# Query Answering over DL ABoxes: How to Pick the Relevant Symbols

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## 1 Introduction

One of the main applications of description logics (DLs) is ontology-based data access: a conceptual model of a domain is formalized using a DL TBox, and this formalization is exploited to obtain complete answers when querying data stored in an ABox. The current availability of professional and comprehensive ontologies for the bio-medical domain such as SNOMED CT, NCI, and Galen allows an easy and inexpensive adoption of this approach in bio-medical applications such as querying electronic medical records [1]. In such applications, it is typical that an "off-the-shelf" ontology such as SNOMED CT is used together with ABoxes that derive from the actual application. Since ontologies such as SNOMED CT are huge, containing more than 400.000 concept names and embracing various areas such as anatomy, diseases, medication, and even social context and geographic location, it is usually the case that many symbols (concept or role names) defined in the ontology are excluded from the signature $\Sigma$ used to formulate ABoxes in the given application. Such a symbol $S$ may be linked to the symbols in $\Sigma$ via the TBox, but it may also be completely unrelated so that $S$ does never contribute to deriving certain answers of queries posed to $\Sigma$-ABoxes. Clearly, symbols of the latter kind are not relevant for formulating queries in this particular application.

The aim of this paper is (i) to propose a notion of *ABox relevance* of a symbol that describes when a symbol $S$ is relevant for ABoxes formulated in a given signature $\Sigma$, with a given background TBox $\mathcal{T}$ in place; and (ii) to study the computational complexity of deciding ABox relevance. This decision problem is of interest for a variety of reasons. First, knowing which symbols are relevant for ABox querying is useful for the construction of meaningful queries because non-relevant symbols can be discarded. When working with TBoxes that have more than 400.000 concept names such as SNOMED CT, support of this type is clearly indispensable. Second, the set of relevant symbols can be used to guide module extraction [2–4]. Recall that module extraction is the problem of extracting a subset $M$ from a TBox $T$ so that $M$ can be used instead of $T$ for a particular application. In most cases, the extraction of $M$ is guided by a signature $\Sigma$ one is interested in and about which the module should "say the same" as the original TBox. Now, having determined the set of ABox relevant symbols of a TBox $T$ for a given ABox schema, we can use this set as the seed

signature $\Sigma$ for module extraction. Symbols not in $\Sigma$ will not be relevant for querying ABoxes (over the ABox schema) and one can work with the module corresponding to this $\Sigma$ instead of the whole TBox.

The notion of ABox relevance depends on the query language used. In this paper, we study instance queries and conjunctive queries, the latter being the class of queries currently generating the most interest in the DL community [5–13]. After introducing preliminaries in Section 2, we present our notion of ABox relevance along with some basic observations in Section 3. We then analyze the complexity of deciding relevance in the $\mathcal{EL}$ family of DLs in Section 4, showing that it ranges from polynomial to ExpTime-complete. Results on the $\mathcal{ALC}$ family of DLs are given in Section 5, showing in particular that ABox relevance is decidable in $\mathcal{ALC}$ and $\mathcal{ALCI}$, but relevance regarding instance queries is undecidable in $\mathcal{ALCF}$. Some longer proofs have been moved to the appendix.

## 2 Preliminaries

We consider various DLs throughout the paper and use standard notation for syntax, semantics, and DL names, see [14]. In particular, we use $\mathsf{N_C}$ and $\mathsf{N_R}$ to denote the sets of concept names and role names, $C, D$ to denote (potentially) composite concepts, $A, B$ for concept names, $r, s$ for role names, and $a, b$ for individual names. When we speak of a *TBox*, we mean a set of *concept inclusions* (CIs) $C \sqsubseteq D$. An *ABox* is a set of *concept assertions* $A(a)$ and $\neg A(a)$ and *role assertions* $r(a, b)$. To distinguish this kind of ABox from ABoxes that admit composite concepts in concept assertions, we sometimes use the term *literal ABox*. We use $\mathsf{Ind}(\mathcal{A})$ to denote the set of individual names used in the ABox $\mathcal{A}$. As usual in the context of query answering, we adopt the unique name assumption (UNA).

We study two query languages: (i) the set $\mathcal{IQ}$ of *instance queries*, which take the form $A(v)$; and (ii) the set $\mathcal{CQ}$ of *conjunctive queries (CQs)*, which take the form $\exists \boldsymbol{v}.\varphi(\boldsymbol{v}, \boldsymbol{u})$ where $\varphi$ is a conjunction of atoms of the form $A(t)$ and $r(t, t')$ with $t, t'$ *terms*, i.e., variables or individual names. Note that we disallow composite concepts in instance queries and conjunctive queries, which is a realistic assumption for many applications. Also note that instance queries can only be used to query concept names, but not role names. This is the traditional definition, which is due to the fact that role assertions in an ABox can only be implied by an ABox if they are explicitly contained in it (and thus querying is trivial). Given a TBox $\mathcal{T}$, an ABox $\mathcal{A}$, and a (conjunctive or instance) query $q$ with $k$ answer variables $v_1, \ldots, v_k$, we write $\mathcal{T}, \mathcal{A} \models q[a_1, \ldots, a_k]$ if the tuple $(a_1, \ldots, a_k)$ of individual names is a *certain answer* to $q$ w.r.t. $\mathcal{A}$ and $\mathcal{T}$ (defined in the usual way). We use $\mathsf{cert}_{\mathcal{T},\mathcal{A}}(q)$ to denote the set of all certain answers to $q$ w.r.t. $\mathcal{A}$ and $\mathcal{T}$.

We use the term *symbol* to refer to a concept name or role name, *signature* to refer to a set of symbols, and $\mathsf{sig}(q)$ to denote the set of symbols used in the query $q$. Given a signature $\Sigma$, a *$\Sigma$-ABox* (resp. *$\Sigma$-concept*) is an ABox (resp. concept) using symbols from $\Sigma$ only.

## 3 The ABox Relevance Problem

The following definition describes the set of symbols $\Sigma_{\mathcal{T}}^{\mathcal{L}}$ that can meaningfully be used in a query posed against ABoxes that are formulated in the signature $\Sigma$, with the TBox $\mathcal{T}$ in the background.

**Definition 1.** *Let $\mathcal{T}$ be a TBox, $\Sigma$ a signature, and $\mathcal{L} \in \{\mathcal{IQ}, \mathcal{CQ}\}$ a query language. A symbol $S$ is* $\mathcal{L}$-relevant for $\Sigma$ given $\mathcal{T}$ *if there exists a $\Sigma$-ABox and $\mathcal{L}$-query $q$ such that $\mathcal{A}$ is consistent w.r.t. $\mathcal{T}$, $S \in \mathsf{sig}(q)$, and $\mathsf{cert}_{\mathcal{T},\mathcal{A}}(q) \neq \emptyset$. The* $\mathcal{L}$-extension *of $\Sigma$ given $\mathcal{T}$ is the following signature:*

$$\Sigma_{\mathcal{T}}^{\mathcal{L}} := \Sigma \cup \{S \in \mathsf{N_C} \cup \mathsf{N_R} \mid S \text{ is } \mathcal{L}\text{-relevant for } \mathcal{T} \text{ and } \Sigma\}.$$

For example, the concept name $A$ is both $\mathcal{IQ}$- and $\mathcal{CQ}$-relevant for $\Sigma = \{r\}$ given $\mathcal{T} = \{\exists r.\top \sqsubseteq A\}$, as witnessed by the query $q = A(v)$ and $\Sigma$-ABox $\{r(a,b)\}$ since $\mathsf{cert}_{\mathcal{T},\mathcal{A}}(q) = \{a\}$. Note that $\Sigma_{\mathcal{T}}^{\mathcal{IQ}}$ can never include any role names since role names cannot occur in an instance query. We are interested in *deciding $\mathcal{L}$-relevance* for $\mathcal{L} \in \{\mathcal{IQ}, \mathcal{CQ}\}$: given a TBox $\mathcal{T}$, a signature $\Sigma$ and a symbol $S$, decide whether $S \in \Sigma_{\mathcal{T}}^{\mathcal{L}}$. Clearly, this problem can be used to compute the signature $\Sigma_{\mathcal{T}}^{\mathcal{L}}$.

It should not be surprising that, in general, we need not have $\Sigma_{\mathcal{T}}^{\mathcal{IQ}} = \Sigma_{\mathcal{T}}^{\mathcal{CQ}}$. For example, if $\mathcal{T} = \{A \sqsubseteq \exists r.B\}$ and $\Sigma = A$, then $B \notin \Sigma_{\mathcal{T}}^{\mathcal{IQ}}$, but $B \in \Sigma_{\mathcal{T}}^{\mathcal{CQ}}$. For the former, it suffices to note that $\mathsf{cert}_{\mathcal{T},\mathcal{A}}(B(v)) = \emptyset$ for all $\Sigma$-ABoxes $\mathcal{A}$. For the latter, note that $\mathsf{cert}_{\mathcal{T},\mathcal{A}}(\exists v.B(v)) = \{()\}$ when $\mathcal{A} = \{A(a)\}$ (and where $()$ is the empty tuple). The following lemma, which is independent of the DL in which TBoxes are formulated, shows that we can always concentrate on CQs of such a simple form. It is an easy consequence of the fact that, since composite concepts are disallowed, CQs are purely positive, existential, and conjunctive.

**Lemma 1.** *$A \in \mathsf{N_C}$ (resp. $r \in \mathsf{N_R}$) is $\mathcal{CQ}$-relevant for $\Sigma$ given $\mathcal{T}$ iff there is an ABox $\mathcal{A}$ with $\mathsf{cert}_{\mathcal{T},\mathcal{A}}(\exists v.A(v)) \neq \emptyset$ (resp. $\mathsf{cert}_{\mathcal{T},\mathcal{A}}(\exists v, v'.r(v,v')) \neq \emptyset$).*

Lemma 1 allows us to consider only queries of the form $\exists v.A(v)$ and $\exists v, v'.r(v,v')$ when dealing with $\mathcal{CQ}$-relevance. From now on, we do this without further notice.

Answering conjunctive queries is typically more difficult than answering instance queries, both regarding the computational complexity and the required algorithms [7, 9]. Thus, it may be a little surprising that, as stated by the following result, $\mathcal{CQ}$-relevance can be polynomially reduced to $\mathcal{IQ}$-relevance. The converse is, in general, not known. In Section 4, we will see that it holds in the $\mathcal{EL}$ family of DLs.

**Theorem 1.** *In any DL with (qualified) existential restrictions, $\mathcal{CQ}$-relevance can be polynomially reduced to $\mathcal{IQ}$-relevance.*

*Proof (sketch).* Let $\mathcal{T}$ be a TBox, $\Sigma$ a signature, $B$ a concept name that does not occur in $\mathcal{T}$ and $\Sigma$, and $s$ a role name that does not occur in $\mathcal{T}$ and $\Sigma$. Then

1. $A$ is $\mathcal{CQ}$-relevant for $\Sigma$ given $\mathcal{T}$ iff $B$ is $\mathcal{IQ}$-relevant for $\Sigma \cup \{s\}$ given the TBox $\mathcal{T}' = \mathcal{T} \cup \mathcal{T}_B \cup \{A \sqsubseteq B\}$, where $\mathcal{T}_B = \{\exists r.B \sqsubseteq B \mid r = s \text{ or } r \text{ occurs in } \mathcal{T}\}$;

2. $r$ is $\mathcal{CQ}$-relevant for $\Sigma$ given $\mathcal{T}$ iff $B$ is $\mathcal{IQ}$-relevant for $\Sigma \cup \{s\}$ given the TBox $\mathcal{T}' = \mathcal{T} \cup \mathcal{T}_B \cup \{\exists r.\top \sqsubseteq B\}$, where $\mathcal{T}_B$ is as above.

The proofs of Points 1 and 2 are similar and we concentrate on Point 1. First suppose that $A$ is $\mathcal{CQ}$-relevant for $\Sigma$ given $\mathcal{T}$. Then there is a $\Sigma$-ABox $\mathcal{A}$ such that $\mathcal{T}, \mathcal{A} \models \exists v.A(v)$. Choose an $a_0 \in \mathsf{Ind}(\mathcal{A})$ and set $\mathcal{A}' := \mathcal{A} \cup \{s(a_0, b) \mid b \in \mathsf{Ind}(\mathcal{A})\}$. Using the fact that $\mathcal{T}, \mathcal{A} \models \exists v.A(v)$ and the definition of $\mathcal{A}'$ and $\mathcal{T}'$, it can be shown that $\mathcal{T}', \mathcal{A}' \models B(a_0)$. For the converse direction, suppose that $B$ is $\mathcal{IQ}$-relevant for $\Sigma \cup \{s\}$ given $\mathcal{T}'$. Then there is a $\Sigma \cup \{s\}$-ABox $\mathcal{A}'$ such that $\mathcal{T}', \mathcal{A}' \models B(a)$ for some $a \in \mathsf{Ind}(\mathcal{A}')$. Let $\mathcal{A}$ be obtained from $\mathcal{A}'$ by removing all assertions $s(a, b)$. Using the fact that $\mathcal{T}', \mathcal{A}' \models B(a)$ and the definition of $\mathcal{A}'$ and $\mathcal{T}'$, it can be shown that $\mathcal{T}, \mathcal{A} \models \exists v.A(v)$. □

In some proofs, it will be convenient to drop the UNA. The following lemma states that this can be done w.l.o.g. in $\mathcal{ALCI}$ (and all its fragments such as $\mathcal{EL}$ and $\mathcal{ALC}$) because the certain answers and thus also the notion of $\mathcal{L}$-relevance does not change. The lemma is easily proved using the fact that, in $\mathcal{ALCI}$, we can easily convert a model $\mathcal{I}$ for an ABox and a TBox that violates the UNA into a model $\mathcal{I}'$ that satisfies the UNA by "duplicating points" and such that $\mathcal{I}$ and $\mathcal{I}'$ are bisimilar.

**Lemma 2.** *Let $\mathcal{T}$ be an $\mathcal{ALCI}$-TBox, $\mathcal{A}$ an ABox, and $q \in \mathcal{L}$. Then $\mathsf{cert}_{\mathcal{T},\mathcal{A}}(q)$ is identical with and without UNA.*

An analogous statement fails, e.g., for $\mathcal{ALCF}$. To see this, take $\mathcal{T} = \{\top \sqsubseteq (\leq 1\ r) \sqcup A\}$ and $\Sigma = \{r\}$. Then $A$ is $\mathcal{IQ}$- and $\mathcal{CQ}$-relevant with UNA due to the ABox $\{r(a, b), r(a, b')\}$, but it is not relevant without UNA.

## 4 The $\mathcal{EL}$ Family

We study ABox relevance in the $\mathcal{EL}$ family of lightweight DLs [15]. In particular, we show that ABox relevance in plain $\mathcal{EL}$ can be decided in polynomial time, whereas it is ExpTime-complete in $\mathcal{ELI}$ and $\mathcal{EL}_\bot$. It is interesting to contrast these results with the complexity of subsumption and instance checking, which can be decided in polynomial time in the case of $\mathcal{EL}$ and $\mathcal{EL}_\bot$ and are ExpTime-complete in $\mathcal{ELI}$.

Throughout this section, we assume that the UNA is not imposed. This can be done w.l.o.g. due to Lemma 2. We start with a technical lemma that will be useful for several proofs later on. The lemma applies to ABoxes which are potentially infinite and *positive*, i.e. in which all concept assertions are of the form $A(a)$ with $A$ a concept name.

**Lemma 3.** *For every $\mathcal{ELI}_\bot$-TBox $\mathcal{T}$ and positive, potentially infinite ABox $\mathcal{A}$ consistent w.r.t. $\mathcal{T}$, there is a model $\mathcal{I}_{\mathcal{T},\mathcal{A}}$ of $\mathcal{T}$ and $\mathcal{A}$ such that the following conditions are satisfied:*

1. *for any $a \in \mathsf{Ind}(\mathcal{A})$ and $\mathcal{ELI}_\bot$-concept $C$, $a^{\mathcal{I}_{\mathcal{T},\mathcal{A}}} \in C^{\mathcal{I}_{\mathcal{T},\mathcal{A}}}$ iff $\mathcal{T}, \mathcal{A} \models C(a)$;*

2. *for any $k$-ary conjunctive query $q$ and $(a_1, \ldots, a_k) \in \mathsf{N}_\mathrm{I}^k$, $\mathcal{I}_{\mathcal{T},\mathcal{A}} \models q[a_1, \ldots, a_k]$ iff $(a_1, \ldots, a_k) \in \mathsf{cert}_{\mathcal{T},\mathcal{A}}(q)$.*

*Proof.* Let $\mathcal{T}$ be an $\mathcal{ELI}_\bot$-TBox and $\mathcal{A}$ an ABox such that $\mathcal{A}$ is consistent w.r.t. $\mathcal{T}$. For $a \in \mathsf{Ind}(\mathcal{A})$, a *path* for $\mathcal{A}$ and $\mathcal{T}$ is a finite sequence $a\, r_1\, C_1\, r_2 C_2 \cdots r_n\, C_n$, $n \geq 0$, where the $C_i$ are concepts from $\mathcal{T}$ (probably occurring as a subconcept) and the $r_i$ are roles such that the following conditions are satisfied:

- $a \in \mathsf{Ind}(\mathcal{A})^\mathcal{I}$,
- $\mathcal{T}, \mathcal{A} \models \exists r_1.C_1(a)$ of $n \geq 1$
- for $1 \leq i < n$, $\mathcal{T} \models C_i \sqsubseteq \exists r_{i+1}.C_{i+1}$.

We use $\mathsf{paths}(\mathcal{T}, \mathcal{A})$ to denote the set of all paths for $\mathcal{A}$ and $\mathcal{T}$. If $p \in \mathsf{paths}(\mathcal{T}, \mathcal{A}) \setminus \mathsf{Ind}(\mathcal{A})$, then $\mathsf{tail}(p)$ denotes the last concept $C_n$ in $p$. The *canonical model* $\mathcal{I}_{\mathcal{T},\mathcal{A}}$ of $\mathcal{T}$ and $\mathcal{A}$ is defined as follows:

$$
\begin{array}{rl}
\Delta^{\mathcal{I}_{\mathcal{T},\mathcal{A}}} & := \mathsf{paths}(\mathcal{T}, \mathcal{A}) \\
A^{\mathcal{I}_{\mathcal{T},\mathcal{A}}} & := \{a \in \mathsf{Ind}(\mathcal{A}) \mid \mathcal{T}, \mathcal{A} \models A(a)\} \cup \\
& \quad \{p \in \mathsf{paths}(\mathcal{T}, \mathcal{A}) \setminus \mathsf{Ind}(\mathcal{A}) \mid \mathcal{T} \models \mathsf{tail}(p) \sqsubseteq A\} \\
r^{\mathcal{I}_{\mathcal{T},\mathcal{A}}} & := \{(p, q) \in \mathsf{paths}(\mathcal{T}, \mathcal{A}) \times \mathsf{paths}(\mathcal{T}, \mathcal{A}) \mid q = p \cdot r\, C \text{ for some concept } C\} \\
a^{\mathcal{I}_{\mathcal{T},\mathcal{A}}} & := a \text{ for all } a \in \mathsf{Ind}(\mathcal{T}, \mathcal{A})
\end{array}
$$

It is standard to verify that $\mathcal{I}_{\mathcal{T},\mathcal{A}}$ satisfies the stated properties. □

Since DLs of the $\mathcal{EL}$ family do not offer negation, it may be deemed unnatural to define ABox relevance based on literal ABoxes, which admit negation. However, as the following lemma demonstrates, there is actually no difference between defining ABox relevance based on literal ABoxes and positive ABoxes. This holds for both $\mathcal{IQ}$- and $\mathcal{CQ}$-relevance and allows us henceforth to restrict our attention to positive ABoxes when working with $\mathcal{ELI}_\bot$ and its fragments.

**Lemma 4.** *For every $\mathcal{ELI}_\bot$ TBox $\mathcal{T}$, literal ABox $\mathcal{A}$ consistent w.r.t. $\mathcal{T}$, and conjunctive query $q$, we have $\mathsf{cert}_{\mathcal{T},\mathcal{A}}(q) = \mathsf{cert}_{\mathcal{T},\mathcal{A}^-}(q)$, where $\mathcal{A}^-$ is the restriction of $\mathcal{A}$ to assertions of the form $A(a)$ and $r(a, b)$.*

*Proof.* Since "$\supseteq$" is trivial, we concentrate on "$\subseteq$". Suppose $(a_1, \ldots, a_k) \notin \mathsf{cert}_{\mathcal{T},\mathcal{A}^-}(q)$. Then there is a model $\mathcal{I}$ of $\mathcal{T}$ and $\mathcal{A}^-$ such that $\mathcal{I} \not\models q[a_1, \ldots, a_k]$. By Point 2 of Lemma 3, $\mathcal{I}_{\mathcal{T},\mathcal{A}^-} \not\models q[a_1, \ldots, a_k]$. To prove that $(a_1, \ldots, a_k) \notin \mathsf{cert}_{\mathcal{T},\mathcal{A}^-}(q)$, it thus suffices to show that $\mathcal{I}_{\mathcal{T},\mathcal{A}}$ satisfies all negative concept assertions in $\mathcal{A}$. Let $\neg A(a) \in \mathcal{A}$. Since $\mathcal{A}$ is consistent w.r.t. $\mathcal{T}$, $\mathcal{T}, \mathcal{A} \not\models A(a)$. By Point 1 of Lemma 3, $a^{\mathcal{I}_{\mathcal{T},\mathcal{A}^-}} \notin A^{\mathcal{I}_{\mathcal{T},\mathcal{A}^-}}$ and we are done. □

We now state the announced converse of Theorem 1.

**Theorem 2.** *In $\mathcal{ELI}_\bot$, $\mathcal{IQ}$-relevance can be polynomially reduced to $\mathcal{CQ}$-relevance.*

*Proof.* We claim that $A$ is $\mathcal{IQ}$-relevant for $\Sigma$ given $\mathcal{T}$ iff $B$ is $\mathcal{CQ}$-relevant for $\Sigma \cup \{X\}$ given the TBox $\mathcal{T}' = \mathcal{T} \cup \{A \sqcap X \sqsubseteq B\}$, where $B$ and $X$ are concept names that do not occur in $\mathcal{T}$.

For the "if" direction, assume that $B$ is $\mathcal{CQ}$-relevant for $\Sigma \cup \{X\}$ given $\mathcal{T}'$ and let $\mathcal{A}'$ be a positive $\Sigma \cup \{X\}$-ABox such that $\mathcal{T}', \mathcal{A}' \models \exists v.B(v)$. By Point 2 of Lemma 3, $\mathcal{I}_{\mathcal{T}',\mathcal{A}'} \models \exists v.B(v)$. We want to show that there is an $a \in \mathsf{Ind}(\mathcal{A}')$ with $a^{\mathcal{I}_{\mathcal{T}',\mathcal{A}'}} \in B^{\mathcal{I}_{\mathcal{T}',\mathcal{A}'}}$. Assume to the contrary that there is no such $a$. Let $\mathcal{I}$ be obtained from $\mathcal{I}_{\mathcal{T}',\mathcal{A}'}$ by setting

$$X^{\mathcal{I}} := \{a^{\mathcal{I}_{\mathcal{T}',\mathcal{A}'}} \mid a \in \mathsf{Ind}(\mathcal{A}')\}$$
$$B^{\mathcal{I}} := B^{\mathcal{I}_{\mathcal{T}',\mathcal{A}'}} \cap X^{\mathcal{I}}$$

It is easy to see that $\mathcal{I}$ is still a model of $\mathcal{T}'$ and $\mathcal{A}'$. By our assumption that there is no $a \in \mathsf{Ind}(\mathcal{A}')$ with $a^{\mathcal{I}_{\mathcal{T}',\mathcal{A}'}} \in B^{\mathcal{I}_{\mathcal{T}',\mathcal{A}'}}$, we have $B^{\mathcal{I}} = \emptyset$, in contradiction to $\mathcal{T}', \mathcal{A}' \models \exists v.B(v)$. Thus, the desired $a \in \mathsf{Ind}(\mathcal{A}')$ exists. By Point 1 of Lemma 3, $a^{\mathcal{I}_{\mathcal{T}',\mathcal{A}'}} \in B^{\mathcal{I}_{\mathcal{T}',\mathcal{A}'}}$ implies that $\mathcal{T}', \mathcal{A}' \models B(a)$. By definition of $\mathcal{T}'$, this implies $\mathcal{T}', \mathcal{A}' \models A(a)$. Again by definition of $\mathcal{T}'$, this clearly implies $\mathcal{T}, \mathcal{A} \models A(a)$, where $\mathcal{A}$ is obtained from $\mathcal{A}'$ by dropping all concept assertions of the form $X(b)$. Since $\mathcal{A}$ is a $\Sigma$-ABox and consistent w.r.t. $\mathcal{T}$ (since $\mathcal{A}'$ is consistent w.r.t. $\mathcal{T}$'), it witnesses that $A$ is $\mathcal{IQ}$-relevant for $\Sigma$ given $\mathcal{T}$.

For the "only if" direction, assume that $A$ is $\mathcal{IQ}$-relevant for $\Sigma$ given $\mathcal{T}$ and let $\mathcal{A}$ be a positive $\Sigma$-ABox such that $\mathcal{T}, \mathcal{A} \models A(a)$ for some $a \in \mathsf{Ind}(\mathcal{A})$. Set $\mathcal{A}' := \mathcal{A} \cup \{X(a)\}$. It is easy to see that $\mathcal{T}', \mathcal{A}' \models \exists v.B(v)$ and thus $B$ is $\mathcal{CQ}$-relevant for $\Sigma \cup \{X\}$ given $\mathcal{T}'$. □

Theorem 2 allow us to choose freely between $\mathcal{IQ}$ and $\mathcal{CQ}$ when proving lower and upper bounds for relevance in the $\mathcal{EL}$ family of DLs. Note that, by the example given in Section 2, these two notions do not coincide even in $\mathcal{EL}$.

**Theorem 3.** *In $\mathcal{EL}$, $\mathcal{IQ}$-relevance and $\mathcal{CQ}$-relevance can be decided in* PTIME.

*Proof.* We consider $\mathcal{IQ}$-relevance. Let $\mathcal{T}$ be an $\mathcal{EL}$-TBox and $\Sigma$ a signature. Define the *total $\Sigma$-ABox* as $\mathcal{A}_\Sigma := \{A(a_\Sigma) \mid A \in \Sigma\} \cup \{r(a_\Sigma, a_\Sigma) \mid r \in \Sigma\}$.

**Claim**. For all concept names $A$ and role names $r$, $A$ is $\mathcal{IQ}$-relevant for $\Sigma$ given $\mathcal{T}$ iff $\mathcal{T}, \mathcal{A}_\Sigma \models A(a_\Sigma)$;

Since the instance problem can be solved in polynomial time in $\mathcal{EL}$ [15], Theorem 3 is an immediate consequence of the claim.

The "if" direction of the above claim is trivial. For the "only if" direction, let $A$ be $\mathcal{CQ}$-relevant for $\Sigma$ given $\mathcal{T}$. By lemma 4, there is a positive $\Sigma$-ABox $\mathcal{A}$ such that $\mathcal{T}, \mathcal{A} \models A(a_0)$ for some $a_0 \in \mathsf{Ind}(\mathcal{A})$. Let $\mathcal{I}$ be a model of $\mathcal{T}$ and $\mathcal{A}_\Sigma$. We have to show that $a_0^{\mathcal{I}} \in A^{\mathcal{I}}$. Modify $\mathcal{I}$ by setting $b^{\mathcal{I}} := a_\Sigma^{\mathcal{I}}$ for all individual names $b$. It is easy to verify that $\mathcal{I}$ is a model of the positive ABox $\mathcal{A}$ and of $\mathcal{T}$. Since $\mathcal{T}, \mathcal{A} \models A(a_0)$, we have $a_0^{\mathcal{I}} \in A^{\mathcal{I}}$ as required. □

Note that we need very little for the proof of Theorem 3 to go through: it suffices that $\mathcal{A}_\Sigma$ is consistent with every TBox and that the DL in question is monotone. It follows that for all DLs of this sort, deciding $\mathcal{IQ}$- and $\mathcal{CQ}$-relevance has the same complexity as subsumption/instance checking (whose complexity coincides for almost every DL). The upper bound is obtained as in the proof of Theorem 3,

based on instance checking. For the lower bound, note that $C$ is subsumed by $D$ w.r.t. $\mathcal{T}$ iff $B$ is $\mathcal{IQ}$-/$\mathcal{CQ}$-relevant for $\mathcal{T} \cup \{A \sqsubseteq C, D \sqsubseteq B\}$ and the signature $\{A\}$, where $A, B \notin \mathsf{sig}(C, D, \mathcal{T})$. We thus obtain the following result for the DL $\mathcal{ELI}$, in which subsumption and instance checking are ExpTime-complete [16].

**Theorem 4.** *In $\mathcal{ELI}$, $\mathcal{IQ}$-relevance and $\mathcal{CQ}$-relevance are* ExpTime*-complete.*

The simplest extension of $\mathcal{EL}$ in which the total ABox $\mathcal{A}_\Sigma$ is not consistent w.r.t. every TBox is $\mathcal{EL}_\bot$. Here, deciding relevance is significantly harder than deciding subsumption/instance checking (which can be decided in polynomial time). We start by proving an NP lower bound for a very simple fragment of $\mathcal{EL}_\bot$: let $\mathcal{L}$ be the DL that admits only CIs of the form $A \sqcap A' \sqsubseteq B$ and $A \sqcap B \sqsubseteq \bot$, with $A$, $A'$, and $B$ concept names. This is a fragment of $\mathcal{EL}_\bot$, but also of those variants of DL-Lite that admit conjunction on the left-hand side of CIs [8].

**Theorem 5.** *In $\mathcal{L}$, $\mathcal{IQ}$-relevance and $\mathcal{CQ}$-relevance are* NP*-hard.*

*Proof.* Reduction from SAT. Let $\varphi$ be a propositional formula in NNF using variables $v_0, \dots, v_n$ and $\mathsf{sub}(\varphi)$ the set of subformulas of $\varphi$. Define a TBox $\mathcal{T}$ as the union of the following:

- $A_{v_i} \sqcap A_{\neg v_i} \sqsubseteq \bot$ for all $i \leq n$;
- $A_\vartheta \sqcap A_\chi \sqsubseteq A_\psi$ for all $\psi = \vartheta \wedge \chi \in \mathsf{sub}(\varphi)$;
- $A_\vartheta \sqsubseteq A_\psi,\ A_\chi \sqsubseteq A_\psi$ for all $\psi = \vartheta \vee \chi \in \mathsf{sub}(\varphi)$.

Let $\Sigma = \{A_{v_i}, A_{\neg v_i} \mid i \leq n\}$. It can be verified that $A_\varphi$ is $\mathcal{IQ}$-relevant for $\Sigma$ given $\mathcal{T}$ iff $\varphi$ is satisfiable. □

For full $\mathcal{EL}_\bot$, Theorem 5 can be improved to an ExpTime lower bound. The idea is to make use of an existing ExpTime lower bound for deciding conservative extensions in $\mathcal{EL}/\mathcal{EL}_\bot$ established in [17]. To implement this, we first establish a technical proposition. Its proof is similar to Lemma 22 (i) in [17].

**Proposition 1.** *If a concept name $B$ is $\mathcal{IQ}$-relevant for a signature $\Sigma$ given an $\mathcal{EL}_\bot$-TBox $\mathcal{T}$, then there is a $\Sigma$-concept $C$ such that $C$ is satisfiable w.r.t. $\mathcal{T}$ and $\mathcal{T} \models C \sqsubseteq B$.*

*Proof.* Refer to Appendix A.

We now prove the lower bound.

**Theorem 6.** *In $\mathcal{EL}_\bot$, $\mathcal{IQ}$-relevance and $\mathcal{CQ}$-relevance are* ExpTime*-hard.*

*Proof.* We consider $\mathcal{IQ}$-relevance. The following result can be established by carefully analyzing the reduction underlying Theorem 36 in [17]: given an $\mathcal{EL}_\bot$-TBox $\mathcal{T}$, a signature $\Sigma$, and a concept name $B$, it is ExpTime-hard to decide if there exist a $\Sigma$-concept $C$ such that $C$ is satisfiable w.r.t. $\mathcal{T}$ and $\mathcal{T} \models C \sqsubseteq B$. Thus it suffices to show that the following conditions are equivalent, for any $\mathcal{EL}_\bot$-TBox $\mathcal{T}$, signature $\Sigma$, and concept name $B$:

1. there exists a $\Sigma$-concept $C$ such that $C$ is satisfiable w.r.t. $\mathcal{T}$ and $\mathcal{T} \models C \sqsubseteq B$;
2. there exists a $\Sigma$-ABox $\mathcal{A}$ such that $(\mathcal{T}, \mathcal{A})$ is consistent and $(\mathcal{T}, \mathcal{A}) \models B(a)$ for some $a \in \mathsf{Ind}(\mathcal{A})$.

The implication from Point 1 to Point 2 is trivial and the reverse direction is established by Proposition 1. □

To prove a matching upper bound for Theorem 6, we first establish a proposition that constrains the shape of ABoxes to be considered when deciding relevance in $\mathcal{EL}_\bot$. Here and in what follows, an ABox $\mathcal{A}$ is *tree-shaped* if

1. the directed graph $(\mathsf{Ind}(\mathcal{A}), \{(a,b) \mid r(a,b) \in \mathcal{A} \text{ for some } r \in \mathsf{N_R}\})$ is a tree and
2. for all $a, b \in \mathsf{Ind}(\mathcal{A})$, there is at most one role name $r$ such that $r(a,b) \in \mathcal{A}$.

The following is a simple consequence of Proposition 1.

**Proposition 2.** *A concept name $A$ is $\mathcal{IQ}$-relevant for a signature $\Sigma$ given an $\mathcal{EL}_\bot$-TBox $\mathcal{T}$ iff there is a tree-shaped ABox $\mathcal{A}$ such that $\mathcal{A}$ is consistent w.r.t. $\mathcal{T}$ and $\mathcal{T}, \mathcal{A} \models A(a_0)$, with $a_0$ the root of $\mathcal{A}$.*

For the upper bound, we use non-deterministic bottom-up automata on finite, ranked trees. Such an automaton is a tuple $\mathfrak{A} = (Q, \mathcal{F}, Q_f, \Theta)$, where $Q$ is a finite set of *states*, $\mathcal{F}$ is a *ranked alphabet*, $Q_f \subseteq Q$ is a set of *final states*, and $\Theta$ is a set of *transition rules* of the form $f(q_1, \ldots, q_n) \to q$, where $n \geq 0$, $f \in \mathcal{F}$ is of rank $n$, and $q_1, \ldots, q_n, q \in Q$. Note that transition rules for symbols of rank 0 replace initial states.

Automata work on finite, node-labeled, ordered trees $T = (V, E, \ell)$, where $V$ is a finite set of nodes, $E \subseteq V \times V$ is a set of edges, and $\ell$ is a node-labeling function the maps each node $v \in V$ with $i$ successors to a symbol $\ell(v) \in \mathcal{F}$ of rank $i$. We assume an implicit total order on the successors of each node. A *run* of the automaton $\mathfrak{A}$ on $T$ is a map $\rho : V \to Q$ such that

- $\rho(\varepsilon) \in Q_f$, with $\varepsilon \in V$ the root of $T$;
- for all $v \in V$ with $\ell(v) = f$ and where $v$ has (ordered) successors $v_1, \ldots, v_n$, $n \geq 0$, we have that $f(\rho(v_1), \ldots, \rho(v_n)) \to \rho(v)$ is a rule in $\Delta$.

An automaton $\mathfrak{A}$ *accepts* a tree $T$ if there is a run of $\mathfrak{A}$ on $T$. We use $L(\mathfrak{A})$ to denote the set of all trees accepted by $\mathfrak{A}$. It can be computed in polynomial time whether $L(\mathfrak{A}) = \emptyset$.

Let $\mathcal{T}$ be an $\mathcal{EL}_\bot$-TBox, $\Sigma$ a signature, and $A_0$ a concept name such that it is to be decided whether $A_0$ is $\mathcal{IQ}$-relevant for $\Sigma$ given $\mathcal{T}$. W.l.o.g., we may assume that $A_0$ occurs in $\mathcal{T}$. We use $\mathsf{sub}(\mathcal{T})$ to denote the set of all subconcepts of concepts occurring in $\mathcal{T}$ and set $\Gamma := \Sigma \cup \mathsf{sub}(\mathcal{T})$. A *$\Sigma$-type* is a finite set $t$ of concept names that occur in $\Sigma$ and such that $\bigsqcap t$ is satisfiable w.r.t. $\mathcal{T}$. A *$\Gamma$-type* is a subset $t$ of $\Gamma$ such that $\bigsqcap t$ is satisfiable w.r.t. $\mathcal{T}$. Given a $\Gamma$-type $t$, we use $\mathsf{cl}_\mathcal{T}(t)$ to denote the set $\{C \in \Gamma \mid \mathcal{T} \models \bigsqcap t \sqsubseteq C\}$. We use $\mathsf{ex}(\mathcal{T})$ to denote the number of concepts of the form $\exists r.C$ that occur in $\mathcal{T}$ (possibly as a subconcept). Define an automaton $\mathfrak{A} = (Q, \mathcal{F}, Q_f, \Delta)$ as follows:

– $\mathcal{F} = \{\langle t, r_1, \ldots, r_n\rangle \mid t \text{ a } \Sigma\text{-type}, i < \mathsf{ex}(\mathcal{T})\}$ with $\langle t, r_1, \ldots, r_n\rangle$ of rank $n$;
– $Q$ is the set of $\Gamma$-types;
– $Q_f = \{q \in Q \mid A_0 \in q\}$;
– $\Delta$ consists of all rules $f(q_1, \ldots, q_n) \to q$ with $f = \langle t, r_1, \ldots, r_n\rangle$ such that

$$q = \mathsf{cl}_{\mathcal{T}}(t \cup \{\exists r.C \in \mathsf{sub}(\mathcal{T}) \mid r = r_i \text{ and } C \in q_i \text{ for some } i \text{ with } 1 \leq i \leq n\}.$$

**Lemma 5.** *$L(\mathfrak{A}) \neq \emptyset$ iff $A_0$ is $\mathcal{IQ}$-relevant for $\Sigma$ given $\mathcal{T}$.*

*Proof.* Refer to Appendix A.

Since $\mathfrak{A}$ is single-exponentially large in $|\mathcal{T}|$ and the emptiness problem can be decided in polynomial time in the size of the automaton, we obtain a single-exponential-time procedure for deciding relevance in $\mathcal{EL}_\bot$.

**Theorem 7.** *In $\mathcal{EL}_\bot$, $\mathcal{IQ}$-relevance and $\mathcal{CQ}$-relevance are* ExpTime*-complete.*

## 5 Expressive DLs

We establish some first results for ABox relevance in $\mathcal{ALC}$ and its extensions. For $\mathcal{ALCI}$, we prove decidability of $\mathcal{IQ}$- (and thus also $\mathcal{CQ}$-) relevance, and a NExpTime$^{\text{NP}}$ upper bound; for $\mathcal{ALCF}$, we prove undecidability of $\mathcal{IQ}$-relevance.

### 5.1 ABox Relevance in $\mathcal{ALC}$ and $\mathcal{ALCI}$

The NExpTime$^{\text{NP}}$ upper bound is based on the following theorem, which places an upper bound on the size of ABoxes that we need to consider.

**Theorem 8.** *Let $\mathcal{T}$ be an $\mathcal{ALCI}$-TBox. If $A \in \mathsf{N_C}$ is $\mathcal{IQ}$-relevant for $\mathcal{T}$ w.r.t. propositional ABoxes, then there is a literal $\Sigma$-ABox $\mathcal{A}$ such that $\mathcal{A}$ is consistent w.r.t. $\mathcal{T}$, $\mathcal{T}, \mathcal{A} \models A(a)$ for some $a \in \mathsf{Ind}(\mathcal{A})$, and $|\mathsf{Ind}(\mathcal{A})| \leq 2^{|\mathcal{T}|+|\Sigma|}$.*

*Proof.* We do not make the UNA. We consider only the case $A \in \mathsf{N_C}$, as the case $r \in \mathsf{N_R}$ is analogous. Assume that $A$ is $\mathcal{IQ}$-relevant for $\Sigma$ given $\mathcal{T}$. Then there is a literal $\Sigma$-ABox $\mathcal{A}$ such that $\mathcal{A}$ is consistent w.r.t. $\mathcal{T}$ and $\mathcal{T}, \mathcal{A} \models A(a_0)$ for some $a_0 \in \mathsf{Ind}(\mathcal{A})$. Let $\mathcal{I}$ be a model of $\mathcal{A}$ and $\mathcal{T}$, and let $\mathcal{J}$ be the filtration of $\mathcal{I}$ w.r.t. $\Gamma = \mathsf{cl}(\mathcal{T}) \cup \{A, \neg A \mid A \in \Sigma\}$, i.e., define an equivalence relation $\sim \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ by setting $d \sim e$ iff

$$\{C \mid C \in \Gamma \wedge d \in C^{\mathcal{I}}\} = \{C \mid C \in \Gamma \wedge e \in C^{\mathcal{I}}\}$$

and set

$$\begin{aligned}
\Delta^{\mathcal{J}} &:= \{[d] \mid d \in \Delta^{\mathcal{I}}\} \\
A^{\mathcal{J}} &:= \{[d] \mid d \in A^{\mathcal{I}}\} \\
r^{\mathcal{J}} &:= \{([d], [e]) \mid \exists d' \in [d], e' \in [e] : (d', e') \in r^{\mathcal{I}}\} \\
a^{\mathcal{J}} &:= [a^{\mathcal{I}}]
\end{aligned}$$

Clearly, $|\Delta^{\mathcal{J}}| \leq 2^{|\mathcal{T}|+|\Sigma|}$. It is routine to prove that $\mathcal{J}$ is a model of $\mathcal{T}$. Define an ABox

$$\begin{aligned}\mathcal{A}_{\mathcal{J}} = &\{A(a_{[d]}) \mid A \in \Sigma \wedge [d] \in A^{\mathcal{J}}\} \cup \\ &\{\neg A(a_{[d]}) \mid A \in \Sigma \wedge [d] \in (\neg A)^{\mathcal{J}}\} \cup \\ &\{r(a_{[d]}, a_{[e]}) \mid r \in \Sigma \wedge ([d],[e]) \in r^{\mathcal{J}}\}.\end{aligned}$$

Clearly, $\mathcal{J}$ is a model of $\mathcal{A}_{\mathcal{J}}$. Thus, $\mathcal{A}_{\mathcal{J}}$ is consistent w.r.t. $\mathcal{T}$. It remains to show that $\mathcal{T}, \mathcal{A}_{\mathcal{J}} \models A(a_{[a_0^{\mathcal{I}}]})$. Let $\mathcal{J}'$ be a model of $\mathcal{A}_{\mathcal{J}}$ and $\mathcal{T}$. Define a model $\mathcal{I}'$ from $\mathcal{J}'$ by setting $a^{\mathcal{I}'} = (a_{[a^{\mathcal{I}}]})^{\mathcal{J}'}$ for all $a \in \mathsf{Ind}(\mathcal{A})$. It is readily checked that $\mathcal{I}'$ is a model of $\mathcal{A}$ and $\mathcal{T}$, and thus $a_0 \in A^{\mathcal{I}'}$, implying $a^{\mathcal{J}'}_{[a_0^{\mathcal{I}}]} \in A^{\mathcal{J}'}$ as required. □

It is interesting to note that the bound from Theorem 8 is tight. To see this, let

$$\begin{aligned}\mathcal{T} := \{ \quad A &\sqsubseteq \neg P_0 \sqcap \cdots \sqcap \neg P_{n-1} \\ \exists r.(P_0 \sqcap \cdots \sqcap P_i) &\sqsubseteq \neg P_i \\ \exists r.(P_0 \sqcap \cdots \sqcap P_{i-1} \sqcap \neg P_i) &\sqsubseteq P_i \\ \exists r.((\neg P_0 \sqcup \cdots \sqcup \neg P_{i-1}) \sqcap P_i) &\sqsubseteq P_i \\ \exists r.((\neg P_0 \sqcup \cdots \sqcup \neg P_{i-1}) \sqcap \neg P_i) &\sqsubseteq \neg P_i \\ P_0 \sqcap \cdots \sqcap P_{n-1} &\sqsubseteq X \ \}\end{aligned}$$

and $\Sigma = \{A, r\}$. Then $X$ is relevant for $\mathcal{T}$ and $\Sigma$, but the smallest witness ABox is an $r$-chain of length $2^n$ whose last element is an instance of $A$. Note that an ABox that has the form of a cycle of length $< 2^n$ is inconsistent w.r.t. $\mathcal{T}$. We now use Theorem 8 to prove membership in NExpTime$^{\text{NP}}$.

**Theorem 9.** *In $\mathcal{ALCI}$, $\mathcal{IQ}$-relevance and $\mathcal{CQ}$-relevance are in* NExpTime$^{\text{NP}}$.

*Proof.* We show the result for $\mathcal{IQ}$-relevance; the upper bound for $\mathcal{CQ}$-relevance follows by Theorem 2. Consider the following nondeterministic algorithm:

**Step 1:** Guess a $\Sigma$-ABox $\mathcal{A}$ such that $|\mathsf{Ind}(\mathcal{A})| = 2^{|\mathcal{T}|+|\Sigma|}$.

**Step 2:** Use an oracle to verify that $\mathcal{A}$ is consistent with $\mathcal{T}$. Reject if not.

**Step 3:** For each $a \in \mathsf{Ind}(\mathcal{A})$, use an oracle to check whether $\mathcal{A} \cup \{\neg A(a)\}$ is consistent with $\mathcal{T}$. Accept if for some $a \in \mathsf{Ind}(\mathcal{A})$ the ABox $\mathcal{A} \cup \{\neg A(a)\}$ is inconsistent with $\mathcal{T}$. Otherwise reject.

If the algorithm accepts, then we have found a $\Sigma$-ABox $\mathcal{A}$ consistent with $\mathcal{T}$ which implies some assertion $A(a)$, i.e. $A$ is $\mathcal{IQ}$-relevant for $\Sigma$ given $\mathcal{T}$. Conversely, if $A$ is $\mathcal{IQ}$-relevant, then by Theorem 8, there must be some $\Sigma$-ABox $\mathcal{A}$ with at most $2^{|\mathcal{T}|+|\Sigma|}$ individuals which is consistent with $\mathcal{T}$ and such that $\mathcal{T}, \mathcal{A} \models A(a)$ for some $a \in \mathsf{Ind}(\mathcal{A})$. We create a new $\Sigma$-ABox from $\mathcal{A}$ as follows: $\mathcal{A}' = \mathcal{A} \cup \{\top(b_i) \mid 1 \leq i \leq 2^{|\mathcal{T}|+|\Sigma|} - |\mathsf{Ind}(\mathcal{A})|\}$. By construction, $\mathcal{A}'$ has precisely $2^{|\mathcal{T}|+|\Sigma|}$ individuals, is consistent with $\mathcal{T}$, and is such that $\mathcal{A}', \mathcal{T} \models A(a)$. If $\mathcal{A}'$ is guessed in Step 1, the algorithm accepts.

We remark that in Steps 2 and 3 of the algorithm, we test the consistency of literal ABoxes that are exponentially larger than the TBox $\mathcal{T}$. Because of this, the standard precompletion approach to deciding ABox consistency w.r.t.

a TBox requires only nondeterministic polynomial time (rather than the usual deterministic single-exponential time). This means that we can use an NP-oracle in Steps 2 and 3, yielding membership in $\text{NExpTime}^{\text{NP}}$. □

We conjecture that $\mathcal{IQ}$- and $\mathcal{CQ}$-relevance are actually $\text{NExpTime}^{\text{NP}}$-complete, but leave the lower bound open for now.

### 5.2 ABox Relevance in $\mathcal{ALCF}$

We show that the simple addition of functional roles to $\mathcal{ALC}$ leads to undecidability of $\mathcal{IQ}$-relevance. We speculate that the same result might hold for $\mathcal{CQ}$-relevance.

The proof is by reduction of the tiling problem of finite (but unbounded) rectangles. An instance of this problem is given by a triple $(\mathfrak{T}, H, V)$ with $\mathfrak{T}$ a non-empty, finite set of *tile types* including an *initial tile* $T_{\mathsf{init}}$ to be placed on the lower left corner and a *final tile* $T_{\mathsf{final}}$ to be placed on the upper right corner, $H \subseteq \mathfrak{T} \times \mathfrak{T}$ a *horizontal matching relation*, and $V \subseteq \mathfrak{T} \times \mathfrak{T}$ a *vertical matching relation*. A *tiling* for $(\mathfrak{T}, H, V)$ is a map $f : \{0, \dots, n\} \times \{0, \dots, m\} \to \mathfrak{T}$ such that $n, m \geq 0$, $f(0,0) = T_{\mathsf{init}}$, $f(n,m) = T_{\mathsf{final}}$, $(f(i,j), f(i+1,j)) \in H$ for all $i < n$, and $(f(i,j), f(i,j+1)) \in v$ for all $i < m$. It is undecidable whether a tiling problem has a tiling.

For the reduction, let $(\mathfrak{T}, H, V)$ be an instance of the tiling problem with $\mathfrak{T} = \{T_1, \dots, T_p\}$. We construct a signature $\Sigma$ and a TBox $\mathcal{T}$ such that $(\mathfrak{T}, H, V)$ has a solution iff a selected concept name $A$ is $\mathcal{IQ}$-relevant for $\Sigma$ given $\mathcal{T}$. More precisely, the ABox $\mathcal{A}$ witnessing $\mathcal{IQ}$-relevance has the form of an $n \times m$-rectangle together with a tiling for $(\mathfrak{T}, H, V)$. W.l.o.g., we concentrate on solutions where $T_{\mathsf{final}}$ occurs nowhere else than in the upper right corner. The ABox signature is

$$\Sigma = \{T_1, \dots, T_p, x, y\}$$

where $T_1, \dots, T_p$ are used as concept names and $x$ and $y$ are *functional* role names representing horizontal and vertical adjacency of points in the rectangle. In $\mathcal{T}$ (and $\mathcal{A}$), we additionally use the concept names $U, R, A, Y, C$, where $U$ and $R$ mark the upper and right border of the rectangle, $A$ is the concept name used in the instance query, and $Y$ and $C$ are used for technical purposes explained below. More precisely, $\mathcal{T}$ is defined as the union of the following CIs, for all $(T_i, T_j) \in H$ and $(T_i, T_\ell) \in V$:

$$\begin{aligned}
T_{\mathsf{final}} &\sqsubseteq Y \sqcap U \sqcap R \\
\exists x.(U \sqcap T_j) \sqcap T_i &\sqsubseteq U \sqcap Y \\
\exists y.(R \sqcap T_\ell) \sqcap T_i &\sqsubseteq R \sqcap Y \\
(\exists x.\exists y.Z \sqcap \exists y.\exists x.Z) \sqcup (\exists x.\exists y.\neg Z \sqcap \exists y.\exists x.\neg Z) &\sqsubseteq C \\
\exists x.(T_j \sqcap Y \sqcap \exists y.Y) \sqcap \exists y.(T_\ell \sqcap Y \sqcap \exists x.Y) \sqcap C \sqcap T_i &\sqsubseteq Y \\
Y \sqcap T_{\mathsf{init}} &\sqsubseteq A \\
\\
U &\sqsubseteq \forall y.\bot \\
R &\sqsubseteq \forall x.\bot \\
\bigsqcup_{1 \leq s < t \leq p} T_s \sqcap T_t &\sqsubseteq \bot
\end{aligned}$$

Observe that the concept name $A$ used in the instance query occurs only once in the TBox, on the right-hand side of a CI. Taken together, the upper part of $\mathcal{T}$ ensures the existence of a tiled $n \times m$-rectangle in a witness ABox. The concept name $Y$ is entailed at every individual name in such an ABox that is part of the rectangle. Observe that the CIs for $Y$ enforce the horizontal and vertical matching conditions. The CI for $C$ enforces confluence, i.e., $C$ is entailed at an individual name $a$ if there is an individual $b$ that is both an $x$-$y$-successor and a $y$-$x$-successor of $a$. This is so because, intuitively, $Z$ is universally quantified: if confluence fails, we can interpret $C$ in a way such that neither of the two disjuncts in the pre-condition of the CI for $C$ is satisfied.

**Lemma 6.** *There is a tiling for $(\mathfrak{T}, H, V)$ iff there exists a $\Sigma$-ABox $\mathcal{A}$ that is consistent with $\mathcal{T}$ and such that $\mathcal{T}, \mathcal{A} \models A(a)$ for some $a$.*

*Proof.* Refer to Appendix A.

Undecidability of $\mathcal{IQ}$-relevance follows directly from Lemma 6.

**Theorem 10.** *In $\mathcal{ALCF}$, $\mathcal{IQ}$-relevance is undecidable.*

## 6 Related Work

Several notions of relevance have been previously proposed in the philosophy and artificial intelligence literatures, but they are rather different in nature from the notion of relevance we study in this paper. For example, in the area of relevant logic [18], it is an inference, rather than a symbol, which is said to be relevant, and in the work of Levy et al. [19] it is a premise of a proof which may or may not be relevant to the deduction of a given formula. Relevance of a *signature* (hence symbol) can be found in Lakemeyer's study of relevance [20], in which he defines relevance of a signature to a formula given a theory as well as relevance of two signatures to each other given a theory. However, Lakemeyer's notions of relevance are defined only for propositional logic, and even in the case of propositional theories, do not appear to bear any relationship to ABox relevance as studied in this paper.

## 7 Conclusion

We have introduced a new notion of relevance that describes when a symbol can be used meaningfully in queries that are posed to ABoxes formulated in a given signature, with a given background TBox in place. We have established a relatively complete picture regarding the complexity of deciding $\mathcal{IQ}$- and $\mathcal{CQ}$-relevance in the $\mathcal{EL}$ family of lightweight DLs, and some first results for DLs of the $\mathcal{ALC}$ family. Some important open questions have been pointed out in the paper, most notably the exact complexity of relevance in $\mathcal{ALC}$ and $\mathcal{ALCI}$, and the decidability of $\mathcal{CQ}$-relevance in $\mathcal{ALCF}$. Another open issue is the formulation of a notion of relevance for queries that may contain composite concepts. This

is not trivial due to the possibility of using tautological concepts in the query. Finally, we are currently investigating whether the set of relevant symbols as defined in this paper can be used to obtain more efficient algorithms for module extraction.

## A Proofs

*Proof of Proposition 1.* Let $A$ be $\mathcal{IQ}$-relevant for $\Sigma$ given $\mathcal{T}$. We start by establishing the following technical claim.

**Claim**. If $\Gamma$ is an (infinite) set of $\mathcal{EL}_\bot$ concepts that is closed under conjunction and satisfiable w.r.t. a TBox $\mathcal{T}$, then there is a model $\mathcal{I}$ of $\mathcal{T}$ and a $d_\mathcal{I} \in \Delta^\mathcal{I}$ such that for all concepts $D$, $d_\mathcal{I} \in D^\mathcal{I}$ iff $\mathcal{T} \models C \sqsubseteq D$ for some $C \in \Gamma$.

To prove the claim, choose an individual name $b$ and let $\mathcal{B}$ be the infinite ABox $\{D(b) \mid D \in \Gamma\}$. By Lemma 3, there is a model $\mathcal{I}_{\mathcal{T},\mathcal{B}}$ of $\mathcal{T}$ and $\mathcal{I}$ such that for any $\mathcal{EL}_\bot$ concept $D$, we have $b^{\mathcal{I}_{\mathcal{T},\mathcal{B}}} \in D^{\mathcal{I}_{\mathcal{T},\mathcal{B}}}$ iff $\mathcal{T}, \mathcal{A} \models D(b)$. By definition of $\mathcal{B}$, closure of $\Gamma$ under conjunction and compactness, we further have $\mathcal{T}, \mathcal{A} \models D(b)$ iff $\mathcal{T} \models C \sqsubseteq D$ for some $C \in \Gamma$. This establishes the claim since we can choose $\mathcal{I} = \mathcal{I}_{\mathcal{T},\mathcal{B}}$ and $d_\mathcal{I} = b^{\mathcal{I}_{\mathcal{T},\mathcal{A}}}$.

Now choose an ABox $\mathcal{A}$ and individual name $a_0 \in \mathsf{Ind}(\mathcal{A})$ such that $\mathcal{T}, \mathcal{A} \models B(a_0)$. Assume to the contrary of what is to be shown that there does not exist a $\Sigma$-concept $C$ which is satisfiable w.r.t. $\mathcal{T}$ and such that $\mathcal{T} \models C \sqsubseteq B$. For each individual name $a$ in $\mathcal{A}$, let $t_a$ denote the set of $\Sigma$-concepts $C$ such that $\mathcal{T}, \mathcal{A} \models C(a)$ and let $\mathcal{I}_a$ be a model as in the above claim with $\Gamma = t_a$. By our assumption and the choice of $\mathcal{I}_a$, we have $d_{\mathcal{I}_{a_0}} \notin B^{\mathcal{I}_{a_0}}$.

We may assume that the $\mathcal{I}_a$ are mutually disjoint. Take the following union $\mathcal{I}$ of the models $\mathcal{I}_a$:

- $\Delta^\mathcal{I} = \bigcup_{a \in \mathsf{Ind}(\mathcal{A})} \Delta^{\mathcal{I}_a}$;
- $A^\mathcal{I} = \bigcup_{a \in \mathsf{Ind}(\mathcal{A})} A^{\mathcal{I}_a}$, for $A \in \mathsf{N_C}$;
- $r^\mathcal{I} = \bigcup_{a \in \mathsf{Ind}(\mathcal{A})} r^{\mathcal{I}_a} \cup \{(d_{\mathcal{I}_a}, d_{\mathcal{I}_b}) \mid r(a,b) \in \mathcal{A}\}$, for $r \in \mathsf{N_R}$;
- $a^\mathcal{I} = d_{\mathcal{I}_a}$, for $a \in \mathsf{Ind}(\mathcal{A})$.

For all $\mathcal{EL}$-concepts $C$, $a \in \mathsf{Ind}(\mathcal{A})$, and $d \in \Delta^{\mathcal{I}_a}$, we have

$(*)$ $d \in C^{\mathcal{I}_a}$ iff $d \in C^\mathcal{I}$.

The proof is by induction on the structure of $C$. The only interesting case is $C = \exists r.D$ and the direction from right to left. Assume $d \in C^\mathcal{I} \cap \Delta^{\mathcal{I}_a}$. For $d \neq d_{\mathcal{I}_a}$, $d \in C^{\mathcal{I}_a}$ follows immediately by IH. Assume $d = d_{\mathcal{I}_a}$. Take $d'$ with $(d, d') \in r^\mathcal{I}$ and $d' \in D^\mathcal{I}$. Again, if $d' \in \Delta^{\mathcal{I}_a}$, then the claim follows immediately from the IH. Now assume $d' \notin \Delta^{\mathcal{I}_a}$. Then $d' = b$ for some $b$ with $r(a,b) \in \mathcal{A}$. By IH, $d' \in D^{\mathcal{I}_b}$. Hence $\mathcal{T} \models E \sqsubseteq D$ for some $E \in t_b$. Then $\mathcal{T}, \mathcal{A} \models E(b)$, Since $r(a,b) \in \mathcal{A}$, we obtain $\mathcal{T}, \mathcal{A} \models \exists r.D(a)$. Therefore, $\exists r.D \in t_a$. Thus, the choice of $\mathcal{I}_a$ yields $d_{\mathcal{I}_a} \in C^{\mathcal{I}_a}$ as required.

By $(*)$ and since the $\mathcal{I}_a$ are models for $\mathcal{T}$, $\mathcal{I}$ is also a model of $\mathcal{T}$. Moreover, $\mathcal{I}$ by definition satisfies all role assertions in $\mathcal{A}$ and all concept assertions are

satisfied by $(*)$ and since $A(a) \in \mathcal{A}$ implies $A \in t_a$ and thus $d_{\mathcal{I}} \in A^{\mathcal{I}}$. Since $d_{\mathcal{I}_{a_0}} \notin B^{\mathcal{I}_{a_0}}$, $(*)$ also yields $a_0^{\mathcal{I}} \notin B^{\mathcal{I}}$. Summing up, we have shown that $\mathcal{T}, \mathcal{A} \not\models B(a_0)$, which is a contradiction. □

*Proof of Lemma 5.* For the "if" direction, assume that $A_0$ is relevant for $\Sigma$ given $\mathcal{T}$. By Proposition 2, there is a tree-shaped and consistent ABox $\mathcal{A}$ with root $a_0$ such that $\mathcal{T}, \mathcal{A} \models A_0(a_0)$. Define a mapping $\rho$ that maps each $a \in \mathsf{Ind}(\mathcal{A})$ to the $\Gamma$-type $\rho(a) := \{C \in \Gamma \mid \mathcal{T}, \mathcal{A} \models C(a)\}$. We now inductively choose a subset $R \subseteq \mathsf{Ind}(\mathcal{A})$ as follows:

- Initially, set $R := \{a_0\}$;
- if $a \in R$ is not a leaf in $\mathcal{A}$ and no successors of $a$ in $\mathcal{A}$ are yet in $R$, then choose for each $\exists r.C \in \rho(a)$ an $a' \in \mathsf{Ind}(\mathcal{A})$ with $r(a, a') \in \mathcal{A}$ and $C \in \rho(a')$ if such an $a'$ exists, and add $a'$ to $R$.

For each node in $\mathcal{A}$, fix a total order on the successors that are in $R$ (call this an "$R$-successor"). For $a \in \mathsf{Ind}(\mathcal{A})$, we use $\sigma_{\mathcal{A}}(a)$ to denote the set $\{A \mid A(a) \in \mathcal{A}\}$. Define a tree $T = (V, E, \ell)$ as follows:

- $V = R$;
- $E = \{(a, b) \in V \times V \mid r(a, b) \in \mathcal{A}\}$ and the order of successor in $T$ agrees with the chosen order on $R$-successors in $\mathcal{A}$;
- $\ell(a) = \langle \sigma_{\mathcal{A}}(a), r_1, \ldots, r_n \rangle$ where $r_i$ is the (unique!) role such that $r_i(a, a_i) \in \mathcal{A}$, with $a_i$ the $i$-th $R$-successor of $a$ .

Let $\rho'$ be the restriction of $\rho$ to $R$. We show that $r'$ is a run of $\mathfrak{A}$ on $T$. First, note that $\mathcal{T}, \mathcal{A} \models A_0(a_0)$ implies $A_0 \in \rho'(a_0)$, and thus $\rho'(a_0) \in Q_f$ (observe that $a_0$ is the root of $T$). It thus remains to show that for all $a \in V$ with successors $a_1, \ldots, a_n$ that are connected in $\mathcal{A}$ via $r_1, \ldots, r_n$, respectively, we have $\rho'(a) = \Omega$, where

$$\Omega = \mathsf{cl}_{\mathcal{T}}(\sigma_{\mathcal{A}}(a) \cup \{\exists r.C \in \mathsf{sub}(\mathcal{T}) \mid r = r_i \text{ and } C \in \rho'(a_i) \text{ for some } 1 \leq i \leq n\}).$$

The "$\supseteq$" direction is immediate by the definitions of $\rho'$, $\sigma_{\mathcal{A}}$, and $\mathsf{cl}_{\mathcal{T}}$ and the semantics. For the "$\subseteq$" direction, assume that $C_0 \in \Gamma \setminus \Omega$. Let $b_1, \ldots, b_m$ be the successors of $a$ in $\mathcal{A}$ (including those not in $R$) and for $1 \leq i \leq m$, let $\mathcal{A}_i$ be the restriction of $\mathcal{A}$ to $b_i$ and all individual names that are reachable from it in $\mathcal{A}$. The proof of the following property is by a relatively straightforward model-theoretic construction based on canonical models. Details are left to the reader.

$(*)$ for all $C \in \Gamma$ and $b \in \mathsf{Ind}(\mathcal{A}_i)$, we have $\mathcal{T}, \mathcal{A} \models C(b)$ iff $\mathcal{T}, \mathcal{A}_i \models C(b)$.

To continue our argument, take the interpretations $\mathcal{I}_{\mathcal{T},\mathcal{A}}$ and $\mathcal{I}_{\mathcal{T},\mathcal{A}_1}, \ldots, \mathcal{I}_{\mathcal{T},\mathcal{A}_m}$. Assume w.l.o.g. that their domains are pairwise disjoint. Define a new interpretation $\mathcal{J}$ as follows:

- $\Delta^{\mathcal{J}} = \Delta^{\mathcal{I}_{\mathcal{T},\mathcal{A}}} \uplus \Delta^{\mathcal{I}_{\mathcal{T},\mathcal{A}_1}} \uplus \cdots \uplus \Delta^{\mathcal{I}_{\mathcal{T},\mathcal{A}_m}} \uplus \{a\}$;

- $A^{\mathcal{J}} = A^{\mathcal{I}_{\mathcal{T},\mathcal{A}}} \cup A^{\mathcal{I}_{\mathcal{T},\mathcal{A}_1}} \cup \cdots \cup A^{\mathcal{I}_{\mathcal{T},\mathcal{A}_m}} \cup S$, where $S = \{a\}$ if $A \in \Omega$ and $S = \emptyset$ otherwise;
- $r^{\mathcal{J}} = r^{\mathcal{I}_{\mathcal{T},\mathcal{A}}} \cup r^{\mathcal{I}_{\mathcal{T},\mathcal{A}_1}} \cup \cdots \cup r^{\mathcal{I}_{\mathcal{T},\mathcal{A}_m}} \cup \{(d,a) \mid (d, a^{\mathcal{I}_{\mathcal{T},\mathcal{A}}}) \in r^{\mathcal{I}_{\mathcal{T},\mathcal{A}}}\} \cup$
  $\{(a, b_i^{\mathcal{I}_{\mathcal{T},\mathcal{A}_i}}) \mid 1 \leq i \leq m \text{ and } r(a,b_i) \in \mathcal{A}\}$;
- $a^{\mathcal{J}} = a$;
- for all $b \in \mathsf{Ind}(\mathcal{A}) \setminus \{a\}$, $b^{\mathcal{J}} = b^{\mathcal{I}_{\mathcal{T},\mathcal{A}_i}}$ if $b \in \mathsf{Ind}(\mathcal{A}_i)$ and $b^{\mathcal{J}} = b^{\mathcal{I}_{\mathcal{T},\mathcal{A}}}$ if $b \notin \mathsf{Ind}(\mathcal{A}_1) \cup \cdots \cup \mathsf{Ind}(\mathcal{A}_m)$.

Observe that the interpretation of individual names is well defined since $\mathcal{A}$ is tree-shaped, which implies that the sets $\mathsf{Ind}(\mathcal{A}_i)$ are pairwise disjoint.

**Claim**. For all $i$ with $1 \leq i \leq m$ and all $C \in \Gamma$, we have:

1. $d \in C^{\mathcal{J}}$ iff $d \in C^{\mathcal{I}_{\mathcal{T},\mathcal{A}_i}}$ for all $d \in \Delta^{\mathcal{I}_{\mathcal{T},\mathcal{A}_i}}$;
2. $a \in C^{\mathcal{J}}$ iff $C \in \Omega$;
3. $d \in C^{\mathcal{J}}$ iff $d \in C^{\mathcal{I}_{\mathcal{T},\mathcal{A}}}$ for all $d \in \Delta^{\mathcal{I}_{\mathcal{T},\mathcal{A}}}$.

All points are proved by induction on the structure of $C$. For Point 1, this is straightforward. For Point 2, the induction start is immediate by definition of $\mathcal{J}$. The induction step is trivial when $C$ is a conjunction. Thus, let $C = \exists r.D$. If $a \in C^{\mathcal{J}}$, then there is a $d \in D^{\mathcal{J}}$ with $(a,d) \in r^{\mathcal{J}}$. By definition of $\mathcal{J}$, we have $d = b_i^{\mathcal{I}_{\mathcal{T},\mathcal{A}_i}}$ for some $i$ with $1 \leq i \leq m$ and $r(a,b_i) \in \mathcal{A}$. By Point 1 of the claim and Point 1 of Lemma 3 and $(*)$, $\mathcal{T},\mathcal{A} \models D(b_i)$. By definition of $R$, there is a $j$ with $1 \leq j \leq n$, $r_j = r$, and $\mathcal{T},\mathcal{A} \models D(a_j)$. It follows that $\exists r.D \in \Omega$ as required. For the converse direction, let $\exists r.D \in \Omega$. Then there is an $i$ with $1 \leq i \leq n$, $r_i = r$, and $\mathcal{T},\mathcal{A} \models D(a_i)$. By $(*)$, Point 1 of Lemma 3, and Point 1 of the claim, we have $a_i^{\mathcal{J}} \in D^{\mathcal{J}}$. By definition of $r^{\mathcal{J}}$, we have $(a,a_i) \in r^{\mathcal{J}}$, thus $a \in (\exists r.D)^{\mathcal{J}}$ as required. In the induction proof of Point 3, only the "only if" direction of the case $C = \exists r.D$ is non-trivial. Let $(d,e) \in r^{\mathcal{J}}$ and $e \in D^{\mathcal{J}}$. If $e \neq a$, we only need to apply the induction hypothesis and use the semantics. Now assume that $e = a$. By Point 2, we get $D \in \Omega$, thus $\mathcal{T},\mathcal{A} \models D(a)$. It follows that $a^{\mathcal{I}_{\mathcal{T},\mathcal{A}}} \in D^{\mathcal{I}_{\mathcal{T},\mathcal{A}}}$. Since $(d,a) \in r^{\mathcal{J}}$, we have $(d, a^{\mathcal{I}_{\mathcal{T},\mathcal{A}}}) \in r^{\mathcal{I}_{\mathcal{T},\mathcal{A}}}$. Thus, $d \in (\exists r.D)^{\mathcal{I}_{\mathcal{T},\mathcal{A}}}$ by the semantics.

Since $\mathcal{I}_{\mathcal{T},\mathcal{A}}$ and the $\mathcal{I}_{\mathcal{T},\mathcal{A}_i}$ are models of $\mathcal{T}$ and $\Omega$ is closed under $\mathcal{T}$-consequence, the claim implies that $\mathcal{J}$ is a model of $\mathcal{T}$. Using the definition of $\mathcal{J}$, it is not hard to verify that $\mathcal{J}$ is also a model of $\mathcal{A}$. Since $C_0 \notin \Omega$, we have $a^{\mathcal{J}} \notin C_0^{\mathcal{J}}$ by Point 2 of the claim. It follows that $\mathcal{T},\mathcal{A} \not\models C_0(a)$. By definition of $\rho'$, we obtain $C_0 \notin \rho'(a)$ as required.

For the "only if" direction, let $T = (V,E,\ell)$ be a tree accepted by $\mathcal{A}$, and $\rho$ be a run of $\mathfrak{A}$ on $T$. Define an ABox

$$\mathcal{A} := \{A(a_v) \mid v \in V \text{ and } \ell(v) = \langle t, r_1, \ldots, r_n\rangle \text{ with } A \in t\} \cup$$
$$\{r(a_v, a_{v_i}) \mid v_i \text{ is } i\text{-th successor of } v \text{ and } \ell(v) = \langle t, r_1, \ldots, r_n\rangle \text{ with } r_i = r\}.$$

We want to show that $\mathcal{A}$ witnesses the $\mathcal{IQ}$-relevance of $A_0$ given $\mathcal{T}$. We begin by proving the consistency of $\mathcal{A}$ with respect to $\mathcal{T}$. Let us define $\Psi$ as the set of concepts $C$ which are satisfiable w.r.t. $\mathcal{T}$ and such that $\exists r.C \in \Gamma$ for some role

$r$. For each $C \in \Psi$, we let $\mathcal{J}_C$ be the canonical model of the ABox $\{B(b)\}$ and TBox $\mathcal{T} \cup \{B \equiv C\}$, and let $x_C$ be the element in $\Delta^{\mathcal{J}_C}$ with $b^{\mathcal{J}_C} = x_C$. Suppose w.l.o.g. that the universes of the $\mathcal{J}_C$ are all disjoint. We use the interpretations $\mathcal{J}_C$ to construct a new interpretation $\mathcal{I}$ as follows:

$$\Delta^{\mathcal{I}} = V \cup \bigcup_{C \in \Psi} \Delta^{\mathcal{J}_C}$$

$$A^{\mathcal{I}} = \{v \in V \mid A \in \rho(v)\} \cup \bigcup_{C \in \Psi} A^{\mathcal{J}_C}$$

$$r^{\mathcal{I}} = \{(v, w) \in E \mid w \text{ is } i\text{-th successor of } v \text{ and } \ell(v) = \langle t, r_1, ..., r_n \rangle \text{ where } r_i = r\}$$
$$\cup \{(v, x_C) \mid v \in V \text{ and } \exists r.C \in \rho(v)\} \cup \bigcup_{C \in \Psi} r^{\mathcal{J}_C}$$

$$a_v^{\mathcal{I}} = v$$

It is easy to see that $\mathcal{I}$ is a model of $\mathcal{A}$. In order to show that it is also a model of $\mathcal{T}$, we will require the following two properties:

1. $C \in \rho(v) \Rightarrow v \in C^{\mathcal{I}}$
2. $v \in C^{\mathcal{I}}$ & $C \in \Gamma \Rightarrow C \in \rho(v)$

If $C$ is an atomic concept, then (1) follows directly from the definition of $A^{\mathcal{I}}$. If $C$ is of the form $\exists r.D$, then we use the fact that $v$ is connected via $r$ to the individual $x_D$ which belongs to $D^{\mathcal{J}_C}$, hence $D^{\mathcal{I}}$. If $C = C_1 \sqcap C_2$, then both $C_1 \in \rho(v)$ and $C_2 \in \rho(v)$ (by definition of the rule set $\Theta$), so the statement follows by structural induction.

For statement (2), the proof is by induction on the co-depth of $v$. The base case is when $v$ is a leaf node. The case where $C = A$ is trivial, and the case where $C$ is a conjunction is easily dealt with by structural induction, so the only interesting case is when $C$ is of the form $\exists r.D$. In this case, $v$ must have some $r$-successor which is in $D^{\mathcal{I}}$, and since $v$ has no successors in $V$, the $r$-successor must be $x_E$ for some $E$ such that $\exists r.E \in \rho(v)$. Now since $x_E \in D^{\mathcal{I}}$, it is easy to see that $x_E \in D^{\mathcal{J}_E}$, too. Using properties of canonical models (Lemma 3), we find that $D(b)$ is entailed by $\{B(b)\}$ and $\mathcal{T} \cup \{B \equiv E\}$, which means that $\mathcal{T} \models E \sqsubseteq D$. But in that case, we must have $\exists r.D \in \rho(v)$, as desired. Now let us consider the case where $v$ is a non-leaf node with label $\langle t, r_1, ..., r_n \rangle$, and suppose that we have already shown statement (2) to hold for all of $v$'s successors. Again, we restrict our attention to the interesting case where $C = \exists r.D$. If $v$'s only $r$-successors satisfying $D$ are outside $V$, then we can proceed as in the base case. Instead suppose that $r_i = r$, the $i$-th successor of $v$ is $w$, and $w \in D^{\mathcal{I}}$. Then by the induction hypothesis, we must have $D \in \rho(w)$. It follows from the definition of the rule set $\Theta$ that $\exists r.D$ belongs to $\rho(v)$.

Now let us suppose that $C \sqcup D \in \mathcal{T}$ and $y \in C^{\mathcal{I}}$. The case where $y \in \Delta^{\mathcal{J}_E}$ for some $E \in \Psi$ is straightforward, so we concentrate on the case where $y \in V$. In this case, we know from statement (2) that $C \in \rho(y)$, which means that $D$ must also belong to $\rho(y)$. It follows then from statement (1) that $y \in D^{\mathcal{I}}$, as desired. We have thus shown that $\mathcal{I}$ is a model of $\mathcal{A}$ and $\mathcal{T}$, so $\mathcal{A}$ is consistent with $\mathcal{T}$.

We now prove that some $A_0$ assertion is entailed by $\mathcal{A}$ and $\mathcal{T}$. We start by establishing the following claim.

**Claim**. For all $v \in V$ and $C \in \rho(v)$, we have $\mathcal{T}, \mathcal{A} \models C(a_v)$.

The proof is by induction on the co-depth of $v$. If $v$ is a leaf and $C \in \rho(v)$, then the definition of $\Delta$ and $\mathcal{A}$ yields that $C \in \mathsf{cl}_{\mathcal{T}}(\sigma_{\mathcal{A}}(a_v))$. A straightforward semantic argument shows that this implies $\mathcal{T}, \mathcal{A} \models C(a_v)$. Now let $v$ be a non-leaf with $\ell(v) = \langle t, r_1, \ldots, r_n \rangle$ and successors $v_1, \ldots, v_n$. Moreover, let $C \in \rho(v)$. Then

$$C \in \mathsf{cl}_{\mathcal{T}}(\sigma_{\mathcal{A}}(a_v) \cup \{\exists r.D \in \mathsf{sub}(\mathcal{T}) \mid r = r_i \text{ and } D \in r(a_{v_i}) \text{ for some } 1 \leq i \leq n\}).$$

By IH, we know that $D \in \rho(a_{v_i})$ implies $\mathcal{T}, \mathcal{A} \models D(a_{v_i})$. Thus, we can use the semantics to show that $\mathcal{T}, \mathcal{A} \models C(a_v)$. This finishes the proof of the claim.

By definition of $Q_f$ and of runs, we have $A_0 \in r(\varepsilon)$ with $\varepsilon$ the root of $T$. The claim thus yields that $\mathcal{T}, \mathcal{A} \models A_0(v_\varepsilon)$. □

*Proof of Lemma 6.*

*Forward direction*: Straightforward. Consider some $n \times m$ solution to the tiling problem. Create individuals $a_{i,j}$ for $0 \leq i \leq n-1$ and $0 \leq j \leq m-1$, and consider the ABox $\mathcal{A}$ composed of the following assertions:

- $x(a_{i,j}, a_{i+1,j})$ for $0 \leq i < n-1$ and $0 \leq j \leq m-1$
- $y(a_{i,j}, a_{i+1,j})$ for $0 \leq j < n-1$ and $0 \leq i \leq m-1$
- $T_k(a_{i,j})$ where $T_k$ is the tile associated with the position $(i, j)$

It can be easily verified that $\mathcal{A}$ is consistent with $\mathcal{T}$ and satisfies $\mathcal{T}, \mathcal{A} \models A(a_{0,0})$.

*Backward direction*: We show that if $\mathcal{A}$ is a $\Sigma$-ABox consistent with $\mathcal{T}$ and such that $\mathcal{T}, \mathcal{A} \models A(a)$, then $\mathcal{A}$ must contain (i) a set of role assertions that describe a rectangle w.r.t. the role names $x$ and $y$, where the lower left corner is an individual name $a_i$ and the upper right corner an individual name $a_f$; (ii) assertions $T_{\mathsf{init}}(a_i)$ and $T_{\mathsf{final}}(a_f)$; and (iii) exactly one assertion of the form $T_i(b)$ for each individual $b$ that participates in the rectangle. Moreover, asserted tiles of adjacent grid elements satisfy the matching conditions. Thus we can use the ABox $\mathcal{A}$ to generate a solution to the tiling problem. the TBox $\mathcal{T}$ and is such that $\mathcal{T}, \mathcal{A} \models A(a)$. We begin by proving some useful properties of $\mathcal{T}$ and $\mathcal{A}$:

1. $\mathcal{T}, \mathcal{A} \models U(b) \quad \Rightarrow \quad \mathcal{T}, \mathcal{A} \models Y(b)$

   *Proof.* Suppose there is some model $\mathcal{I}$ of $\mathcal{T}, \mathcal{A}$ where $b^{\mathcal{I}} \notin Y^{\mathcal{I}}$. Let $\mathcal{I}'$ be the model obtained from $\mathcal{I}$ by redefining $U$ such that $U^{\mathcal{I}'} = U^{\mathcal{I}} \setminus \{b^{\mathcal{I}}\}$. One can easily check that $\mathcal{I}'$ is still a model of $\mathcal{T}, \mathcal{A}$ (this is because the axioms with $U$ on the rhs also have $Y$ on the rhs).
2. $\mathcal{T}, \mathcal{A} \models R(b) \quad \Rightarrow \quad \mathcal{T}, \mathcal{A} \models Y(b)$

   *Proof.* Entirely similar to the previous proof.

3. If $\mathcal{T}, \mathcal{A} \models Y(b)$, then we must have $T_i(b) \in \mathcal{A}$ for some $T_i$

   *Proof.* Suppose $T_i(b) \notin \mathcal{A}$ for every $i$, and let $\mathcal{I}$ be a model of $\mathcal{T}$ and $\mathcal{A}$. Create a model $\mathcal{I}'$ from $\mathcal{I}$ by setting $T_i^{\mathcal{I}'} = T_i^{\mathcal{I}} \setminus \{b^{\mathcal{I}}\}$ for every $i$, and $Y^{\mathcal{I}'} = Y^{\mathcal{I}} \setminus \{b^{\mathcal{I}}\}$. It is easily verified that $\mathcal{I}'$ is still a model of $\mathcal{T}$ and $\mathcal{A}$.

4. If $\mathcal{T}, \mathcal{A} \models Y(b)$ and $T_i(b) \in \mathcal{A}$, then one of the following must hold:
   (a) $T_i = T_{\mathsf{final}}$
   (b) there exists an individual $c$ such that $x(b, c) \in \mathcal{A}$, $T_j(c) \in \mathcal{A}$ for some $(T_i, T_j) \in H$, and $\mathcal{T}, \mathcal{A} \models U(c)$
   (c) there exists an individual $d$ such that $y(b, d) \in \mathcal{A}$, $T_j(d) \in \mathcal{A}$ for some $(T_i, T_j) \in \mathcal{V}$, and $\mathcal{T}, \mathcal{A} \models R(d)$
   (d) there exist individuals $c$, $d$, and $e$ such that $x(b, c)$, $y(b, d)$, $x(d, e)$, $y(c, e)$ are all in $\mathcal{A}$, there exist $j$ and $l$ such that $(T_i, T_j) \in H$, $(T_i, T_l) \in \mathcal{V}$, $T_j(c) \in \mathcal{A}$, and $T_l(d) \in \mathcal{A}$, and $\mathcal{A}, \mathcal{T} \models \{Y(c), Y(d), Y(e)\}$

   *Proof.* Suppose $\mathcal{T}, \mathcal{A} \models Y(b)$ and $T_i(b) \in \mathcal{A}$. We distinguish four cases:

   **Case 1** there are no assertions in $\mathcal{A}$ of the form $x(b, _)$ or $y(b, _)$

   **Case 2** there is an assertion $x(b, c)$ in $\mathcal{A}$, but no assertion of type $y(b, _)$

   **Case 3** there is an assertion $y(b, d)$ in $\mathcal{A}$, but no assertion of type $x(b, _)$

   **Case 4** there are assertions $x(b, c)$ and $y(b, d)$ in $\mathcal{A}$

   We will only give the proof for case 4, which is the most involved. We first remark that since we have assertions $x(b, c)$ and $y(b, d)$ in $\mathcal{A}$, we must also have $\mathcal{T}, \mathcal{A} \models \neg U(b)$ and $\mathcal{T}, \mathcal{A} \models \neg R(b)$. Using this and the fact that $T_i(b) \in \mathcal{A}$, we can show that for all models $\mathcal{I}$ of $\mathcal{T}$ and $\mathcal{A}$ we have

   $$b^{\mathcal{I}} \in (\exists x.(T_{j_1} \sqcap Y \sqcap \exists y.Y) \sqcap \exists y.(T_{j_2} \sqcap Y \sqcap \exists x.Y) \sqcap C \sqcap T_i)^{\mathcal{I}} \qquad (1)$$

   for some $j_1$ and $j_2$ satisfying $(T_i, T_{j_1}) \in H$ and $(T_i, T_{j_2}) \in V$. Since $x$ and $y$ are functional, this implies that $\mathcal{T}, \mathcal{A} \models Y(c)$ and $\mathcal{T}, \mathcal{A} \models Y(d)$. But then according to statement 3, we can find $T_c$ and $T_d$ such that $T_c(c) \in \mathcal{A}$ and $T_d(d) \in \mathcal{A}$. Since the concepts $T_i$ are mutually exclusive, it follows that in Equation 1, we must have $T_{j_1} = T_c$ and $T_{j_2} = T_d$, and hence $(T_i, T_c) \in H$ and $(T_i, T_d) \in \mathcal{V}$. It also follows from Equation 1 that $c^{\mathcal{I}}$ (resp. $d^{\mathcal{I}}$) has an $y$-successor (resp. $x$-successor) in every model $\mathcal{I}$. Since the TBox does not have any existential restrictions on the rhs of axioms, this implies the existence of assertions $y(c, g)$ and $x(d, h)$ in $\mathcal{A}$. It follows from Equation 1 and functionality of $x$ and $y$ that $\mathcal{T}, \mathcal{A} \models Y(g)$ and $\mathcal{T}, \mathcal{A} \models Y(h)$. All that remains to be shown is that $g = h$. Suppose for a contradiction that this is not the case, and consider the interpretation $\mathcal{I}$ defined as follows:

   - $\Delta^{\mathcal{I}} = \mathsf{Ind}(\mathcal{A})$
   - $(p^{\mathcal{I}}, q^{\mathcal{I}}) \in x^{\mathcal{I}}$ iff $x(p, q) \in \mathcal{A}$
   - $(p^{\mathcal{I}}, q^{\mathcal{I}}) \in x^{\mathcal{I}}$ iff $x(p, q) \in \mathcal{A}$
   - $p^{\mathcal{I}} \in T_j^{\mathcal{I}}$ iff $T_j(p) \in \mathcal{A}$

– Define $U^{\mathcal{I}}$ inductively as follows:

$$\begin{aligned} U_0 &= T_{\mathsf{final}}^{\mathcal{I}} \\ U_{k+1} \setminus U_k &= \bigcup_i \{t \in T_i^{\mathcal{I}} \mid \text{ there exists } u \in \Delta^{\mathcal{I}} \text{ such that } (t,u) \in x^{\mathcal{I}} \text{ and} \\ &\qquad u \in U^{\mathcal{I}} \text{ and } u \in T_j^{\mathcal{I}} \text{ for some } T_j \text{ such that } (T_i, T_j) \in H\} \\ U^{\mathcal{I}} &= \bigcup_k U_k \end{aligned}$$

– Define $R^{\mathcal{I}}$ in an analogous manner to $U^{\mathcal{I}}$
– $Z^{\mathcal{I}} = \{g^{\mathcal{I}}\}$
– $C^{\mathcal{I}} = ((\exists x.\exists y.Z \sqcap \exists y.\exists x.Z) \sqcup (\exists x.\exists y.\neg Z \sqcap \exists y.\exists x.\neg Z))^{\mathcal{I}}$
– $Y^{\mathcal{I}}$ is defined inductively as follows:

$$\begin{aligned} Y_0 &= U^{\mathcal{I}} \cup R^{\mathcal{I}} \\ Y_{k+1} \setminus Y_k &= \bigcup_i \{t \in (C \sqcap T_i)^{\mathcal{I}} \mid \text{ there exists } u, v, w \in Y_i \text{ such that} \\ &\qquad (t,u) \in x^{\mathcal{I}}, (t,v) \in y^{\mathcal{I}}, (u,w) \in y^{\mathcal{I}}, (v,w) \in x^{\mathcal{I}}, \\ &\qquad u \in T_{j_1}^{\mathcal{I}} \text{ and } v \in T_{j_2} \text{ where } (T_i, T_{j_1}) \in H \text{ and } (T_i, T_{j_2}) \in V\} \\ Y^{\mathcal{I}} &= \bigcup_i Y_i \end{aligned}$$

– $A^{\mathcal{I}} = Y^{\mathcal{I}} \cap T_{\mathsf{init}}^{\mathcal{I}}$

Note that since we assumed $g \neq h$, we have $g^{\mathcal{I}} \in Z^{\mathcal{I}}$ and $h^{\mathcal{I}} \notin Z^{\mathcal{I}}$. It follows that $b^{\mathcal{I}} \notin C^{\mathcal{I}}$ and from that $b^{\mathcal{I}} \notin Y^{\mathcal{I}}$. It can be verified however that $\mathcal{I}$ is a model of $\mathcal{T}$ and $\mathcal{A}$, yielding $\mathcal{T}, \mathcal{A} \models Y(b)$, a contradiction. We thus conclude that $g = h$, as desired.

5. If $\mathcal{T}, \mathcal{A} \models Y(b)$, then there exist individuals $b = c_1, ..., c_n = f$ such that $T_{\mathsf{final}}(f) \in \mathcal{A}$ and either $x(c_i, c_{i+1}) \in \mathcal{A}$ or $y(c_i, c_{i+1}) \in \mathcal{A}$ for every $1 \leq i < n$

   *Proof.* Suppose that $b$ is not linked via role assertions in $\mathcal{A}$ to any individual $f$ such that $T_{\mathsf{final}}(f) \in \mathcal{A}$. Consider some model $\mathcal{I}$ of $\mathcal{T}$ and $\mathcal{A}$, and let $\mathcal{I}'$ be the model which is the same as $\mathcal{I}$ except we set $T_{\mathsf{final}}$, $U$, $R$, and $Y$ to false at all points reachable from $b$. It can be verified that $\mathcal{I}'$ is a model of $\mathcal{T}$ and $\mathcal{A}$ in which $b^{\mathcal{I}} \notin Y^{\mathcal{I}'}$.

6. If $\mathcal{T}, \mathcal{A} \models Y(b)$ and there exist individuals $b = c_1, ..., c_n = d$ such that either $x(c_i, c_{i+1}) \in \mathcal{A}$ or $y(c_i, c_{i+1}) \in \mathcal{A}$ for every $1 \leq i < n$, then $\mathcal{T}, \mathcal{A} \models Y(d)$

   *Proof.* Follows easily from statement 4.

7. If $\mathcal{T}, \mathcal{A} \models Y(b)$ and there exist individuals $b = c_1, ..., c_{n+1}$ and roles $r_1, ..., r_n$ such that $r_i(c_{i-1}, c_i) \in \mathcal{A}$ for every $1 \leq i \leq n$ ($r_i \in \{x, y\}$), then for every permutation $r'_1, ..., r'_n$ of $r_1, ..., r_n$ we can find individuals $b = d_1, ..., d_{n+1} = c_{n+1}$ such that $r'_i(d_{i-1}, d_i) \in \mathcal{A}$ for every $1 \leq i \leq n$

   *Proof.* By induction on $n$. Base cases where $n = 0$ or $n = 1$ are trivial. Suppose then that the statement holds whenever $n \leq k$, and consider some

$b$ such that $\mathcal{T}, \mathcal{A} \models Y(b)$ and there exist individuals $b = c_1, ..., c_{k+2}$ and roles $r_1, ..., r_{k+1}$ such that $r_i(c_{i-1}, c_i) \in \mathcal{A}$ for every $1 \leq i \leq k+1$ ($r_i \in \{x, y\}$). Let $r'_1, ..., r'_{k+1}$ be a permutation of $r_1, ..., r_{k+1}$. If $r_{k+1} = r'_{k+1}$, the result follows easily from the induction hypothesis. Otherwise, suppose we have $r_{k+1} = x$ and $r'_{k+1} = y$ (the case where $r_{k+1} = x$ and $r'_{k+1} = y$ is treated analogously). Because of the induction hypothesis, we can suppose without loss of generality that $r_k = y$. Now by statement 6, we know that $\mathcal{T}, \mathcal{A} \models Y(c_k)$. That means that $c_k$ must belong to one of the four cases in statement 4. Since $c_k$ has a $y$-successor, we cannot have $\mathcal{T}, \mathcal{A} \models U(c_k)$, so $c_k$ cannot belong to cases (a) and (b). It also cannot belong to case (c), since then we would have $\mathcal{T}, \mathcal{A} \models R(c_{k+1})$, contradicting the fact that $c_{k+1}$ has an $x$-successor. Thus, $c_k$ must belong to case (d), which means that there are $xy$ and $yx$ paths in the ABox starting from $c_k$. In particular, it follows that there exists an individual $e$ such that $x(c_k, e) \in \mathcal{A}$. The sequence of individuals $b = c_1, ..., c_k, e$ and sequences of roles $r_1, ..., r_{k-1}, x$ and $r'_1, ..., r'_k$ can be shown to satisfy the conditions of the induction hypothesis. This means that we can find individuals $b = d_1, ..., d_{k+1} = e$ such that $r'_i(d_{i-1}, d_i) \in \mathcal{A}$ for every $1 \leq i \leq k$. Also, we know from (d) that the $xy$ and $yx$ paths starting from $c_k$ culminate in a single point, which must be $c_{k+2}$. Using the functionality of $y$, we can conclude that the assertion $r'_{k+1}(e, c_{k+2}) = y(e, c_{k+2})$ belongs to $\mathcal{A}$, completing the proof.

Since $\mathcal{T}, \mathcal{A} \models A(a)$, it follows that both $\mathcal{T}, \mathcal{A} \models T_{\mathsf{init}}(a)$ and $\mathcal{T}, \mathcal{A} \models Y(a)$. From property 5 above, we know that there must exist individuals $a = c_0, ..., c_n = f$ and roles $r_1, ..., r_n$ ($r_i \in \{x, y\}$) such that $T_{\mathsf{final}}(f) \in \mathcal{A}$ and $r_i(c_{i-1}, c_i) \in \mathcal{A}$ for every $1 \leq i < n$. Let $l$ (resp. $m$) denote the number of $r_i$ such that $r_i = x$ (resp. $r_i = y$). Now since $\mathcal{T}, \mathcal{A} \models Y(f)$, we know from property 7 that every permutation $r'_1, ..., r'_n$ of $r_1, ..., r_n$ there are individuals $a = d_0, ..., d_n = f$ such that $r'_i(d_{i-1}, d_i) \in \mathcal{A}$ for every $1 \leq i \leq n$. In particular, this means that for every sequence of roles $s_1, ..., s_p$ containing at most $l$ occurrences of $x$ and at most $m$ occurrences of $y$, there is a sequence of individuals $a = e_0, ..., e_p$ such that $s_i(e_{i-1}, e_i) \in \mathcal{A}$ for every $1 \leq i \leq p$. Note also that we must have $\mathcal{T}, \mathcal{A} \models Y(e_p)$ (by 6), so by 7 we know that every permutation of $s_1, ..., s_p$ will also yield a path in $\mathcal{A}$ from $a$ to $e_p$. For $0 \leq i \leq l$ and $0 \leq j \leq m$, let us denote by $b_{i,j}$ the unique individual which is reached from $a$ using $i$ role assertions of type $x$ and $j$ of type $y$ (uniqueness follows from the functionality of $x$ and $y$). Note that it follows from statement 7 above that:

- For all $1 \leq i < n$ and $1 \leq j \leq n$: $x(b_{i,j}, b_{i+1,j}) \in \mathcal{A}$
- For all $1 \leq i \leq n$ and $1 \leq j < n$: $y(b_{i,j}, b_{i,j+1}) \in \mathcal{A}$

All of the $b_{i,j}$ are such that $\mathcal{T}, \mathcal{A} \models Y(b_{i,j})$ (by statement 6), so we know from statement 3 that for each $b_{i,j}$ there must be some $T_k$ with $T_k(a_{i,j}) \in \mathcal{A}$. Let us then consider the $l + 1 \times m + 1$-tiling in which the tile at position $(i, j)$ (for $0 \leq i \leq l$, $0 \leq j \leq m$) is defined to be the unique tile $T_k$ such that $T_k(a_{i,j}) \in \mathcal{A}$. As $b_{0,0} = a$ and $T_{\mathsf{init}}(a) \in \mathcal{A}$, it follows that the lower left corner tile is $T_{\mathsf{init}}$. Likewise, we have $b_{l,m} = f$ and $T_{\mathsf{final}}(f) \in \mathcal{A}$, so the upper-right corner tile must

be $T_{\mathsf{final}}$. The matching of the tiles follows from the fact that the individual $b_{i,j}$ is linked via $x$ to $b_{i+1,j}$ and via $y$ to $b_{i,j+1}$, together with statement 4, which ensures that $x$- and $y$-successors of $Y$-individuals have matching tiles. □

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