathsf{obj}(\mathcal{A})} A^{\mathcal{I}_a}$, for $A \in \mathsf{N_C}$;
- $r^{\mathcal{I}} = \bigcup_{a \in \mathsf{obj}(\mathcal{A})} r^{\mathcal{I}_a} \cup \{(d_a, d_b) \mid r'(a, b) \in \mathcal{A}, r' \sqsubseteq_{\mathcal{T}_1} r\}$, for $r \in \mathsf{N_R}$;
- $a^{\mathcal{I}} = d_a$, for $a \in \mathsf{obj}(\mathcal{A})$.

We show that $\mathcal{I}$ is a model of $(\mathcal{T}_1, \mathcal{A})$ and that there exist the required $(\mathsf{obj}(\mathcal{A}), n, \Sigma)$-homomorphisms. First observe the following:

Claim 1. For all $\mathcal{C}^{\sqcap}$-concepts C and $d \in \Delta^{\mathcal{I}_a}$:

$$d \in C^{\mathcal{I}} \Leftrightarrow d \in C^{\mathcal{I}_a}$$

The proof is by induction on the construction of C. The interesting case is $C = \exists r_1 \sqcap \cdots \sqcap r_m.D$ and the direction from left to right. Assume that $C = \exists r_1 \sqcap \cdots \sqcap r_m.D$ and $d \in C^{\mathcal{I}} \cap \Delta^{\mathcal{I}_a}$. For $d \neq d_a$, $d \in C^{\mathcal{I}_a}$ follows immediately by IH. Assume $d = d_a$. Take d' with $(d, d') \in r_i^{\mathcal{I}}$ for $1 \leq i \leq m$ and $d' \in D^{\mathcal{I}}$. Again, if $d' \in \Delta^{\mathcal{I}_a}$, then the claim follows immediately from the IH. Now assume $d' \notin \Delta^{\mathcal{I}_a}$. Then $d' = b$ for some b with $r'(a, b) \in \mathcal{A}$ and $r' \sqsubseteq_{\mathcal{T}_1} r_i$ for $1 \leq i \leq m$. By IH, $d' \in D^{\mathcal{I}_b}$. Hence $\mathcal{T}_1 \cup t_{\mathcal{I}'}(b) \models D$. By compactness, there exists a concept $E \in t_{\mathcal{I}'}(b)$ such that $\mathcal{T}_1 \models E \sqsubseteq D$. We obtain $\exists r'.E \in t_{\mathcal{I}'}(a)$. But then $\mathcal{T}_1 \models \exists r'.E \sqsubseteq \exists r'.D$ and we obtain $d_a \in C^{\mathcal{I}_a}$ using $r' \sqsubseteq_{\mathcal{T}_1} r_i$, for $1 \leq i \leq m$.

Claim 2. For all $\mathcal{C}^{\mathsf{ran}}_{\Sigma}$-concepts C and $d \in \Delta^{\mathcal{I}}$:

- if $d = d_a$, then $d \in C^{\mathcal{I}}$ implies $\mathcal{T}_1 \cup t_{\mathcal{I}'}(a) \models C$;
- if $d \in \Delta^{\mathcal{I}_a}$ and $d \neq d_a$, then $d \in C^{\mathcal{I}}$ iff $d \in C^{\mathcal{I}_a}$.

The second claim is trivial. Consider the first claim. Assume first that $C = \mathsf{ran}(r)$ and let $d = d_a$. Take d' with $(d', d) \in r^{\mathcal{I}}$. Then $d' = b$ for some b with $r'(b, a) \in \mathcal{A}$ and $r' \sqsubseteq_{\mathcal{T}_1} r$. Then $\mathsf{ran}(r') \in t_{\mathcal{I}'}(a)$. Hence $\mathcal{T}_1 \cup t_{\mathcal{I}'}(a) \models \mathsf{ran}(r)$. Now suppose $C = \exists r.D$, let $d = d_a$, and assume $d \in C^{\mathcal{I}}$. Take d' with $(d, d') \in r^{\mathcal{I}}$ and $d' \in D^{\mathcal{I}}$. If $d' \in \Delta^{\mathcal{I}_a}$, then $d' \in D^{\mathcal{I}_a}$ and $\mathcal{T}_1 \cup t_{\mathcal{I}'}(a) \models \exists r.D$ follows from the definition. Now assume $d' \notin \Delta^{\mathcal{I}_a}$. Then $d' = b$ for some b with $r'(a, b) \in \mathcal{A}$ and $r' \sqsubseteq_{\mathcal{T}_1} r$. By IH, $\mathcal{T}_1 \cup t_{\mathcal{I}'}(b) \models D$. By compactness, there exists a concept $E \in t_{\mathcal{I}'}(b)$ such that $\mathcal{T}_1 \models E \sqsubseteq D$. We obtain $\exists r'.E \in t_{\mathcal{I}'}(a)$. But then $\mathcal{T}_1 \cup t_{\mathcal{I}'}(a) \models \exists r.D$.

Claim 3. For all $\mathcal{C}^{\sqcap}$-concepts C: there exists $d \in C^{\mathcal{I}}$ iff there exists $a \in \mathsf{obj}(\mathcal{A})$ such that $\mathcal{T}_1 \cup t_{\mathcal{I}'}(a) \models \exists u.C$.

Suppose $C^{\mathcal{I}} \neq \emptyset$. Then there exists $a \in \mathsf{obj}(\mathcal{A})$ such that $d \in \Delta^{\mathcal{I}_a} \cap C^{\mathcal{I}}$. By Claim 1, $d \in C^{\mathcal{I}_a}$, and therefore, $\mathcal{T}_1 \cup t_{\mathcal{I}'}(a) \models \exists u.C$. Conversely, suppose there exists $a \in \mathsf{obj}(\mathcal{A})$ such that $\mathcal{T}_1 \cup t_{\mathcal{I}'}(a) \models \exists u.C$. Then $\Delta^{\mathcal{I}_a} \cap C^{\mathcal{I}_a} \neq \emptyset$. Hence $C^{\mathcal{I}} \neq \emptyset$, by Claim 1.

Claim 4. $\mathcal{I}$ is an $\mathsf{obj}(\mathcal{A})$-forest and a model of $(\mathcal{T}_1, \mathcal{A})$.

The first claim is clear by construction. It follows from Claim 1 to 3, that $\mathcal{I}$ is a model of $(\mathcal{T}_1, \mathcal{A})$: for the role inclusions of $\mathcal{T}_1$, this is clear from the definition. Suppose $C_1 \sqsubseteq C_2 \in \mathcal{T}_1$ is an $\mathcal{ELH}^r$-inclusion. If C_1 is an $\mathcal{EL}$-concept, then $\mathcal{I} \models C_1 \sqsubseteq C_2$ follows from Claim 1 and the condition that the $\mathcal{I}_a$ are models of $\mathcal{T}_1$. If $C_1 = \mathsf{ran}(r)$, then $\mathcal{I} \models C_1 \sqsubseteq C_2$ follows from Claim 2, the construction of the $\mathcal{I}_a$, and the condition that the $\mathcal{I}_a$ are models of $\mathcal{T}_1$: for suppose $d \in \mathsf{ran}(r)^{\mathcal{I}}$. If $d \neq d_a$ for any a, then $d \in C_2^{\mathcal{I}}$ since the $\mathcal{I}_a$ are models of $\mathcal{T}_1$. If $d = d_a$, there exists $r'(b, a) \in \mathcal{A}$ with $r' \sqsubseteq_{\mathcal{T}_1} r$. We have $\mathcal{T}_1 \cup t_{\mathcal{I}'}(a) \models \mathsf{ran}(r')$, and so $\mathcal{T}_1 \cup t_{\mathcal{I}'}(a) \models \mathsf{ran}(r)$. Hence $d_a \in C_2^{\mathcal{I}_a}$.

Claim 5. For every $n > 0$ there exists an $(\mathsf{obj}(\mathcal{A}), n, \Sigma')$-homomorphism from $\mathcal{I}'$ to $\mathcal{I}$.

By Lemma 3, it is sufficient to show (in1) and (in2). For (in1) suppose $a \in \mathsf{obj}(\mathcal{A})$. Let $C \in t_{\mathcal{I}}^{n,\Sigma',\sqcap}(a^{\mathcal{I}})$. By Claim 1, $a^{\mathcal{I}} \in C^{\mathcal{I}_a}$. Hence $\mathcal{T}_1 \cup t_{\mathcal{I}'}(a) \models C$. By compactness and since $\mathcal{T}_1 \equiv_{\Sigma'}^{\mathsf{ran},\sqcap,u} \mathcal{T}_2$, $\mathcal{T}_2 \cup t_{\mathcal{I}'}(a) \models C$. But then $C \in t_{\mathcal{I}'}^{n,\Sigma',\sqcap}(a^{\mathcal{I}'})$, as required.

For (in2), let $d \in \Delta^{\mathcal{I}}$ and $C = \bigsqcap_{D \in \in t_{\mathcal{I}}^{n,\Sigma',\sqcap}(d)} D$. By Claim 3, there exists $a \in \mathsf{obj}(\mathcal{A})$ such that $\mathcal{T}_1 \cup t_{\mathcal{I}'}(a) \models \exists u.C$. By compactness and since $\mathcal{T}_1 \equiv_{\Sigma'}^{\mathsf{ran},\sqcap,u} \mathcal{T}_2$, $\mathcal{T}_2 \cup t_{\mathcal{I}'}(a) \models \exists u.C$. Take $d' \in \Delta^{\mathcal{I}'}$ with $d' \in C^{\mathcal{I}'}$. Then $C \in t_{\mathcal{I}'}^{n,\Sigma',\sqcap}(d')$ and so $t_{\mathcal{I}}^{n,\Sigma',\sqcap}(d) \subseteq t_{\mathcal{I}'}^{n,\Sigma',\sqcap}(d')$, as required.

G Computing Σ-interpolants

In this section, we give a proof of theorems 3, 4, and 5. The set Atom consists of all concepts in $\mathsf{N_C}$, all concepts of the form $\mathsf{ran}(r)$, and all concepts of the form $\mathsf{dom}(r)$. The following lemma is the first main step towards a proof of the theorems. It states that it is suffient to consider inclusions of a rather simple form only; namely inclusions in which either a member of Atom is on the left hand side or a concept name on the right hand side.

Lemma 8. *Let $\mathcal{T}$ be an $\mathcal{ELH}^r$-terminology and Σ a finite signature.*
(i) An $\mathcal{EL}^{\mathsf{ran},0}$-TBox $\mathcal{T}'$ is a concept Σ-interpolant for $\mathcal{T}$ if $\mathsf{sig}(\mathcal{T}') \subseteq \overline{\Sigma}$, $\mathcal{T} \models \mathcal{T}'$ and $\mathcal{T}' \models \alpha$ whenever $\mathcal{T} \models \alpha$, for all $\overline{\Sigma}$-assertions of the form

$$r \sqsubseteq s, \quad C' \sqsubseteq A, \quad D \sqsubseteq C,$$

where C' is an $\mathcal{C}^{\mathsf{ran},0}$-concept, $D \in \mathsf{Atom}$, and C an $\mathcal{EL}$-concept.
(ii) An $\mathcal{EL}^{\mathsf{ran}}$-TBox $\mathcal{T}'$ is a instance Σ-interpolant for $\mathcal{T}$ if $\mathsf{sig}(\mathcal{T}') \subseteq \overline{\Sigma}$, $\mathcal{T} \models \mathcal{T}'$ and $\mathcal{T}' \models \alpha$ whenever $\mathcal{T} \models \alpha$, for all $\overline{\Sigma}$-assertions of the form

$$r \sqsubseteq s, \quad C' \sqsubseteq A, \quad D \sqsubseteq C,$$

where C' is an $\mathcal{C}^{\mathsf{ran}}$-concept, $D \in \mathsf{Atom}$, and C an $\mathcal{EL}$-concept.
(iii) An $\mathcal{EL}^{\mathsf{ran},\sqcap,u}$-TBox $\mathcal{T}'$ is a query Σ-interpolant for $\mathcal{T}$ if $\mathsf{sig}(\mathcal{T}') \subseteq \overline{\Sigma}$, $\mathcal{T} \models \mathcal{T}'$ and $\mathcal{T}' \models \alpha$ whenever $\mathcal{T} \models \alpha$, for all $\overline{\Sigma}$-assertions of the form

$$r \sqsubseteq s, \quad C' \sqsubseteq A, \quad D \sqsubseteq C,$$

where C' is an $\mathcal{C}^{\mathsf{ran}}$-concept, $D \in \mathsf{Atom}$, and C an $\mathcal{C}^{\sqcap,u}$-concept.

Proof. First recall that it follows from Theorem 2 that it is sufficient to show that $\mathcal{T}'$ with the properties stated in (i) (respectively, (ii) and (iii)) implies the same $\mathcal{EL}_{\overline{\Sigma}}^{\mathsf{ran},0}$-inclusions as $\mathcal{T}$ (respectively, the same $\mathcal{EL}_{\overline{\Sigma}}^{\mathsf{ran}}$-inclusions and the same $\mathcal{EL}_{\overline{\Sigma}}^{\mathsf{ran},\sqcap,u}$-inclusions).

We first consider the claim for concept Σ-interpolants. Suppose $\mathcal{T} \models C_1 \sqsubseteq C_2$, where $C_1 \sqsubseteq C_2$ is an $\mathcal{EL}^{\mathsf{ran},0}$-inclusion with $\mathsf{sig}(C_1 \sqsubseteq C_2) \cap \Sigma = \emptyset$. We prove by induction on the construction of C_2 that $\mathcal{T}' \models C_1 \sqsubseteq C_2$.

Case 1. C_2 is a concept name. Then this follows immediately from the condition that $\mathcal{T}' \models C \sqsubseteq A$ whenever $\mathcal{T} \models C \sqsubseteq A$, for all $\mathcal{EL}^{\mathsf{ran},0}$-inclusions $C \sqsubseteq A$ with $\mathsf{sig}(C \sqsubseteq A) \subseteq \overline{\Sigma}$.

Case 2. $C_2 = D_1 \sqcap D_2$. Then $\mathcal{T} \models C_1 \sqsubseteq D_1$ and $\mathcal{T} \models C_1 \sqsubseteq D_2$. By IH, $\mathcal{T}' \models C_1 \sqsubseteq D_1$ and $\mathcal{T}' \models C_1 \sqsubseteq D_2$. Hence $\mathcal{T}' \models C_1 \sqsubseteq C_2$.

Case 3. $C_2 = \exists r.D$. The role-depth $\mathsf{rd}(C)$ of an $\mathcal{EL}^{\mathsf{ran},0}$-concept C is the number of nestings of $\exists r.D$ in C. Now, the proof is by induction on the role-depth $\mathsf{rd}(C_1)$ of C_1. Suppose $\mathsf{rd}(C_1) = 0$. Then

$$C_1 = \mathsf{ran}(s) \sqcap A_1 \sqcap \cdots \sqcap A_m$$

By Theorem 10, there exists a conjunct $E \in \mathsf{Atom}$ of C_1 such that $\mathcal{T} \models E \sqsubseteq \exists r.D$. But then $\mathcal{T}' \models E \sqsubseteq \exists r.D$ and so $\mathcal{T}' \models C_1 \sqsubseteq \exists r.D$.

Now assume that $\mathsf{rd}(C_1) = n + 1$. Let

$$C_1 = \mathsf{ran}(s) \sqcap A_1 \sqcap \cdots \sqcap A_m \sqcap \exists r_1.D_1 \sqcap \cdots \sqcap \exists r_n.D_n.$$

By Theorem 10 above, there are two possibilities: (i) there exists $E \in \mathsf{Atom}$ such that $\models C_1 \sqsubseteq E$ and $\mathcal{T} \models E \sqsubseteq \exists r.D$ or (ii) there exists r_i with $\mathcal{T} \models r_i \sqsubseteq r$ and

$$\mathcal{T} \models \mathsf{ran}(r_i) \sqcap D_i \sqsubseteq D$$

If (i), then $\mathcal{T}' \models E \sqsubseteq \exists r.D$, and so $\mathcal{T}' \models C_1 \sqsubseteq \exists r.D$. If (ii), then, by IH, $\mathcal{T}' \models \mathsf{ran}(r_i) \sqcap D_i \sqsubseteq D$. Moreover, we have $\mathcal{T}' \models r_i \sqsubseteq r$. Hence $\mathcal{T}' \models \exists r_i.D_i \sqsubseteq \exists r.D$ and we obtain $\mathcal{T}' \models C_1 \sqsubseteq C_2$, as required.

Now we consider the claim for instance Σ-interpolants. Suppose $\mathcal{T} \models C_1 \sqsubseteq C_2$, where $C_1 \sqsubseteq C_2$ is an $\mathcal{EL}^{\mathsf{ran}}$-inclusion with $\mathsf{sig}(C_1 \sqsubseteq C_2) \cap \Sigma = \emptyset$. We prove by induction on the construction of C_2 that $\mathcal{T}' \models C_1 \sqsubseteq C_2$.

Case 1. C_2 is a concept name. Then this follows immediately from the condition that $\mathcal{T}' \models C \sqsubseteq A$ whenever $\mathcal{T} \models C \sqsubseteq A$, for all $\mathcal{EL}^{\mathsf{ran}}$-inclusions $C \sqsubseteq A$ with $\mathsf{sig}(C \sqsubseteq A) \subseteq \overline{\Sigma}$.

Case 2. $C_2 = D_1 \sqcap D_2$. Then $\mathcal{T} \models C_1 \sqsubseteq D_1$ and $\mathcal{T} \models C_1 \sqsubseteq D_2$. By IH, $\mathcal{T}' \models C_1 \sqsubseteq D_1$ and $\mathcal{T}' \models C_1 \sqsubseteq D_2$. Hence $\mathcal{T}' \models C_1 \sqsubseteq C_2$.

Case 3. $C_2 = \exists r.D$. The proof is by induction on the role-depth $\mathsf{rd}(C_1)$ of C_1. Suppose $\mathsf{rd}(C_1) = 0$. Then

$$C_1 = \mathsf{ran}(s_1) \sqcap \cdots \mathsf{ran}(s_l) \sqcap A_1 \sqcap \cdots \sqcap A_m$$

By Theorem 10, there exists a conjunct $E \in \mathsf{Atom}$ of C_1 such that $\mathcal{T} \models E \sqsubseteq \exists r.D$. But then $\mathcal{T}' \models E \sqsubseteq \exists r.D$ and so $\mathcal{T}' \models C_1 \sqsubseteq \exists r.D$.

Now assume that $\mathsf{rd}(C_1) = n + 1$. Let

$$C_1 = \mathsf{ran}(s_1) \sqcap \cdots \mathsf{ran}(s_l) \sqcap A_1 \sqcap \cdots \sqcap A_m \sqcap \exists r_1.D_1 \sqcap \cdots \sqcap \exists r_n.D_n.$$

By Theorem 10 above, there are two possibilities: (i) there exists $E \in \mathsf{Atom}$ such that $\models C_1 \sqsubseteq E$ and $\mathcal{T} \models E \sqsubseteq \exists r.D$ or (ii) there exists r_i with $\mathcal{T} \models r_i \sqsubseteq r$ and

$$\mathcal{T} \models \mathsf{ran}(r_i) \sqcap D_i \sqsubseteq D$$

If (i), then $\mathcal{T}' \models E \sqsubseteq \exists r.D$. If (ii), then, by IH, $\mathcal{T}' \models \mathsf{ran}(r_i) \sqcap D_i \sqsubseteq D$. Moreover, we have $\mathcal{T}' \models r_i \sqsubseteq r$. Hence $\mathcal{T}' \models \exists r_i.D_i \sqsubseteq \exists r.D$ and we obtain $\mathcal{T}' \models C_1 \sqsubseteq C_2$, as required.

Now we consider the claim for query Σ-interpolants. Suppose $\mathcal{T} \models C_1 \sqsubseteq C_2$, where $C_1 \sqsubseteq C_2$ is an $\mathcal{EL}^{\mathsf{ran},\sqcap,0}$-inclusion with $\mathsf{sig}(C_1 \sqsubseteq C_2) \cap \Sigma = \emptyset$. We prove by induction on the construction of C_2 that $\mathcal{T}' \models C_1 \sqsubseteq C_2$.

Case 1. C_2 is a concept name. Then this follows immediately from the condition that $\mathcal{T}' \models C \sqsubseteq A$ whenever $\mathcal{T} \models C \sqsubseteq A$, for all $\mathcal{EL}^{\mathsf{ran}}$-inclusions $C \sqsubseteq A$ with $\mathsf{sig}(C \sqsubseteq A) \subseteq \overline{\Sigma}$.

Case 2. $C_2 = D_1 \sqcap D_2$. Then $\mathcal{T} \models C_1 \sqsubseteq D_1$ and $\mathcal{T} \models C_1 \sqsubseteq D_2$. By IH, $\mathcal{T}' \models C_1 \sqsubseteq D_1$ and $\mathcal{T}' \models C_1 \sqsubseteq D_2$. Hence $\mathcal{T}' \models C_1 \sqsubseteq C_2$.

Case 3. $C_2 = \exists R.D$, where $R = r_1' \sqcap \cdots \sqcap r_k'$. The proof is by induction on the role-depth $\mathsf{rd}(C_1)$ of C_1.

By Theorem 9, there exists s such that $\mathcal{T} \models C_1 \sqsubseteq \exists s.D$ and $\mathcal{T} \models s \sqsubseteq r_i'$ for $i \leq k$. Suppose $\mathsf{rd}(C_1) = 0$. Then

$$C_1 = \mathsf{ran}(s_1) \sqcap \cdots \mathsf{ran}(s_l) \sqcap A_1 \sqcap \cdots \sqcap A_m$$

By Theorem 10, there exists a conjunct $E \in \mathsf{Atom}$ of C_1 such that $\mathcal{T} \models E \sqsubseteq \exists s.D$. Therefore, $\mathcal{T} \models E \sqsubseteq \exists R.D$. But then $\mathcal{T}' \models E \sqsubseteq \exists R.D$ and so $\mathcal{T}' \models C_1 \sqsubseteq \exists R.D$.

Now assume that $\mathsf{rd}(C_1) = n + 1$. Let

$$C_1 = \mathsf{ran}(s_1) \sqcap \cdots \mathsf{ran}(s_l) \sqcap A_1 \sqcap \cdots \sqcap A_m \sqcap \exists r_1.D_1 \sqcap \cdots \sqcap \exists r_n.D_n.$$

By Theorem 10 above, there are two possibilities: (i) there exists $E \in \mathsf{Atom}$ such that $\models C_1 \sqsubseteq E$ and $\mathcal{T} \models E \sqsubseteq \exists s.D$ or (ii) there exists r_i with $\mathcal{T} \models r_i \sqsubseteq s$ and

$$\mathcal{T} \models \mathsf{ran}(r_i) \sqcap D_i \sqsubseteq D$$

If (i), then $\mathcal{T} \models E \sqsubseteq \exists R.D$ and so $\mathcal{T}' \models E \sqsubseteq \exists R.D$. If (ii), then by IH, $\mathcal{T}' \models \mathsf{ran}(r_i) \sqcap D_i \sqsubseteq D$. Moreover, we have $\mathcal{T}' \models r_i \sqsubseteq r_j'$ for $j \leq k$. Hence $\mathcal{T}' \models \exists r_i.D_i \sqsubseteq \exists R.D$ and we obtain $\mathcal{T}' \models C_1 \sqsubseteq C_2$, as required.

Case 4. $C_2 = \exists u.D$. If $\mathcal{T} \models C_1 \sqsubseteq D$, then we are done, by IH. Otherwise, by Theorem 9,

(a) there exists a subconcept $\mathsf{ran}(r) \sqcap C'$ of C_1 such that there exists a sequence $r_1', \ldots, r_n'$ such that $\mathcal{T} \models \exists r.C' \sqsubseteq \exists r_1'. \cdots \exists r_n'.D$ or
(b) there exists a sequence $r_1', \ldots, r_n'$ such that $\mathcal{T} \models C_1 \sqsubseteq \exists r_1'. \cdots \exists r_n'.D$.

For (a), by Lemma 10, we have

- $\mathcal{T} \models \mathsf{dom}(r) \sqsubseteq \exists r_1'. \cdots \exists r_n'.D$, or
- $\mathcal{T} \models r \sqsubseteq r_1'$ and $\mathcal{T} \models \mathsf{ran}(r) \sqcap C' \sqsubseteq \exists r_2'. \cdots \exists r_n'.D$.

Observe that, in the second case,

$$\mathcal{T} \models C_1 \sqsubseteq \exists r_1''. \cdots \exists r_m''.\exists r_2'. \cdots \exists r_n'.D$$

for some roles $r''_1, \ldots, r''_m$, and so we are in case (b). In the first case, $\mathcal{T} \models \mathsf{dom}(r) \sqsubseteq \exists u.D$ and so $\mathcal{T}' \models \mathsf{dom}(r) \sqsubseteq \exists u.D$. Moreover $\models C_1 \sqsubseteq \exists u.\mathsf{dom}(r)$. Hence $\mathcal{T}' \models C_1 \sqsubseteq C_2$, as required.

We now consider (b). Take a non-empty sequence $r'_1, \ldots, r'_k$ such that $\mathcal{T} \models C_1 \sqsubseteq \exists r'_1.\cdots\exists r'_k.D$. Suppose $\mathsf{rd}(C_1) = 0$. Then

$$C_1 = \mathsf{ran}(s_1) \sqcap \cdots \mathsf{ran}(s_l) \sqcap A_1 \sqcap \cdots \sqcap A_m$$

By Theorem 10, there exists a conjunct $E \in \mathsf{Atom}$ of C_1 such that $\mathcal{T} \models E \sqsubseteq \exists u.D$. But then $\mathcal{T}' \models E \sqsubseteq \exists u.D$ and so $\mathcal{T}' \models C_1 \sqsubseteq \exists u.D$.

Now assume that $\mathsf{rd}(C_1) = n + 1$. Let

$$C_1 = \mathsf{ran}(s_1) \sqcap \cdots \mathsf{ran}(s_l) \sqcap A_1 \sqcap \cdots \sqcap A_m \sqcap \exists r_1.D_1 \sqcap \cdots \sqcap \exists r_n.D_n.$$

By Theorem 10 above, there are two possibilities: (i) there exists $E \in \mathsf{Atom}$ such that $\models C_1 \sqsubseteq E$ and $\mathcal{T} \models E \sqsubseteq \exists u.D$ or (ii) there exists r_i with $\mathcal{T} \models r_i \sqsubseteq r'_1$ and

$$\mathcal{T} \models \mathsf{ran}(r_i) \sqcap D_i \sqsubseteq \exists u.D$$

If (i), then $\mathcal{T}' \models E \sqsubseteq \exists u.D$. If (ii), then, by IH, $\mathcal{T}' \models \mathsf{ran}(r_i) \sqcap D_i \sqsubseteq \exists u.D$. Hence $\mathcal{T}' \models \exists r_i.D_i \sqsubseteq \exists u.D$ and we obtain $\mathcal{T}' \models C_1 \sqsubseteq C_2$, as required.

We start with restating and proving Theorem 4.

Theorem 13. *Let Σ be a finite signature and $\mathcal{T}$ a normalized $\mathcal{ELH}^r$-terminology without Σ-loops.*

Then the algorithms computing $P_\Sigma(A)$ and $Q_\Sigma(A)$ (in the instance case) in Figures 3 and 4 terminate for all $A \in \mathsf{sig}(\mathcal{T})$. Let $\mathcal{T}^i_\Sigma$ consist of all inclusions, where A, r, and s range over $\mathsf{sig}(\mathcal{T}) \cap \overline{\Sigma}$:

- *$r \sqsubseteq s$, for all $r \sqsubseteq_\mathcal{T} s$;*
- *$D \sqsubseteq A$, for all $D \in P_\Sigma(A)$;*
- *$A \sqsubseteq D$, for all $D \in Q_\Sigma(A)$;*
- *$\mathsf{ran}(r) \sqsubseteq D$, for all $D \in Q_\Sigma(B)$ such that $\mathsf{ran}(r) \sqsubseteq B \in \mathcal{T}$ and $B \in \Sigma$;*
- *$\mathsf{dom}(r) \sqsubseteq D$, for all $D \in Q_\Sigma(B)$ such that $\mathsf{dom}(r) \sqsubseteq B \in \mathcal{T}$ and $B \in \Sigma$.*

Then $\mathcal{T}^i_\Sigma$ is an instance Σ-interpolant of $\mathcal{T}$.

Proof. Termination of the computation of $P_\Sigma(A)$ and $Q_\Sigma(A)$ is readily checked and left to the reader. Moreover, it is straighforward to show that $\mathcal{T} \models \alpha$ for every $\alpha \in \mathcal{T}^i_\Sigma$. Thus, by Lemma 8, it is sufficient to show for all $A \in \mathsf{sig}(\mathcal{T})$:

$(*)$ If $\mathcal{T} \models D \sqsubseteq A$ for an $\mathcal{C}^{\mathsf{ran}}_{\overline{\Sigma}}$-concept D, then there exists $D' \in P_\Sigma(A)$ such that $\mathcal{T}^i_\Sigma \models D \sqsubseteq D'$.

$(**)$ If $\mathcal{T} \models A \sqsubseteq C$ for $\mathcal{EL}_{\overline{\Sigma}}$-concepts C, then

$$\mathcal{T}^i_\Sigma \models \bigsqcap_{D \in Q_\Sigma(A)} D \sqsubseteq C.$$

We start with the proof of $(*)$. The proof is by induction on the construction of D. Suppose $\mathcal{T} \models D \sqsubseteq A$.

– $D \in \mathsf{Atom}$. Then $D \in P_\Sigma(A)$ and so the claim follows from the definition.
– $D = D_1 \sqcap D_2$. Assume first that A has no definition of the form $A \equiv B_1 \sqcap \cdots \sqcap B_n$ in $\mathcal{T}$. Then, by Lemma 10, $\mathcal{T} \models D_i \sqsubseteq A$ for some $i = 1, 2$. Assume wlog. that $D_i = D_1$. By IH, there exists $D' \in P_\Sigma(A)$ such that $\mathcal{T}^i_\Sigma \models D_1 \sqsubseteq D'$. But then $\mathcal{T}^i_\Sigma \models D \sqsubseteq D'$ and the claim follows.
  Assume now that $A \equiv B_1 \sqcap \cdots \sqcap B_n \in \mathcal{T}$. Then $\mathcal{T} \models D_1 \sqcap D_2 \sqsubseteq B_i$, for all B_i. We may assume that none of the B_i has a definition of the form $B_i \equiv B'_1 \sqcap \cdots B'_k$ in $\mathcal{T}$. Thus, by Lemma 10, for every i there exists $E_i \in \{D_1, D_2\}$ with $\mathcal{T} \models E_i \sqsubseteq B_i$. Hence, by IH, for every B_i there exists $E'_i \in P_\Sigma(B_i)$ with $\mathcal{T}^i_\Sigma \models E_i \sqsubseteq E'_i$. By definition,
$$D' = \bigsqcap_{B_i \in \overline{\Sigma}} B_i \sqcap \bigsqcap_{B_i \in \Sigma} E'_i \in P_\Sigma(A)$$
  Moreover, $\mathcal{T}^i_\Sigma \models D \sqsubseteq D'$. Thus, the claim is proved.
– Let $D = \exists r.D'$. Assume first that $A \equiv \exists s.A' \in \mathcal{T}$. Then, by Lemma 10, we have the following cases:
  - $\mathcal{T} \models r \sqsubseteq s$ and $\mathcal{T} \models \mathsf{ran}(r) \sqcap D' \sqsubseteq A'$.
  - $\mathcal{T} \models \mathsf{dom}(r) \sqsubseteq A$.

  The second case is trivial: we have $r \not\in \Sigma$; so we have $\mathsf{dom}(r) \in P_\Sigma(A)$ and, clearly, $\models D \sqsubseteq \mathsf{dom}(r)$. Consider the first case. Assume first that $A' \in \overline{\Sigma}$. Then $\exists r.A' \in P_\Sigma(A)$. We have $\mathcal{T}^i_\Sigma \models \exists r.D' \sqsubseteq \exists r.A'$, and so we are done. Assume now that $A' \in \Sigma$. Then, if A' has no definition of the form $A' \equiv B_1 \sqcap \cdots \sqcap B_n$, then we have (a) $\mathcal{T} \models \mathsf{ran}(r) \sqsubseteq A'$ or (b) $\mathcal{T}' \models D' \sqsubseteq A'$. Let (a) hold. Then $\mathsf{ran}(r) \in P_\Sigma(A')$. Thus, $\exists r.\top \in P_\Sigma(A)$ and $\mathcal{T}' \models \exists r.D' \sqsubseteq \exists r.\top$, and we are done. If (b), then, by IH, there exists $D'' \in P_\Sigma(A')$ such that $\mathcal{T}^i_\Sigma \models D' \sqsubseteq D''$. But then $\exists r.D'' \in P_\Sigma(A)$ and $\mathcal{T}^i_\Sigma \models \exists r.D' \sqsubseteq \exists r.D''$, and we are done.
  Now let $A' \equiv B_1 \sqcap \cdots \sqcap B_n \in \mathcal{T}$. Then, for each B_i, we may assume that $\mathcal{T} \models \mathsf{ran}(r) \sqsubseteq B_i$ or $\mathcal{T} \models D' \sqsubseteq B_i$, by Lemma 10. Now the proof is similar to the proof above and left to the reader.
  If A is pseudo-primitive or $A \equiv B_1 \sqcap \cdots \sqcap B_n \in \mathcal{T}$, then the proof is straightforward and left to the reader.

Proof of $(**)$ The proof is by induction on the construction of C.

Assume C is a concept name and $\mathcal{T} \models A \sqsubseteq C$. Then $(**)$ follows immediately from the definition of $Q_\Sigma(A)$.

Assume $C = C_1 \sqcap C_2$ and $\mathcal{T} \models A \sqsubseteq C$. Then $\mathcal{T} \models A \sqsubseteq C_1$ and $\mathcal{T} \models A \sqsubseteq C_2$. By IH,
$$\mathcal{T}^i_\Sigma \models \bigsqcap_{D \in Q_\Sigma(A)} D \sqsubseteq C_1, \quad \mathcal{T}^i_\Sigma \models \bigsqcap_{D \in Q_\Sigma(A)} D \sqsubseteq C_2.$$
But then
$$\mathcal{T}^i_\Sigma \models \bigsqcap_{D \in Q_\Sigma(A)} D \sqsubseteq C_1 \sqcap C_2,$$
as required.

Assume $C = \exists s.C'$ and $\mathcal{T} \models A \sqsubseteq C$. We prove this case by induction on the proof depth of A. Suppose $\mathsf{pd}_{\exists s.C'}(A) = 0$. Then $A \bowtie \exists r.A' \in \mathcal{T}$ for some $A' \in \mathsf{N_C}$ and $\mathcal{T} \models r \sqsubseteq s$ such that $\mathcal{T} \models \mathsf{ran}(r) \sqcap A' \sqsubseteq C'$. Since $s \in \overline{\Sigma}$, there exists $s' \in \mathcal{C}^{\Sigma}_{\mathcal{T}}(r)$ such that $r \sqsubseteq_{\mathcal{T}} s'$ and $s' \sqsubseteq_{\mathcal{T}} s$. Take such an s'.

Now $Q_\Sigma(A)$ contains $\exists s'.E$, where

$$E = \bigsqcap_{\substack{B \in \Sigma,\, \mathsf{ran}(r) \sqsubseteq B \in \mathcal{T} \\ \mathsf{ran}(s') \sqsubseteq B \notin \mathcal{T},\, D \in Q_\Sigma(B)}} D \quad \sqcap \bigsqcap_{\substack{D \in Q_\Sigma(A') \\ A' \in \Sigma}} D \quad \sqcap \bigsqcap_{\substack{B \in \overline{\Sigma} \\ \mathcal{T} \models \mathsf{ran}(r) \sqcap A' \sqsubseteq B}} B.$$

Assume $C' = A_1 \sqcap \cdots \sqcap A_k \sqcap \exists r_1.C_1 \sqcap \cdots \exists r_m.C_m$. Then

$$\models \bigsqcap_{B \in \overline{\Sigma}, \mathcal{T} \models \mathsf{ran}(r) \sqcap A' \sqsubseteq B} B \sqsubseteq A_1 \sqcap \cdots \sqcap A_k$$

because every A_i is a conjunct of the concept to the left. By considering the last conjunct of E, it follows that $\models E \sqsubseteq A_i$ for all A_i.

For each $\exists r_i.C_i$ we have

– (a) $\mathcal{T} \models \mathsf{ran}(r) \sqsubseteq \exists r_i.C_i$ or (b) $\mathcal{T} \models A' \sqsubseteq \exists r_i.C_i$, by Lemma 10.

Fix an $\exists r_i.C_i$. Assume first that (b) holds. Since $\exists r_i.C_i$ is a sub-concept of C, by IH, $\mathcal{T}^i_\Sigma \models \bigsqcap_{D \in Q_\Sigma(A')} D \sqsubseteq \exists r_i.C_i$. It can be easity seen that $\mathcal{T}^i_\Sigma \models E \sqsubseteq \exists r_i.C_i$. Now assume that (a) holds. By IH and Lemma 10, there exists a conjunct D' of

$$\bigsqcap_{B \in \Sigma, \mathsf{ran}(r) \sqsubseteq B \in \mathcal{T}, D \in Q_\Sigma(B)} D \sqcap \bigsqcap_{\mathsf{ran}(r) \sqsubseteq B \in \mathcal{T}, B \in \overline{\Sigma}} B$$

such that $\mathcal{T}^i_\Sigma \models D' \sqsubseteq \exists r_i.C_i$. Notice, however, that D' is not necesserily a conjunct of E. We consider the following cases. If

– $\mathsf{ran}(r) \sqsubseteq D' \in \mathcal{T}$ and $D' \in \overline{\Sigma}$ or
– $D' \in Q_\Sigma(B)$ with $\mathsf{ran}(r) \sqsubseteq B \in \mathcal{T}$ and $\mathsf{ran}(s') \sqsubseteq B \notin \mathcal{T}$,

then D' is a conjunct of E and, therefore, $\mathcal{T}^i_\Sigma \models E \sqsubseteq \exists r_i.C_i$. On the other hand, if neither Point 1 not Point 2 holds, then there exists $B \in \Sigma$ such that

– $\mathsf{ran}(s') \sqsubseteq B \in \mathcal{T}$ and $D' \in Q_\Sigma(B)$.

We have

$$\mathsf{ran}(s') \sqsubseteq \bigsqcap_{D'' \in Q_\Sigma(B)} D'' \in \mathcal{T}^i_\Sigma,$$

and, therefore, $\mathcal{T}^i_\Sigma \models \exists s'.\top \sqsubseteq \exists s.\exists r_i.C_i$.

Summarising we obtain that that $\mathcal{T}^i_\Sigma \models \exists s'.E \sqsubseteq \exists s.C'$, as required.

Assume now that $\mathsf{pd}_{\exists s.C'}(A) = n$ and we proved $(**)$ for all B's whose proof depth wrt. $\exists s.C'$ is less than n. Then either $A \bowtie B_1 \sqcap \cdots \sqcap B_n \in \mathcal{T}$ and $\mathcal{T} \models B_i \sqsubseteq \exists s.C'$ such that $\mathsf{pd}_{\exists s.C'}(B_i) < \mathsf{pd}_{\exists s.C'}(A)$ or $A \bowtie \exists r.A' \in \mathcal{T}$ and $\mathsf{dom}(r) \sqsubseteq B \in \mathcal{T}$ and

$\mathcal{T} \models B \sqsubseteq \exists s.C$ and $\mathsf{pd}_{\exists s.C'}(B) < \mathsf{pd}_{\exists s.C'}(A)$. We consider the second case only and leave the first to the reader. Notice that, by IH, $\mathcal{T}^i_\Sigma \models \bigsqcap_{D \in Q_\Sigma(B)} D \sqsubseteq C$.

If $B \in \overline{\Sigma}$, then $B \in Q_\Sigma(A)$. By definition,

$$\mathcal{T}^i_\Sigma \models B \sqsubseteq \bigsqcap_{D \in Q_\Sigma(B)} D$$

Thus,

$$\mathcal{T}^i_\Sigma \models \bigsqcap_{D \in Q_\Sigma(A)} D \sqsubseteq C.$$

If $B \in \Sigma$ and $r \in \Sigma$, then $Q_\Sigma(B) \subseteq Q_\Sigma(A)$ and the claim follows by induction. If $B \in \Sigma$ and $r \in \overline{\Sigma}$, then

$$\mathsf{dom}(r) \sqsubseteq \bigsqcap_{D \in Q_\Sigma(B)} D \in \mathcal{T}^i_\Sigma$$

and, clearly,

$$\mathcal{T}^i_\Sigma \models \bigsqcap_{D \in Q_\Sigma(A)} D \sqsubseteq \exists r.\top.$$

Thus

$$\mathcal{T}^i_\Sigma \models \bigsqcap_{D \in Q_\Sigma(A)} D \sqsubseteq \bigsqcap_{D \in Q_\Sigma(B)} D,$$

and, by IH,

$$\mathcal{T}^i_\Sigma \models \bigsqcap_{D \in Q_\Sigma(A)} D \sqsubseteq C.$$

We now restate and show Theorem 3.

Theorem 14. *Let Σ be a finite signature and $\mathcal{T}$ a normalized $\mathcal{ELH}^r$-terminology without Σ-loops. Define $\mathcal{T}^C_\Sigma$ in the same way as $\mathcal{T}^i_\Sigma$ except that $P_\Sigma(A)$ is defined using the algorithm in Figure 2. Then $\mathcal{T}^C_\Sigma$ is a concept Σ-interpolant of $\mathcal{T}$.*

Proof. Clearly, by Lemma 8, the first comments of the proof of Theorem 13 apply to this case as well. As $Q_\Sigma(A)$ has not changed, it is sufficient to prove for all $A \in \mathsf{sig}(\mathcal{T})$:

$(*)$ If $\mathcal{T} \models D \sqsubseteq A$ for a $\mathcal{C}^{\mathsf{ran},0}_{\overline{\Sigma}}$-concept D, then there exists $D' \in P'_\Sigma(A)$ such that $\mathcal{T}^C_\Sigma \models D \sqsubseteq D'$.

To prove $(*)$, we require the following property of $P_\Sigma(A)$:

(A) If $\mathsf{ran}(r) \sqcap C \in P_\Sigma(A)$, $s \sqsubseteq_\mathcal{T} r$, and $s \in \overline{\Sigma}$, then $\mathsf{ran}(s) \sqcap C \in P_\Sigma(A)$.

(A) is easily proved by induction on the construction of $P_\Sigma(A)$. Clearly, (A) holds as well if we replace $P_\Sigma(A)$ by $P'_\Sigma(A)$.

Now the proof of $(*)$ is by induction on the construction of D. The proof is similar to the proof of $(*)$ for Theorem 13 and we focus on the new steps.

– $D \in \mathsf{Atom}$. Then $D \in P_\Sigma(A)$ and so the claim follows from the definition.

- $D = D_1 \sqcap D_2$. If A has no definition of the form $A \equiv B_1 \sqcap \cdots \sqcap B_n$ in $\mathcal{T}$, then the proof is the same as for Theorem 13 and left to the reader.
  Assume now that $A \equiv B_1 \sqcap \cdots \sqcap B_n \in \mathcal{T}$. Then $\mathcal{T} \models D_1 \sqcap D_2 \sqsubseteq B_i$, for all B_i. We may assume that none of the B_i has a definition of the form $B_i \equiv B_1' \sqcap \cdots B_k'$ in $\mathcal{T}$. Thus, by Lemma 10, for every i there exists $E_i \in \{D_1, D_2\}$ with $\mathcal{T} \models E_i \sqsubseteq B_i$. Hence, by IH, for every B_i there exists $E_i' \in P_\Sigma'(B_i)$ with $\mathcal{T}_\Sigma^C \models E_i \sqsubseteq E_i'$. Recall that $D_1 \sqcap D_2$ is a $\mathcal{C}^{\mathsf{ran},0}$-concept. Thus, there is at most one conjunct of the form $\mathsf{ran}(r)$ in $D_1 \sqcap D_2$ (which might have more than one occurrence). Assume this conjunct is $\mathsf{ran}(r)$ (the case where there is none is simpler and left to the reader). Let
  $$\Gamma = \{\mathsf{ran}(s) \mid \exists i \leq n \text{ such that } \mathsf{ran}(s) \text{ is a conjunct of } E_i'\}.$$
  As $\mathcal{T}_\Sigma^C \models D_1 \sqcap D_2 \sqsubseteq E_i'$ for $i \leq n$ and by the tree-model property, we obtain that $\mathcal{T} \models r \sqsubseteq s$, for all $\mathsf{ran}(s) \in \Gamma$. As $r \in \overline{\Sigma}$, we obtain by (A) that $E_i'' \in P_\Sigma'(B_i)$ for the concept E_i'' obtained from E_i by replacing $\mathsf{ran}(s)$ by $\mathsf{ran}(r)$. Observe that we still have $\mathcal{T}_\Sigma^C \models E_i \sqsubseteq E_i''$. But now
  $$D' = \bigsqcap_{B_i \in \overline{\Sigma}} B_i \sqcap \bigsqcap_{B_i \in \Sigma} E_i'' \in P_\Sigma'(A)$$
  as the only possible conjunct of the form $\mathsf{ran}(s)$ in any E_i'' is $\mathsf{ran}(r)$. Finally, we still have $\mathcal{T}_\Sigma^C \models D \sqsubseteq D'$. Thus, the claim is proved.
- Let $D = \exists r.D'$. Assume that $A \equiv \exists s.A' \in \mathcal{T}$; the other cases are left to the reader. Then, by Lemma 10, we have the following cases:
  - $\mathcal{T} \models r \sqsubseteq s$ and $\mathcal{T} \models \mathsf{ran}(r) \sqcap D' \sqsubseteq A'$.
  - $\mathcal{T} \models \mathsf{dom}(r) \sqsubseteq A$.

  As in the proof of Theorem 13, the second case and the case $A' \in \overline{\Sigma}$ are straighforward. Consider the first case under the assumption $A' \in \Sigma$. Then, if A' has no definition of the form $A' \equiv B_1 \sqcap \cdots \sqcap B_n$, then we have (a) $\mathcal{T} \models \mathsf{ran}(r) \sqsubseteq A'$ or (b) $\mathcal{T}' \models D' \sqsubseteq A'$. Let (a) hold. Then $\mathsf{ran}(r) \in P_\Sigma'(A')$. Thus, $\exists r.\top \in P_\Sigma'(A)$ and $\mathcal{T}_\Sigma^C \models \exists r.D' \sqsubseteq \exists r.\top$, and we are done. If (b), then, by IH, there exists $D'' \in P_\Sigma'(A')$ such that $\mathcal{T}_\Sigma^C \models D' \sqsubseteq D''$. As, by definition, D' does not cantain any conjunct of the form $\mathsf{ran}(r)$, if follows that D'' cannot contain any such conjunct. But then $\exists r.D'' \in P_\Sigma'(A)$ and $\mathcal{T}_\Sigma^C \models \exists r.D' \sqsubseteq \exists r.D''$, and we are done.
  Now let $A' \equiv B_1 \sqcap \cdots \sqcap B_n \in \mathcal{T}$. Then, for each B_i, we may assume that $\mathcal{T} \models \mathsf{ran}(r) \sqsubseteq B_i$ or $\mathcal{T} \models D' \sqsubseteq B_i$, by Lemma 10. Now the proof is similar to the proof above and left to the reader.

Extend the relation $\prec_\Sigma$ to $\prec_\Sigma^u$ by adding (A, B) to $\prec_\Sigma$ if there are $A \bowtie \exists r.A' \in \mathcal{T}$ such that $\mathcal{C}_\mathcal{T}^\Sigma(r) = \emptyset$ and $\mathsf{ran}(r) \sqsubseteq B \in \mathcal{T}$. $\mathcal{T}$ contains Σ-u-loops if $\prec_\Sigma^u$ contains a cycle.

Now, we extend the notion of a proof depth to $\mathcal{C}^\sqcap$-concepts. Let $A \in \mathsf{N_C}$ and C be a $\mathcal{C}^\sqcap$-concept of the form $C = \exists S.C_1$, where S is either a role name or a conjunction of roles, such that for a $\mathcal{ELH}^r$-terminology $\mathcal{T}$ we have $\mathcal{T} \models A \sqsubseteq C$. By Theorem 9, whenever $\exists(t_1 \sqcap \cdots \sqcap t_l).D$ is a sub-concept of C for some $t_i \in \mathsf{N_R}$ and D a $\mathcal{C}^\sqcap$-concept, there exists $r \in \mathsf{N_R}$ such that $\mathcal{T} \models r \sqsubseteq t_i$, $1 \leq i \leq l$ and $\mathcal{T} \models A \sqsubseteq C'$, where C' is obtained from C by replacing $\exists(t_1 \sqcap \cdots \sqcap t_l).D$ with $\exists r.D$. Let C^* be the result of recursive replacing of all expressions of the form $\exists(t_1 \sqcap \cdots \sqcap t_l).D$ with $\exists r.D$ for an appropriate r. Then we define $\mathsf{pd}_C(A) = \mathsf{pd}_{C^*}(A)$.

Finally, we restate and proof Theorem 5.

Theorem 15. *Let Σ be a finite signature and $\mathcal{T}$ a normalized $\mathcal{ELH}^r$-terminology without Σ-u-loops. Define $\mathcal{T}^q_\Sigma$ in the same way as $\mathcal{T}^i_\Sigma$ with the exception that $Q_\Sigma(A)$ is defined using the algorithm in Figure 5. Then $\mathcal{T}^q_\Sigma$ is a query Σ-interpolant of $\mathcal{T}$.*

Proof. Termination of the computation of $P_\Sigma(A)$ and $Q_\Sigma(A)$ is readily checked and left to the reader. Moreover, it is straighforward to show that $\mathcal{T} \models \alpha$ for every $\alpha \in \mathcal{T}^q_\Sigma$. Thus, by Lemma 8, it is sufficient to show for all $A \in \mathsf{sig}(\mathcal{T})$:

$(*)$ If $\mathcal{T} \models D \sqsubseteq A$ for a $\mathcal{C}^{\mathsf{ran}}_{\overline{\Sigma}}$-concept D, then there exists $D' \in P_\Sigma(A)$ such that $\mathcal{T}^q_\Sigma \models D \sqsubseteq D'$.

$(**)$ If $\mathcal{T} \models A \sqsubseteq C$ for $\mathcal{C}^{\sqcap,u}_{\overline{\Sigma}}$-concept C, then

$$\mathcal{T}^q_\Sigma \models \bigsqcap_{D \in Q_\Sigma(A)} D \sqsubseteq C.$$

The proof of $(*)$ is the same as the proof of $(*)$ for Theorem 13, so we consider $(**)$ only. Again, the proof is similar to $(**)$ for Theorem 13.

The proof is by induction on the construction of C.

Assume C is a concept name and $\mathcal{T} \models A \sqsubseteq C$. Then $(**)$ follows immediately from the definition of $Q_\Sigma(A)$.

Assume $C = C_1 \sqcap C_2$ and $\mathcal{T} \models A \sqsubseteq C$. Then $\mathcal{T} \models A \sqsubseteq C_1$ and $\mathcal{T} \models A \sqsubseteq C_2$. By IH,

$$\mathcal{T}^q_\Sigma \models \bigsqcap_{D \in Q_\Sigma(A)} D \sqsubseteq C_1, \quad \mathcal{T}^q_\Sigma \models \bigsqcap_{D \in Q_\Sigma(A)} D \sqsubseteq C_2.$$

But then

$$\mathcal{T}^q_\Sigma \models \bigsqcap_{D \in Q_\Sigma(A)} D \sqsubseteq C_1 \sqcap C_2,$$

as required.

Assume $C = \exists S.C'$, where $S = s'_1 \sqcap \cdots \sqcap s'_k$, and $\mathcal{T} \models A \sqsubseteq C$. Notice that we may assume wlog. that C is a $\mathcal{C}^{\sqcap}_{\overline{\Sigma}}$-concept. We prove this case by induction on the proof depth of A wrt. $\exists S.C'$.

Assume $\mathsf{pd}_{\exists S.C'}(A) = 0$. Then $A \bowtie \exists r.A' \in \mathcal{T}$ for some $A \in \mathsf{N_C}$ and $\mathcal{T} \models r \sqsubseteq S$ such that $\mathcal{T} \models \mathsf{ran}(r) \sqcap A' \sqsubseteq C'$.

Since $s'_1, \ldots, s'_k \in \overline{\Sigma}$, for each s'_i there exists $s' \in \mathcal{C}^\Sigma_\mathcal{T}(r)$ such that $r \sqsubseteq_\mathcal{T} s'$ and $s' \sqsubseteq_\mathcal{T} s'_i$. Let $S' = \bigsqcap_{s' \in \mathcal{C}^\Sigma_\mathcal{T}(r)} s'$.

Now $Q_\Sigma(A)$ contains $\exists S'.E$, where

$$E = \bigsqcap_{\substack{B \in \Sigma,\, \mathsf{ran}(r) \sqsubseteq B \in \mathcal{T} \\ \forall s' \in S' (\mathsf{ran}(s') \sqsubseteq B \notin \mathcal{T}),\, D \in Q_\Sigma(B)}} D \quad \sqcap \bigsqcap_{\substack{D \in Q_\Sigma(A') \\ A' \in \Sigma}} D \quad \sqcap \bigsqcap_{\substack{B \in \overline{\Sigma} \\ \mathcal{T} \models \mathsf{ran}(r) \sqcap A' \sqsubseteq B}} B.$$

Assume $C' = A_1 \sqcap \cdots \sqcap A_k \sqcap \exists r_1.C_1 \sqcap \cdots \exists r_m.C_m$. Then

$$\models \bigsqcap_{B \in \overline{\Sigma}, \mathcal{T} \models \mathsf{ran}(r) \sqcap A' \sqsubseteq B} B \sqsubseteq A_1 \sqcap \cdots \sqcap A_k$$

because every A_i is a conjunct of the concept to the left. By considering the last conjunct of E, it follows that $\models E \sqsubseteq A_i$ for all A_i.

For each $\exists r_i.C_i$ we have

– (a) $\mathcal{T} \models \mathsf{ran}(r) \sqsubseteq \exists r_i.C_i$ or (b) $\mathcal{T} \models A' \sqsubseteq \exists r_i.C_i$, by Lemma 10.

Fix an $\exists r_i.C_i$. Assume first that (b) holds. If $A' \in \overline{\Sigma}$, then A' is a conjunct of $\bigsqcap_{B \in \overline{\Sigma}, \mathcal{T} \models \mathsf{ran}(r) \sqcap A' \sqsubseteq B} B$ and, therefore, $\models E \sqsubseteq \exists r_i.C_i$. If $A' \notin \overline{\Sigma}$, then by IH,

$$\mathcal{T} \models \bigsqcap_{D \in Q_\Sigma(A')} D \sqsubseteq \exists r_i.C_i.$$

By considering the second conjunct of E, we have $\models E \sqsubseteq \exists r_i.C_i$. Now assume that (a) holds. By IH and Lemma 10, there exists a conjunct D' of

$$\bigsqcap_{B \in \Sigma, \mathsf{ran}(r) \sqsubseteq B \in \mathcal{T}, D \in Q_\Sigma(B)} D \sqcap \bigsqcap_{\mathsf{ran}(r) \sqsubseteq B \in \mathcal{T}, B \in \overline{\Sigma}} B$$

such that $\mathcal{T}^q_\Sigma \models D' \sqsubseteq \exists r_i.C_i$. Similarly to the proof of Theorem 13, we consider the following cases. If

– $\mathsf{ran}(r) \sqsubseteq D' \in \mathcal{T}$ or
– $D' \in Q_\Sigma(B)$ with $\mathsf{ran}(r) \sqsubseteq B \in \mathcal{T}$ and $\mathsf{ran}(s') \sqsubseteq B \notin \mathcal{T}$ for all $s' \in S'$,

then D' is a conjunct of E and, therefore, $\mathcal{T}^q_\Sigma \models E \sqsubseteq \exists r_i.C_i$. On the other hand, if neither Point 1 not Point 2 holds, then there exists $B \in \Sigma$ such that

– $\mathsf{ran}(s') \sqsubseteq B \in \mathcal{T}$ for some $s' \in S'$ and $D' \in Q_\Sigma(B)$.

We have

$$\mathsf{ran}(s') \sqsubseteq \bigsqcap_{D'' \in Q_\Sigma(B)} D'' \in \mathcal{T}^q_\Sigma,$$

for some $s' \in S'$ and, therefore, $\mathcal{T}^q_\Sigma \models \exists S'.\top \sqsubseteq \exists S.\exists r_i.C_i$.

Summarising we obtain that that $\mathcal{T}^q_\Sigma \models \exists S'.E \sqsubseteq \exists S.C'$, as required.

The proofs of the case when $\mathsf{pd}_{\exists S.C'}(A) > 0$ is exactly the same as in the proof of Theorem 13 and left to the reader.

Assume $C = \exists u.C'$ and $\mathcal{T} \models A \sqsubseteq C$. Then, by Theorem 9, there exists $r'_1, \ldots, r'_n$ such that $\mathcal{T} \models A \sqsubseteq \exists r'_1 \ldots \exists r'_n.C'$.

We prove by induction on the sequence length k that whenever $\mathcal{T} \models A \sqsubseteq \exists t_1 \ldots \exists t_k C'$ for some $A \in \mathsf{N_C} \cap \mathsf{sig}(\mathcal{T})$ and $t_i \in \mathsf{N_R} \cap \mathsf{sig}(\mathcal{T})$ we have $\mathcal{T}^q_\Sigma \models \bigsqcap_{D \in Q_\Sigma(A)} D \sqsubseteq \exists u.C'$. If $k = 0$ the claim is trivial. Assume now we proved the claim for all $k < n$. Let $\mathcal{T} \models A \sqsubseteq \exists r'_1 \ldots \exists r'_n.C'$. We proceed by induction on the proof depth of A wrt. $\exists r'_1 \ldots \exists r'_n.C'$.

Assume $\mathsf{pd}_{\exists r'_1 \ldots \exists r'_n . C'}(A) = 0$. Then $A \bowtie \exists r.A' \in \mathcal{T}$ for some $A \in \mathsf{N_C}$ and $\mathcal{T} \models r \sqsubseteq r'_1$ such that $\mathcal{T} \models \mathsf{ran}(r) \sqcap A' \sqsubseteq \exists r'_2 \ldots \exists r'_n . C'$.

Let $S' = \bigsqcap_{s' \in \mathcal{C}^{\Sigma}_{\mathcal{T}}(r)} s'$, if $\mathcal{C}^{\Sigma}_{\mathcal{T}}(r) \neq \emptyset$, or u otherwise. Note that if $r'_1 \in \overline{\Sigma}$ we have $\mathcal{C}^{\Sigma}_{\mathcal{T}}(r) \neq \emptyset$ and for some $s' \in S'$ we have $\mathcal{T} \models s' \sqsubseteq r'_1$.

Now $Q_\Sigma(A)$ contains $\exists S'.E$, where

$$E = \bigsqcap_{\substack{B \in \Sigma,\, \mathsf{ran}(r) \sqsubseteq B \in \mathcal{T} \\ \forall s' \in S' (\mathsf{ran}(s') \sqsubseteq B \notin \mathcal{T}),\, D \in Q_\Sigma(B)}} D \quad \sqcap \bigsqcap_{\substack{D \in Q_\Sigma(A') \\ A' \in \Sigma}} D \quad \sqcap \bigsqcap_{\substack{B \in \overline{\Sigma} \\ \mathcal{T} \models \mathsf{ran}(r) \sqcap A' \sqsubseteq B}} B.$$

Assume first that $n \geq 2$. Then we have

- (a) $\mathcal{T} \models \mathsf{ran}(r) \sqsubseteq \exists r'_2 \ldots \exists r'_n . C'$ or (b) $\mathcal{T} \models A' \sqsubseteq \exists r'_2 \ldots \exists r'_n . C'$, by Lemma 10.

Assume first that (b) holds. By IH $\mathcal{T}^q_\Sigma \models \bigsqcap_{D \in Q_\Sigma(A')} D \sqsubseteq \exists u.C'$. Thus $\mathcal{T}^q_\Sigma \models E \sqsubseteq \exists u.C'$. Now assume that (a) holds. By IH and Lemma 10, there exists a conjunct D' of

$$\bigsqcap_{B \in \Sigma, \mathsf{ran}(r) \sqsubseteq B \in \mathcal{T}, D \in Q_\Sigma(B)} D \sqcap \bigsqcap_{\mathsf{ran}(r) \sqsubseteq B \in \mathcal{T}, B \in \overline{\Sigma}} B$$

such that $\mathcal{T}^q_\Sigma \models D' \sqsubseteq \exists u.C'$. Again, similarly to the proof of Theorem 13, if

- $\mathsf{ran}(r) \sqsubseteq D' \in \mathcal{T}$ or
- $D' \in Q_\Sigma(B)$ with $\mathsf{ran}(r) \sqsubseteq B \in \mathcal{T}$ and either $S' = u$ or $\mathsf{ran}(s') \sqsubseteq B \notin \mathcal{T}$ for all $s' \in S'$,

then D' is a conjunct of E and, therefore, $\mathcal{T}^q_\Sigma \models E \sqsubseteq \exists u.C'$. On the other hand, if neither Point 1 not Point 2 holds, then there exists $B \in \Sigma$ such that

- $\mathsf{ran}(s') \sqsubseteq B \in \mathcal{T}$ for some $s' \in S'$ and $D' \in Q_\Sigma(B)$.

We have

$$\mathsf{ran}(s') \sqsubseteq \bigsqcap_{D'' \in Q_\Sigma(B)} D'' \in \mathcal{T}^q_\Sigma,$$

for some $s' \in S'$ and, therefore, $\mathcal{T}^q_\Sigma \models \exists S'.\top \sqsubseteq \exists S.\exists u.C'$.

Summarising we obtain that that $\mathcal{T}^q_\Sigma \models \exists S'.E \sqsubseteq \exists u.C'$, as required.

The case of $n = 1$ can be proved by combining the reasoning above and the one needed for proving the case $C = \exists s'_1 \sqcap \cdots \sqcap s'_k$.

The proofs of the case when $\mathsf{pd}_{\exists s.C'}(A) > 0$ is exactly the same as in the proof of Theorem 13 and left to the reader.

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