Decidability in First-Order Modal Logic with Non-Rigid Constants and Definite Descriptions

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Abstract

While modal extensions of decidable fragments of first-order logic are usually undecidable, their monodic counterparts, in which formulas in the scope of modal operators have at most one free variable, are typically decidable. This only holds, however, under the provision that non-rigid constants, definite descriptions and non-trivial counting are not admitted. Indeed, several monodic fragments having at least one of these features are known to be undecidable. We investigate these features systematically and show that fundamental monodic fragments such as the two-variable fragment with counting and the guarded fragment of standard first-order modal logics $\mathbf{K}_n$ and $\mathbf{S5}_n$ are decidable. Tight complexity bounds are established as well. Under the expanding-domain semantics, we show decidability of the basic modal logic extended with the transitive closure operator on finite acyclic frames; this logic, however, is Ackermann-hard.

Keywords: First-Order Modal Logic, First-Order Temporal Logic, Monodicity, Decidability, Definite Descriptions, Non-Rigid Designators

1. Introduction

First-order modal logics are notorious for their bad computational behaviour, with modal extensions of decidable fragments of first-order logic (FO) being typically undecidable. A well-known example is the monadic fragment of FO which admits only unary predicates and no function symbols. While the monadic fragment of FO is known to be decidable [1], the monadic fragment of almost any standard first-order modal logic is undecidable. This applies, in particular, to the basic modal logic $\mathbf{K}_n$ of all Kripke frames with n accessibility relations and the basic epistemic modal logic $\mathbf{S5}_n$ determined by the class of Kripke frames with n equivalence relations, for all $n \geq 1$ [2, 3]. In fact, over the past fifty years, numerous powerful undecidability results have been obtained, reflecting the fact that the two-dimensional shape of combinations of Kripke frames with first-order domains often provides enough structure to encode Turing machines

or other models of computation with undecidable halting problem [4, 5, 6, 7]. While new families of decidable fragments of non-modal FO have been discovered on a regular basis (for instance, variants of the two-variable, the guarded, and the fluted fragments [8]), progress has been slower in first-order modal logic.

In the early 2000s the *monodic* fragment of first-order modal and temporal logic was introduced, in which modal operators are only applied to formulas with at most one free variable. This fragment turned out to behave much better in the sense that very often the monodic fragment of a modal extension of a decidable fragment of FO is decidable again. Prominent examples include the monadic monodic, two-variable monodic, and guarded monodic fragments of first-order modal logics $\mathbf{K}_n$ and $\mathbf{S5}_n$, even if extended with modal operators for transitive closure [9]. Other examples are various monodic fragments of first-order temporal logics, including linear temporal logic LTL [10, 11].

These positive results hold independently from whether expanding or constant domains are assumed, but they come with a crucial provision: the language should not admit *non-rigid designators*, in the form of constants or definite descriptions denoting different individuals in different worlds, or non-trivial *counting*, expressing, for instance, that a unary predicate has at most n elements. In the sequel, we say that a language has *NRDC features* if it contains non-rigid designators and/or non-trivial counting. Extended with NRDC features, even the one-variable fragment of first-order LTL with constant or expanding domains becomes undecidable [12, 13], and so does the one-variable fragment of first-order $\mathbf{K}_n$ with the universal modality on constant-domain models [14].

NRDC features are, however, often the main reason for using first-order modal logic and thus are fundamental for many applications. In philosophy, they have been instrumental in shaping the analysis of *referential opacity* in modal contexts, related to the failure of the substitutivity principle for equality in the presence of modal operators, witnessed, for instance, by 'the number of planets is necessarily greater than 7' as opposed to '8 is necessarily greater than 7' [15, 16]. Furthermore, the development of an adequate syntactic and semantic treatment of non-rigid definite descriptions, such as 'the Evening Star', in contrast to rigid names, such as 'Venus', has provided a formal background to the debate between *descriptivist* theories, on the one hand, and *direct reference* theories of proper names, on the other [17]. Recently, definite descriptions have received a renewed attention in free logic [18, 19, 20], description logic [21, 22, 23], as well as modal and hybrid logic literature [24, 25, 26, 27]. Under the *epistemic* reading of modalities, non-rigid terms have been deployed as a device for denoting distinct individuals in alternative scenarios considered conceivable by an agent [28, 29, 30, 31]. Similar applications have also been found in the context of quantified *temporal* logics, with formalisms involving individual symbols that are allowed to change their denotation over time [32, 33, 34].

As discussed, despite these applications, mostly negative results have been obtained so far regarding the computational behaviour of first-order modal logics with NRDC features. *The aim of this paper is to initiate a systematic investigation of decidability and complexity of monodic fragments of first-order modal*

frames $\mathcal{C}$	dom.	$\mathcal{C}$-validity			global $\mathcal{C}$-consequence		
		$\mathsf{Q}^{1=}\mathsf{ML}_\iota$	$\mathsf{C}^2_{\boxed{1}}\mathsf{ML}_\iota$	$\mathsf{GF}^{=}_{\boxed{1}}\mathsf{ML}_\iota$	$\mathsf{Q}^{1=}\mathsf{ML}_\iota$	$\mathsf{C}^2_{\boxed{1}}\mathsf{ML}_\iota$	$\mathsf{GF}^{=}_{\boxed{1}}\mathsf{ML}_\iota$
S5	$=$	coNExp [Th. 19]	coNExp [Th. 27]	2Exp [Th. 21]	coNExp [Rem. 5]	coNExp [Rem. 5]	2Exp [Rem. 5]
$\mathbf{S5}_n$, $n \geq 2$	$=$	coNExp [Th. 19]	coNExp [Th. 27]	2Exp [Th. 21]	undecidable [Th. 4]		
$\mathbf{K}_n$	$=$	coNExp [Th. 19]	coNExp [Th. 27]	2Exp [Th. 21]	undecidable [Th. 4]		
	$\subseteq$	PSpace [Th. 41]	coNExp [Th. 27]	2Exp [Th. 21]	?		
$\mathbf{K}_{*n}$, $\mathbf{LTL}^{(\Diamond)}$	$=$	Σ^1_1 [L. 43 + Th. 42 (1)]					
	$\subseteq$	undecidable [L. 43 + Th. 42 (2)]					
$\mathbf{K}\boldsymbol{f}_{*n}$, $\mathbf{LTL}\boldsymbol{f}^{(\Diamond)}$	$=$	undecidable [L. 43 + Th. 42 (1)]					
	$\subseteq$	decidable, Ackermann-hard [L. 43 + Th. 37, 42 (2)]					

Table 1: $\mathcal{C}$-validity and global $\mathcal{C}$-consequence in fragments of $\mathsf{Q}^{=}_{\boxed{1}}\mathsf{ML}_\iota$ (complexity bounds are tight and, unless otherwise stated, $n \geq 1$).

logics extended with NRDC features. In particular, we show that such extensions of many fundamental monodic fragments remain decidable.

We now discuss our results and techniques in detail. We consider standard Kripke semantics for first-order modal logic with expanding or constant domains (and use the fact that reasoning in the former can be reduced in polynomial time to reasoning in the latter). Our most general language admits equality, non-rigid and possibly non-denoting constants, along with definite descriptions. We observe that neither non-denoting constants nor definite descriptions increase the complexity of reasoning, and so show our main technical results for the language with non-rigid constants and equality. We focus on the standard modal logics $\mathbf{K}_n$ and $\mathbf{S5}_n$ and the extension $\mathbf{K}_{*n}$ of $\mathbf{K}_n$ with a modal operator interpreted by the transitive closure of the remaining n accessibility relations. Of particular interest for mapping out the border between decidability and undecidability is the modal logic $\mathbf{K}\boldsymbol{f}_{*n}$, obtained from $\mathbf{K}_{*n}$ by considering only Kripke frames without infinite ascending chains.

We investigate decidability and complexity of monodic fragments of these first-order modal logics restricted to decidable fragments of FO. While our techniques can in principle be applied to any decidable fragment of FO, the focus here is on the following three fragments, a minimal one and two maximal ones.

- The one-variable fragment of first-order modal logic, $\mathsf{Q}^{1=}\mathsf{ML}_\iota$, is underpinned by the one-variable fragment FO^1 of FO with equality and constants. Validity in FO^1 is coNP-complete, so it behaves exactly as propositional logic.

- The monodic two-variable fragment with counting, $\mathsf{C}^2_{\boxed{1}}\mathsf{ML}_\iota$, is underpinned by the two-variable fragment C^2 of FO with counting quantifiers, which

has received considerable attention as a decidable fragment of FO supporting quantitative reasoning [35, 36, 37]. Validity in C^2 is coNExpTime-complete.

- The monodic guarded fragment, $\mathsf{GF}^{=}_{\boxdot}\mathsf{ML}_{\iota}$, is underpinned by the guarded fragment GF of FO, which generalises standard modal and description logics [38, 39]. Validity in GF is 2ExpTime-complete, but it is incompatible with C^2 in the sense that GF with counting quantifiers is undecidable.

Our main results are shown in Table 1. In this table, the frame classes are listed in the first column, with **LTL*f*** denoting the class of *all finite strict linear orders* and **LTL** denoting $\{(\mathbb{N}, <)\}$. The superscript $(\Diamond)$ indicates that the results are the same for the modal languages with both $\Diamond$ and $\bigcirc$ and with $\Diamond$ only. The "$=$" and "$\subseteq$" in the second column indicate constant or expanding domain, respectively. One can see that decidability does not depend on which fragment is considered, while, not so surprisingly, the computational complexity does depend on the fragment. We note that, without the NRDC features, all these logics are decidable in at most 2ExpTime [9, 4], so NRDC features have a significant impact. There are, however, positive results to emphasise: we obtain elementary decidability for the fundamental logics $\mathbf{K}_n$ and **S5**; moreover, on finite acyclic expanding-domain models, even transitive closure can be added while still retaining decidability.

The main steps of our proofs are as follows. We adapt the machinery developed for monodic fragments without the NRDC features. The basic abstraction needed is *quasimodels*: Kripke models in which first-order domains are replaced by quasistates (sets of types), while sets of runs (functions from worlds to quasistates) are used to represent individual domain elements. To deal with the NRDC features, quasistates now become *multisets* of types and, similarly, sets of runs become multisets to reflect the need to count domain elements. Our first basic result shows that, when dealing with monodic fragments, Kripke models can be replaced by such 'quantitative' quasimodels. In contrast to the case without the NRDC features, quasimodels are typically not yet well-behaved. For instance, if the first-order fragment does not have the finite model property, one can easily enforce infinite branching of Kripke frames. Hence, we replace quasimodels by *weak* quasimodels in which the $\Diamond$-saturation conditions on runs are weakened. We show that one can reconstruct quasimodels from weak quasimodels, but only by expanding the underpinning Kripke frame. In particular, we show that, on $\mathbf{K}_n$ and $\mathbf{S5}_n$ frames, monodic fragments are determined by weak quasimodels containing only exponentially many worlds. This approach underpins our decidability results for $\mathbf{K}_n$ and $\mathbf{S5}_n$. To obtain tight complexity bounds, we employ techniques and results developed for GF [39, 40] and C^2 [8]. The decidability proof for expanding-domain models with transitive closure also relies on weak quasimodels and shows, using Dickson's Lemma, a (non-primitive recursive) bound on their size. The lower bounds are all shown by reduction from known results for related fragments [4, 13].

This article significantly extends ideas first developed by the authors in the context of modal and temporal description logics [41, 42].

2. Preliminaries

In this section we first introduce the language $\mathsf{Q}^{=}\mathsf{ML}_{\iota}$ of first-order modal logic with equality, constants and definite descriptions and then define its semantics, where constants and definite descriptions are interpreted non-rigidly. We then define a number of monodic fragments of $\mathsf{Q}^{=}\mathsf{ML}_{\iota}$ and formulate the main decision problems for the fragments.

2.1. Syntax and Semantics

Let A be a finite set of *modalities*. The alphabet of the *first-order modal language with equality and definite descriptions*, $\mathsf{Q}^{=}\mathsf{ML}_{\iota}$, consists of countably infinite and pairwise disjoint sets of predicate symbols Pred (each of a fixed non-negative arity), constants Con and variables Var, equality $=$, Boolean connectives $\neg$ and $\wedge$, the existential quantifier $\exists$, the definite description operator ι, and modal operators $\Diamond_a$ (diamond), for $a \in A$. *Terms* τ and *formulas* φ of $\mathsf{Q}^{=}\mathsf{ML}_{\iota}$ are defined by mutual induction:

$$\tau ::= x \mid c \mid \iota x.\varphi,$$
$$\varphi ::= P(\tau_1, \ldots, \tau_m) \mid \tau_1 = \tau_2 \mid \neg\varphi \mid (\varphi_1 \wedge \varphi_2) \mid \exists x\, \varphi \mid \Diamond_a \varphi,$$

where $x \in \mathsf{Var}$, $c \in \mathsf{Con}$, $P \in \mathsf{Pred}$ (m-ary, for $m \geq 0$) and $a \in A$. Formulas of the form $P(\tau_1, \ldots, \tau_m)$ and $\tau_1 = \tau_2$ are called *atomic*. Other standard syntactic abbreviations are assumed: in particular, $\varphi_1 \vee \varphi_2$ stands for $\neg(\neg\varphi_1 \wedge \neg\varphi_2)$, $\varphi_1 \to \varphi_2$ for $\neg\varphi_1 \vee \varphi_2$, $\varphi_1 \leftrightarrow \varphi_2$ for $(\varphi_1 \to \varphi_2) \wedge (\varphi_2 \to \varphi_1)$; $\forall x\, \varphi$ abbreviates $\neg\exists x \neg\varphi$, and $\Box_a \varphi$ abbreviates $\neg\Diamond_a \neg\varphi$.

The *free variables* in terms and formulas are defined in the standard way by mutual induction (cf. [43]): in particular, the free variables of $P(\tau_1, \ldots, \tau_m)$ are those of $\tau_1, \ldots, \tau_m$, while the free variables of $\iota x.\varphi$ and $\exists x\, \varphi$ are the free variables of φ, with the exception of x. A $\mathsf{Q}^{=}\mathsf{ML}_{\iota}$ *sentence* is a $\mathsf{Q}^{=}\mathsf{ML}_{\iota}$ formula without free variables, and a $\mathsf{Q}^{=}\mathsf{ML}_{\iota}$ *theory* is a finite set of $\mathsf{Q}^{=}\mathsf{ML}_{\iota}$ sentences. The set of constants occurring in a formula φ is denoted by $\mathsf{Con}(\varphi)$, and for a theory Γ we set $\mathsf{Con}(\Gamma) = \bigcup_{\varphi \in \Gamma} \mathsf{Con}(\varphi)$.

The *size* of terms and formulas is introduced by mutual induction, setting $|x| = |c| = 1$, $|\iota x.\varphi| = 1 + |\varphi|$, $|P(\tau_1, \ldots, \tau_m)| = 1 + |\tau_1| + \ldots + |\tau_m|$, and the other cases given in the natural way (we treat the Boolean connectives other than $\neg$ and $\wedge$ as abbreviations). The sets of *subformulas* of terms and formulas are also defined by mutual induction, with $sub(x) = sub(c) = \emptyset$, $sub(\iota x.\varphi) = sub(\varphi)$, $sub(P(\tau_1, \ldots, \tau_m)) = \{P(\tau_1, \ldots, \tau_m)\} \cup sub(\tau_1) \cup \cdots \cup sub(\tau_m)$, and the remaining cases as usual. Finally, the *modal depth* of terms and formulas is the maximum number of nested modal operators, defined again by mutual induction: $d(x) = d(c) = 0$, $d(\iota x.\varphi) = d(\varphi)$, $d(P(\tau_1, \ldots, \tau_m)) = \max\{d(\tau_1), \ldots, d(\tau_m)\}$, $d(\Diamond_a \varphi) = d(\varphi) + 1$, and with the other cases given in the obvious way.

A *partial interpretation with expanding domains* is a structure $\mathfrak{M} = (\mathfrak{F}, \Delta, \cdot)$, where

- $\mathfrak{F} = (W, \{R_a\}_{a \in A})$ is a *frame*, with W being a non-empty set of *worlds* and $R_a \subseteq W \times W$ being an *accessibility relation* on W, for each modality $a \in A$ (we say that $\mathfrak{M}$ is *based on* $\mathfrak{F}$);
- Δ is a function associating with every $w \in W$ a non-empty set, Δ_w, called the *domain of* w *in* $\mathfrak{M}$, such that $\Delta_w \subseteq \Delta_v$, whenever wR_av, for some $a \in A$;
- $\cdot$ is a function associating with each $w \in W$ a *partial* first-order interpretation $\mathfrak{M}(w)$ with domain Δ_w so that $P^{\mathfrak{M}(w)} \subseteq \Delta_w^m$, for each predicate $P \in \mathsf{Pred}$ of arity m, and $c^{\mathfrak{M}(w)} \in \Delta_w$, for each constant symbol c in *some subset* of Con.

Hence, every $\cdot^{\mathfrak{M}(w)}$ is a total function on Pred but a *partial* function on Con. If $\mathfrak{M}(w)$ is defined on $c \in \mathsf{Con}$, then we say that c *designates at* w. If every $c \in \mathsf{Con}$ designates at $w \in W$, then $\mathfrak{M}(w)$ is called *total*. We say that $\mathfrak{M} = (\mathfrak{F}, \Delta, \cdot)$ is a *total* interpretation if every $\mathfrak{M}(w)$, for $w \in W$, is a *total* interpretation. In the remainder of this work, we refer to partial interpretations as 'interpretations' and add explicitly the adjective 'total' when this is the case.

An *interpretation with constant domains* is a special case of an interpretation with expanding domains where the function Δ is such that $\Delta_w = \Delta_v$, for every $w, v \in W$. With an abuse of notation, we denote the common domain by Δ and call it the *domain of* $\mathfrak{M}$.

Given an interpretation $\mathfrak{M} = (\mathfrak{F}, \Delta, \cdot)$, an *assignment at* w is a function $\mathfrak{a}$ from Var to Δ_w. An x*-variant* of an assignment $\mathfrak{a}$ at w is an assignment $\mathfrak{a}'$ at w that can differ from $\mathfrak{a}$ only on x. Observe that, if wR_av for some $a \in A$, then an assignment at w is also an assignment at v. The definitions of the *value* $\tau^{\mathfrak{M}(w),\mathfrak{a}}$ of a term τ under assignment $\mathfrak{a}$ at world w of $\mathfrak{M}$, and of *satisfaction* $\mathfrak{M}, w \models^{\mathfrak{a}} \varphi$ of a formula φ at world w of $\mathfrak{M}$ under assignment $\mathfrak{a}$ are defined by mutual induction. We set

$$\tau^{\mathfrak{M}(w),\mathfrak{a}} = \begin{cases} \mathfrak{a}(x), & \text{if } \tau \text{ is } x \in \mathsf{Var}; \\ c^{\mathfrak{M}(w)}, & \text{if } \tau \text{ is } c \in \mathsf{Con} \text{ and } c^{\mathfrak{M}(w)} \text{ is defined}; \\ \mathfrak{a}'(x), & \text{if } \tau \text{ is } \iota x.\varphi \text{ and } \mathfrak{M}, w \models^{\mathfrak{a}'} \varphi, \text{ for } \textit{exactly one} \\ & \qquad x\text{-variant } \mathfrak{a}' \text{ of } \mathfrak{a} \text{ at } w; \\ \text{undefined}, & \text{otherwise}; \end{cases}$$

and define

$$\begin{array}{lll} \mathfrak{M}, w \models^{\mathfrak{a}} P(\tau_1, \ldots, \tau_m) & \text{iff} & \tau_1^{\mathfrak{M}(w),\mathfrak{a}}, \ldots, \tau_m^{\mathfrak{M}(w),\mathfrak{a}} \text{ are defined and} \\ & & (\tau_1^{\mathfrak{M}(w),\mathfrak{a}}, \ldots, \tau_m^{\mathfrak{M}(w),\mathfrak{a}}) \in P^{\mathfrak{M}(w)}; \\ \mathfrak{M}, w \models^{\mathfrak{a}} \tau_1 = \tau_2 & \text{iff} & \text{both } \tau_i^{\mathfrak{M}(w),\mathfrak{a}} \text{ are defined and } \tau_1^{\mathfrak{M}(w),\mathfrak{a}} = \tau_2^{\mathfrak{M}(w),\mathfrak{a}}; \\ \mathfrak{M}, w \models^{\mathfrak{a}} \neg\varphi & \text{iff} & \mathfrak{M}, w \not\models^{\mathfrak{a}} \varphi; \\ \mathfrak{M}, w \models^{\mathfrak{a}} \varphi_1 \wedge \varphi_2 & \text{iff} & \mathfrak{M}, w \models^{\mathfrak{a}} \varphi_1 \text{ and } \mathfrak{M}, w \models^{\mathfrak{a}} \varphi_2; \\ \mathfrak{M}, w \models^{\mathfrak{a}} \exists x\, \varphi & \text{iff} & \mathfrak{M}, w \models^{\mathfrak{a}'} \varphi, \text{ for some } x\text{-variant } \mathfrak{a}' \text{ of } \mathfrak{a} \text{ at } w; \\ \mathfrak{M}, w \models^{\mathfrak{a}} \Diamond_a \varphi & \text{iff} & \mathfrak{M}, v \models^{\mathfrak{a}} \varphi, \text{ for some } v \in W \text{ such that } wR_av. \end{array}$$

If $\tau^{\mathfrak{M}(w),\mathfrak{a}}$ is defined, then we say that τ *designates under* $\mathfrak{a}$ *at* w in $\mathfrak{M}$. Observe that variables always designate under an assignment $\mathfrak{a}$ at w, and that assignments play no role in constants' designation (hence, we will simply say that a constant *designates at* w *in* $\mathfrak{M}$). To simplify the notation, when there is no risk of confusion, we will often write P^w, c^w and $\tau^{w,\mathfrak{a}}$ in place, respectively, of $P^{\mathfrak{M}(w)}$, $c^{\mathfrak{M}(w)}$ and $\tau^{\mathfrak{M}(w),\mathfrak{a}}$.

We say that a formula φ is *true in* $\mathfrak{M}$, written $\mathfrak{M} \models \varphi$, iff φ is satisfied under every assignment $\mathfrak{a}$ at every world w of $\mathfrak{M}$. Dually, we say that φ is *satisfied in* $\mathfrak{M}$ iff φ is satisfied under some assignment $\mathfrak{a}$ at some world w of $\mathfrak{M}$. A theory Γ is *true in* $\mathfrak{M}$, written $\mathfrak{M} \models \Gamma$, if every sentence in Γ is true in $\mathfrak{M}$.

Given a class of frames $\mathcal{C}$, we say that φ is *valid on* $\mathcal{C}$ (or $\mathcal{C}$*-valid*) if φ is true in every interpretation $\mathfrak{M}$ based on a frame $\mathfrak{F} \in \mathcal{C}$. Dually, we say that φ is *satisfiable in* $\mathcal{C}$ (or $\mathcal{C}$*-satisfiable*) if there exists an interpretation $\mathfrak{M}$ based on a frame $\mathfrak{F} \in \mathcal{C}$ such that φ is satisfied in $\mathfrak{M}$. A formula φ is said to be a *global* $\mathcal{C}$*-consequence of a theory* Γ if φ is true in any interpretation $\mathfrak{M}$ based on a frame in $\mathcal{C}$ such that $\mathfrak{M} \models \Gamma$; recall that theories have no free variables.

In the following, the class of all frames with n accessibility relations, for $n \geq 1$, is denoted by $\mathbf{K}_n$. The class of frames with n equivalence relations is $\mathbf{S5}_n$, and we write $\mathbf{S5}$ in place of $\mathbf{S5}_1$.

Example 1 (Vulcan and Venus)**.** Let us consider some examples of $\mathsf{Q}^{=}\mathsf{ML}_{\iota}$ formulas: first, "it is conceivable *that* Vulcan is the planet orbiting between the Sun and Mercury" is represented as

$$\Diamond\big(\mathsf{vulcan} = \iota z.\mathsf{OrbitsBetween}(z, \mathsf{sun}, \mathsf{mercury})\big),$$

while "even though such a planet does not exist" can be written as

$$\neg\exists x\,\big(x = \mathsf{vulcan}\big) \wedge \neg\exists x\,\big(x = \iota z.\mathsf{OrbitsBetween}(z, \mathsf{sun}, \mathsf{mercury})\big).$$

Here, neither the constant vulcan nor the definite description $\iota z.\mathsf{OrbitsBetween}(z, \mathsf{sun}, \mathsf{mercury})$ designate in the current world.

Second, "it is known *of* the planet orbiting between Mercury and Earth that it is Venus" can be rendered as

$$\exists x\,\big(x = \iota z.\mathsf{OrbitsBetween}(z, \mathsf{mercury}, \mathsf{earth}) \;\wedge\; \Box(x = \mathsf{venus})\big).$$

Note that, in $\mathbf{S5}$ frames, the above formula implies that venus is a rigid designator; see [43, Proposition 10.2.5].

2.2. Monodic Fragments and Decision Problems

We next introduce various fragments of $\mathsf{Q}^{=}\mathsf{ML}_{\iota}$. A $\mathsf{Q}^{=}\mathsf{ML}_{\iota}$ formula φ is called *monodic* if every subformula of φ of the form $\Diamond_a\psi$ has at most one free variable. We denote by $\mathsf{Q}^{=}_{[1]}\mathsf{ML}_{\iota}$ the set of monodic $\mathsf{Q}^{=}\mathsf{ML}_{\iota}$ formulas, and call it the *monodic fragment* of $\mathsf{Q}^{=}\mathsf{ML}_{\iota}$. The monodic fragment contains full first-order logic and so is undecidable. To obtain potentially decidable fragments of $\mathsf{Q}^{=}_{[1]}\mathsf{ML}_{\iota}$, we restrict its first-order component.

A minimal language we consider in this paper extends the *one-variable fragment* of first-order logic. Denote by $\mathsf{Q}^{1=}\mathsf{ML}_\iota$ the set of all formulas in $\mathsf{Q}^{=}\mathsf{ML}_\iota$ that use a single variable and only predicates of arity at most one (note that equality, constants, and definite descriptions are allowed). Clearly, all $\mathsf{Q}^{1=}\mathsf{ML}_\iota$ formulas are monodic by definition. Variants of this fragment have been investigated extensively as product modal logics (see, e.g., [44, 4, 45, 14]), and we point out specific relevant results below when discussing $\mathsf{Q}^{1=}\mathsf{ML}_\iota$.

At the other end of the spectrum, we consider two maximal languages. First, we extend C^2, the *two-variable fragment of first-order logic with counting quantifiers* $\exists^{\geq k}x$, $k \geq 0$. Denote by $\mathsf{C}^2_{\boxed{1}}\mathsf{ML}_\iota$ the set of all formulas in $\mathsf{Q}^{=}_{\boxed{1}}\mathsf{ML}_\iota$ extended by $\exists^{\geq k}$ and constructed using only *two variables* and predicates of arity at most two. We also have equality, constants and definite descriptions. The semantics of the counting quantifiers is given in the obvious way by modifying the quantifier case as follows:

$$\mathfrak{M}, w \models^{\mathfrak{a}} \exists^{\geq k} x\, \varphi \quad \text{iff} \quad \mathfrak{M}, w \models^{\mathfrak{a}'} \varphi, \text{ for at least } k \text{ distinct } x\text{-variants } \mathfrak{a}' \text{ of } \mathfrak{a} \text{ at } w.$$

It should be clear that counting can be expressed in first-order logic with equality, but $\exists^{\geq k}$ would require k variables. We use standard abbreviations such as $\exists^{\leq k} x\, \varphi = \neg\exists^{\geq k+1} x\, \neg\varphi$ and $\exists^{=k} x\, \varphi = \exists^{\leq k} x\, \varphi \wedge \exists^{\geq k} x\, \varphi$.

Example 2 (The number of planets)**.** We illustrate the interaction between modal operators and counting quantifiers. Let $\varphi_1 = \Diamond\exists^{\leq 9} x\, \mathsf{Planet}(\mathsf{x})$ and $\varphi_2 = \exists^{\leq 9} x\, \Diamond\mathsf{Planet}(\mathsf{x})$. Then, on arbitrary frames, φ_1 does not entail φ_2 under expanding or constant domains and φ_2 entails φ_1 under constant domains but not under expanding domains.

Observe that definite descriptions add expressive power to the non-modal two-variable fragment without counting and equality. For instance, it can be seen that $\forall x\, F(x, \iota y.F(x, y))$ ensures that binary predicate F is interpreted as a partial function (that is, the formula is equivalent to $\forall x \exists^{\leq 1} y\, F(x, y)$).

The second maximal language extends GF, the *guarded fragment of* first-order logic. Define the fragment $\mathsf{GF}^{=}_{\boxed{1}}\mathsf{ML}_\iota$ of $\mathsf{Q}^{=}_{\boxed{1}}\mathsf{ML}_\iota$ by defining its terms and formulas by mutual induction as follows. Terms in $\mathsf{GF}^{=}_{\boxed{1}}\mathsf{ML}_\iota$ are limited to variables, constants and *closed* definite descriptions, which are of the form $\iota x.\chi(x)$ for a $\mathsf{GF}^{=}_{\boxed{1}}\mathsf{ML}_\iota$ formula $\chi(x)$ with a single free variable x. Formulas in $\mathsf{GF}^{=}_{\boxed{1}}\mathsf{ML}_\iota$ are constructed as in $\mathsf{Q}^{=}_{\boxed{1}}\mathsf{ML}_\iota$ except that quantifiers are required to be guarded:

$$\exists x_1 \cdots \exists x_k\, (\alpha \wedge \varphi),$$

where α is a predicate or equality atom that contains all free variables of φ. Note that if φ has only one free variable, x, then $\exists x\, \varphi(x)$ can be considered guarded: it is equivalent to $\exists x\, ((x = x) \wedge \varphi(x))$, with $x = x$ as its guard. Observe that in the definition of $\mathsf{GF}^{=}_{\boxed{1}}\mathsf{ML}_\iota$ the restriction to closed definite descriptions is necessary to guarantee decidability of the logic, even without modal operators. We have seen that the formula $\forall x\, ((x = x) \to F(x, \iota y.F(x, y))$ ensures that F

is a graph of a partial function, but the non-modal guarded fragment with such functionality statements is undecidable [39, Theorem 5.8].

We also consider fragments that do not contain definite descriptions or constants: if in a fragment name we replace the ι subscript by c, then we refer to its restriction that admits constant symbols but not definite descriptions, and if we drop the subscript altogether, then we refer to the restriction without definite descriptions and constant symbols (i.e., having only variables as terms of the language).

Let $\mathcal{L}$ be a fragment of $\mathsf{Q}^{=}\mathsf{ML}_{\iota}$ with n modalities and $\mathcal{C}$ a class of frames with n accessibility relations. We consider the following decision problems.

$\mathcal{C}$-Validity in $\mathcal{L}$: Given an $\mathcal{L}$-formula φ, is φ valid on $\mathcal{C}$?

Global $\mathcal{C}$-Consequence in $\mathcal{L}$: Given an $\mathcal{L}$-formula φ and an $\mathcal{L}$-theory Γ, is φ a global $\mathcal{C}$-consequence of Γ?

In the sequel, we can add qualifiers to the problems above when we intend to refer to the corresponding problem restricted to certain classes of interpretations: for instance, "total $\mathcal{C}$-validity in $\mathcal{L}$" is the problem of validity of $\mathcal{L}$-formulas in total interpretations based on frames from $\mathcal{C}$.

Our main concern is decidability and complexity of the above reasoning problems for the standard classes of frames.

The *signature* $sig(\varphi)$ of a formula φ is the set of predicate symbols and constants in φ; the signature $sig(\Gamma)$ of a theory Γ is defined similarly. Let $\mathcal{C}$ be a class of frames. For sentences φ and φ', we write $\varphi \leq_{\mathcal{C}} \varphi'$ if every interpretation based on a frame from $\mathcal{C}$ and satisfying φ can be made to satisfy φ' by modifying the interpretation of symbols in $sig(\varphi') \setminus sig(\varphi)$, while preserving the interpretation of symbols in $sig(\varphi)$. A sentence φ' is called a *$\mathcal{C}$-model-conservative extension* of a sentence φ if $\varphi \leq_{\mathcal{C}} \varphi'$ and every interpretation based on a frame from $\mathcal{C}$ and satisfying φ' also satisfies φ. It should be clear that any polynomial-time transformation τ of formulas φ such that $\tau(\varphi)$ is a $\mathcal{C}$-conservative extension of φ is a polytime-reduction of $\mathcal{C}$-validity of formulas of the form $\neg\varphi$ to formulas of the form $\neg\tau(\varphi)$.

We also define the notion of $\mathcal{C}$-model-conservative extensions for pairs of the form (φ, Γ), for a sentence φ and a theory Γ: we write $(\varphi, \Gamma) \leq_{\mathcal{C}} (\varphi, \Gamma')$ if, for any interpretation $\mathfrak{M}$ based on a frame in $\mathcal{C}$ such that $\mathfrak{M} \models \Gamma$ and $\mathfrak{M}, w \models \varphi$, for some $w \in W$, there is an extension $\mathfrak{M}'$ that coincides with $\mathfrak{M}$ on $sig(\varphi) \cup sig(\Gamma)$ such that $\mathfrak{M}' \models \Gamma'$ and $\mathfrak{M}', w \models \varphi'$. Then (φ, Γ') is a *$\mathcal{C}$-conservative extension* of (φ, Γ) if $(\varphi, \Gamma) \leq_{\mathcal{C}} (\varphi, \Gamma')$ and, for any interpretation $\mathfrak{M}'$ based on a frame from $\mathcal{C}$ such that $\mathfrak{M}' \models \Gamma'$ and $\mathfrak{M}', w \models \varphi'$, for some $w \in W$, we have $\mathfrak{M}' \models \Gamma$ and $\mathfrak{M}', w \models \varphi$.

3. Reductions and Related Formalisms

In this section, we first observe that reasoning in $\mathsf{Q}^{1=}\mathsf{ML}_{c}$ and a variant in which constants are replaced by an "elsewhere" quantifier are mutually polytime-reducible. This enables us to transfer existing undecidability results to $\mathsf{Q}^{1=}\mathsf{ML}_{c}$.

We also establish polytime reductions that allow us to eliminate definite descriptions and partial designators. Finally, we remind the reader of a reduction of reasoning in expanding-domain models to reasoning in constant-domain models.

3.1. Non-Rigid Constants and Difference Operator

We start by introducing $\mathsf{Q}^{1\neq}\mathsf{ML}$, the language obtained from $\mathsf{Q}^{1=}\mathsf{ML}_c$ by replacing constants by the "elsewhere" quantifier. It was introduced and investigated in [14, 13, 46]. Formulas in $\mathsf{Q}^{1\neq}\mathsf{ML}$ are defined by the following grammar:

$$\varphi ::= P(x) \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \exists x\, \varphi \mid \exists^{\neq} x\, \varphi \mid \Diamond_a \varphi$$

for unary $P \in \mathsf{Pred}$ and a single fixed $x \in \mathsf{Var}$, where $\exists^{\neq}$ is called an *elsewhere* or *difference quantifier*. The semantics is given as in Section 2.1, with the following additional clause:

$$\mathfrak{M}, w \models^{\mathfrak{a}} \exists^{\neq} x\, \varphi \quad \text{iff} \quad \mathfrak{M}, w \models^{\mathfrak{a}'} \varphi, \text{ for some } x\text{-variant } \mathfrak{a}' \text{ of } \mathfrak{a} \text{ at } w \text{ } \textit{different} \text{ from } \mathfrak{a}.$$

We emphasise that, unlike for $\exists x\, \varphi$, the truth value of $\exists^{\neq} x\, \varphi$ depends on $\mathfrak{a}(x)$ and so x is 'free' in $\exists^{\neq} x\, \varphi$.

Theorem 3. *Let $\mathcal{C}$ be any class of frames. $\mathcal{C}$-validity in $\mathsf{Q}^{1=}\mathsf{ML}_c$ and $\mathsf{Q}^{1\neq}\mathsf{ML}$ are mutually polytime-reducible, with both constant and expanding domains. The same applies to global $\mathcal{C}$-consequence.*

Proof. The proof lifts to our setting the results by Gargov and Goranko [47, Section 4.1] for propositional modal logic, which show that the difference modality has the same expressive power as nominals with the universal modality.

Reasoning in $\mathsf{Q}^{1=}\mathsf{ML}_c$ is polytime-reducible to reasoning in $\mathsf{Q}^{1\neq}\mathsf{ML}$. Without loss of generality, we assume constants occur only in atoms of the form $x = c$: replacing each $P(c)$ by $\exists x \big(P(x) \wedge (x = c)\big)$, each $c = d$ by $\exists x \big((x = c) \wedge (x = d)\big)$ and each $c = x$ by $x = c$ results in an equivalent formula of linear size. We also consider only sentences, as validity of formulas coincides with validity of their universal closures.

Given a $\mathsf{Q}^{1=}\mathsf{ML}_c$ sentence φ, we take a fresh unary predicate Q_c, for every $c \in \mathsf{Con}(\varphi)$, and define $\varphi^\dagger$ as the $\mathsf{Q}^{1\neq}\mathsf{ML}$ sentence obtained by replacing every $x = c$ by $Q_c(x) \wedge \neg\exists^{\neq} x\, Q_c(x)$. For a $\mathsf{Q}^{1=}\mathsf{ML}_c$ theory Γ, we set $\Gamma^\dagger = \{\gamma^\dagger \mid \gamma \in \Gamma\}$. We now show that $(\varphi, \Gamma) \leq_{\mathcal{C}} (\varphi^\dagger, \Gamma^\dagger)$ and $(\varphi^\dagger, \Gamma^\dagger) \leq_{\mathcal{C}} (\varphi, \Gamma)$.

($\Rightarrow$) Given $\mathfrak{M}$, we extend it to $\mathfrak{M}^\dagger$ that additionally interprets, for every w,

$$Q_c^{\mathfrak{M}^\dagger(w)} = \begin{cases} \{c^{\mathfrak{M}(w)}\}, & \text{if } c^{\mathfrak{M}(w)} \text{ is defined,} \\ \emptyset, & \text{otherwise.} \end{cases}$$

It is enough to show that, for any $\mathsf{Q}^{1=}\mathsf{ML}_c$ formula ψ, we have $\mathfrak{M}, w \models^{\mathfrak{a}} \psi$ iff $\mathfrak{M}^\dagger, w \models^{\mathfrak{a}} \psi^\dagger$, for every assignment $\mathfrak{a}$ at every world w. We proceed by induction on the structure of ψ. For the base case of $x = c$, we have $\mathfrak{M}, w \models^{\mathfrak{a}} x = c$ iff

$c^{\mathfrak{M}(w)}$ is defined and $\mathfrak{a}(x) = c^{\mathfrak{M}(w)}$. By the definition of $\mathfrak{M}^\dagger$, this is equivalent to $\mathfrak{M}^\dagger, w \models^{\mathfrak{a}} Q_c(x) \wedge \neg\exists^{\neq} x\, Q_c(x)$, meaning $\mathfrak{M}^\dagger, w \models^{\mathfrak{a}} (x = c)^\dagger$. The remaining base case of $P(x)$ is trivial, and the inductive cases follow by a straightforward application of the induction hypothesis.

($\Leftarrow$) Given $\mathfrak{M}^\dagger$, we define $\mathfrak{M}$ as an extension of $\mathfrak{M}^\dagger$ such that, for every w,

$$c^{\mathfrak{M}(w)} \text{ is } \begin{cases} d, & \text{if } Q_c^{\mathfrak{M}^\dagger(w)} = \{d\}, \text{ for some } d \in \Delta, \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

As above, for any $\mathsf{Q}^{1=}\mathsf{ML}_c$ formula ψ, it can be seen that $\mathfrak{M}^\dagger, w \models^{\mathfrak{a}} \psi^\dagger$ iff $\mathfrak{M}, w \models^{\mathfrak{a}} \psi$, for every assignment $\mathfrak{a}$ at every world w.

Global $\mathcal{C}$-consequence in $\mathsf{Q}^{1\neq}\mathsf{ML}$ is polytime-reducible to that in $\mathsf{Q}^{1=}\mathsf{ML}_c$. Given a $\mathsf{Q}^{1\neq}\mathsf{ML}$ sentence φ, we recursively replace every subformula of the form $\exists^{\neq} x\, \psi$, where ψ does not contain other occurrences of $\exists^{\neq} x$, by $(\exists^{\neq} x\, \psi)^\ddagger$ defined by

$$(\exists^{\neq} x\, \psi)^\ddagger \;\; = \;\; \exists x\, P_\psi(x) \;\wedge\; \big(x = c_\psi \to \exists x \left(\neg(x = c_\psi) \wedge P_\psi(x)\right)\big),$$

where c_ψ is a fresh constant and P_ψ is a fresh unary predicate symbol (which plays the role of a surrogate for ψ and is used to avoid an exponential blow-up in the transformation), and denote the resulting $\mathsf{Q}^{1=}\mathsf{ML}_c$ sentence by $\varphi^\ddagger$. Moreover, we denote by $\Gamma_\varphi^\ddagger$ the set of $\mathsf{Q}^{1=}\mathsf{ML}_c$ sentences singl_ψ defined by

$$\mathsf{singl}_\psi \;\; = \;\; \forall x \left(\psi(x) \to P_\psi(x)\right) \;\; \wedge \;\; \forall x \left(P_\psi(x) \to \psi(x) \wedge \psi(c_\psi)\right),$$

for each subformula $\exists^{\neq} x\, \psi$ replaced in the transformation of φ.

Claim 1. *For any $\mathsf{Q}^{1\neq}\mathsf{ML}$ sentence χ and any $\mathfrak{M}^\ddagger$ such that $\mathfrak{M}^\ddagger \models \Gamma_\chi^\ddagger$, we have $\mathfrak{M}^\ddagger, w \models \chi$ iff $\mathfrak{M}^\ddagger, w \models \chi^\ddagger$, for any world w of $\mathfrak{M}^\ddagger$.*

Proof. The proof is by induction on the structure of χ: we show only the case of $\exists^{\neq} x\, \psi$, where ψ does not contain other occurrences of $\exists^{\neq} x$, since the others follow by straightforward applications of the inductive hypothesis.

($\Rightarrow$) Suppose $\mathfrak{M}^\ddagger, w \models^{\mathfrak{a}} \exists^{\neq} x\, \psi$. Then $\mathfrak{M}^\ddagger, w \models^{\mathfrak{a}'} \psi$, for some x-variant $\mathfrak{a}'$ of $\mathfrak{a}$ at w different from $\mathfrak{a}$. Thus, $\mathfrak{M}^\ddagger, w \models^{\mathfrak{a}} \exists x\, \psi$. It remains to satisfy the second conjunct of $(\exists^{\neq} x\, \psi)^\ddagger$. As $\mathfrak{M}^\ddagger$ satisfies singl_ψ, we have $\mathfrak{M}^\ddagger, w \models^{\mathfrak{a}''} \psi$, for an x-variant $\mathfrak{a}''$ of $\mathfrak{a}$ and $\mathfrak{a}'$ at w with $\mathfrak{a}''(x) = c_\psi^{\mathfrak{M}^\ddagger(w)}$. We distinguish two cases. If $\mathfrak{a}(x) \neq \mathfrak{a}''(x)$, then it is satisfied vacuously. Otherwise, $\mathfrak{M}^\ddagger, w \models^{\mathfrak{a}'} \neg(x = c_\psi) \wedge \psi$ and, by the first conjunct in singl_ψ, $\mathfrak{M}^\ddagger, w \models^{\mathfrak{a}'} \neg(x = c_\psi) \wedge P_\psi(x)$, as required.

($\Leftarrow$) The converse direction is shown similarly. $\dashv$

Now, given a $\mathsf{Q}^{1\neq}\mathsf{ML}$ theory Γ, we set $\Gamma^\ddagger = \{\gamma^\ddagger \mid \gamma \in \Gamma\} \cup \bigcup_{\gamma\in\Gamma} \Gamma_\gamma^\ddagger$ and show that $(\varphi^\ddagger, \Gamma_\varphi^\ddagger \cup \Gamma^\ddagger)$ is $\mathcal{C}$-model-conservative extension of (φ, Γ).

($\Rightarrow$) We extend $\mathfrak{M}$ to $\mathfrak{M}^\ddagger$ that interprets the constants c_ψ, for subformulas $\exists^{\neq} x\, \psi$ of φ and Γ, as follows, for every w: we take any x-variant $\mathfrak{a}'$ of $\mathfrak{a}$ at w such that $\mathfrak{M}, w \models^{\mathfrak{a}'} \psi$ (if it exists) and set $c_\psi^{\mathfrak{M}^\ddagger(w)} = \mathfrak{a}'(x)$; otherwise, if such

an x-variant does not exist, we leave $c_\psi^{\mathfrak{M}^\ddagger(w)}$ undefined. Let $\mathfrak{M}, v \models \varphi$ and $\mathfrak{M} \models \Gamma$. By the definition of $\mathfrak{M}^\ddagger$, we have $\mathfrak{M}^\ddagger \models \Gamma^\ddagger_\varphi \cup \Gamma^\ddagger_\gamma$, whence, by Claim 1, $\mathfrak{M}^\ddagger, v \models \varphi^\ddagger$ and $\mathfrak{M}^\ddagger \models \gamma^\ddagger$, for each $\gamma \in \Gamma$. Thus, $\mathfrak{M}^\ddagger \models \Gamma^\ddagger$.

($\Leftarrow$) Let $\mathfrak{M}^\ddagger, v \models \varphi^\ddagger$ and $\mathfrak{M}^\ddagger \models \Gamma^\ddagger_\varphi \cup \Gamma^\ddagger$. By the definition of $\Gamma^\ddagger$ and Claim 1, we have $\mathfrak{M}^\ddagger \models \Gamma$. Finally, since $\mathfrak{M}^\ddagger \models \Gamma^\ddagger_\varphi$, by Claim 1, we obtain $\mathfrak{M}^\ddagger, v \models \varphi$.

$\mathcal{C}$-validity in $\mathsf{Q}^{1\neq}\mathsf{ML}$ is polytime-reducible to that in $\mathsf{Q}^{1=}\mathsf{ML}_c$. We modify the proof above by carefully selecting sequences of boxes to avoid an exponential blowup of the formula. To this end, for a sentence φ and a subformula ψ of φ, we define the set of *ψ-relevant paths in φ*, denoted by $\mathsf{rp}(\varphi, \psi)$, as the set of sequences $a_1, \ldots, a_n$ of the modalities under which ψ occurs in φ. For instance, for $\varphi = \Diamond_1 \neg P(c) \wedge \Diamond_2 \Diamond_3 P(c)$, we have $\mathsf{rp}(\varphi, P(c)) = \{\langle 1\rangle, \langle 2, 3\rangle\}$ and $\mathsf{rp}(\varphi, \neg P(c)) = \{\langle 1\rangle\}$. Clearly, the maximum length of a path in $\mathsf{rp}(\varphi, \psi)$ is $d(\varphi)$. Moreover, for any path π, we recursively define $\Box^\epsilon \chi = \chi$ and $\Box^{a\cdot\pi}\chi = \Box_a \Box^\pi \chi$, where ϵ denotes the empty path. Then, we can again replace any occurrence of $\exists^{\neq} x\, \psi$ in φ not containing another $\exists^{\neq}$ by $(\exists^{\neq} x\, \psi)^\ddagger$, while adding to φ the conjunct $\Box^\pi \mathsf{singl}_\psi$ for every $\pi \in \mathsf{rp}(\varphi, \exists^{\neq} x\, \psi)$. As above, by repeatedly applying this procedure in a bottom-up way, we obtain in polynomial time a sentence $\varphi^\ddagger$ that is a $\mathcal{C}$-model-conservative extension of φ. $\dashv$

We use Theorem 3 to show how our first undecidability result follows in a straightforward way from earlier work on products of modal logics [14].

Theorem 4. *For constant-domain models, global $\mathbf{K}_n$-consequence with $n \geq 1$ and global $\mathbf{S5}_n$-consequence with $n \geq 2$ in $\mathsf{Q}^{1=}\mathsf{ML}_c$ are undecidable.*

Proof. (sketch) We give the proof for $\mathbf{K}_n$, $n \geq 1$. Then undecidability for $\mathbf{S5}_n$, $n \geq 2$, can be shown using standard reductions of modal logic on $\mathbf{K}_n$ frames for $n = 1$ to modal logic on $\mathbf{S5}_n$ frames for $n = 2$ [4, Theorem 2.37]. To show undecidability of the global $\mathbf{K}_n$-consequence in $\mathsf{Q}^{1=}\mathsf{ML}_c$ it suffices to show undecidability of the global $\mathbf{K}_n$-consequence in $\mathsf{Q}^{1\neq}\mathsf{ML}$. This is shown in [14], where the result is formulated in terms of a product modal logic that corresponds to the extension of $\mathsf{Q}^{1\neq}\mathsf{ML}$ with the universal modality on $\mathbf{K}_n$ frames with constant domain. It is straightforward to see that one can replace the universal modality by the global $\mathbf{K}_n$-consequence in the undecidability proof. $\dashv$

Remark 5. Note that Theorem 4 does not hold for **S5**. In fact, the decidability and complexity results we show below for **S5**-validity also hold for global **S5**-consequence since, for any relevant Γ and φ, the following holds: φ is a global **S5**-consequence of Γ iff $\Box \bigwedge \Gamma \to \varphi$ is **S5**-valid.

3.2. Simplifying the Problem Landscape

We first observe that the distinction between partial and total interpretations and the availability of definite descriptions does not affect the complexity of our decision problems for the fragments we are concerned with. In addition, the standard reduction from expanding to constant domains still holds in our setting with partial interpretations.

Theorem 6. *Let $\mathcal{L}$ be $\mathsf{Q}^{1=}\mathsf{ML}_\iota$, $\mathsf{C}^2_{\boxed{1}}\mathsf{ML}_\iota$, $\mathsf{GF}^{=}_{\boxed{1}}\mathsf{ML}_\iota$ or $\mathsf{Q}^{=}_{\boxed{1}}\mathsf{ML}_\iota$, and let $\mathcal{C}$ be any class of frames.*

a) *$\mathcal{C}$-validity in $\mathcal{L}$ is polytime-reducible to $\mathcal{C}$-validity in $\mathcal{L}$ without definite descriptions, with both constant and expanding domains.*

b) *In $\mathcal{L}$, the problems of $\mathcal{C}$-validity in partial and total interpretations are mutually polytime-reducible, with both constant and expanding domains.*

c) *In $\mathcal{L}$, $\mathcal{C}$-validity with expanding domains is polytime-reducible to constant-domain $\mathcal{C}$-validity.*

Items a)–c) *also apply to global $\mathcal{C}$-consequence in $\mathcal{L}$.*

Proof. Without loss of generality, we consider the problems for sentences.

Item a). First, we can assume that definite descriptions contain only atomic formulas. This is achieved by the following normalisation procedure.

For the case of global $\mathcal{C}$-consequence, given an $\mathcal{L}$-sentence φ and an $\mathcal{L}$-theory Γ, let $\iota x.\psi(x,\boldsymbol{y})$ be a definite description occurring in φ or Γ, where $\psi(x,\boldsymbol{y})$ is a formula with free variables x and $\boldsymbol{y}$. We replace any occurrence of $\iota x.\psi(x,\boldsymbol{y})$ in φ or Γ with $\iota x.P_\psi(x,\boldsymbol{y})$, where P_ψ is a fresh $(1+|\boldsymbol{y}|)$-ary predicate symbol, and add to the resulting theory sentence norm_ψ defined by taking

$$\mathsf{norm}_\psi \quad = \quad \forall x \forall \boldsymbol{y}\, \big(P_\psi(x,\boldsymbol{y}) \leftrightarrow \psi(x,\boldsymbol{y})\big).$$

Observe that the definite description is of the form $\iota x.\psi(x)$ if $\mathcal{L}$ is either $\mathsf{Q}^{1=}\mathsf{ML}_\iota$ or $\mathsf{GF}^{=}_{\boxed{1}}\mathsf{ML}_\iota$. It follows that in these cases $\boldsymbol{y}$ is empty and so the additional conjunct belongs to $\mathcal{L}$. By repeated applications of this procedure, always starting from innermost definite descriptions (that is, definite descriptions that do not contain other definite descriptions), it can be seen that we obtain in polynomial time a $\mathcal{C}$-model-conservative extension of (φ,Γ).

For the case of $\mathcal{C}$-validity, we adjust the normalisation just described by selecting relevant paths of modalities, as done in the proof of Theorem 3 above: we replace any occurrence of $\iota x.\psi(x,\boldsymbol{y})$ in φ with $\iota x.P_\psi(x,\boldsymbol{y})$, while adding to the conjunct $\Box^\pi \mathsf{norm}_\psi$ for every $\pi \in \mathsf{rp}(\varphi,\psi)$ to the formula. By repeatedly applying this procedure in a bottom-up way, we obtain in polynomial time a model-conservative extension of φ.

Now, using the normalisation procedures described above, we can assume that definite descriptions occurring in an $\mathcal{L}$-sentence φ are of the form $\iota x.Q(x,\boldsymbol{y})$, where $\boldsymbol{y}$ are variables and Q is a predicate symbol. To eliminate definite descriptions from φ, we replace atoms of the form $\alpha(\iota x.Q(x,\boldsymbol{y}),\boldsymbol{\tau})$, by the following:

- $\exists x\,\big(\alpha(x,\boldsymbol{\tau}) \wedge Q(x,\boldsymbol{y}) \wedge \forall x'\,(Q(x',\boldsymbol{y}) \to x' = x)\big)$, in $\mathcal{L} = \mathsf{Q}^{=}_{\boxed{1}}\mathsf{ML}_\iota$;
- $\exists x\,\big(\alpha(x,\boldsymbol{\tau}) \wedge Q(x,\boldsymbol{y})\big) \wedge \exists^{=1} x\, Q(x,\boldsymbol{y})$, in $\mathcal{L} = \mathsf{C}^2_{\boxed{1}}\mathsf{ML}_\iota$;
- $\alpha(c_{\iota x.Q(x)},\boldsymbol{\tau}) \wedge Q(c_{\iota x.Q(x)}) \wedge \forall x\,\big(Q(x) \to x = c_{\iota x.Q(x)}\big)$, where $c_{\iota x.Q(x)}$ is a fresh constant symbol, in $\mathcal{L} = \mathsf{Q}^{1=}\mathsf{ML}_\iota$ or $\mathcal{L} = \mathsf{GF}^{=}_{\boxed{1}}\mathsf{ML}_\iota$.

Note that the first two formulas are equivalent, while the third is obtained by introducing a Skolem constant, $c_{\iota x.Q(x)}$, for the existential quantifier in the first formula (this, however, relies on $\boldsymbol{y}$ being empty). Denote by φ^* the result of applying the above transformations to φ. It can be seen that φ^*, which is polynomial in the size of φ, is a model-conservative extension of φ. Similarly, (φ^*, Γ^*) is a $\mathcal{C}$-model-conservative extension of (φ, Γ).

Item b). Due to Item a), we can assume that every term τ occurring in a formula φ is either a variable or a constant.

From partial to total interpretations. Let φ be an $\mathcal{L}$-sentence and Γ an $\mathcal{L}$-theory. For a constant c, take a fresh 0-ary predicate p_c (i.e., a propositional letter): it represents the statement that c designates at the world. We define φ' by replacing each $P(\tau_1, \ldots, \tau_m)$ and each $\tau_1 = \tau_2$ in φ with

$$\bigwedge_{c_i \in \{\tau_1, \ldots, \tau_m\} \cap \mathsf{Con}} p_{c_i} \quad \wedge \quad P(\tau_1, \ldots, \tau_m) \qquad \text{and} \qquad \bigwedge_{c_i \in \{\tau_1, \tau_2\} \cap \mathsf{Con}} p_{c_i} \quad \wedge \quad (\tau_1 = \tau_2),$$

respectively. Let $\Gamma' = \{\gamma' \mid \gamma \in \Gamma\}$. We show that φ is satisfied in $\mathfrak{M}$ based on $\mathfrak{F}$ (provided that $\mathfrak{M} \models \Gamma$) iff φ' is satisfied in a total interpretation $\mathfrak{M}'$ based on $\mathfrak{F}$ (respectively, provided that $\mathfrak{M}' \models \Gamma'$); moreover, $\mathfrak{M}$ and $\mathfrak{M}'$ share the same domains. The claim for both problems follows.

($\Rightarrow$) Given an interpretation $\mathfrak{M}$, define a total interpretation $\mathfrak{M}'$ as $\mathfrak{M}$, with the addition of the following, for every $c_i \in \mathsf{Con}(\varphi) \cup \mathsf{Con}(\Gamma)$ and every world w:

$$p_{c_i}^{\mathfrak{M}'(w)} \text{ is } \begin{cases} \text{true}, & \text{if } c_i \text{ designates at } w \text{ in } \mathfrak{M}; \\ \text{false}, & \text{otherwise.} \end{cases}$$

We show, by induction on a formula ψ, that $\mathfrak{M}, w \models^{\mathfrak{a}} \psi$ iff $\mathfrak{M}', w \models^{\mathfrak{a}} \psi'$, for every assignment $\mathfrak{a}$ at every world w. For the base case $P(\tau_1, \ldots, \tau_m)$, we have $\mathfrak{M}, w \models^{\mathfrak{a}} P(\tau_1, \ldots, \tau_m)$ iff each $c_i \in \{\tau_1, \ldots, \tau_m\} \cap \mathsf{Con}$ designates at w in $\mathfrak{M}$ and $(\tau_1^{\mathfrak{M}(w), \mathfrak{a}}, \ldots, \tau_m^{\mathfrak{M}(w), \mathfrak{a}}) \in P^{\mathfrak{M}(w)}$. By construction of $\mathfrak{M}'$, this is equivalent to $p_{c_i}^{\mathfrak{M}'(w)}$ being true, for each $c_i \in \{\tau_1, \ldots, \tau_m\} \cap \mathsf{Con}$, and $(\tau_1^{\mathfrak{M}'(w), \mathfrak{a}}, \ldots, \tau_m^{\mathfrak{M}'(w), \mathfrak{a}}) \in P^{\mathfrak{M}'(w)}$, meaning that $\mathfrak{M}', w \models^{\mathfrak{a}} \bigwedge_{c_i \in \{\tau_1, \ldots, \tau_m\} \cap \mathsf{Con}} p_{c_i} \wedge P(\tau_1, \ldots, \tau_m)$. The base case $\tau_1 = \tau_2$ is similar, and the inductive cases follow by a straightforward application of the inductive hypothesis. It follows that $\mathfrak{M} \models \Gamma$ and $\mathfrak{M}, v \models \varphi$ imply that $\mathfrak{M}' \models \Gamma'$ and $\mathfrak{M}', v \models \varphi'$.

($\Leftarrow$) Given a total interpretation $\mathfrak{M}'$, we define $\mathfrak{M}$ that coincides with $\mathfrak{M}'$, except for the following, for every $c_i \in \mathsf{Con}(\varphi) \cup \mathsf{Con}(\Gamma)$ and every world w:

$$c_i^{\mathfrak{M}(w)} \text{ is } \begin{cases} c_i^{\mathfrak{M}'(w)}, & \text{if } p_{c_i}^{\mathfrak{M}'(w)} \text{ is true;} \\ \text{undefined}, & \text{otherwise;} \end{cases}$$

We show, by induction on a formula ψ, that $\mathfrak{M}', w \models^{\mathfrak{a}} \psi'$ iff $\mathfrak{M}, w \models^{\mathfrak{a}} \psi$, for every assignment $\mathfrak{a}$ at every world w. For the base case $P(\tau_1, \ldots, \tau_m)$, we have $\mathfrak{M}', w \models^{\mathfrak{a}} \bigwedge_{c_i \in \{\tau_1, \ldots, \tau_m\} \cap \mathsf{Con}} p_{c_i} \wedge P(\tau_1, \ldots, \tau_m)$ iff $p_{c_i}^{\mathfrak{M}'(w)}$ is true, for every

$c_i \in \{\tau_1, \ldots, \tau_m\} \cap \mathsf{Con}$, and $(\tau_1^{\mathfrak{M}(w),\mathfrak{a}}, \ldots, \tau_m^{\mathfrak{M}(w),\mathfrak{a}}) \in P^{\mathfrak{M}'(w)}$. By construction of $\mathfrak{M}$, this is equivalent to $c_i^{\mathfrak{M}(w)}$ being defined, with $c_i^{\mathfrak{M}(w)} = c_i^{\mathfrak{M}'(w)}$, for every $c_i \in \{\tau_1, \ldots, \tau_m\} \cap \mathsf{Con}$, and $(\tau_1^{\mathfrak{M}(w),\mathfrak{a}}, \ldots, \tau_m^{\mathfrak{M}(w),\mathfrak{a}}) \in P^{\mathfrak{M}(w)}$. By definition, it means that $\mathfrak{M}, w \models^{\mathfrak{a}} P(\tau_1, \ldots, \tau_m)$. The base case $\tau_1 = \tau_2$ is similar, and the inductive cases follow again by a straightforward application of the inductive hypothesis. In conclusion, $\mathfrak{M}' \models \Gamma'$ and $\mathfrak{M}', v \models \varphi'$ imply $\mathfrak{M} \models \Gamma$ and $\mathfrak{M}, v \models \varphi$.

From total to partial interpretations. First, consider global $\mathcal{C}$-consequence. Let φ be an $\mathcal{L}$-sentence and Γ an $\mathcal{L}$-theory. Define Γ' by adding to Γ the sentences

$$\exists x\,(x = c),$$

for every constant symbol c occurring in Γ or φ. We first show that φ is satisfied in a total interpretation $\mathfrak{M}$ based on $\mathfrak{F}$ with $\mathfrak{M} \models \Gamma$ iff φ is satisfied in $\mathfrak{M}'$ based on $\mathfrak{F}$ with $\mathfrak{M}' \models \Gamma'$. Since the additional sentences in Γ' are trivially satisfied in any total interpretation, if $\mathfrak{M}$ is total and such that $\mathfrak{M} \models \Gamma$ and $\mathfrak{M}, v \models \varphi$, then clearly $\mathfrak{M} \models \Gamma'$ and $\mathfrak{M}, v \models \varphi$. Conversely, every interpretation $\mathfrak{M}'$ for which $\mathfrak{M}' \models \Gamma'$ and $\mathfrak{M}', v \models \varphi$ can be extended to a total interpretation $\mathfrak{M}$ such that $\mathfrak{M} \models \Gamma'$, hence $\mathfrak{M} \models \Gamma$, and $\mathfrak{M}, v \models \varphi$, as it only remains to arbitrarily fix the interpretation of constant symbols not occurring in φ and Γ.

Now, to show that $\mathcal{C}$-satisfiability in total interpretations is polytime-reducible to $\mathcal{C}$-satisfiability, we modify the reduction above by taking, for an $\mathcal{L}$-sentence φ, the conjunction φ' of φ together with all the sentences $\Box^{\pi}\exists x\,(x = c)$, where $\mathsf{rp}(\varphi, \psi)$, for an atomic ψ such that $c \in \mathsf{Con}(\psi)$. It can be seen that φ is satisfied in a total interpretation $\mathfrak{M}$ based on $\mathfrak{F}$ iff φ' is satisfied in an interpretation $\mathfrak{M}'$ based on $\mathfrak{F}$; moreover, $\mathfrak{M}$ and $\mathfrak{M}'$ share the same domains.

Item c). By Items a) and b), it is sufficient to reduce, in $\mathcal{L}$ *without* definite descriptions, *total* $\mathcal{C}$-validity with expanding domains to *total* $\mathcal{C}$-validity with constant domain. The reduction is analogous to the one given in [4, Proposition 3.20 (ii)]. The same applies to the case of global $\mathcal{C}$-consequence. ⊣

Infinite branching models. As the following example shows, we can easily enforce infinite branching using equality or counting quantifiers.

Example 7. Let φ_0 be a C^2-sentence that has only infinite models, e.g.,

$$\forall x \exists^{=1} y\, P(x, y) \quad \wedge \quad \forall x \exists^{\leq 1} z\, P(z, x) \quad \wedge \quad \exists x \neg \exists z\, P(z, x).$$

Then consider the following $\mathsf{C}^2_{\Box}\mathsf{ML}_{\iota}$-sentence:

$$\varphi \quad = \quad \varphi_0 \quad \wedge \quad \forall x \Diamond_a A(x) \quad \wedge \quad \Box_a \exists^{\leq 1} x\, A(x).$$

Figure 1 illustrates an interpretation satisfying φ: in world w_0, sentence φ_0 forces an infinite chain $d_0, d_1, d_2, \ldots$ of domain elements connected by P: every element in the chain has a unique P-successor, and every element but d_0 has a unique P-predecessor, while d_0 has no P-predecessors. By the remaining two conjuncts of φ, for each d_i, $i \in \mathbb{N}$, world w_0 has an R_a-successor world v_i that contains one and only one element in A, thus resulting in infinite branching.

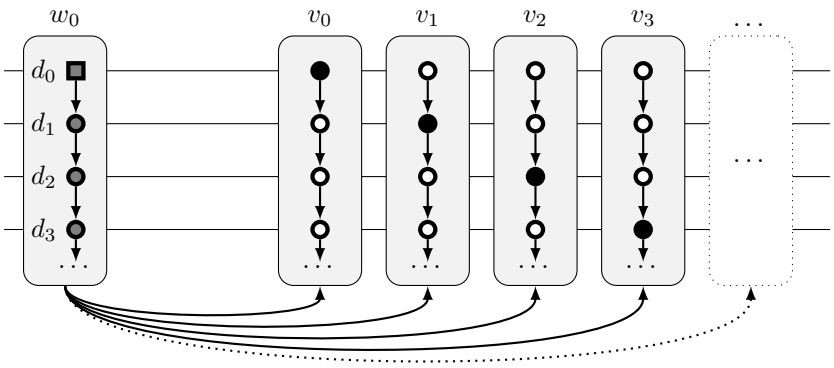

Figure 1: An interpretation satisfying φ in Example 7.

4. Quasimodels for $\mathsf{Q}^{=}_{\boxed{1}}\mathsf{ML}_c$

In this section we present a straightforward generalisation of the standard quasimodel technique (see, e.g., [4]) to the languages with constants, equality and/or counting quantifiers.

For every monodic formula of the form $\Diamond_a\psi(x)$ with one free variable x, we reserve a unary predicate $R_{\Diamond_a\psi}(x)$, and, for every monodic sentence of the form $\Diamond_a\psi$, a propositional variable $p_{\Diamond_a\psi}$. Symbols $R_{\Diamond_a\psi}(x)$ and $p_{\Diamond_a\psi}$ are called the *surrogates* of $\Diamond_a\psi(x)$ and $\Diamond_a\psi$, respectively. For clarity of presentation, we assume that these surrogates are *not* in the original signature.

Given a $\mathsf{Q}^{=}_{\boxed{1}}\mathsf{ML}_c$-formula φ, we denote by $\overline{\varphi}$ the result of replacing all its subformulas of the form $\Diamond_a\psi(x)$ or $\Diamond_a\psi$, which are not in the scope of another modal operator, by their surrogates. Thus, $\overline{\varphi}$ contains no occurrences of modal operators at all. Observe that, for all monodic formulas ψ_1 and ψ_2, we have

$$\overline{\psi_1 \wedge \psi_2} = \overline{\psi_1} \wedge \overline{\psi_2}, \qquad \overline{\neg\psi_1} = \neg\overline{\psi_1} \quad \text{and} \quad \overline{\forall x\, \psi_1} = \forall x\, \overline{\psi_1}.$$

Let x be a variable not occurring in φ. Put

$$sub_x(\varphi) = \big\{\psi\{x/y\}, \neg\psi\{x/y\} \mid \psi(y) \in sub(\varphi)\big\},$$

where, as usual, we write $\psi(y)$ to indicate that ψ has at most one free variable, y, and $\psi\{x/y\}$ denotes the result of replacing all free occurrences of y in ψ with x. Clearly, $|sub_x(\varphi)| \leq 2|\varphi|$. By a *type* for a $\mathsf{Q}^{=}_{\boxed{1}}\mathsf{ML}_c$-sentence φ we mean any Boolean-saturated subset t of $sub_x(\varphi)$, that is,

- $\psi_1 \wedge \psi_2 \in t$ iff $\psi_1 \in t$ and $\psi_2 \in t$, for every $\psi_1 \wedge \psi_2 \in sub_x(\varphi)$;
- $\neg\psi \in t$ iff $\psi \notin t$, for every $\neg\psi \in sub_x(\varphi)$.

For a type t for φ, we write $\overline{t}$ to denote the set $\{\overline{\psi} \mid \psi \in t\}$. We will often identify such a set with the conjunction of formulas in it.

We use quasistates to describe worlds in interpretations. With the equality in the language, the description of each world needs to specify not only the types realised in that world, but also the number of domain elements that realise the type. To deal with this issue, multisets of types are useful.

A multiset $\boldsymbol{X}$ is a pair, (X, card_X), consisting of an underlying set, X, and a function, card_X that assigns to each element its multiplicity. To represent infinite domains of interpretations, the multiplicities need to range over the set $\mathbb{N}_\infty = \mathbb{N} \cup \{\aleph_0\}$, with the usual assumptions that $\aleph_0 + m = m + \aleph_0 = \aleph_0$, for each $m \in \mathbb{N}$, and $\aleph_0 + \aleph_0 = \aleph_0$. Note that we assume $0 \in \mathbb{N}$, and so $\mathit{card}_X(x)$ can in principle be 0, indicating that x does *not* belong to the multiset $\boldsymbol{X}$. Equally, we can always take a larger underlying set and set $\mathit{card}_X(x) = 0$ for any element x outside the original underlying set. In the sequel, we often omit the underlying set, assuming that it is clear from the context, and slightly abuse notation by identifying a multiset $\boldsymbol{X}$ with its multiplicity function card_X and write $\boldsymbol{X}(x)$ for $\mathit{card}_X(x)$. As a shortcut, we also write $x \in \boldsymbol{X}$ whenever $\boldsymbol{X}(x) > 0$ and so that, for example, a multiset $\boldsymbol{X}$ is non-empty if there is $x \in \boldsymbol{X}$. Given a multiset $\boldsymbol{X}$, we denote by $|\boldsymbol{X}|$ its cardinality, that is, $\sum_{x \in X} \boldsymbol{X}(x)$.

Formally, a *quasistate candidate for* φ is a non-empty multiset $\boldsymbol{n}$ of types for φ. We say that $\boldsymbol{n}$ is *realised in* a first-order (non-modal) structure $\mathfrak{B}$ if

$$\boldsymbol{n}(t) = |\{b \in \mathfrak{B} \mid \mathfrak{B} \models \overline{t}[b]\}|, \text{ for all types } t \text{ for } \varphi.$$

If $\mathfrak{B} \models \overline{t}[b]$, then we say that t is *realised by* b *in* $\mathfrak{B}$, and if $\boldsymbol{n}$ is realised in some $\mathfrak{B}$, it is also said to be *realisable* (without referring to $\mathfrak{B}$), and a realisable quasistate candidate is simply called a *quasistate for* φ. Note that the realisability condition allows us to deal with inconsistency and constants in a uniform way: for example, if a type t contains $\forall y\, A(y)$ and $\neg A(x)$, then $\boldsymbol{n}(t) = 0$, for any quasistate $\boldsymbol{n}$; also, if $x = c$, for $c \in \mathsf{Con}$, is a subformula of φ, then, any quasistate $\boldsymbol{n}$ will contain a type t with $(x = c) \in t$ and $\boldsymbol{n}(t) = 1$ because $\forall x\, (x \neq c)$ is unsatisfiable (we assume first-order structures interpret all constant symbols) but no other type t' with $\boldsymbol{n}(t') > 0$ can contain $x = c$ because only one element, $c^{\mathfrak{B}}$, can satisfy $x = c$ in any $\mathfrak{B}$.

Let $\mathfrak{F} = (W, \{R_a\}_{a \in A})$ be a frame. A *basic structure for* φ *based on* $\mathfrak{F}$ is a pair $(\mathfrak{F}, \boldsymbol{q})$, where $\boldsymbol{q}$ is a function associating with every $w \in W$ a quasistate $\boldsymbol{q}(w)$ for φ. In the sequel, we write $\boldsymbol{q}(w, t)$ for the multiplicity $\boldsymbol{q}(w)(t)$ of type t in $\boldsymbol{q}(w)$.

We say that $W' \subseteq W$ is *upward-closed* if W' contains all v with wR_av for $w \in W'$ and $a \in A$. In any expanding-domains interpretation $(\mathfrak{F}, \Delta, \cdot)$, the set of worlds where a domain element d exists, $\{w \in W \mid d \in \Delta_w\}$, is an upward-closed set. Runs in quasimodels correspond to domain elements, which motivates the following definition. A *run through* $(\mathfrak{F}, \boldsymbol{q})$ is a function ρ mapping each world w in an upward-closed subset W' of W to a type $\rho(w) \in \boldsymbol{q}(w)$ satisfying the following coherence and saturation conditions for every $w \in W'$:

(r-coh) $\Diamond_a \psi \in \rho(w)$ if there exists $v \in W$ with wR_av and $\psi \in \rho(v)$, for every $\Diamond_a \psi \in \mathit{sub}_x(\varphi)$;

(r-sat) if $\Diamond_a \psi \in \rho(w)$, then there exists $v \in W$ with $wR_a v$ and $\psi \in \rho(v)$, for every $\Diamond_a \psi \in sub_x(\varphi)$.

The domain, W', of ρ is denoted by $\mathsf{dom}\rho$. We say that a run ρ is *full* if $\mathsf{dom}\rho = W$.

In our quasimodels we will need to count not only the number of times a type is realised at a certain world in an interpretation, but also the number of domain elements that give rise to the same run. In other words, instead of sets of runs in standard quasimodels, we use *multisets* of runs. The following notation will be useful in the sequel. Let $\mathfrak{R}$ be a multiset of runs through a basic structure $(\mathfrak{F}, \boldsymbol{q})$. For a world $w \in W$ in $\mathfrak{F}$, we denote by $\mathfrak{R}_w$ the *w-slice of* $\mathfrak{R}$, that is, the multiset of runs ρ in $\mathfrak{R}$ such that $w \in \mathsf{dom}\rho$, with each run in the w-slice having the same multiplicity as in $\mathfrak{R}$: formally,

$$\mathfrak{R}_w(\rho) = \begin{cases} \mathfrak{R}(\rho), & \text{if } w \in \mathsf{dom}\rho, \\ 0, & \text{otherwise.} \end{cases}$$

Similarly, for a world $w \in W$ and a type t for φ, we denote by $\mathfrak{R}_{w,t}$ the *(w,t)-slice of* $\mathfrak{R}$, the multiset of runs ρ in $\mathfrak{R}$ such that $w \in \mathsf{dom}\rho$ and $\rho(w) = t$:

$$\mathfrak{R}_{w,t}(\rho) = \begin{cases} \mathfrak{R}(\rho), & \text{if } w \in \mathsf{dom}\rho \text{ and } \rho(w) = t, \\ 0, & \text{otherwise.} \end{cases}$$

An (*expanding-domains*) *quasimodel for φ based on* $\mathfrak{F}$ is a triple $\mathfrak{Q} = (\mathfrak{F}, \boldsymbol{q}, \mathfrak{R})$, where $(\mathfrak{F}, \boldsymbol{q})$ is a basic structure for φ based on $\mathfrak{F}$ and $\mathfrak{R}$ is a multiset of runs through $(\mathfrak{F}, \boldsymbol{q})$ such that

(card) $\boldsymbol{q}(w,t) = |\mathfrak{R}_{w,t}|$, for every $w \in W$ and every type t for φ.

We say that $\mathfrak{Q}$ is a *constant-domain quasimodel* if $\mathfrak{R}$ consists of full runs. A quasimodel $\mathfrak{Q} = (\mathfrak{F}, \boldsymbol{q}, \mathfrak{R})$ is said to *satisfy* φ if

(b) $\varphi \in t$ for some $w_0 \in W$ and some type $t \in \boldsymbol{q}(w_0)$.

In the sequel it will often be convenient to represent a multiset $\mathfrak{R}$ of runs as a set $\hat{\mathfrak{R}}$ of *indexed runs*:

$$\hat{\mathfrak{R}} = \{(\rho, \ell) \in \mathfrak{R} \times \mathbb{N} \mid 0 \leq \ell < \mathfrak{R}(\rho)\};$$

note that if ρ has infinite multiplicity ($\aleph_0$) in $\mathfrak{R}$, then by definition $\hat{\mathfrak{R}}$ contains all pairs of the form (ρ, ℓ), for $\ell \in \mathbb{N}$.

Example 8. We continue with our Example 7 and consider a quasimodel depicted in Fig. 2. There are two types for φ at w_0: t_0 contains $\neg\exists z\, P(z,x)$ and is shown by a grey rectangle, while t_0' contains $\exists z\, P(z,x)$ and is shown by grey circles; both types also include $\exists^{=1} y\, P(x,y)$, $\exists^{\leq 1} z\, P(z,x)$, $\Diamond_a A(x)$, $\Box_a \exists^{\leq 1} x\, A(x)$ and their universal closures and suitable conjunctions, but no other positive formula from $sub_x(\psi)$. Then there are two types for φ at each of the v_i, for $i \in \mathbb{N}$:

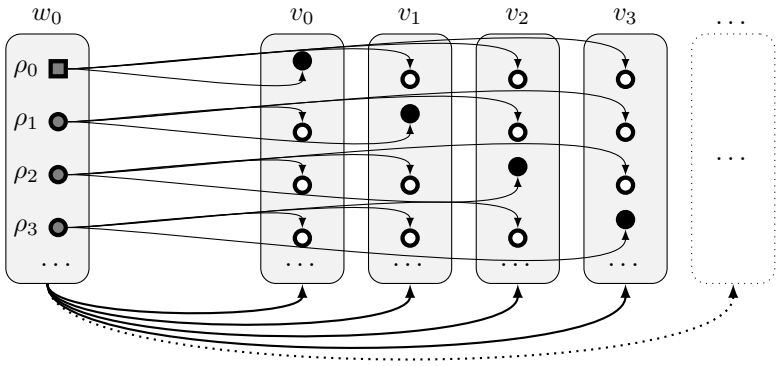

Figure 2: A quasimodel for the interpretation in Fig. 1.

t_1, depicted by black circles, contains $A(x)$ and $\exists^{\leq 1} x\, A(x)$, while t_2, depicted by white circles, contains $\neg A(x)$ and $\exists^{\leq 1} x\, A(x)$.

In the quasimodel in Fig. 2, quasistate $\boldsymbol{q}(w_0)$ contains one t_0 and $\aleph_0$-many t_0', while quasistate $\boldsymbol{q}(v_i)$, for $i \in \mathbb{N}$, contains one t_1 and $\aleph_0$-many t_2. Each run ρ_i, for $i \in \mathbb{N}$, in the multiset $\mathfrak{R}$ has multiplicity 1 and is depicted as a tree with types in quasistates being its vertices and the edges matching the accessibility relation of the frame: for each $i \in \mathbb{N}$, we have

$$\rho_i(w_0) = \begin{cases} t_0, & \text{if } i = 0, \\ t_0', & \text{otherwise,} \end{cases} \qquad \rho_i(v_i) = t_1 \quad \text{and} \quad \rho_i(v_j) = t_2, \text{ for } j \in \mathbb{N} \setminus \{i\}.$$

The next lemma provides a link between quasimodels and interpretations.

Lemma 9. *For both constant and expanding domains, a $\mathsf{Q}^{=}_{\boxed{1}}\mathsf{ML}_c$-sentence φ is satisfiable in an interpretation based on a frame $\mathfrak{F}$ iff there is a quasimodel satisfying φ and based on $\mathfrak{F}$.*

Proof. ($\Rightarrow$) Let $\mathfrak{M} = (\mathfrak{F}, \Delta, \cdot)$ be an interpretation based on $\mathfrak{F} = (W, \{R_a\}_{a \in A})$ with $\mathfrak{M}, w_0 \models \varphi$ for some $w_0 \in W$. Let

$$t^{\mathfrak{M}(w)}(d) = \big\{\psi \in sub_x(\varphi) \mid \mathfrak{M}, w \models \psi[d]\big\},$$

for every every $w \in W$ and $d \in \Delta_w$. Clearly, $t^{\mathfrak{M}(w)}(d)$ is Boolean-saturated and so is a type for φ. We now define a triple $\mathfrak{Q} = (\mathfrak{F}, \boldsymbol{q}, \mathfrak{R})$, where

- $\boldsymbol{q}$ is the function from W to the set of quasistate candidates for φ defined, for every $w \in W$, by setting $\boldsymbol{q}(w, t) = \min(\aleph_0, |\{d \in \Delta_w \mid t^{\mathfrak{M}(w)}(d) = t\}|)$, for every type t for φ;
- $\mathfrak{R}$ is the multiset of functions ρ_d, $d \in \bigcup_{w \in W} \Delta_w$, from upward-closed subsets of W to the set of types for φ defined by $\rho_d(w) = t^{\mathfrak{M}(w)}(d)$, for each $w \in W$ with $d \in \Delta_w$. Note that if k distinct $d_0, d_2, \ldots, d_{k-1}$ give rise

to the same function ρ, that is, $\rho = \rho_{d_i}$, for $0 \leq i < k$, then $\mathfrak{R}(\rho) = k$; if infinitely many distinct $d_0, d_1, \ldots$ give rise to the same function ρ, that is, $\rho = \rho_{d_i}$, for all $i \in \mathbb{N}$, then $\mathfrak{R}(\rho) = \aleph_0$.

It is easy to see that each $\boldsymbol{q}(w)$ is realisable. The elements of $\mathfrak{R}$, by construction, satisfy **(r-coh)** and **(r-sat)**. By the definition of $\boldsymbol{q}$ and $\mathfrak{R}$, the triple $\mathfrak{Q}$ satisfies **(card)**. Finally, $\mathfrak{Q}$ satisfies φ because $\varphi \in t$, for all (some) types $t \in \boldsymbol{q}(w_0)$.

($\Leftarrow$) Suppose there is a quasimodel $\mathfrak{Q} = (\mathfrak{F}, \boldsymbol{q}, \mathfrak{R})$ satisfying φ. For each $w \in W$, we consider the set $\hat{\mathfrak{R}}_w$ of indexed runs in the w-slice of $\mathfrak{R}$. Since the domains of runs are upward-closed, we have $\hat{\mathfrak{R}}_w \subseteq \hat{\mathfrak{R}}_v$, for each $w, v \in W$ with wR_av, for some $a \in A$. By **(card)**, we have $|\hat{\mathfrak{R}}_w| = \sum_t \boldsymbol{q}(w,t)$, for each $w \in W$; moreover, by the definition of quasistate, $|\hat{\mathfrak{R}}_w| > 0$. So, each $\hat{\mathfrak{R}}_w$ is either countably infinite or non-empty and finite, and $|\hat{\mathfrak{R}}_w| \leq |\hat{\mathfrak{R}}_v|$, for each $w, v \in W$ with wR_av, for some $a \in A$. It then follows that, for each $w \in W$, we can take a first-order structure $\mathfrak{B}_w$ realising $\boldsymbol{q}(w)$ with the domain of cardinality $|\hat{\mathfrak{R}}_w|$. So, for each $w \in W$, there is a bijection f_w between the set $\hat{\mathfrak{R}}_w$ and the domain of $\mathfrak{B}_w$ satisfying

$$\mathfrak{B}_w \models \overline{\rho(w)}[f_w(\rho, \ell)], \text{ for every } w \in W \text{ and every } (\rho, \ell) \in \hat{\mathfrak{R}}_w. \tag{1}$$

Define a total interpretation $\mathfrak{M} = (\mathfrak{F}, \Delta, \cdot)$, by setting $\Delta_w = \hat{\mathfrak{R}}_w$, $c^{\mathfrak{M}(w)} = f_w^{-1}(c^{\mathfrak{B}_w})$, for any constant c in φ (recall that the $\mathfrak{B}_w$ interpret all constant symbols), $P^{\mathfrak{M}(w)} = f_w^{-1}(P^{\mathfrak{B}_w})$, for any k-ary predicate symbol P in φ, where f_w^{-1} is extended to sets of k-tuples ($k \geq 0$) of indexed runs in a component-wise way. We show that, for each subformula ψ of φ, we have

$$\mathfrak{B}_w \models^{f_w \circ \mathfrak{b}} \overline{\psi} \quad \text{iff} \quad \mathfrak{M}, w \models^{\mathfrak{b}} \psi, \tag{2}$$

for all assignments $\mathfrak{b}$ and all $w \in W$, where $f_w \circ \mathfrak{b} \colon x \mapsto f_w(\mathfrak{b}(x))$, for all variables x. The basis of induction, the case of atomic formulas, including equalities, is immediate from the definition (for equalities, we use the fact that f_w is a bijection). The cases of the Boolean connectives $\wedge$ and $\neg$ and the quantifier $\exists$ easily follow from IH.

So, it remains to consider the case of subformulas of the form $\Diamond_a\psi$. First, note that ψ contains at most one free variable, say, x (for a sentence, pick any variable). Consider $(\rho, \ell) = \mathfrak{b}(x)$. Suppose first $\mathfrak{B}_w \models^{f_w \circ \mathfrak{b}} \overline{\Diamond_a\psi}$, that is, $\mathfrak{B}_w \models \overline{\Diamond_a\psi}[f_w(\rho, \ell)]$. By (1), we have $\Diamond_a\psi \in \rho(w)$, whence, by **(r-coh)**, $\psi \in \rho(v)$, for some $v \in W$ with wR_av. By (1), we have $\mathfrak{B}_v \models \overline{\rho(v)}[f_v(\rho, \ell)]$ and thus $\mathfrak{B}_v \models \overline{\psi}[f_v(\rho, \ell)]$. By IH, $\mathfrak{M}, v \models \psi[(\rho, \ell)]$, whence $\mathfrak{M}, w \models \Diamond_a\psi[(\rho, \ell)]$. Conversely, suppose $\mathfrak{M}, w \models \Diamond_a\psi[(\rho, \ell)]$. Then we have $\mathfrak{M}, v \models \psi[(\rho, \ell)]$, for some $v \in W$ with wR_av. By IH, we obtain $\mathfrak{B}_v \models \overline{\psi}[f_v(\rho, \ell)]$. By (1), we have $\psi \in \rho(v)$, whence, by **(r-sat)**, $\Diamond_a\psi \in \rho(w)$. By (1), we obtain $\mathfrak{B}_w \models \overline{\Diamond_a\psi}[f_w(\rho, \ell)]$, as required.

Now we can easily finish the proof of Lemma 9 for expanding domains by observing that, by **(b)**, there is a world $w_0 \in W$ and a type t with $\boldsymbol{q}(w_0, t) > 0$ such that $\varphi \in t$. Thus, by **(card)**, there exists $(\rho_0, \ell_0) \in \hat{\mathfrak{R}}_{w_0}$ with $\rho_0(w_0) = t$ and $\mathfrak{B}_{w_0} \models \overline{\varphi}[f_{w_0}(\rho_0, \ell_0)]$, whence, by (2), we obtain $\mathfrak{M}, w_0 \models \varphi$.

For the case of constant domains, observe that (i) constant domains mean that the runs are full and (ii) full runs give rise to constant domains. $\dashv$

Example 10. It can be seen that any quasimodel satisfying φ from our running Example 7 will have infinite branching. Indeed, it should be clear that a type containing the same positive formulas as t'_0 in Example 8 has infinite multiplicity ($\aleph_0$) in the quasistate, w_0, required by **(b)**. Then, by **(card)**, the multiset $\mathfrak{R}$ of runs contains $\aleph_0$ runs that go through t'_0 at w_0. By **(r-sat)**, there is an a-successor, v_1, of w_0 with a type, say t_1, containing $A(x)$. But since, by **(card)**, only one run can go through t_1 at v_1, each of the $\aleph_0$ runs in $\mathfrak{R}$ requires a separate a-successor, as depicted in Fig. 2.

5. Weak Quasimodels for $\mathsf{Q}^{=}_{\boxed{1}}\mathsf{ML}_c$ over $\mathbf{K}_n$ Frames

As we have seen in Section 4, standard quasimodels can require infinite frames in the presence of equality. We now define a weaker notion of quasimodel that will allow us to consider only finite frames. To this end, we first need to define a class of frames with simpler structure.

A $\mathbf{K}_n$ frame is called *tree-shaped* if there exists a $w_0 \in W$, called the *root of* $\mathfrak{F}$, such that the domain W of $\mathfrak{F}$ is a prefix-closed set of words of the form

$$\boldsymbol{w} = w_0 a_0 w_1 a_1 \cdots a_{m-1} w_m, \tag{3}$$

where $a_j \in A$, for $0 \leq j < m$, and each R_a, $a \in A$, is the smallest relation containing all pairs of the form $(\boldsymbol{w}, \boldsymbol{w}aw) \in W \times W$. For a word $\boldsymbol{w} \in W$, its last component, w_m, is denoted by $\mathit{tail}(\boldsymbol{w})$. Also, we denote m by $d(\boldsymbol{w})$ and call it the *depth of* $\boldsymbol{w}$. The *depth of* $\mathfrak{F}$ as the maximum depth $d(\boldsymbol{w})$ of $\boldsymbol{w} \in W$. Let $\mathbf{Tree}^d_n$ denote the set of all tree-shaped $\mathbf{K}_n$ frames of depth bounded by d. The following result is shown by using the standard modal logic unfolding technique lifted to the first-order modal language with constants and equality.

Lemma 11. *For both constant and expanding domains, every* $\mathbf{K}_n$*-satisfiable* $\mathsf{Q}^{=}\mathsf{ML}_c$*-formula* φ *is also* $\mathbf{Tree}^{d(\varphi)}_n$*-satisfiable.*

Proof. Let $\mathfrak{M} = (\mathfrak{F}, \Delta, \cdot)$ with $\mathfrak{F} = (W, \{R_a\}_{a \in A})$ and $\mathfrak{M}, w_0 \models^{\mathfrak{a}} \varphi$, for some $\mathfrak{a}$, be given. Unfold $\mathfrak{F}$ into a tree-shaped $\mathbf{K}_n$ frame $\mathfrak{F}^* = (W^*, \{R^*_a\}_{a \in A})$, where W^* is the set of all words $\boldsymbol{w}$ of the form (3) with $d(\boldsymbol{w}) \leq d(\varphi)$ and $w_j R_{a_j} w_{j+1}$, for each $0 \leq j < d(\boldsymbol{w})$, and $R^*_a = \{(\boldsymbol{w}, \boldsymbol{w}aw) \in W^* \times W^*\}$, for each $a \in A$. Clearly, $d(\mathfrak{F}^*) \leq d(\varphi)$. Define an interpretation $\mathfrak{M}^* = (\mathfrak{F}^*, \Delta^*, \cdot^*)$ by taking $\Delta^*_{\boldsymbol{w}} = \Delta_{\mathit{tail}(\boldsymbol{w})}$, $c^{\mathfrak{M}^*(\boldsymbol{w})} = c^{\mathfrak{M}(\mathit{tail}(\boldsymbol{w}))}$ and $P^{\mathfrak{M}^*(\boldsymbol{w})} = P^{\mathfrak{M}(\mathit{tail}(\boldsymbol{w}))}$ for every constant c, predicate P and $\boldsymbol{w} \in W^*$. Observe that $\mathfrak{M}^*$ has constant domains whenever $\mathfrak{M}$ also has constant domains. By induction on the structure of φ, one can show that

$$\mathfrak{M}^*, \boldsymbol{w} \models^{\mathfrak{b}} \psi \quad \text{iff} \quad \mathfrak{M}, \mathit{tail}(\boldsymbol{w}) \models^{\mathfrak{b}} \psi, \tag{4}$$

for all assignments $\mathfrak{b}$, all subformulas ψ of φ and all words $\boldsymbol{w} \in W^*$ such that $d(\boldsymbol{w}) + d(\psi) \leq d(\varphi)$. The claim of the lemma will then be immediate. So, we proceed with the proof of (4).

The base case of atomic formulas, including equalities, which are subformulas of modal depth 0, is immediate from the definition of $\mathfrak{M}^*$. The cases of the Boolean connectives $\wedge$ and $\neg$ and the quantifier $\exists$ are standard. It remains to consider the case of $\Diamond_a\psi$. First, note that $d(\psi) = d(\Diamond_a\psi) - 1$. Suppose $\mathfrak{M}^*, \boldsymbol{w} \models^{\mathfrak{b}} \Diamond_a\psi$. Then there is $\boldsymbol{v} \in W^*$ with $\boldsymbol{w}R_a^*\boldsymbol{v}$ and $\mathfrak{M}^*, \boldsymbol{v} \models^{\mathfrak{b}} \psi$. Since $d(\boldsymbol{v}) = d(\boldsymbol{w}) + 1$, by IH, we have $\mathfrak{M}, \mathit{tail}(\boldsymbol{v}) \models^{\mathfrak{b}} \psi$, whence, due to $\mathit{tail}(\boldsymbol{w})R_a\mathit{tail}(\boldsymbol{v})$, we obtain $\mathfrak{M}, \mathit{tail}(\boldsymbol{w}) \models^{\mathfrak{b}} \Diamond_a\psi$. Conversely, suppose that $\mathfrak{M}, \mathit{tail}(\boldsymbol{w}) \models^{\mathfrak{b}} \Diamond_a\psi$. Then there is $v \in W$ with $\mathit{tail}(\boldsymbol{w})R_a v$ and $\mathfrak{M}, v \models^{\mathfrak{b}} \psi$. Consider $\boldsymbol{v} = \boldsymbol{w}av \in W^*$. Since $d(\boldsymbol{v}) = d(\boldsymbol{w}) + 1$, we have, by IH, $\mathfrak{M}^*, \boldsymbol{v} \models^{\mathfrak{b}} \psi$ and so $\mathfrak{M}^*, \boldsymbol{w} \models^{\mathfrak{b}} \Diamond_a\psi$. $\dashv$

Lemmas 11 and 9 provide a bound on the depth of frames in quasimodels, but such frames can still have infinite branching; see Example 10. We now define weak quasimodels, which will allow us to deal with this issue. Let φ be a $\mathsf{Q}^{=}_{\boxed{1}}\mathsf{ML}_c$-sentence and $\mathfrak{F} = (W, \{R_a\}_{a\in A})$ a tree-shaped $\mathbf{K}_n$ frame. We call a function ρ from an upward-closed subset of W to the set of types for φ a *weak run* if, for all $a \in A$,

(a-r-coh) $\rho(\boldsymbol{w}) \to_a \rho(\boldsymbol{v})$, for every $\boldsymbol{w}, \boldsymbol{v} \in \mathsf{dom}\rho$ with $\boldsymbol{w}R_a\boldsymbol{v}$, where $t \to_a t'$ denotes the relation on types t, t' for φ defined as follows:

$$\psi \in t' \quad \text{implies} \quad \Diamond_a\psi \in t, \text{ for all } \Diamond_a\psi \in \mathit{sub}_x(\varphi).$$

Note that these conditions, for $a \in A$, taken together coincide with the run coherence condition **(r-coh)**, but weak runs are not required to be saturated in general. A weak run ρ is said to be *a-saturated at $\boldsymbol{w} \in W$*, for $a \in A$, if

(a-$\boldsymbol{w}$-r-sat) for every $\Diamond_a\psi \in \rho(\boldsymbol{w})$, there is $\boldsymbol{v} \in W$ with $\boldsymbol{w}R_a\boldsymbol{v}$ and $\psi \in \rho(\boldsymbol{v})$.

A *weak $\mathbf{K}_n$ quasimodel for φ* is a quadruple $\mathfrak{Q} = (\mathfrak{F}, \boldsymbol{q}, \mathfrak{R}, \mathfrak{p})$, where $(\mathfrak{F}, \boldsymbol{q})$ is a basic structure for φ based on a tree-shaped $\mathbf{K}_n$ frame, $\mathfrak{R}$ is a multiset of weak runs through $(\mathfrak{F}, \boldsymbol{q})$ such that **(card)** holds for weak runs in $\mathfrak{R}$ and $\mathfrak{p}$ is a *prototype function* satisfying the following:

(wq-sat) $\mathfrak{p}(\boldsymbol{w}, t)$ is defined for every world $\boldsymbol{w} \in W$ and every type $t \in \boldsymbol{q}(\boldsymbol{w})$ so that $\mathfrak{p}(\boldsymbol{w}, t) \in \mathfrak{R}_{\boldsymbol{w},t}$ is a-saturated at $\boldsymbol{w}$, for each $a \in A$.

As before, we say that $\mathfrak{Q}$ is *constant-domain* if its every weak run is full. A weak quasimodel $\mathfrak{Q}$ *satisfies* φ if **(b)** holds.

It follows that every quasimodel satisfying φ trivially gives rise to a weak quaismodel satisfying φ; moreover, as Lemma 13 below shows, from any quasimodel satisfying φ we can extract a weak quasimodel satisfying φ based on a frame of small (exponential) size. Conversely, Lemma 14 below shows that any weak quasimodel can be "saturated" to obtain a quasimodel.

Example 12. In our running Example 8, the frame is already tree-shaped and we use $\boldsymbol{w}_0$ and $\boldsymbol{v}_i$ to refer to its worlds: let $\boldsymbol{w}_0 = w_0$ and $\boldsymbol{v}_i = w_0av_i$, for $i \in \mathbb{N}$. First, we pick, say, ρ_0 from $\mathfrak{R}_{\boldsymbol{w}_0,t_0}$ and ρ_1 from $\mathfrak{R}_{\boldsymbol{w}_0,t_0'}$, with $\boldsymbol{v}_0$ and $\boldsymbol{v}_1$ being the $\Diamond_a A(x)$-witness at $\boldsymbol{w}_0$ for ρ_0 and ρ_1, respectively (note that

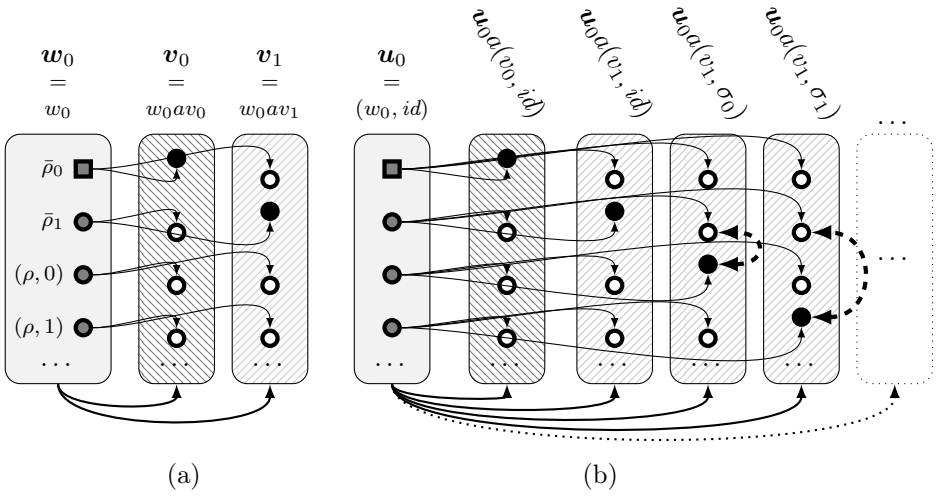

Figure 3: (a) A weak quasimodel in Example 12 and (b) its saturated quasimodel constructed in the proof of Lemma 13, where σ_i swaps (ρ, i) with the only copy of $\bar{\rho}_1$ and preserves all other indexed runs.

ρ_0 is uniquely determined, while ρ_1 could be any of the ρ_i, for $i \geq 1$). Then the set of worlds W' comprises $\boldsymbol{w}_0$, $\boldsymbol{v}_0$ and $\boldsymbol{v}_1$, and the multiset $\mathfrak{R}'$ of weak runs consists of the restrictions $\bar{\rho}_0$ and $\bar{\rho}_1$ of the chosen ρ_0 and ρ_1 to W' together with $\aleph_0$ copies of the "unsaturated" weak run ρ that maps $\boldsymbol{w}_0$ to t_0' and both $\boldsymbol{v}_0$ and $\boldsymbol{v}_1$ to t_2 (these unsaturated weak runs are the restrictions of the ρ_i, for $i \geq 2$, to W'). For our prototype weak runs, $\mathfrak{p}(\boldsymbol{w}_0, t_0)$ and $\mathfrak{p}(\boldsymbol{w}_0, t_0')$, we take $\bar{\rho}_0$ and $\bar{\rho}_1$, respectively; for $\mathfrak{p}(\boldsymbol{v}_i, t_1)$ and $\mathfrak{p}(\boldsymbol{v}_i, t_2)$, with $i = 0, 1$, we take any of the weak runs—the choice is irrelevant as t_1 and t_2 have no positive $\Diamond_a$-formulas. As Lemma 13 below shows, this is a finitely representable weak quasimodel satisfying the formula; see Fig. 3 (a). Note, however, that the weak quasimodel is not finite as, for example, t_0' still has infinite multiplicity in $\boldsymbol{w}_0$.

On the other hand, given this small weak quasimodel, we can restore the original quasimodel by creating a separate a-successor of $\boldsymbol{w}_0$ for each of the copies of the unsaturated weak run ρ and using the prototype weak run $\bar{\rho}_1$ as a template for witnessing $\Diamond_a A(x)$ in that successor; more details are given in the proof of Lemma 14 below.

Lemma 13. *Let φ be a $\mathsf{Q}^{=}_{\boxed{1}}\mathsf{ML}_c$-sentence. For both constant and expanding domains, for every quasimodel satisfying φ based on a $\mathbf{Tree}^d_n$ frame, there is a weak quasimodel satisfying φ based on a $\mathbf{Tree}^d_n$ frame of size $2^{O(d \cdot |\varphi|)}$.*

Proof. Assume a quasimodel $\mathfrak{Q} = (\mathfrak{F}, \boldsymbol{q}, \mathfrak{R}, \mathfrak{p})$ satisfying φ with tree-shaped $\mathfrak{F} = (W, \{R_a\}_{a \in A})$ of finite depth is given. We inductively define a sequence $W_0, W_1, \cdots W_{d(\mathfrak{F})} \subseteq W$ of sets of worlds with $d(\boldsymbol{w}) = i$ for all $\boldsymbol{w} \in W_i$ as follows. Set $W_0 = \{w_0\}$, where w_0 is the root of $\mathfrak{F}$, and assume W_i has been defined. For

each $\boldsymbol{w} \in W_i$ and each $t \in \boldsymbol{q}(\boldsymbol{w})$, we fix a run $\rho = \mathfrak{p}(\boldsymbol{w}, t) \in \mathfrak{R}_{\boldsymbol{w},t}$. By **(r-sat)**, for every $\Diamond_a \psi \in \rho(\boldsymbol{w})$, there exists $\boldsymbol{v} \in W$ such that $\boldsymbol{w} R_a \boldsymbol{v}$ and $\psi \in \rho(\boldsymbol{v})$, so we add $\boldsymbol{v}$ to W_{i+1}. Thus, $|W_{i+1}| = |W_i| \cdot 2^{|\varphi|} \cdot |\varphi|$. Let $\mathfrak{Q}' = (\mathfrak{F}', \boldsymbol{q}', \mathfrak{R}', \mathfrak{p}')$ be the restriction of $\mathfrak{Q}$ to $W' = \bigcup_{i=0}^{d(\mathfrak{F})} W_i$. It can be seen that $\mathfrak{Q}'$ is a weak quasimodel required by the lemma: in particular, by construction, each $\mathfrak{p}'(\boldsymbol{w}, t)$, which is the restriction of $\mathfrak{p}(\boldsymbol{w}, t)$ to W', is a-saturated at $\boldsymbol{w}$, for each $a \in A$. Moreover, weak quasimodel $\mathfrak{Q}'$ contains only full runs whenever quasimodel $\mathfrak{Q}$ does. $\dashv$

Lemma 14. *Let φ be a $\mathsf{Q}^{=}_{[1]}\mathsf{ML}_c$-sentence. For both constant and expanding domains, for any weak quasimodel $(\mathfrak{F}, \boldsymbol{q}, \mathfrak{R}, \mathfrak{p})$ satisfying φ based on a $\mathbf{Tree}^d_n$ frame, there is a quasimodel satisfying φ based on a $\mathbf{Tree}^d_n$ frame of size $O(|\mathfrak{F}| \cdot |\mathfrak{R}|^d)$.*

Proof. Let $\mathfrak{Q} = (\mathfrak{F}, \boldsymbol{q}, \mathfrak{R}, \mathfrak{p})$ be a weak quasimodel satisfying φ based on a tree-shaped frame $\mathfrak{F} = (W, \{R_a\}_{a \in A})$ of depth d. We construct a new frame using copies of the worlds and extend the existing weak runs to the world copies in such a way that each run becomes saturated, that is, satisfying **(r-sat)**. More precisely, by **(wq-sat)**, for each $\boldsymbol{w} \in W$ and each $t \in \boldsymbol{q}(\boldsymbol{w})$, we take the prototype weak run $\mathfrak{p}(\boldsymbol{w}, t) \in \mathfrak{R}_{\boldsymbol{w},t}$, which is a-saturated at $\boldsymbol{w}$, for all $a \in A$: for every $\Diamond_a \psi \in t$, there is a $\Diamond_a \psi$-witness world $\boldsymbol{v}_{\boldsymbol{w},t,\Diamond_a\psi} \in W$ such that $\boldsymbol{w} R_a \boldsymbol{v}_{\boldsymbol{w},t,\Diamond_a\psi}$ and $\psi \in \mathfrak{p}(\boldsymbol{w}, t)(\boldsymbol{v}_{\boldsymbol{w},t,\Diamond_a\psi})$. Then, for each indexed weak run (ρ, ℓ) that coincides with $\mathfrak{p}(\boldsymbol{w}, t)$ on $\boldsymbol{w}$, we create a copy $\boldsymbol{v}_{\rho,\ell}$ of the witness world $\boldsymbol{v}_{\boldsymbol{w},t,\Diamond_a\psi}$ as an a-successor for $\boldsymbol{w}$ and extend (ρ, ℓ) to this world so that it coincides with the prototype $\mathfrak{p}(\boldsymbol{w}, t)$ on the $\Diamond_a \psi$-witness world, thus making the extended (ρ, ℓ) saturated for $\Diamond_a \psi$ at $\boldsymbol{w}$; all other runs are extended to $\boldsymbol{v}_{\rho,\ell}$ in exactly the same way as they are defined on $\boldsymbol{v}_{\boldsymbol{w},t,\Diamond_a\psi}$. In order to preserve the multiplicity constraints, we use bijections that 'swap' the types of (ρ, ℓ) and $(\mathfrak{p}(\boldsymbol{w}, t), 0)$ on $\boldsymbol{v}_{\rho,\ell}$, while preserving all other types.

Formally, we consider the set $\hat{\mathfrak{R}}$ of indexed weak runs (ρ, ℓ) associated with the multiset $\mathfrak{R}$ and denote the set of all bijections on $\hat{\mathfrak{R}}$ by $B(\hat{\mathfrak{R}})$. Then, for each $\boldsymbol{w} \in W$, we define the *$\boldsymbol{w}$-repair set* $\mathrm{Rep}(\boldsymbol{w})$ by taking

$$\mathrm{Rep}(\boldsymbol{w}) = \big\{\sigma_{\boldsymbol{w},\rho,\ell} \in B(\hat{\mathfrak{R}}) \mid (\rho, \ell) \in \hat{\mathfrak{R}}_{\boldsymbol{w}}\big\},$$

where $\sigma_{\boldsymbol{w},\rho,\ell}$ is the bijection on $\hat{\mathfrak{R}}$ that swaps (ρ, ℓ) with the indexed prototype weak run $(\mathfrak{p}(\boldsymbol{w}, t), 0)$ for $\boldsymbol{w}$ and $t = \rho(\boldsymbol{w})$ and maps the remaining indexed weak runs to themselves. Observe that, for any $\sigma \in \mathrm{Rep}(\boldsymbol{w})$ and any indexed run $(\rho, \ell) \in \mathfrak{R}$, either both the domain of ρ and the domain of its image $\sigma(\rho, \ell)$ contain $\boldsymbol{w}$ or neither contains $\boldsymbol{w}$. Note that, as each $\hat{\mathfrak{R}}_{\boldsymbol{w}}$ is non-empty, the set $\mathrm{Rep}(\boldsymbol{w})$ is also non-empty and contains the identity function *id* on $\hat{\mathfrak{R}}$, which swaps *any* indexed prototype weak run with itself.

Now, for each word

$$\boldsymbol{w} = w_0 a_0 w_1 a_1 \cdots a_{m-1} w_m \in W,$$

we construct words of the form

$$\boldsymbol{u} = (w_0, \sigma_0) a_0 (w_1, \sigma_1) a_1 \cdots a_{m-1} (w_m, \sigma_m), \tag{5}$$

where $\sigma_0 = id$ and $\sigma_{j+1} \in \mathrm{Rep}(\boldsymbol{w}_j)$ with $\boldsymbol{w}_j = w_0 a_0 w_1 a_1 \cdots a_{j-1} w_j$, for each $0 \leq j < m$. We denote $\boldsymbol{w}$ by $\tilde{\boldsymbol{u}}$ and $\sigma_m \circ \cdots \circ \sigma_1 \in B(\hat{\mathfrak{R}})$ by $\tau_{\boldsymbol{u}}$. Let W' be the set of all words of the form (5) for all $\boldsymbol{w} \in W$. Consider a tree-shaped frame $\mathfrak{F}' = (W', \{R'_a\}_{a \in A})$, where $R'_a = \{(\boldsymbol{u}, \boldsymbol{u}a(w,\sigma)) \in W' \times W'\}$, for $a \in A$. Clearly, it is as required by the lemma. Define $\boldsymbol{q}'$ by setting $\boldsymbol{q}'(\boldsymbol{u}) = \boldsymbol{q}(\tilde{\boldsymbol{u}})$ for any $\boldsymbol{u} \in W'$.

We now define a multiset $\mathfrak{R}'$ of runs through $\mathfrak{F}'$. Denote by *rep* the function mapping any $(\rho, \ell) \in \hat{\mathfrak{R}}$ to a weak run ρ' on W' defined by setting

$$\rho'(\boldsymbol{u}) = \tau_{\boldsymbol{u}}(\rho, \ell)(\tilde{\boldsymbol{u}}), \text{ for any } \boldsymbol{u} \in W' \text{ with } \tilde{\boldsymbol{u}} \in \mathsf{dom}\rho.$$

By induction on the depth of $\boldsymbol{u} \in W'$, we can show that $\rho'(\boldsymbol{u})$ is well-defined. Indeed, this is trivial for the only $\boldsymbol{u}$ of depth 0 as it is of the form (w_0, id). Suppose $\rho'(\boldsymbol{u})$ is well-defined for all $\boldsymbol{u} \in W'$ of depth m. Consider any $\boldsymbol{v} \in W'$ of depth $m+1$, which is of the form $\boldsymbol{u}a(w,\sigma)$ for some $\boldsymbol{u} \in W'$, w and $\sigma \in \mathrm{Rep}(\tilde{\boldsymbol{u}})$. By definition, we have $\tilde{\boldsymbol{v}} = \tilde{\boldsymbol{u}}aw$ and $\tau_{\boldsymbol{v}} = \sigma \circ \tau_{\boldsymbol{u}}$. If $\tilde{\boldsymbol{u}} \in \mathsf{dom}\rho$, then, by IH, $\tau_{\boldsymbol{u}}(\rho, \ell)$ is defined on $\tilde{\boldsymbol{u}}$. As we observed above, for any $\sigma \in \mathrm{Rep}(\tilde{\boldsymbol{u}})$, the run $\sigma(\tau_{\boldsymbol{u}}(\rho, \ell))$ is defined on $\tilde{\boldsymbol{u}}$ and so on $\tilde{\boldsymbol{v}}$ (as $\tilde{\boldsymbol{u}} R_a \tilde{\boldsymbol{v}}$). If $\tilde{\boldsymbol{u}} \notin \mathsf{dom}\rho$, then $\sigma = id$ and in fact $\tau_{\boldsymbol{u}} = id$, whence $\tau_{\boldsymbol{v}}(\rho, \ell) = (\rho, \ell)$ and so, $\tau_{\boldsymbol{v}}(\rho, \ell)(\tilde{\boldsymbol{v}})$ is defined whenever $\tilde{\boldsymbol{v}} \in \mathsf{dom}\rho$.

Let $\mathfrak{R}'$ be the multiset of weak runs ρ' such that

$$\mathfrak{R}'(\rho') = \big|\{(\rho, \ell) \in \hat{\mathfrak{R}} \mid rep(\rho, \ell) = \rho'\}\big|.$$

Due to this multiplicity condition, *rep* can be extended to a bijection from $\hat{\mathfrak{R}}$ onto the set $\hat{\mathfrak{R}}'$ of indexed weak runs in $\mathfrak{R}'$ (note, however, that the indexes of $\rho' = rep(\rho, \ell)$ will in general be unrelated to ℓ).

We show that $\mathfrak{Q}' = (\mathfrak{F}', \boldsymbol{q}', \mathfrak{R}')$ is a required quasimodel. Condition **(b)** is straightforward. We prove **(card)** for (weak) runs in $\mathfrak{R}'$ using **(card)** for weak runs in $\mathfrak{R}$. To this end, we show that the composition $\tau_{\boldsymbol{u}} \circ rep^{-1}$ is a bijection from $\hat{\mathfrak{R}}'_{\boldsymbol{u},t}$ onto $\hat{\mathfrak{R}}_{\tilde{\boldsymbol{u}},t}$, for any $\boldsymbol{u} \in W'$ and any $t \in \boldsymbol{q}'(\boldsymbol{u})$. First, for any $(\rho', \ell') \in \hat{\mathfrak{R}}'_{\boldsymbol{u},t}$, we have $\tau_{\boldsymbol{u}}(rep^{-1}(\rho', \ell')) \in \hat{\mathfrak{R}}_{\tilde{\boldsymbol{u}},t}$: denote $(\rho, \ell) = rep^{-1}(\rho', \ell')$; then $\tau_{\boldsymbol{u}}(rep^{-1}(\rho', \ell'))(\tilde{\boldsymbol{u}}) = \tau_{\boldsymbol{u}}(\rho, \ell)(\tilde{\boldsymbol{u}}) = \rho'(\boldsymbol{u}) = t$, as required. The composition is injective as both $\tau_{\boldsymbol{u}}$ and *rep* are bijections. Finally, the composition is surjective: for any $(\rho, \ell) \in \hat{\mathfrak{R}}_{\tilde{\boldsymbol{u}},t}$, consider $(\rho', \ell') = rep(\tau_{\boldsymbol{u}}^{-1}(\rho, \ell))$; we have $\rho'(\boldsymbol{u}) = \tau_{\boldsymbol{u}}(\tau_{\boldsymbol{u}}^{-1}(\rho, \ell))(\tilde{\boldsymbol{u}}) = \rho(\tilde{\boldsymbol{u}}) = t$, and so $(\rho', \ell') \in \hat{\mathfrak{R}}'_{\boldsymbol{u},t}$.

We next show that the elements of $\mathfrak{R}'$ are indeed runs, that is, they satisfy **(r-coh)** and **(r-sat)**. Let $\Diamond_a \psi \in sub_x(\varphi)$. Consider $\rho' \in \mathfrak{R}'$ and $\boldsymbol{u} \in W'$ of the form (5). We have $\rho' = rep(\rho, \ell)$ for some $(\rho, \ell) \in \hat{\mathfrak{R}}$.

For **(r-coh)**, assume there exists $\boldsymbol{v} \in W'$ such that $\boldsymbol{u} R'_a \boldsymbol{v}$ and $\psi \in \rho'(\boldsymbol{v})$. By construction, $\tilde{\boldsymbol{u}} R_a \tilde{\boldsymbol{v}}$. Since $\rho'(\boldsymbol{v}) = \tau_{\boldsymbol{v}}(\rho, \ell)(\tilde{\boldsymbol{v}})$, by coherence of the weak run $\tau_{\boldsymbol{v}}(\rho, \ell)$, we obtain $\Diamond_a \psi \in \tau_{\boldsymbol{v}}(\rho, \ell)(\tilde{\boldsymbol{u}})$. By definition, $\tau_{\boldsymbol{v}} = \sigma(\tau_{\boldsymbol{u}})$, for some $\sigma \in \mathrm{Rep}(\tilde{\boldsymbol{u}})$, and so $\tau_{\boldsymbol{u}}(\rho, \ell)$ coincides with $\tau_{\boldsymbol{v}}(\rho, \ell)$ on $\tilde{\boldsymbol{u}}$. It follows that $\Diamond_a \psi \in \tau_{\boldsymbol{u}}(\rho, \ell)(\tilde{\boldsymbol{u}}) = \rho'(\boldsymbol{u})$.

For **(r-sat)**, assume $\Diamond_a \psi \in \rho'(\boldsymbol{u})$. Let $t = \tau_{\boldsymbol{u}}(\rho, \ell)(\tilde{\boldsymbol{u}})$. We have $\Diamond_a \psi \in t$. Condition **(wq-sat)** provides a prototype weak run $\mathfrak{p}(\tilde{\boldsymbol{u}}, t)$ for $\tilde{\boldsymbol{u}}$ and t and a

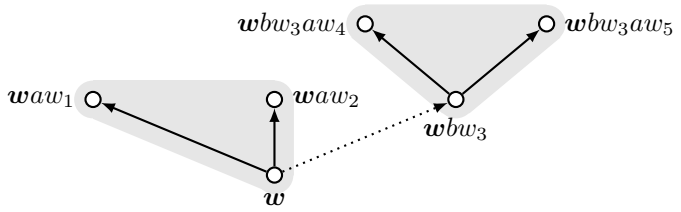

Figure 4: An example of a tree-shaped $\mathbf{S5}_n$ frame.

witness $\boldsymbol{w} \in W$ such that $\tilde{\boldsymbol{u}}R_a\boldsymbol{w}$ and $\psi \in \mathfrak{p}(\tilde{\boldsymbol{u}}, t)(\boldsymbol{w})$. By construction, we have $\boldsymbol{w} = \tilde{\boldsymbol{u}}aw$, for some w. Consider $\boldsymbol{v} = \boldsymbol{u}a(w, \sigma)$, where $\sigma \in \mathrm{Rep}(\tilde{\boldsymbol{u}})$ swaps $\tau_{\boldsymbol{u}}(\rho, \ell)$ with $(\mathfrak{p}(\tilde{\boldsymbol{u}}, t), 0)$. By definition, we have $\tilde{\boldsymbol{v}} = \boldsymbol{w}$, $\boldsymbol{u}R'_a\boldsymbol{v}$ and $\rho'(\boldsymbol{v}) = \sigma(\tau_{\boldsymbol{u}}(\rho, \ell))(\boldsymbol{w}) = \mathfrak{p}(\tilde{\boldsymbol{u}}, t)(\boldsymbol{w})$. So, $\psi \in \rho'(\boldsymbol{v})$, as required.

This completes the proof of Lemma 14. ⊣

6. Weak Quasimodels for $\mathsf{Q}^{=}_{\boxed{1}}\mathsf{ML}_c$ over $\mathbf{S5}_n$ Frames

In this section, we consider interpretations constructed from $\mathbf{S5}_n$ frames. Recall that a frame $\mathfrak{F} = (W, \{R_a\}_{a\in A})$ is an $\mathbf{S5}_n$ frame if all R_a are equivalence relations. Since the R_a are symmetric, any interpretation over an $\mathbf{S5}_n$ frame by definition has constant domains (so, we shall omit the constant-domain qualifier for the rest of the section).

An $\mathbf{S5}_n$ frame is called *tree-shaped* if there exists a $w_0 \in W$, the root of $\mathfrak{F}$, such that the domain W of $\mathfrak{F}$ is a prefix-closed set of words of the form

$$\boldsymbol{w} = w_0a_0w_1a_1 \cdots a_{m-1}w_m, \tag{6}$$

where $a_j \in A$, $a_j \neq a_{j+1}$, and each R_a is the smallest equivalence relation containing all pairs of the form $(\boldsymbol{w}, \boldsymbol{w}aw) \in W \times W$. We define the immediate a-successor relation $\prec_a$, for $a \in A$, by taking $\boldsymbol{w} \prec_a \boldsymbol{w}'$ iff $\boldsymbol{w}'$ is of the form $\boldsymbol{w}aw$, for some w. We also write $\boldsymbol{w} \preceq_a \boldsymbol{w}'$ iff either $\boldsymbol{w} = \boldsymbol{w}'$ or $\boldsymbol{w} \prec_a \boldsymbol{w}'$. Observe that, by definition, such frames have no $\prec_a$-chains of length greater than one; in other words, each world has no $\prec_a$-predecessor or has no $\prec_a$-successors or has neither. It follows that

$$\begin{aligned}\boldsymbol{w}R_a\boldsymbol{w}' \quad \text{iff} \quad \boldsymbol{w} \prec_a \boldsymbol{w}' \quad \text{or} \quad \boldsymbol{w}' \prec_a \boldsymbol{w} \quad \text{or} \quad \boldsymbol{w} = \boldsymbol{w}' \\ \text{or there is } \boldsymbol{v} \in W \text{ with } \boldsymbol{v} \prec_a \boldsymbol{w} \text{ and } \boldsymbol{v} \prec_a \boldsymbol{w}'.\end{aligned} \tag{7}$$

An example is shown in Fig. 4, where relations $\prec_a$ and $\prec_b$ are depicted by solid and dotted arrows, respectively. Then $\boldsymbol{w}$, $\boldsymbol{w}aw_1$ and $\boldsymbol{w}aw_2$ are in the same a-equivalence class, with $\boldsymbol{w}$ having no $\prec_a$-predecessor and the other two having no $\prec_a$-successors, but $\boldsymbol{w}bw_3$, $\boldsymbol{w}bw_3aw_4$ and $\boldsymbol{w}bw_3aw_5$ form another a-equivalence class, with $\boldsymbol{w}bw_3$ having no $\prec_a$-predecessor and the other two no $\prec_a$-successors; $\boldsymbol{w}$ and $\boldsymbol{w}bw_3$ form a b-equivalence class.

Let $\mathbf{S5Tree}_n^d$ denote the set of all tree-shaped $\mathbf{S5}_n$ frames of depth bounded by d, where the notion of depth is defined as in Section 5. We then have the following counterpart of Lemma 11.

Lemma 15. *Every $\mathbf{S5}_n$-satisfiable $\mathsf{Q}^{=}\mathsf{ML}_c$-formula φ is $\mathbf{S5Tree}_n^{d(\varphi)}$-satisfiable.*

Proof. Let $\mathfrak{M} = (\mathfrak{F}, \Delta, \cdot)$ with $\mathfrak{F} = (W, \{R_a\}_{a\in A})$ and $\mathfrak{M}, w_0 \models^{\mathfrak{a}} \varphi$, for some $\mathfrak{a}$, be given. Unfold $\mathfrak{F}$ into $\mathfrak{F}^* = (W^*, \{R_a^*\}_{a\in A})$, where W^* is the set of all words $\boldsymbol{w}$ of the form (6) with $d(\boldsymbol{w}) \leq d(\varphi)$ and $(w_j, w_{j+1}) \in R_{a_j}$, $w_j \neq w_{j+1}$ and $a_j \neq a_{j+1}$, for all $0 \leq j < m$, and R_a^* is the smallest equivalence relation containing all pairs $(\boldsymbol{w}, \boldsymbol{w}aw) \in W^* \times W^*$. Clearly, $\mathfrak{F}^*$ is a tree-shaped $\mathbf{S5}_n$ frame of depth bounded by $d(\varphi)$. As in Lemma 11, define $\mathfrak{M}^* = (\mathfrak{F}^*, \Delta^*, \cdot^*)$ by taking $\Delta^*_{\boldsymbol{w}} = \Delta_{tail(\boldsymbol{w})}$, $c^{\mathfrak{M}^*(\boldsymbol{w})} = c^{\mathfrak{M}(tail(\boldsymbol{w}))}$ and $P^{\mathfrak{M}^*(\boldsymbol{w})} = P^{\mathfrak{M}(tail(\boldsymbol{w}))}$ for every constant c, predicate P and $\boldsymbol{w} \in W^*$. By induction on the structure of φ, one can show that

$$\mathfrak{M}^*, \boldsymbol{w} \models^{\mathfrak{b}} \psi \quad \text{iff} \quad \mathfrak{M}, tail(\boldsymbol{w}) \models^{\mathfrak{b}} \psi, \tag{8}$$

for all assignments $\mathfrak{b}$, subformulas ψ of φ and words $\boldsymbol{w} \in W^*$ such that $d(\boldsymbol{w}) + d(\psi) \leq d(\varphi)$. The claim of the lemma will then be immediate.

In (8), the base case and the cases of the Boolean connectives and the quantifier are as in Lemma 11. It remains to consider the case of $\Diamond_a\psi$. Suppose first $\mathfrak{M}^*, \boldsymbol{w} \models^{\mathfrak{b}} \Diamond_a\psi$. Then there is $\boldsymbol{v} \in W$ with $\boldsymbol{w}R_a^*\boldsymbol{v}$ and $\mathfrak{M}^*, \boldsymbol{v} \models^{\mathfrak{b}} \psi$. By (7), $d(\boldsymbol{v}) \leq d(\boldsymbol{w})+1$. Then, as $d(\psi) = d(\Diamond_a\psi)-1$, by IH, $\mathfrak{M}, tail(\boldsymbol{v}) \models^{\mathfrak{b}} \psi$. Since R_a is reflexive, we have $tail(\boldsymbol{w})R_a tail(\boldsymbol{v})$, whence $\mathfrak{M}, tail(\boldsymbol{w}) \models^{\mathfrak{b}} \Diamond_a\psi$. Conversely, assume that $\mathfrak{M}, tail(\boldsymbol{w}) \models^{\mathfrak{b}} \Diamond_a\psi$. Then there is $v \in W$ such that $tail(\boldsymbol{w})R_a v$ and $\mathfrak{M}, v \models^{\mathfrak{b}} \psi$. If $tail(\boldsymbol{w}) = v$, then $\mathfrak{M}^*, \boldsymbol{w} \models^{\mathfrak{b}} \psi$ by IH and $\mathfrak{M}^*, \boldsymbol{w} \models^{\mathfrak{b}} \Diamond_a\psi$ due to reflexivity of R_a^*. Otherwise, there are two further cases to consider. If $\boldsymbol{w}$ has no $\prec_a$-predecessor, then, as $tail(\boldsymbol{w}) \neq v$, we have $\boldsymbol{w}R_a^*\boldsymbol{w}av$, whence $\mathfrak{M}^*, \boldsymbol{w} \models^{\mathfrak{b}} \Diamond_a\psi$ follows again by IH as $d(\psi) = d(\Diamond_a\psi) - 1$. Finally, we have $\boldsymbol{w}'$ with $\boldsymbol{w}' \prec_a \boldsymbol{w}$ and $tail(\boldsymbol{w}) \neq v$. It follows that $tail(\boldsymbol{w}')R_a tail(\boldsymbol{w})$, which together with $tail(\boldsymbol{w})R_a v$ by transitivity implies $tail(\boldsymbol{w}')R_a v$. Thus, we have either $tail(\boldsymbol{w}') = v$ or $\boldsymbol{w}'R_a^*\boldsymbol{w}'av$. In the former case, by symmetry of R_a^*, we have $\boldsymbol{w}R_a^*\boldsymbol{w}'$, whence, by IH, $\mathfrak{M}^*, \boldsymbol{w}' \models^{\mathfrak{b}} \psi$ and thus $\mathfrak{M}^*, \boldsymbol{w} \models^{\mathfrak{b}} \Diamond_a\psi$. In the latter case, by symmetry of R_a^*, we obtain $\boldsymbol{w}R_a^*\boldsymbol{w}'$, whence, by transitivity, $\boldsymbol{w}R_a^*\boldsymbol{w}'av$. Then, by IH, $\mathfrak{M}^*, \boldsymbol{w}'av \models^{\mathfrak{b}} \psi$ and so $\mathfrak{M}^*, \boldsymbol{w} \models^{\mathfrak{b}} \Diamond_a\psi$. ⊣

We now define a notion of $\mathbf{S5}_n$ weak quasimodels, which differ from $\mathbf{K}_n$ weak quasimodels. Let φ be a $\mathsf{Q}^{=}_{\boxed{1}}\mathsf{ML}_c$-sentence and $\mathfrak{F} = (W, \{R_a\}_{a\in A})$ a tree-shaped $\mathbf{S5}_n$ frame. The notion of coherence between types reflects the fact that the R_a are equivalence relations: we write $t \leftrightarrow_a t'$ if, for all $\Diamond_a\psi \in sub_x(\varphi)$,

$$\begin{array}{c} \psi \in t \cup t' \quad \text{implies} \quad \Diamond_a\psi \in t, t', \\ \Diamond_a\psi \in t \quad \text{iff} \quad \Diamond_a\psi \in t'. \end{array}$$

It follows that each $\leftrightarrow_a$ is an equivalence relation on types for φ. Thus, the run coherence condition **(r-coh)** in $\mathbf{S5}_n$ is equivalent to the following: for all $a \in A$,

(a-r-coh-s5) $\rho(\boldsymbol{w}) \leftrightarrow_a \rho(\boldsymbol{v})$, for every $\boldsymbol{w}, \boldsymbol{v} \in W$ with $\boldsymbol{w}R_a\boldsymbol{v}$.

A function ρ from W to the set of types for φ is called a *weak* $\mathbf{S5}_n$ *run* if $\rho(\boldsymbol{w}) \leftrightarrow_a \rho(\boldsymbol{v})$, for every $a \in A$ and every $\boldsymbol{w}, \boldsymbol{v} \in W$ with $\boldsymbol{w} R_a \boldsymbol{v}$. A *weak* $\mathbf{S5}_n$ *quasimodel for* φ is a quadruple $\mathfrak{Q} = (\mathfrak{F}, \boldsymbol{q}, \mathfrak{R}, \mathfrak{p})$, where $(\mathfrak{F}, \boldsymbol{q})$ is a basic structure for φ based on a tree-shaped $\mathbf{S5}_n$ frame, $\mathfrak{R}$ is a multiset of weak $\mathbf{S5}_n$ runs through $(\mathfrak{F}, \boldsymbol{q})$ such that **(card)** holds for weak runs in $\mathfrak{R}$ and $\mathfrak{p}$ is a prototype function satisfying the following:

(wq-sat-s5) for every $\boldsymbol{w} \in W$ and $t \in \boldsymbol{q}(\boldsymbol{w})$, there exists $\mathfrak{p}(\boldsymbol{w}, t) \in \mathfrak{R}_{\boldsymbol{w},t}$ which is a-saturated at $\boldsymbol{w}$, for every $a \in A$ such that $\boldsymbol{w}$ has no $\prec_a$-predecessor.

When clear from the context, we will omit the $\mathbf{S5}_n$ qualifier for these notions. Note that **(wq-sat-s5)** does *not* require witnesses for $\Diamond_a \psi$ at worlds $\boldsymbol{w}$ that have $\prec_a$-predecessors, which means that, due to the definition of R_a, each witness $\boldsymbol{v}$ for a $\Diamond_a \psi$ satisfies $\boldsymbol{w} \preceq_a \boldsymbol{v}$ (rather than just $\boldsymbol{w} R_a \boldsymbol{v}$ as in $\mathbf{K}_n$ quasimodels, which due to symmetry of R_a may mean $\boldsymbol{v} \prec_a \boldsymbol{w}$). As before, every quasimodel is by definition a weak quaismodel, but we can also extract a weak quasimodel of small (exponential) size. And, conversely, any weak quasimodel can be "saturated" to obtain a quasimodel.

Lemma 16. *Let φ be a $\mathsf{Q}^{=}_{\boxed{1}}\mathsf{ML}_c$-sentence. For every quasimodel satisfying φ based on an $\mathbf{S5Tree}^d_n$ frame, there is a weak quasimodel satisfying φ based on an $\mathbf{S5Tree}^d_n$ frame of size $2^{O(d \cdot |\varphi|)}$.*

Proof. Let $\mathfrak{Q} = (\mathfrak{F}, \boldsymbol{q}, \mathfrak{R})$, for a tree-shaped $\mathbf{S5}_n$ frame $\mathfrak{F} = (W, \{R_a\}_{a \in A})$ of finite depth, be a quasimodel satisfying φ. The construction is nearly identical to the proof of Lemma 13: we construct a sequence $W_0, \dots, W_{d(\mathfrak{F})} \subseteq W$ of sets of worlds by including witnesses $\boldsymbol{v}$ for subformulas of the form $\Diamond_a \psi$, except that a $\Diamond_a \psi$-witness is picked only for a $\boldsymbol{w}$ without a $\prec_a$-predecessor. It follows then that $d(\boldsymbol{w}) = i$ for all $\boldsymbol{w} \in W_i$, which is required to ensure that all relevant worlds are included after $d(\mathfrak{F})$ steps. $\dashv$

Lemma 17. *Let φ be a $\mathsf{Q}^{=}_{\boxed{1}}\mathsf{ML}_c$-sentence. For every weak quasimodel $(\mathfrak{F}, \boldsymbol{q}, \mathfrak{R}, \mathfrak{p})$ satisfying φ based on an $\mathbf{S5Tree}^d_n$ frame, there is a quasimodel satisfying φ based on an $\mathbf{S5Tree}^d_n$ frame of size $O(|\mathfrak{F}| \cdot |\mathfrak{R}|^d)$.*

Proof. Let $\mathfrak{Q} = (\mathfrak{F}, \boldsymbol{q}, \mathfrak{R}, \mathfrak{p})$ be a weak quasimodel satisfying φ based on a tree-shaped $\mathbf{S5}_n$ frame $\mathfrak{F} = (W, \{R_a\}_{a \in A})$ of depth d. The construction is essentially the same as in Lemma 14: let W' be the set of all words

$$\boldsymbol{u} = (w_0, \sigma_0) a_0 (w_1, \sigma_1) a_1 \cdots a_{m-1} (w_m, \sigma_m), \tag{9}$$

for $\boldsymbol{w} = w_0 a_0 w_1 a_1 \cdots a_{m-1} w_m \in W$, $\sigma_0 = id$ and $\sigma_{j+1} \in \mathrm{Rep}(\boldsymbol{w_j})$ with $\boldsymbol{w_j}$ denoting the j-prefix of $\boldsymbol{w}$, for each $0 \leq j < m$. For $\boldsymbol{u}, \boldsymbol{u'} \in W'$, we write $\boldsymbol{u} \prec'_a \boldsymbol{u'}$ if $\boldsymbol{u'}$ is an immediate a-successor of $\boldsymbol{u}$, that is, if $\boldsymbol{u'}$ is of the form $\boldsymbol{u} a (\boldsymbol{w}, \sigma)$. Consider $\mathfrak{F}' = (W', \{R'_a\}_{a \in A})$, where each R'_a is the smallest equivalence relation containing $\prec'_a$, for $a \in A$. Clearly, it is as required by the lemma. Define $\boldsymbol{q'}$ by setting $\boldsymbol{q'}(\boldsymbol{u}) = \boldsymbol{q}(\tilde{\boldsymbol{u}})$ for any $\boldsymbol{u} \in W'$. The multiset $\mathfrak{R}'$ of *full* runs through $\mathfrak{F}'$ is

defined as in the proof of Lemma 14: denote by *rep* the function mapping any $(\rho, \ell) \in \hat{\mathfrak{R}}$ to a weak run ρ' on W' defined by

$$\rho'(\boldsymbol{u}) = \tau_{\boldsymbol{u}}(\rho, \ell)(\tilde{\boldsymbol{u}}), \text{ for any } \boldsymbol{u} \in W'.$$

Let $\mathfrak{R}'$ be such that $\mathfrak{R}'(\rho') = |\{(\rho, \ell) \in \hat{\mathfrak{R}} \mid rep(\rho, \ell) = \rho'\}|$, for each ρ'.

We show that $\mathfrak{Q}' = (\mathfrak{F}', \boldsymbol{q}', \mathfrak{R}')$ is a required quasimodel. Conditions **(b)** and **(card)** are as in the proof of Lemma 14. It remains to prove that the elements of $\mathfrak{R}'$ are indeed $\mathbf{S5}_n$ runs, that is, they satisfy **(r-coh)** and **(r-sat)**. Let $\Diamond_a\psi \in sub_x(\varphi)$. Consider $\rho' \in \mathfrak{R}'$ and $\boldsymbol{u} \in W'$ of the form (9). We have $\rho' = rep(\rho, \ell)$ for some $(\rho, \ell) \in \hat{\mathfrak{R}}$.

For **(r-coh)**, assume there exists $\boldsymbol{v} \in W'$ such that $\boldsymbol{u}R'_a\boldsymbol{v}$ and $\psi \in \rho'(\boldsymbol{v})$. We have to show that $\Diamond_a\psi \in \rho'(\boldsymbol{u})$. We distinguish the following four cases.

1. If $\boldsymbol{v} = \boldsymbol{u}$, then we have $\Diamond_a\psi \in \rho'(\boldsymbol{u})$ since $\rho'(\boldsymbol{u}) \leftrightarrow_a \rho'(\boldsymbol{u})$.

2. If $\boldsymbol{u} \prec'_a \boldsymbol{v}$, then $\tilde{\boldsymbol{u}} \prec_a \tilde{\boldsymbol{v}}$. Since $\tau_{\boldsymbol{v}}(\rho, \ell)(\tilde{\boldsymbol{u}}) \leftrightarrow_a \tau_{\boldsymbol{v}}(\rho, \ell)(\tilde{\boldsymbol{v}}) = \rho'(\boldsymbol{v})$, we get $\Diamond_a\psi \in \tau_{\boldsymbol{v}}(\rho, \ell)(\tilde{\boldsymbol{u}})$. As $\tau_{\boldsymbol{v}} = \sigma \circ \tau_{\boldsymbol{u}}$, for some $\sigma \in \mathrm{Rep}(\tilde{\boldsymbol{u}})$, weak runs $\tau_{\boldsymbol{v}}(\rho, \ell)$ and $\tau_{\boldsymbol{u}}(\rho, \ell)$ coincide on $\tilde{\boldsymbol{u}}$. So, $\Diamond_a\psi \in \tau_{\boldsymbol{u}}(\rho, \ell)(\tilde{\boldsymbol{u}}) = \rho'(\boldsymbol{u})$.

3. If $\boldsymbol{v} \prec'_a \boldsymbol{u}$, then $\tilde{\boldsymbol{v}} \prec_a \tilde{\boldsymbol{u}}$. As $\tau_{\boldsymbol{u}} = \sigma \circ \tau_{\boldsymbol{v}}$, for some $\sigma \in \mathrm{Rep}(\tilde{\boldsymbol{v}})$, weak runs $\tau_{\boldsymbol{v}}(\rho, \ell)$ and $\tau_{\boldsymbol{u}}(\rho, \ell)$ coincide on $\tilde{\boldsymbol{v}}$. So, we obtain $\psi \in \tau_{\boldsymbol{u}}(\rho, \ell)(\tilde{\boldsymbol{v}})$ from $\psi \in \rho'(\boldsymbol{v}) = \tau_{\boldsymbol{v}}(\rho, \ell)(\tilde{\boldsymbol{v}})$. Since $\tau_{\boldsymbol{u}}(\rho, \ell)(\tilde{\boldsymbol{v}}) \leftrightarrow_a \tau_{\boldsymbol{u}}(\rho, \ell)(\tilde{\boldsymbol{u}})$, we have $\Diamond_a\psi \in \tau_{\boldsymbol{u}}(\rho, \ell)(\tilde{\boldsymbol{u}}) = \rho'(\boldsymbol{u})$.

4. If $\boldsymbol{v}' \prec'_a \boldsymbol{v}$ and $\boldsymbol{v}' \prec'_a \boldsymbol{u}$, for some $\boldsymbol{v}'$, then we can use the argument in Point 2 to show that $\Diamond_a\psi \in \rho'(\boldsymbol{v}')$; next, as in Point 3, we can show that $\Diamond_a\psi \in \rho'(\boldsymbol{u})$.

For **(r-sat)**, assume $\Diamond_a\psi \in \rho'(\boldsymbol{u})$. We have to show that there exists $\boldsymbol{v} \in W'$ such that $\boldsymbol{u}R'_a\boldsymbol{v}$ and $\psi \in \rho'(\boldsymbol{v})$. We have $\rho'(\boldsymbol{u}) = \tau_{\boldsymbol{u}}(\rho, \ell)(\tilde{\boldsymbol{u}})$ and distinguish the following two cases.

1. If $\tilde{\boldsymbol{u}}$ has no $\prec_a$-predecessor, then, by **(wq-sat)**, there is a prototype weak run $\mathfrak{p}(\tilde{\boldsymbol{u}}, t)$ for $\tilde{\boldsymbol{u}}$ and $t = \tau_{\boldsymbol{u}}(\rho, \ell)(\tilde{\boldsymbol{u}})$ and a witness $\boldsymbol{w} \in W$ such that $\tilde{\boldsymbol{u}} \preceq_a \boldsymbol{w}$ and $\psi \in \mathfrak{p}(\tilde{\boldsymbol{u}}, t)(\boldsymbol{w})$. Let $\boldsymbol{v} \in W'$ be such that $\tilde{\boldsymbol{v}} = \boldsymbol{w}$ and $\tau_{\boldsymbol{v}} = \sigma \circ \tau_{\boldsymbol{u}}$ for $\sigma \in \mathrm{Rep}(\tilde{\boldsymbol{u}})$ that swaps $\tau_{\boldsymbol{u}}(\rho, \ell)$ with $(\mathfrak{p}(\tilde{\boldsymbol{u}}, t), 0)$. We have $\boldsymbol{u}R'_a\boldsymbol{v}$ and $\tau_{\boldsymbol{v}}(\rho, \ell)(\tilde{\boldsymbol{v}}) = \mathfrak{p}(\tilde{\boldsymbol{u}}, t)(\boldsymbol{w})$, whence $\psi \in \rho'(\boldsymbol{v})$.

2. If $\boldsymbol{w} \prec_a \tilde{\boldsymbol{u}}$, for some $\boldsymbol{w} \in W$, then we consider $\boldsymbol{u}' \in W'$ with $\tilde{\boldsymbol{u}'} = \boldsymbol{w}$. Since $\boldsymbol{u}' \prec'_a \boldsymbol{u}$, we have $\tau_{\boldsymbol{u}}(\rho, \ell)(\tilde{\boldsymbol{u}'}) \leftrightarrow_a \tau_{\boldsymbol{u}}(\rho, \ell)(\tilde{\boldsymbol{u}})$, whence $\Diamond_a\psi \in \tau_{\boldsymbol{u}}(\rho, \ell)(\tilde{\boldsymbol{u}'})$. As $\tau_{\boldsymbol{u}}(\rho, \ell)$ coincides with $\tau_{\boldsymbol{u}'}(\rho, \ell)$ on $\tilde{\boldsymbol{u}'}$, we then obtain $\Diamond_a\psi \in \tau_{\boldsymbol{u}'}(\rho, \ell)(\tilde{\boldsymbol{u}'})$. We now apply the argument in Point 1 to $\tilde{\boldsymbol{u}'}$, which has no $\prec_a$-predecessor, and obtain $\boldsymbol{v} \in W'$ with $\boldsymbol{u}'R'_a\boldsymbol{v}$ and $\psi \in \rho'(\boldsymbol{v})$. Since R'_a is an equivalence relation, $\boldsymbol{u}R'_a\boldsymbol{v}$, as required.

This completes the proof of Lemma 17. ⊣

7. Decidability for Monodic Fragments in Constant Domains

We have reduced the problem of deciding the existence of a model satisfying formula φ to the problem of deciding the existence of a weak quasimodel satisfying φ based on a frame of exponential size. In general, the latter problem is not yet trivially decidable, however, since we have no bound on the size of quasistates (and so also no bound on the number of weak runs). In fact, as C^2 does not have the finite model property, no finite bound exists for $\mathsf{C}^2_{\boxed{1}}\mathsf{ML}_c$, for example.

For some languages with limited counting, in particular the one-variable and guarded fragments, we show that one can directly use weak quasimodels to obtain tight complexity bounds. For the two-variable fragment with counting we develop further machinery.

7.1. One-Variable Fragment

We first consider a $\mathsf{Q}^{1=}\mathsf{ML}_c$-sentence φ and begin with a characterisation of quasistates. We assume that it contains a subformula $x = c$ for any constant $c \in \mathsf{Con}$ that occurs in φ; this can trivially be achieved by adding conjuncts of the form $\exists x\,(x = c)$ to the sentence. We then say that a type t for φ *contains a constant* if it contains $x = c$, for some $c \in \mathsf{Con}$; note that the same type can contain multiple $x = c_i$. Then quasistate candidates $\boldsymbol{n}$ and $\boldsymbol{n}'$ for φ (not necessarily realisable) are *simply compatible* (written $\boldsymbol{n} \sim_0 \boldsymbol{n}'$) if

- $\boldsymbol{n}'(t) = \boldsymbol{n}(t) = 1$, for all t containing a constant and
- $\boldsymbol{n}'(t) = 0$ iff $\boldsymbol{n}(t) = 0$, for all types t for φ.

It can be easily seen that this condition defines a closure property for quasistates in $\mathsf{Q}^{1=}\mathsf{ML}_c$:

Lemma 18. *If $\boldsymbol{n}$ is a quasistate for a $\mathsf{Q}^{1=}\mathsf{ML}_c$-sentence φ, then any quasistate candidate $\boldsymbol{n}'$ for φ with $\boldsymbol{n} \sim_0 \boldsymbol{n}'$ is realisable (thus, a quasistate for φ).*

Using Lemma 18, we can obtain a tight complexity result for $\mathsf{Q}^{1=}\mathsf{ML}_\iota$.

Theorem 19. *For constant domains, $\mathbf{K}_n$-validity and $\mathbf{S5}_n$-validity in $\mathsf{Q}^{1=}\mathsf{ML}_c$ are* coNExpTime*-complete. In fact, every satisfiable sentence is satisfiable in a frame of exponential size.*

Proof. We consider $\mathbf{K}_n$, the proof for $\mathbf{S5}_n$ is similar. Assume φ is satisfiable in some $\mathbf{K}_n$ frame. By Lemmas 11, 9 and 13, there is a weak quasimodel $\mathfrak{Q} = (\mathfrak{F}, \boldsymbol{q}, \mathfrak{R}, \mathfrak{p})$ satisfying φ based on a $\mathbf{Tree}_n^{d(\varphi)}$ frame $\mathfrak{F} = (W, \{R_a\}_{a\in A})$ with $|W| \leq 2^{O(d(\varphi)\cdot|\varphi|)}$. We prune $\mathfrak{Q}$ to obtain an exponential-size weak quasimodel $\mathfrak{Q}' = (\mathfrak{F}, \boldsymbol{q}', \mathfrak{R}', \mathfrak{p})$ by picking for $\mathfrak{R}'$ a prototype weak run $\mathfrak{p}(\boldsymbol{w}, t) \in \mathfrak{R}$ for every $\boldsymbol{w} \in W$ and $t \in \boldsymbol{q}(\boldsymbol{w})$ and setting $\boldsymbol{q}'(\boldsymbol{w}, t) = |\mathfrak{R}'_{\boldsymbol{w},t}|$. By Lemma 18, the $\boldsymbol{q}'(\boldsymbol{w})$ are quasistates for φ, and so $\mathfrak{Q}'$ is a weak quasimodel satisfying φ. We obtain an exponential-size interpretation satisfying φ by Lemmas 14 and 9.

The coNExpTime upper bound for validity follows immediately. The matching lower bound follows from NExpTime-hardness of satisfiability of the one-variable modal logics **K** and **S5** in constant domains, even without equality and constants [44]. ⊣

We note that the coNExpTime-hardness result [44], proved in the context of product modal logics, is contingent on the domains of the models being constant and does not translate to expanding domains. Indeed, we show in Section 8.2 that $\mathsf{Q}^{1=}\mathsf{ML}_c$-validity on $\mathbf{K}_n$ frames in *expanding* domains is PSpace-complete.

7.2. Guarded Fragment

We next show that $\mathbf{K}_n$- and $\mathbf{S5}_n$-validity in $\mathsf{GF}^{=}_{\boxed{1}}\mathsf{ML}_c$ is in 2ExpTime. Let φ be a $\mathsf{GF}^{=}_{\boxed{1}}\mathsf{ML}_c$-sentence. We assume it contains a subformula $x = c$ for any constant $c \in \mathsf{Con}$ that occurs in φ. For quasistate candidates $\boldsymbol{n}$ and $\boldsymbol{n}'$ for φ, define the product ordering as usual by setting $\boldsymbol{n} \leq \boldsymbol{n}'$ if $\boldsymbol{n}(t) \leq \boldsymbol{n}'(t)$ for all types t. We set $\boldsymbol{n} \leq_{gf} \boldsymbol{n}'$ if $\boldsymbol{n} \leq \boldsymbol{n}'$ and $\boldsymbol{n} \sim_0 \boldsymbol{n}'$. We require the following properties of the guarded fragment [40, 39, 11].

Lemma 20 (Guarded Fragment)**.** *Let φ be a $\mathsf{GF}^{=}_{\boxed{1}}\mathsf{ML}_c$-sentence.*

(1) *If $\boldsymbol{n}$ is a quasistate for φ, then any quasistate candidate $\boldsymbol{n}'$ with $\boldsymbol{n} \leq_{gf} \boldsymbol{n}'$ is also a quasistate for φ.*

(2) *If $\boldsymbol{n}$ is a quasistate for φ, then there is a quasistate $\boldsymbol{n}'$ for φ with $\boldsymbol{n}(t) > 0$ iff $\boldsymbol{n}'(t) > 0$ for all types t and such that $\boldsymbol{n}'(t) \leq 2^{2^{|\varphi|^{O(1)}}}$, for all types t.*

(3) *Deciding whether a quasistate candidate $\boldsymbol{n}$ for φ such that, for all types t for φ, either $\boldsymbol{n}(t) \leq 2^{|\varphi|^{O(1)}}$ or $\boldsymbol{n}(t) = \aleph_0$ is realisable is in 2ExpTime (in $|\varphi|$).*

Proof. (1) It suffices to show that for every t not containing a constant and with $\boldsymbol{n}(t) > 0$, the quasistate candidate $\boldsymbol{n}'$ defined in the same way as $\boldsymbol{n}$ except that $\boldsymbol{n}'(t) = \boldsymbol{n}(t) + 1$ is also a quasistate for φ. Consider a first-order structure $\mathfrak{B}$ realising $\boldsymbol{n}$ and, in particular, t by domain element d. Define a new first-order structure $\mathfrak{B}'$ by adding a fresh domain element d' to it and adding a tuple to any R if it is obtained from a tuple in R by replacing uniformly d by d'. The first-order structures $\mathfrak{B}'$ and $\mathfrak{B}$ are guarded bisimilar [38], and therefore, $\boldsymbol{n}'$ is realised in $\mathfrak{B}'$.

(2) Follows from the proof of the double exponential finite model property of the guarded fragment [40]. More precisely, given a quasistate $\boldsymbol{n}$ for φ, we consider the following conjunction ψ of guarded sentences

$$\mathsf{only}_{\boldsymbol{n}} \;=\; \forall x \bigvee_{\boldsymbol{n}(t)>0} \bar{t}(x), \tag{10}$$

$$\mathsf{exists}_{\boldsymbol{n}} \;=\; \bigwedge_{\boldsymbol{n}(t)>0} \exists x\, \bar{t}(x). \tag{11}$$

It should be clear that each first-order structure satisfying ψ gives rise to a quasistate $\boldsymbol{n}'$ for φ such that $\boldsymbol{n}(t) > 0$ iff $\boldsymbol{n}'(t) > 0$, for all types t for φ. Now observe that even though $|\psi|$ is exponential in $|\varphi|$, the sentence has the same

signature as φ and so, by [40, Theorem 1.2], can be satisfied in a structure of double-exponential size in the size of φ.

(3) We assume that no type t with $\boldsymbol{n}(t) = \aleph_0$ contains constants (otherwise, such $\boldsymbol{n}$ would trivially be non-realisable). Then we consider the conjunction ψ of the following guarded sentences: $\mathsf{only}_{\boldsymbol{n}}$ and $\mathsf{exists}_{\boldsymbol{n}}$ defined by (10) and (11) together with $\mathsf{atleast}^t_{\boldsymbol{n}}$ and $\mathsf{atmost}^t_{\boldsymbol{n}}$, for all types t for φ with $0 < \boldsymbol{n}(t) < \aleph_0$, where

$$\mathsf{atleast}^t_{\boldsymbol{n}} \;=\; \bigwedge_{1 \le i \le \boldsymbol{n}(t)} \bar{t}(c_{t,i}) \;\wedge \bigwedge_{1 \le i < j \le \boldsymbol{n}(t)} (c_{t,i} \neq c_{t,j}), \tag{12}$$

$$\mathsf{atmost}^t_{\boldsymbol{n}} \;=\; \forall x \, \big(\bar{t}(x) \to \bigvee_{1 \le i \le \boldsymbol{n}(t)} (x = c_{t,i})\big), \tag{13}$$

for fresh constant symbols $c_{t,i}$. It should be clear that ψ is satisfiable iff $\boldsymbol{n}$ is realisable; in particular, due to Point (1) we can always ensure that types with $\boldsymbol{n}(t) = \aleph_0$ are realised by infinitely many domain elements. Observe that ψ contains $2^{|\varphi|^{O(1)}}$ constants, but the same variables and the same predicate symbols as φ. It follows that its satisfiability can be decided in time $2^{O(|\varphi|)\cdot(l+k)^k}$, where k is the number of variables and l is the number of constants in ψ; see, e.g., [11, Proposition 3.5]. Thus, we obtain the required complexity bound. ⊣

Theorem 21. *For both constant and expanding domains,* $\mathbf{K}_n$*-validity and* $\mathbf{S5}_n$*-validity in* $\mathsf{GF}^{=}_{\boxed{1}}\mathsf{ML}_c$ *are* 2ExpTime*-complete.*

Proof. We consider validity on $\mathbf{K}_n$-frames with constant domains; the proof for $\mathbf{S5}_n$ is similar. 2ExpTime-hardness and the result for expanding domains (by Theorem 6 (c)) then follow from the complexity of GF [39].

Assume φ is given. By Lemmas 9–13, φ is satisfiable in some $\mathbf{K}_n$ frame iff there is a weak quasimodel $\mathfrak{Q} = (\mathfrak{F}, \boldsymbol{q}, \mathfrak{R}, \mathfrak{p})$ satisfying φ based on a $\mathbf{Tree}^{d(\varphi)}_n$ frame of size $2^{O(d(\varphi)\cdot|\varphi|)}$. Observe that the number of weak runs ρ such that $\rho(\boldsymbol{w})$ contains some constant for some $\boldsymbol{w} \in W$ is bounded by $|W| \times |\varphi|$. Let $\mathfrak{R}_c$ be the multiset of these weak runs. By Lemma 20 (1), the tuple $\mathfrak{Q}' = (\mathfrak{F}, \boldsymbol{q}', \mathfrak{R}', \mathfrak{p})$ defined by setting

$$\boldsymbol{q}'(\boldsymbol{w}, t) = \begin{cases} \aleph_0, & \text{if } \rho(\boldsymbol{w}) = t \text{ for some } \rho \in \mathfrak{R} \setminus \mathfrak{R}_c, \\ \boldsymbol{q}(\boldsymbol{w}, t), & \text{otherwise,} \end{cases} \quad \text{for } \boldsymbol{w} \in W \text{ and } t,$$

$$\mathfrak{R}'(\rho) = \begin{cases} \aleph_0, & \text{if } \rho \in \mathfrak{R} \setminus \mathfrak{R}_c, \\ \mathfrak{R}(\rho), & \text{otherwise,} \end{cases} \quad \text{for } \rho \in \mathfrak{R},$$

is again a weak quasimodel satisfying φ. In $\mathfrak{Q}'$, we have, for any $\boldsymbol{w} \in W$ and type t, either $\boldsymbol{q}'(\boldsymbol{w}, t) \le |W| \times |\varphi|$ or $\boldsymbol{q}'(\boldsymbol{w}, t) = \aleph_0$. We can enumerate in double exponential time all such $\mathfrak{Q}'$ if we admit $\boldsymbol{q}'(\boldsymbol{w})$ to be (not necessarily realisable) quasistate candidates for φ. Next, we can use Lemma 20 (3) to check in double exponential time for each $\boldsymbol{w}$ whether $\boldsymbol{q}'(\boldsymbol{w})$ is realisable and so $\mathfrak{Q}'$ is a quasimodel. Thus, we obtain a decision procedure in double exponential time. ⊣

7.3. Two-Variable Fragment with Counting and Presburger Arithmetic

To prove decidability and obtain tight complexity bounds for the two-variable fragment with counting, we now give a different description of constant-domain weak quasimodels which replaces the multiset of weak runs with a *set* of locally saturated weak runs, the number of which has only an *upper bound* (in contrast to **(card)**) along with local constraints on the relationship between quasistates. This allows us to use known, or easily derived, characterisations of quasistates, which together with the local constraints on the relationship between quasistates can then be translated to constraints in decidable fragments of arithmetic over natural numbers extended with infinity ($\aleph_0$).

Let φ be a $\mathsf{Q}^{=}_{\boxed{1}}\mathsf{ML}_c$-sentence and $\mathfrak{F} = (W, \{R_a\}_{a\in A})$ a tree-shaped $\mathbf{K}_n$ frame. Consider a weak quasimodel $(\mathfrak{F}, \boldsymbol{q}, \mathfrak{R}, \mathfrak{p})$ for φ based on $\mathfrak{F}$. We begin by observing that the prototype function $\mathfrak{p}$ gives rise to a *set* $\mathfrak{R}^{\mathfrak{p}}$ of weak runs defined as follows:

$$\mathfrak{R}^{\mathfrak{p}}(\rho) = \begin{cases} 1, & \text{if } \rho = \mathfrak{p}(\boldsymbol{w}, t), \text{ for some } \boldsymbol{w} \in W \text{ and } t \in \boldsymbol{q}(\boldsymbol{w}), \\ 0, & \text{otherwise;} \end{cases}$$

note that $\mathfrak{R}^{\mathfrak{p}}$ does not contain duplicates. It should be clear that $\mathfrak{R}^{\mathfrak{p}}$ is included in $\mathfrak{R}$ as a multiset, and therefore satisfies the following weakening of **(card)**:

(card′) $\boldsymbol{q}(\boldsymbol{w}, t) \geq |\mathfrak{R}^{\mathfrak{p}}_{\boldsymbol{w},t}|$, for every $\boldsymbol{w} \in W$ and every type t for φ,

In order to be able to recover $\mathfrak{R}$ from a given $\mathfrak{R}^{\mathfrak{p}}$, however, we will require some additional local constraints.

A *link* between quasistates $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ is a function L that assigns a multiplicity $L(t_1, t_2) \in \mathbb{N}_\infty$ to any pair t_1, t_2 of types for φ such that

$$\sum_t L(t, t_2) = \boldsymbol{n}_2(t_2), \qquad \text{for all types } t_2,$$

$$\sum_t L(t_1, t) = \boldsymbol{n}_1(t_1), \qquad \text{for all types } t_1.$$

Links describe runs locally in the sense that $L(t_1, t_2)$-many weak runs simultaneously have type t_1 in $\boldsymbol{n}_1$ and type t_2 in $\boldsymbol{n}_2$.

For uniformity with $\mathbf{S5}_n$ tree-shaped frames, in the sequel we shall use the a-successor relation $\prec_a$, for $a \in A$, defined by taking $\boldsymbol{w} \prec_a \boldsymbol{w}'$ iff $\boldsymbol{w}'$ is of the form $\boldsymbol{w}aw$, for some w; clearly, R_a coincides with $\prec_a$ for tree-shaped $\mathbf{K}_n$ frames; in $\mathbf{S5}_n$ frames, each R_a is the reflexive and transitive closure of $\prec_a$.

A $\mathbf{K}_n$ *weak pre-quasimodel for* φ is a quadruple $\mathfrak{Q} = (\mathfrak{F}, \boldsymbol{q}, \mathfrak{p}, \mathfrak{L})$ such that $(\mathfrak{F}, \boldsymbol{q})$ is a basic structure for φ, $\mathfrak{p}$ is a prototype function satisfying **(wq-sat)** for the set $\mathfrak{R}^{\mathfrak{p}}$ of weak runs and $\mathfrak{L}$ is a set of links $L^a_{\boldsymbol{w},\boldsymbol{v}}$ between quasistates $\boldsymbol{q}(\boldsymbol{w})$ and $\boldsymbol{q}(\boldsymbol{v})$, for $a \in A$ and worlds $\boldsymbol{w}, \boldsymbol{v} \in W$ with $\boldsymbol{w} \prec_a \boldsymbol{v}$, such that

(link$_1$) $t_1 \to_a t_2$ whenever $L^a_{\boldsymbol{w},\boldsymbol{v}}(t_1, t_2) > 0$, for all types t_1, t_2 for φ;

(link$_2$) $L^a_{\boldsymbol{w},\boldsymbol{v}}(t_1, t_2) \geq |\mathfrak{R}^{\mathfrak{p}}_{\boldsymbol{w},t_1,\boldsymbol{v},t_2}|$, for all types t_1, t_2 for φ,

where $\mathfrak{R}^{\mathfrak{p}}_{\boldsymbol{w},t_1,\boldsymbol{v},t_2}$ is the set of weak runs $\rho \in \mathfrak{R}^{\mathfrak{p}}$ with $\rho(\boldsymbol{w}) = t_1$ and $\rho(\boldsymbol{v}) = t_2$ (again, this is a set simply because $\mathfrak{R}^{\mathfrak{p}}$ does not contain duplicates). We say $\mathfrak{Q}$ satisfies φ if **(b)** holds.

Lemma 22. *For constant domains, the following conditions are equivalent for any basic structure $(\mathfrak{F}, \boldsymbol{q})$ based on a finite tree-shaped $\mathbf{K}_n$ frame $\mathfrak{F}$:*

1. *$(\mathfrak{F}, \boldsymbol{q})$ can be expanded to a $\mathbf{K}_n$ weak quasimodel satisfying φ;*
2. *$(\mathfrak{F}, \boldsymbol{q})$ can be expanded to a $\mathbf{K}_n$ weak pre-quasimodel satisfying φ with the number of weak runs bounded by $|W| \times 2^{|\varphi|}$.*

Proof. (1) $\Rightarrow$ (2) Assume a weak quasimodel $(\mathfrak{F}, \boldsymbol{q}, \mathfrak{R}, \mathfrak{p})$ is given. To define a weak pre-quasimodel, we take $\mathfrak{L}$ that contains, for each $a \in A$ and $\boldsymbol{w}, \boldsymbol{v} \in W$ with $\boldsymbol{w} \prec_a \boldsymbol{v}$, a link $L^a_{\boldsymbol{w},\boldsymbol{v}}$ defined by taking

$$L^a_{\boldsymbol{w},\boldsymbol{v}}(t_1, t_2) = |\mathfrak{R}_{\boldsymbol{w},t_1,\boldsymbol{v},t_2}|, \text{ for all types } t_1, t_2 \text{ for } \varphi.$$

It is readily checked that $(\mathfrak{F}, \boldsymbol{q}, \mathfrak{p}, \mathfrak{L})$ is a $\mathbf{K}_n$ weak pre-quasimodel satisfying φ and the set $\mathfrak{R}^{\mathfrak{p}}$ contains at most $|W| \times 2^{|\varphi|}$ weak runs.

(2) $\Rightarrow$ (1) Let $(\mathfrak{F}, \boldsymbol{q}, \mathfrak{p}, \mathfrak{L})$ be a weak pre-quasimodel satistying φ. To define a weak quasimodel $(\mathfrak{F}, \boldsymbol{q}, \mathfrak{R}, \mathfrak{p})$ satisfying φ, we extend the set $\mathfrak{R}^{\mathfrak{p}}$ to a multiset $\mathfrak{R}$ of weak runs so that **(card)** holds. To this end, we take an enumeration $(\boldsymbol{w}_1, t_1, \ell_1), (\boldsymbol{w}_2, t_2, \ell_2), \ldots$, where $\boldsymbol{w}_i \in W$, $t_i \in \boldsymbol{q}(\boldsymbol{w}_i)$ and $\ell_i \in \mathbb{N}$ with $\ell_i < \boldsymbol{q}(\boldsymbol{w}, t)$, for all $i > 0$, and ensure inductively that there is an indexed weak run $(\rho_i, \ell_i) \in \hat{\mathfrak{R}}_{\boldsymbol{w}_i,t_i}$ for any $\boldsymbol{w}_i \in W$ and type $t_i \in \boldsymbol{q}(\boldsymbol{w}_i)$.

Assume we have constructed $\hat{\mathfrak{R}}^0 \subseteq \ldots \subseteq \hat{\mathfrak{R}}^{i-1}$ with $\hat{\mathfrak{R}}^0 = \{(\rho, 0) \mid \rho \in \mathfrak{R}^{\mathfrak{p}}\}$ so that $\hat{\mathfrak{R}}^{i-1}$ contains indexed witness weak runs for the first $(i-1)$ members of the enumeration and also satisfies **(card′)** and **(link$_2$)** for all $a \in A$ and $\boldsymbol{w}, \boldsymbol{v} \in W$ with $\boldsymbol{w} \prec_a \boldsymbol{v}$. Consider the ith element $(\boldsymbol{w}_i, t_i, \ell_i)$ of the enumeration. If $\hat{\mathfrak{R}}^{i-1}$ already contains (ρ, ℓ_i) with $\rho(\boldsymbol{w}_i) = t_i$, then we set $\hat{\mathfrak{R}}^i = \hat{\mathfrak{R}}^{i-1}$, thus satisfying **(card′)** and **(link$_2$)**. Otherwise, we have $|\mathfrak{R}^{i-1}_{\boldsymbol{w}_i,t_i}| < \boldsymbol{q}(\boldsymbol{w}_i, t_i)$ and define a weak run ρ_i such that $\rho_i(\boldsymbol{w}_i) = t_i$ and the result $\hat{\mathfrak{R}}^i$ of extending $\hat{\mathfrak{R}}^{i-1}$ with (ρ_i, ℓ_i) satisfies **(card′)** and **(link$_2$)**.

We define ρ_i by induction on the structure of the $\prec_a$, for $a \in A$, starting from $W^0 = \{\boldsymbol{w}_i\}$, ensuring **(link$_2$)**, for $\mathfrak{R}^{i-1}$ and every $\boldsymbol{w}, \boldsymbol{v} \in W^j$ with $\boldsymbol{w} \prec_a \boldsymbol{v}$, and

$$|\mathfrak{R}^{i-1}_{\boldsymbol{w},t}| < \boldsymbol{q}(\boldsymbol{w}, t), \text{ where } t = \rho_i(\boldsymbol{w}), \tag{14}$$

for every $\boldsymbol{w} \in W^j$; cf. **(card′)**. For $W^0 = \{\boldsymbol{w}_i\}$ with $\rho_i(\boldsymbol{w}_i) = t_i$ these conditions hold. Suppose ρ_i is defined on W^j. We have the following cases.

- If $\boldsymbol{w} \in W^j$ but $\boldsymbol{v} \notin W^j$ with $\boldsymbol{w} \prec_a \boldsymbol{v}$, then we consider $t = \rho_i(\boldsymbol{w})$. By (14), as $\boldsymbol{q}(\boldsymbol{w}, t) = \sum_{t'} L^a_{\boldsymbol{w},\boldsymbol{v}}(t, t')$, we can find a type t' for φ with $|\mathfrak{R}^{i-1}_{\boldsymbol{w},t,\boldsymbol{v},t'}| < L^a_{\boldsymbol{w},\boldsymbol{v}}(t, t')$. We extend ρ_i from W^j to $W^{j+1} = W^j \cup \{\boldsymbol{v}\}$ by setting $\rho_i(\boldsymbol{v}) = t'$. Observe that (14) holds for $\boldsymbol{v}$ due to **(link$_2$)** and $\sum_{t''} L^a_{\boldsymbol{w},\boldsymbol{v}}(t'', t') = \boldsymbol{q}(\boldsymbol{v}, t')$.

- If $\boldsymbol{w} \in W^j$ but $\boldsymbol{v} \notin W^j$ with $\boldsymbol{v} \prec_a \boldsymbol{w}$, then the definition of ρ_i on $\boldsymbol{v}$ is similar and left to the reader.

Observe that when ρ_i is defined on W, it is a weak run due to **(link$_1$)**.

Finally, let $\mathfrak{R}$ be such that $\hat{\mathfrak{R}} = \bigcup_i \hat{\mathfrak{R}}^i$. It should be clear that the limit $\mathfrak{R}$ satisfies **(card)** rather than **(card′)**, which holds for the individual stages $\hat{\mathfrak{R}}^i$. Thus, $(\mathfrak{F}, \boldsymbol{q}, \mathfrak{R}, \mathfrak{p})$ is a weak quasimodel satisfying φ. $\dashv$

The above construction can be easily adapted for an $\mathbf{S5}_n$ tree-shaped frame $\mathfrak{F} = (W, \{R_a\}_{a \in A})$. An $\mathbf{S5}_n$ *weak pre-quasimodel for* φ is a quadruple $\mathfrak{Q} = (\mathfrak{F}, \boldsymbol{q}, \mathfrak{p}, \mathfrak{L})$ such that $\mathfrak{p}$ is a prototype function with full weak runs satisfying **(wq-sat-s5)** with $\mathfrak{R}^{\mathfrak{p}}$ and **(card′)**, and $\mathfrak{L}$ is a set of links $L^a_{\boldsymbol{w},\boldsymbol{v}}$ between quasistates $\boldsymbol{q}(\boldsymbol{w})$ and $\boldsymbol{q}(\boldsymbol{v})$, for $a \in A$ and worlds $\boldsymbol{w}, \boldsymbol{v} \in W$ with $\boldsymbol{w} \prec_a \boldsymbol{v}$, satisfying **(link$_2$)** and the following counterpart of **(link$_1$)**:

(link$_1$-s5) $t_1 \leftrightarrow_a t_2$ whenever $L^a_{\boldsymbol{w},\boldsymbol{v}}(t_1, t_2) > 0$, for all types t_1, t_2 for φ.

We then have the following analogue of Lemma 22:

Lemma 23. *The following conditions are equivalent for any basic structure* $(\mathfrak{F}, \boldsymbol{q})$ *based on a finite tree-shaped* $\mathbf{S5}_n$ *frame* $\mathfrak{F}$:

1. $(\mathfrak{F}, \boldsymbol{q})$ *can be expanded to an* $\mathbf{S5}_n$ *weak quasimodel satisfying* φ;
2. $(\mathfrak{F}, \boldsymbol{q})$ *can be expanded to an* $\mathbf{S5}_n$ *weak pre-quasimodel satisfying* φ *with the number of weak runs bounded by* $|W| \times 2^{|\varphi|}$.

We are in a position now to give rather quick decidability proofs for monodic fragments over $\mathbf{K}_n$ and $\mathbf{S5}_n$ frames. We consider $\mathbf{K}_n$ with constant domains: the approach for $\mathbf{S5}_n$ is similar, and decidability for expanding domains then follows by Theorem 6 (c). Fix a sentence φ. The idea is to decide the existence of weak pre-quasimodel for a sentence φ by encoding into a decidable but powerful fragment of arithmetic, for instance, Presburger Arithmetic with infinity [48]. To implement this idea, we require some notation for numbers. By $\mathbb{N}^k_\infty$ we denote the set of k-tuples over $\mathbb{N}_\infty$. We fix an ordering $t_1, \dots, t_{k_\varphi}$ of the types for φ. Then a quasistate candidate $\boldsymbol{n}$ for φ can be identified with a k_φ-tuple $\boldsymbol{n} = (n_1, \dots, n_{k_\varphi}) \in \mathbb{N}^{k_\varphi}_\infty$ by setting $n_i = \boldsymbol{n}(t_i)$, for $1 \leq i \leq k_\varphi$. In this case we also say the tuple *represents* the quasistate candidate; the same applies to quasistates. Now, by Lemmas 9–13, φ is satisfiable in some $\mathbf{K}_n$ frame iff there is a weak quasimodel $\mathfrak{Q} = (\mathfrak{F}, \boldsymbol{q}, \mathfrak{R}, \mathfrak{p})$ satisfying φ based on a $\mathbf{Tree}^{d(\varphi)}_n$ frame $\mathfrak{F} = (W, \{R_a\}_{a \in A})$ with $|W| \leq 2^{O(d(\varphi) \cdot |\varphi|)}$. By Lemma 22, this is the case iff there is a weak pre-quasimodel $\mathfrak{Q} = (\mathfrak{F}, \boldsymbol{q}, \mathfrak{p}, \mathfrak{L})$ satisfying φ such that in addition $|\mathfrak{R}^{\mathfrak{p}}| \leq |W| \times 2^{|\varphi|}$. It follows that the only objects in $\mathfrak{Q}$ that do not have an exponential upper bound are the quasistates $\boldsymbol{q}(\boldsymbol{w})$ and the links $\mathfrak{L}$.

Assume now that we have a monodic fragment L such that we can construct, for any sentence $\varphi \in \mathsf{L}$, a formula $\mathit{Pres}_\varphi(x_1, \dots, x_{k_\varphi})$ in Presburger Arithmetic with infinity (PAI) such that

(encode) a k_φ-tuple $\boldsymbol{n} \in \mathbb{N}^{k_\varphi}_\infty$ represents a quasistate for φ iff PAI $\models \mathit{Pres}_\varphi(\boldsymbol{n})$.

This is known, for instance, if L is $\mathsf{C}^2_{\boxed{1}}\mathsf{ML}_c$ (see below). To decide satisfiability of φ by checking the existence of a weak pre-quasimodel $\mathfrak{Q}$, guess finite $\mathfrak{F}$ and $\mathfrak{p}$ (and so also $\mathfrak{R}^{\mathfrak{p}}$), take for each $\boldsymbol{w} \in W$ individual variables $x_1^{\boldsymbol{w}}, \dots, x_{k_\varphi}^{\boldsymbol{w}}$ and the formulas $\mathit{Pres}_\varphi(x_1^{\boldsymbol{w}}, \dots, x_{k_\varphi}^{\boldsymbol{w}})$, and also take individual variables $x_{a,\boldsymbol{w},\boldsymbol{v},i,j}$ representing $L^a_{\boldsymbol{w},\boldsymbol{v}}(t_i, t_j)$ for $a \in A$, $\boldsymbol{w}, \boldsymbol{v} \in W$ with $\boldsymbol{w} R_a \boldsymbol{v}$ and $1 \leq i, j \leq k_\varphi$, and the conjunction $\mathit{Link}_{\mathfrak{F},\mathfrak{p}}$ of the following equalities and inequalities:

$$
\begin{aligned}
&\sum_{i=1}^{k_\varphi} x_{a,\boldsymbol{w},\boldsymbol{v},i,j} = x_j^{\boldsymbol{v}}, && \text{for all } 1 \leq j \leq k_\varphi;\\
&\sum_{j=1}^{k_\varphi} x_{a,\boldsymbol{w},\boldsymbol{v},i,j} = x_i^{\boldsymbol{w}}, && \text{for all } 1 \leq i \leq k_\varphi;\\
&x_{a,\boldsymbol{w},\boldsymbol{v},i,j} = 0, && \text{for all } 1 \leq i, j \leq k_\varphi \text{ with } t_i \not\to_a t_j;\\
&x_{a,\boldsymbol{w},\boldsymbol{v},i,j} \geq |\mathfrak{R}^{\mathfrak{p}}_{\boldsymbol{w},t_i,\boldsymbol{v},t_j}|, && \text{for all } 1 \leq i, j \leq k_\varphi.
\end{aligned}
$$

Then, by the definition of the above formulas, we have the following equivalence.

Lemma 24. *There exists a weak pre-quasimodel for φ based on a given $\mathfrak{F}$ with a given prototype function $\mathfrak{p}$ iff $\bigwedge_{\boldsymbol{w} \in W} \mathit{Pres}_\varphi(x_1^{\boldsymbol{w}}, \dots, x_{k_\varphi}^{\boldsymbol{w}}) \wedge \mathit{Link}_{\mathfrak{F},\mathfrak{p}}$ is satisfiable.*

It follows that $\mathbf{K}_n$- and $\mathbf{S5}_n$-validity in $\mathsf{C}^2_{\boxed{1}}\mathsf{ML}_c$ are decidable. This straightforward general encoding, however, is not sufficient to obtain the tight complexity upper bound for the two-variable fragment with countring. To this end a more subtle combination with the upper bound proofs for the first-order fragments is needed. We introduce some notation for inequalities. *Linear extended-Diophantine equations and inequalities* take the form

$$a_1 x_1 + \dots + a_k x_k = b, \quad a_1 x_1 + \dots + a_k x_k \leq b, \quad a_1 x_1 + \dots + a_k x_k \geq b$$

with coefficients $a_1, \dots, a_k, b \in \mathbb{N}_\infty$; as usual, $m \leq \aleph_0$, for all $m \in \mathbb{N}_\infty$, $\aleph_0 \cdot 0 = 0 \cdot \aleph_0 = 0$, $\aleph_0 \cdot m = m \cdot \aleph_0 = \aleph_0$, for all $m \in \mathbb{N} \setminus \{0\}$, and $\aleph_0 \cdot \aleph_0 = \aleph_0$. We use the following bounds from [8, Corollary 7.11]:

Theorem 25 ([8])**.** *Let $\mathcal{E}$ be a system of m linear extended-Diophantine equations in k variables, and let M be the maximum value of all the finite coefficients of $\mathcal{E}$. If $\mathcal{E}$ has a solution, then it has a solution in which the finite values are bounded by $(2(k + 2m + 1)M - 1)^{2m}$.*

Now, instead of assuming the computability of a PAI formula Pres_φ in a blackbox manner, we exploit directly the NExpTime upper bound proof for the two-variable fragment with counting in [8, Chapter 8]. Namely, we use the following consequence of the results presented in [8, Chapter 8].

Lemma 26. *Let φ be a $\mathsf{C}^2_{\boxed{1}}\mathsf{ML}_c$-sentence. There is a set $\mathcal{C}$ of sets $\mathcal{E}$ of linear extended-Diophantine equations with variables $x_1, \dots, x_{k_\varphi}$, $|\mathcal{E}|$ at most exponential in $|\varphi|$, and coefficients of at most double exponential size in $|\varphi|$ such that a vector $\boldsymbol{n} \in \mathbb{N}_\infty^{k_\varphi}$ represents a quasistate for φ iff $\boldsymbol{n}$ is a solution to some $\mathcal{E} \in \mathcal{C}$. Moreover, $\mathcal{E} \in \mathcal{C}$ can be decided in exponential time.*

Proof. We give a short sketch. The proof of this lemma is based on [8, Sections 8.4 and 8.5] and is rather cumbersome, but straightforward in principle. The following two observation should be sufficient. (1) The normal form used in [8] does not preserve satisfiability in small models. This can be treated by introducing new sets of equations in $\mathcal{C}$ dealing with satisfiability in small models. (2) The equations in [8] encode satisfiability in terms of the number of domain elements satisfying star-types and not the number of domain elements satisfying "our" types. This can be dealt with by expressing "our" types as disjunctions of star-types. The solutions to the equations in [8] are then used as coefficients in the equations of our equations. This leads to coefficients of double exponential size. ⊣

Theorem 27. *For both constant- and expanding-domain models,* $\mathbf{K}_n$*-validity and* $\mathbf{S5}_n$*-validity in* $\mathsf{C}^2_{[1]}\mathsf{ML}_c$ *are* coNExpTime*-complete.*

Proof. We consider $\mathbf{K}_n$ with constant domains; the proof for $\mathbf{S5}_n$ is similar. Hardness and the result for expanding domains (by Theorem 6 (c)) follow from the complexity of C^2. So, it remains to show the upper complexity bound.

Assume φ is given. We follow the proof schema above but define and use Pres_φ differently. By Lemmas 9–13 and 22, φ is satisfiable in some $\mathbf{K}_n$ frame with constant domain iff there is a weak pre-quasimodel $\mathfrak{Q} = (\mathfrak{F}, \boldsymbol{q}, \mathfrak{p}, \mathfrak{L})$ satisfying φ based on a $\mathbf{Tree}_n^{d(\varphi)}$ frame $\mathfrak{F} = (W, \{R_a\}_{a\in A})$ such that $|W| \leq 2^{O(d(\varphi)\cdot|\varphi|)}$ and $|\mathfrak{R}^{\mathfrak{p}}| \leq |W| \times 2^{|\varphi|}$. The algorithm checking the existence of such a weak pre-quasimodel $\mathfrak{Q}$ now guesses $\mathfrak{F}$ and $\mathfrak{p}$ and, for each $\boldsymbol{w} \in W$, a set $\mathcal{E}_{\boldsymbol{w}}$ of linear extended-Diophantine equations with variables $x_1^{\boldsymbol{w}}, \dots, x_{k_\varphi}^{\boldsymbol{w}}$ satisfying the conditions of Lemma 26 and finally guesses a solution of the set $\mathcal{E}$ of equations comprising $\mathcal{E}_{\boldsymbol{w}}$, for $\boldsymbol{w} \in W$, and $\mathit{Link}_{\mathfrak{F},\mathfrak{p}}$. By our definitions, a solution for $\mathcal{E}$ exists iff the weak pre-quasimodel $\mathfrak{Q}$ introduced above exists. By Theorem 25, a solution for $\mathcal{E}$ exists iff there exists one in which the finite values are bounded by a double exponential function in $|\varphi|$ (and so can be represented using exponential space in $|\varphi|$). It can also be checked in exponential time whether the guessed values actually represent a solution, as required. ⊣

8. Decidability for Monodic Fragments in Expanding Domains

In this section we study the complexity of reasoning in expanding domains. Recall that, by Theorem 6 (c), both our reasoning problems in expanding domains can be reduced to their counterparts in constant domains (by using relativisation of quantifiers). It turns out, however, that in some cases, the case of expanding domains is considerably simpler than the case of constant domains.

First, note that both our main constructions, quasimodels (Lemma 9) and weak quasimodels (Lemmas 11, 13 and 14), work for expanding domains as well.

The following additional assumption on (weak) quasimodels will simplify the constructions in this section: whenever the frame $\mathfrak{F}. = (W, \{R_a\}_{a\in A})$ is tree-shaped, we will assume that all (weak) runs are rooted in the following

sense. For any $w \in W$, we set $W_{\downarrow \boldsymbol{w}} = \{\boldsymbol{w}\} \cup \{\boldsymbol{v} \in W \mid \boldsymbol{w} R_* \boldsymbol{v}\}$ and denote the restriction of a (weak) run $\rho \in \mathfrak{R}_{\boldsymbol{w}}$ to $W_{\downarrow \boldsymbol{w}}$ by $\rho_{\downarrow \boldsymbol{w}}$. A run through $(\mathfrak{F}, \boldsymbol{q})$ is called *rooted* if there is $\boldsymbol{w} \in W$, called the *root* of ρ, such that $W_{\downarrow \boldsymbol{w}} = \mathsf{dom}\rho$.

Lemma 28. *For any quasimodel $\mathfrak{Q} = (\mathfrak{F}, \boldsymbol{q}, \mathfrak{R})$ based on a tree-shaped frame $\mathfrak{F}$, there is a quasimodel $\mathfrak{Q} = (\mathfrak{F}, \boldsymbol{q}, \mathfrak{R}')$ with rooted runs $\mathfrak{R}'$ that satisfies the same formulas and such that $|\mathfrak{R}'| \leq |\mathfrak{R}| \cdot |W|$.*

Similarly, for any weak quasimodel $\mathfrak{Q} = (\mathfrak{F}, \boldsymbol{q}, \mathfrak{R}, \mathfrak{p})$, there is a weak quasimodel $\mathfrak{Q} = (\mathfrak{F}, \boldsymbol{q}, \mathfrak{R}', \mathfrak{p}')$ with rooted runs $\mathfrak{R}'$ that satisfies the same formulas and such that $|\mathfrak{R}'| \leq |\mathfrak{R}| \cdot |W|$.

Proof. Let $\rho \in \mathfrak{R}$ have no root. Let $\boldsymbol{w}_1, \dots, \boldsymbol{w}_k \in W$ be a minimal set such that $\mathsf{dom}\rho = W_{\downarrow \boldsymbol{w}_1} \uplus \dots \uplus W_{\downarrow \boldsymbol{w}_k}$; note that since the frame is tree-shaped, the union is disjoint. Then we replace, in $\mathfrak{R}$, the (weak) run ρ by its restrictions $\rho_{\downarrow \boldsymbol{w}_i}$ to $W_{\boldsymbol{w}_i}$, for $1 \leq i \leq k$. Formally, $\mathfrak{R}'$ is defined as $\mathfrak{R}$ except that

$$\mathfrak{R}'(\rho) = 0 \quad \text{and} \quad \mathfrak{R}'(\rho_{\downarrow \boldsymbol{w}_i}) = \mathfrak{R}(\rho_{\downarrow \boldsymbol{w}_i}) + \mathfrak{R}(\rho), \quad \text{for } 1 \leq i \leq k;$$

note that $\rho_{\downarrow \boldsymbol{w}_i}$ could already exist in $\mathfrak{R}$ as a separate run, which is taken into account by the first component in the sum.

For a weak quasimodel, if $\mathfrak{p}(\boldsymbol{w}, t) = \rho$, for some $\boldsymbol{w} \in W$ and $t \in \boldsymbol{q}(\boldsymbol{w})$, then $\mathfrak{p}(\boldsymbol{w}, t)$ is replaced with the $\rho_{\downarrow \boldsymbol{w}_i}$ such that $\boldsymbol{w} \in W_{\downarrow \boldsymbol{w}_i}$ (which is uniquely determined as the frame is tree-shaped). ⊣

It can be seen that the quasimodel constructed in the proof of Lemma 9 has rooted runs if the frame is tree-shaped; also, the constructions in Lemmas 13 and 14 preserve rootedness of (weak) runs in (weak) quasimodels.

8.1. Logics with Transitive Closure

We consider modal logics with sets of modalities of the form $A = A_0 \cup \{*\}$, where $*$ denotes the modality interpreted by the transitive closure of the union of the remaining modalities in A_0. Formally, a $\mathbf{K}_{*n}$ frame is of the form $\mathfrak{F} = (W, \{R_a\}_{a \in A})$, where $|A_0| = n$ and R_* is the transitive closure of $\bigcup_{a \in A_0} R_a$. If, in addition, R_* contains no infinite ascending chains, that is, no infinite sequence $w_0, w_1, \dots \in W$ with $w_i R_* w_{i+1}$ for all $i \geq 0$, then $\mathfrak{F}$ is called a $\mathbf{K}\boldsymbol{f}_{*n}$ frame. Observe that in contrast to $\mathbf{K}_n$ and $\mathbf{S5}_n$, with $n \geq 2$, global $\mathbf{K}_{*n}$-consequence and global $\mathbf{K}\boldsymbol{f}_{*n}$-consequence are polytime-reducible to $\mathbf{K}_{*n}$- and, respectively, $\mathbf{K}\boldsymbol{f}_{*n}$-validity since φ is a global $\mathcal{C}$-consequence of Γ iff $(\bigwedge \Gamma \wedge \Box_* \bigwedge \Gamma) \to \varphi$ is $\mathcal{C}$-valid for $\mathcal{C} \in \{\mathbf{K}_{*n}, \mathbf{K}\boldsymbol{f}_{*n}\}$; see also Remark 5.

Lemma 29. *For both constant and expanding domains, for $\mathcal{C} \in \{\mathbf{K}_{*n}, \mathbf{K}\boldsymbol{f}_{*n}\}$ and all fragments $\mathcal{L}$ considered, global $\mathcal{C}$-consequence in $\mathcal{L}$ is polytime-reducible to $\mathcal{C}$-validity in $\mathcal{L}$.*

A $\mathbf{K}\boldsymbol{f}_{*n}$-frame is called *tree-shaped* if the restriction $\mathfrak{F}_{|A_0} = (W, \{R_a\}_{a \in A_0})$ of $\mathfrak{F}$ to A_0 is a tree-shaped $\mathbf{K}_n$ frame. We use the notation from Section 5 for these frames, in particular worlds in tree-shaped frames take the form (3) with $a_j \in A_0$ for $0 \leq j < m$.

Lemma 30. *For both constant and expanding domains, every* $\mathbf{Kf}_{*n}$*-satisfiable* $\mathsf{Q}^{=}\mathsf{ML}_c$*-formula is also satisfiable in tree-shaped* $\mathbf{Kf}_{*n}$ *frames.*

Proof. The construction is exactly the same as in the proof of Lemma 11 except that we use the modalities in A_0 only and do not impose the bound $d(\varphi)$ on the depth $d(\boldsymbol{w})$ of worlds $\boldsymbol{w}$. $\dashv$

When working with (weak) quasimodels based on $\mathbf{Kf}_{*n}$ frames, it is useful to extend the set of subformulas $sub_x(\varphi)$ that occur in types. Similar extensions have been introduced for many modal logics with operators for the transitive closure, for instance the Fischer-Ladner closure for PDL [49]. In detail, define $sub^*_x(\varphi)$ by extending $sub_x(\varphi)$ with all formulas of the form $\Diamond_a\psi$ and $\Diamond_a\Diamond_*\psi$, for $\Diamond_*\psi \in sub_x(\varphi)$ and $a \in A_0$. In what follows we always work with $sub^*_x(\varphi)$ instead of $sub_x(\varphi)$. We may assume that $|sub^*_x(\varphi|) \leq s(\varphi)$, where s is defined by taking $s(\varphi) = 2|\varphi|^2$.

A $\mathbf{Kf}_{*n}$*-type* t *for* φ is a Boolean-saturated subset of $sub^*_x(\varphi)$ satisfying

(kf$_*$-type) $\Diamond_*\psi \in t$ iff there is $a \in A_0$ with either $\Diamond_a\psi \in t$ or $\Diamond_a\Diamond_*\psi \in t$, for all $\Diamond_*\psi \in sub^*_x(\varphi)$.

Let $\mathfrak{M} = (\mathfrak{F}, \Delta, \cdot)$ be an interpretation with a $\mathbf{Kf}_{*n}$ frame $\mathfrak{F}$. Then it is straightforward to show that $t^{\mathfrak{M}(w)}(d) = \{\psi \in sub^*_x(\varphi) \mid \mathfrak{M}, w \models \psi[d]\}$ is a $\mathbf{Kf}_{*n}$-type for φ, for every $w \in W$ and $d \in \Delta_w$. A $\mathbf{Kf}_{*n}$ *basic structure for* φ is a basic structure for φ based on a $\mathbf{Kf}_{*n}$ frame in which types for φ are in fact $\mathbf{Kf}_{*n}$-types for φ. A (weak) run and quasimodel are now defined as before and called a (weak) $\mathbf{Kf}_{*n}$ run and $\mathbf{Kf}_{*n}$ quasimodel, respectively. Then Lemma 9 still holds for $\mathbf{Kf}_{*n}$ quasimodels.

We now provide an equivalent description of (weak) $\mathbf{Kf}_{*n}$ runs in which the modality $\Diamond_*$ is considered only implicitly. This simplifies the constructions needed later.

Lemma 31. *Let* ρ *be a function mapping each world* $\boldsymbol{w}$ *in an upward-closed subset* W' *of* W *to a* $\mathbf{Kf}_{*n}$*-type* $\rho(\boldsymbol{w}) \in \boldsymbol{q}(\boldsymbol{w})$*. Then* ρ *satisfies* **(a-r-coh)** *for all* $a \in A$ *iff* ρ *satisfies* **(a-r-coh)** *for all* $a \in A_0$ *and*

(kf$_*$-r-coh) $\rho(\boldsymbol{w}) \Rightarrow_a \rho(\boldsymbol{v})$*, for every* $\boldsymbol{w}, \boldsymbol{v} \in \mathsf{dom}\rho$ *with* $\boldsymbol{w}R_a\boldsymbol{v}$ *and* $a \in A_0$*, where* $t \Rightarrow_a t'$ *is defined as follows*:

$$\Diamond_a\psi \in t' \quad \textit{implies} \quad \Diamond_*\psi \in t, \textit{ for all } \Diamond_*\psi \in sub^*_x(\varphi).$$

Proof. Assume ρ satisfies **(a-r-coh)** for all $a \in A$. To show that ρ satisfies **(kf$_*$-r-coh)**, let $a \in A_0$ and $\boldsymbol{w}, \boldsymbol{v} \in \mathsf{dom}\rho$ with $\boldsymbol{w}R_a\boldsymbol{v}$. Consider $\Diamond_*\psi \in sub^*_x(\varphi)$. By definition, $\Diamond_a\Diamond_*\psi \in sub^*_x(\varphi)$. Suppose now $\Diamond_a\psi \in \rho(\boldsymbol{v})$. Then, by **(kf$_*$-type)**, $\Diamond_*\psi \in \rho(\boldsymbol{v})$. By **($a$-r-coh)**, $\Diamond_a\Diamond_*\psi \in \rho(\boldsymbol{w})$, whence, by **(kf$_*$-type)**, $\Diamond_*\psi \in \rho(\boldsymbol{w})$, as required.

Conversely, assume ρ satisfies **(a-r-coh)** for all $a \in A_0$ and **(kf$_*$-r-coh)**. We show **(a-r-coh)** for $a = *$. Consider $\Diamond_*\psi \in sub^*_x(\varphi)$. By definition, $\Diamond_a\Diamond_*\psi \in sub^*_x(\varphi)$, for all $a \in A_0$. Assume $\boldsymbol{w}, \boldsymbol{v} \in \mathsf{dom}\rho$ with $\boldsymbol{w}R_*\boldsymbol{v}$. Then there is a finite

path $\boldsymbol{w} = \boldsymbol{w}_0 R_{a_1} \cdots R_{a_m} \boldsymbol{w}_m = \boldsymbol{v}$ with $m > 0$ and $a_1, \ldots, a_m \in A_0$. Suppose $\psi \in \rho(\boldsymbol{v}) = \rho(\boldsymbol{w}_m)$. Then, by **($a$-r-coh)** for $a = a_m$, we get $\Diamond_{a_m} \psi \in \rho(\boldsymbol{w}_{m-1})$ and so, by **($\mathbf{kf}_*$-r-coh)**, $\Diamond_* \psi \in \rho(\boldsymbol{w}_{m-1})$. We show that $\Diamond_* \psi \in \rho(\boldsymbol{w}_i)$ for all $0 \leq i < m$ by induction on i. The basis of induction, $i = m - 1$, is done. For the inductive step, let $\Diamond_* \psi \in \rho(\boldsymbol{w}_i)$. Since $\Diamond_{a_i} \Diamond_* \psi \in sub^*_x(\varphi)$, by **($a$-r-coh)** for $a = a_i$, we have $\Diamond_{a_i} \Diamond_* \psi \in \rho(\boldsymbol{w}_{i-1})$, whence, by **($\mathbf{kf}_*$-type)**, $\Diamond_* \psi \in \rho(\boldsymbol{w}_{i-1})$. Thus, $\Diamond_* \psi \in \rho(\boldsymbol{w}_0) = \rho(\boldsymbol{w})$, as required. ⊣

Lemma 32. *A weak $\mathbf{Kf}_{*n}$ run ρ satisfies* **(a-$\boldsymbol{w}$-r-sat)** *for all $a \in A$ and all $\boldsymbol{w} \in W$ iff ρ satisfies* **(a-$\boldsymbol{w}$-r-sat)** *for all $a \in A_0$ and all $\boldsymbol{w} \in W$.*

Proof. We show the direction from right to left. Let $\Diamond_* \psi \in \rho(\boldsymbol{w})$. By **($\mathbf{kf}_*$-type)**, $\Diamond_a \psi \in \rho(\boldsymbol{w})$ or $\Diamond_a \Diamond_* \psi \in \rho(\boldsymbol{w})$, for some $a \in A_0$. In the former case, by **(a-$\boldsymbol{w}$-r-sat)**, there is $\boldsymbol{v} \in W$ with $\boldsymbol{w} R_a \boldsymbol{v}$ and $\psi \in \rho(\boldsymbol{v})$, as required. In the latter case, we can argue inductively and construct a finite path $\boldsymbol{w} = \boldsymbol{w}_0 R_{a_1} \cdots R_{a_m} \boldsymbol{w}_m = \boldsymbol{v}$ with $a_1, \ldots, a_m \in A_0$ such that $\psi \in \rho(\boldsymbol{v})$. This is possible since the frame has no infinite paths. ⊣

It follows that coherence and saturation conditions for the modality $*$ in $\mathbf{Kf}_{*n}$ quasimodels are fully covered by using relations $\to_a$ and $\Rightarrow_a$ for $a \in A_0$. We can thus define weak $\mathbf{Kf}_{*n}$ quasimodels $\mathfrak{Q} = (\mathfrak{F}, \boldsymbol{q}, \mathfrak{R}, \mathfrak{p})$ accordingly by replacing condition **(wq-sat)** by the following weakening to A_0:

(wq-sat*) for every $\boldsymbol{w} \in W$ and $t \in \boldsymbol{q}(\boldsymbol{w})$, there is a *prototype weak $\mathbf{Kf}_{*n}$ run* $\mathfrak{p}(\boldsymbol{w}, t) \in \mathfrak{R}_{\boldsymbol{w},t}$, which is a-saturated at $\boldsymbol{w}$, for each $a \in A_0$.

Lemma 33. *Let φ be a $\mathsf{Q}^{=}_{\boxed{1}}\mathsf{ML}_c$-sentence. For every $\mathbf{Kf}_{*n}$ quasimodel satisfying φ based on a tree-shaped frame, there exists a weak $\mathbf{Kf}_{*n}$ quasimodel satisfying φ based on a (finite) frame of outdegree bounded by $s(\varphi)2^{s(\varphi)}$.*

Conversely, for every weak $\mathbf{Kf}_{*n}$ quasimodel satisfying φ based on a finite frame, there exists a $\mathbf{Kf}_{*n}$ quasimodel satisfying φ.

Proof. By straightforward adaptation of the proofs of Lemmas 13 and 14. ⊣

By Lemma 33, to show decidability of a fragment of $\mathsf{Q}^{=}_{\boxed{1}}\mathsf{ML}_c$ over $\mathbf{Kf}_{*n}$ frames, it suffices to show that satisfiable sentences φ are satisfied in weak quasimodels based on Kripke frames and quasistates of recursive size in $|\varphi|$. To this end, we aim to show that it suffices to consider weak quasimodels with a recursive bound (depending on φ) on the length of sequences $\boldsymbol{n}_1 R_{a_1} \boldsymbol{n}_2 R_{a_2} \ldots$ of quasistates.

To obtain such a bound, we apply Dickson's Lemma to the quasistates with the product ordering. Call a pair $\boldsymbol{n}, \boldsymbol{n}' \in \mathbb{N}^k_\infty$ with $\boldsymbol{n} \leq \boldsymbol{n}'$ an *increasing pair*. A (finite or infinite) sequence $\boldsymbol{n}_1, \boldsymbol{n}_2, \ldots \in \mathbb{N}^k_\infty$ is *bad* if is contains no increasing pair $\boldsymbol{n}_i, \boldsymbol{n}_j$ with $i < j$. By Dickson's Lemma, no infinite sequence $\boldsymbol{n}_1, \boldsymbol{n}_2, \ldots \in \mathbb{N}^k_\infty$ is bad. It is easy to see that there are arbitrarily long finite bad sequences, however. To bound also the length of bad sequences we introduce *control functions* [50] providing a bound on the size of the $\boldsymbol{n}_i$. Define a *norm* $|\boldsymbol{n}|_f \in \mathbb{N}$ on $\mathbb{N}^k_\infty$ by setting $|\boldsymbol{n}|_f = \sum_{n_i < \aleph_0} x_i$, for $\boldsymbol{n} \in \mathbb{N}^k_\infty$. This norm is *proper*

since $\{\boldsymbol{n} \in \mathbb{N}_\infty^k \mid |\boldsymbol{n}|_f < N\}$ is finite for all $N \in \mathbb{N}$. The following can be proved along the lines of proofs of Dickson's Lemma [51, 50].

Theorem 34. *Let $g\colon \mathbb{N} \to \mathbb{N}$ be a recursive function, and let length(m) be the maximal length of a bad sequence $\boldsymbol{n}_1, \boldsymbol{n}_2, \ldots \in \mathbb{N}_\infty^k$ with $|\boldsymbol{n}_i|_f \leq g^i(m)$ for all i, where g^i denoted the i iterated applications of g. Then length(m) is bounded by a recursive function that can be obtained from g.*

Theorem 34 can be refined in various way by providing bounds on *length*(m) depending on f if f is primitive recursive. We note, however, that *length*(m) grows very fast. In our application of Theorem 34, m is given by the size $|\varphi|$ of the input formula φ.

Lemma 35. *There is a primitive recursive function g such that, for each $\mathsf{GF}^=_{\boxed{1}}\mathsf{ML}_c$-sentence φ, each quasistate $\boldsymbol{m}$ and each* finite *quasistate candidate $\boldsymbol{n} \leq \boldsymbol{m}$ for φ, there exists a* finite *quasistate $\boldsymbol{m}'$ for φ such that $\boldsymbol{n} \leq \boldsymbol{m}' \leq \boldsymbol{m}$ and $|\boldsymbol{m}'| \leq g(|\boldsymbol{n}| + |\varphi|)$.*

The statement above also holds for $\mathsf{C}^2_{\boxed{1}}\mathsf{ML}_c$-sentences without the condition that $\boldsymbol{n}$ and $\boldsymbol{m}'$ are finite but with $|\cdot|$ replaced by $|\cdot|_f$.

Proof. Assume $\boldsymbol{m}$ and $\boldsymbol{n}$ are given and assume first that φ is a $\mathsf{GF}^=_{\boxed{1}}\mathsf{ML}_c$-sentence. It is known that there is a double exponential function f such that every satisfiable sentence in the guarded fragment is satisfiable in a model of size at most $f(|\psi|)$ [40]. For any number $N \geq |\boldsymbol{n}| + |\varphi|$, we define ψ_N as a conjunction of $\mathsf{only}_{\boldsymbol{m}}$ together with

- $\mathsf{atleast}^t_{\boldsymbol{n}}$, for each $\mathbf{K}\boldsymbol{f}_{*n}$-type t for φ with $\boldsymbol{n}(t) > 0$, and
- $\mathsf{atmost}^t_{\boldsymbol{m}}$, for each $\mathbf{K}\boldsymbol{f}_{*n}$-type t for φ with $\boldsymbol{m}(t) \leq N$;

see (10)–(13). It should be clear that every first-order structure satisfying ψ_N realises only the types $t \in \boldsymbol{m}$ with the minimum of $\boldsymbol{n}(t)$ times (by assumption, $\boldsymbol{n}(t) \leq N$) and the maximum of $\boldsymbol{m}(t)$ times provided that it does not exceed N (in particular, there is no upper bound when $\boldsymbol{m}(t) = \aleph_0$). Now let $N_0 \geq |\boldsymbol{n}|+|\varphi|$ be minimal such that, for all t,

$$\text{either} \quad \boldsymbol{m}(t) \leq N_0 \quad \text{or} \quad \boldsymbol{m}(t) > f(|\psi_{N_0}|). \tag{15}$$

Such an N_0 partitions the types t for φ into those with the multiplicity we need to take account of in $\mathsf{atmost}^t_{\boldsymbol{m}}$ and those with the required multiplicity exceeding the size of 'small' finite models for ψ_{N_0}. We can obtain N_0 by iterating: we start with $N_0 = |\boldsymbol{n}| + |\varphi|$ and check whether it satisfies (15). If it does, then we are done. Otherwise, we take the new N_0 to be $f(|\psi_{N_0}|)$ and repeat the check. Since at each step at least one more type will satisfy $\boldsymbol{m}(t) \leq N_0$, the iteration stops after at most $2^{|sub_x(\varphi)|}$ times. It follows that N_0 is bounded by a primitive recursive function in $|\boldsymbol{n}| + |\varphi|$ (but in general not elementary as the height of the exponent depends on the number of types). Finally, as $\boldsymbol{m}$ is a quasistate, the sentence ψ_{N_0} is satisfiable, and the quasistate $\boldsymbol{m}'$ realised by a first-order structure satisfying ψ_{N_0} of size bounded by $f(|\psi_{N_0}|)$ is as required.

Now assume that φ is a $\mathsf{C}^2_{\boxed{1}}\mathsf{ML}_c$ sentence. There is a double exponential function f such that if a C^2-sentence ψ is satisfiable, then it is satisfiable in a first-order structure where the number of domain elements realising a type t for ψ that is realised by finitely many domain elements only is bounded by $f(|\psi|)$ [8]. For any number $N \geq |\boldsymbol{n}|_f + |\varphi|$, define ψ'_N as a conjunction of $\mathsf{only}_{\boldsymbol{m}}$ together with

- $\mathsf{atleast}^t_{\boldsymbol{n}}$, for each $\mathbf{K}\boldsymbol{f}_{*n}$-type t with $0 < \boldsymbol{n}(t) < \aleph_0$,
- $\mathsf{infinite}^t$, for each $\mathbf{K}\boldsymbol{f}_{*n}$-type t for φ with $\boldsymbol{n}(t) = \aleph_0$, which are C^2-sentences expressing (using an additional binary relation, see Example 7) that t is realised infinitely many times,
- $\mathsf{atmost}^t_{\boldsymbol{m}}$, for each $\mathbf{K}\boldsymbol{f}_{*n}$-type t with $\boldsymbol{m}(t) \leq N$.

(Note that $\mathsf{atleast}^t_{\boldsymbol{n}}$ and $\mathsf{atmost}^t_{\boldsymbol{m}}$ can now also be expressed using counting quantifiers instead of constants.) Now let $N_0 \geq |\boldsymbol{n}|_f + |\varphi|$ be minimal such that either $\boldsymbol{m}(t) \leq N_0$ or $\boldsymbol{m}(t) > f(|\psi'_{N_0}|)$, for all t with $\boldsymbol{m}(t) < \aleph_0$. Then ψ'_{N_0} is satisfiable and we can take a first-order structure with the number of domain elements realising a type for ψ'_{N_0} realised by finitely many domain elements is bounded by $f(|\psi'_{N_0}|)$. The quasistate $\boldsymbol{m}'$ realised by this structure is as required, and $|\boldsymbol{m}'|_f$ is bounded by a primitive recursive function in $|\boldsymbol{n}|_f + |\varphi|$. ⊣

Lemma 36. *For expanding domains, there is a primitive recursive function g such that, for each $\mathsf{GF}^=_{\boxed{1}}\mathsf{ML}_c$-sentence φ, there exists a $\mathbf{K}\boldsymbol{f}_{*n}$ quasimodel satisfying φ based on a tree-shaped frame iff there is a weak quasimodel $\mathfrak{Q} = (\mathfrak{F}, \boldsymbol{q}, \mathfrak{R}, \mathfrak{p})$ satisfying φ with a (finite) tree-shaped $\mathbf{K}\boldsymbol{f}_{*n}$ frame $\mathfrak{F}$ such that, for all worlds $\boldsymbol{w} \in W$,*

1. *$|\boldsymbol{q}(\boldsymbol{w})| \leq g^{d(\boldsymbol{w})+1}(|\varphi|)$;*
2. *$\boldsymbol{q}(\boldsymbol{w}) \not\leq \boldsymbol{q}(\boldsymbol{v})$, for all $\boldsymbol{v} \in W$ with $\boldsymbol{w}R_*\boldsymbol{v}$;*
3. *the outdegree of $\boldsymbol{w}$ is bounded by $s(\varphi)2^{s(\varphi)}$.*

The statement above holds for $\mathsf{C}^2_{\boxed{1}}\mathsf{ML}_c$-sentences with $|\cdot|$ replaced by $|\cdot|_f$.

Proof. We consider $\mathsf{C}^2_{\boxed{1}}\mathsf{ML}_c$ first. Let $g\colon \mathbb{N} \to \mathbb{N}$ be defined by taking $g(n) = g_0(2^{5n})$, for $n \in \mathbb{N}$, where g_0 is given by Lemma 35; we assume $g_0(n) \geq n$, for all $n \in \mathbb{N}$. Let φ be a $\mathsf{C}^2_{\boxed{1}}\mathsf{ML}_c$-sentence. The ($\Leftarrow$) direction is by Lemma 14.

($\Rightarrow$) By Lemma 33, we obtain a weak quasimodel $\mathfrak{Q} = (\mathfrak{F}, \boldsymbol{q}, \mathfrak{R}, \mathfrak{p})$ satisfying φ with $\mathfrak{F} = (W, \{R_a\}_{a \in A_0})$ rooted in $\boldsymbol{w}_0$ such that the outdegree of every world $\boldsymbol{w}$ is bounded by $s(\varphi)2^{s(\varphi)}$, thus satisfying Item 3. By Lemma 28, we assume that the runs in $\mathfrak{R}$ are rooted.

Next, we manipulate $\mathfrak{R}$ so that, unless ρ is a prototype weak run at $\boldsymbol{w}$, infinite multiplicity of $\rho(\boldsymbol{w})$ implies infinite multiplicity of $\rho(\boldsymbol{v})$, for all $\boldsymbol{v} \in W_{\downarrow \boldsymbol{w}}$. Formally, we define the following two sets of weak runs for each $\boldsymbol{w} \in W$:

$$\mathfrak{p}(\boldsymbol{w}) = \{\, \mathfrak{p}(\boldsymbol{w}, t) \mid t \in \boldsymbol{q}(\boldsymbol{w}) \,\} \quad \text{and} \quad \mathfrak{fin}(\boldsymbol{w}) = \{\, \rho \in \mathfrak{R}_{\boldsymbol{w}} \mid \boldsymbol{q}(\boldsymbol{w}, \rho(\boldsymbol{w})) < \aleph_0 \,\}.$$

We show that we can assume that

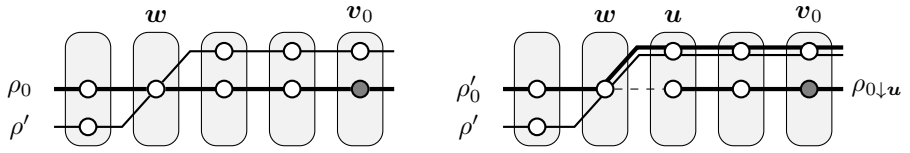

Figure 5: Replacing ρ_0 with ρ'_0 and the $\rho_{0\downarrow\boldsymbol{u}}$ to satisfy $(\dagger_\infty)$: types with infinite multiplicity are white circles, types with finite multiplicity are grey circles.

$(\dagger_\infty)$ for all $\rho \in \mathfrak{R}_{\boldsymbol{w}}$ with $\rho \notin \mathfrak{p}(\boldsymbol{w})$, if $\rho \notin \mathfrak{fin}(\boldsymbol{w})$, then $\rho \notin \mathfrak{fin}(\boldsymbol{v})$, for all $\boldsymbol{v} \in W_{\downarrow\boldsymbol{w}}$.

We prove $(\dagger_\infty)$ by induction on the depth of $\boldsymbol{w} \in W$, updating $\mathfrak{R}$ and $\mathfrak{p}$ at each step. Let $\boldsymbol{w} \in W$ and assume $(\dagger_\infty)$ holds for $\mathfrak{R}$ and $\mathfrak{p}$ at all $\boldsymbol{w}' \in W$ with $\boldsymbol{w}' R_* \boldsymbol{w}$. We modify $\mathfrak{R}$ and $\mathfrak{p}$ so that $(\dagger_\infty)$ also holds for $\boldsymbol{w}$. Suppose $\rho_0 \in \mathfrak{R}_{\boldsymbol{w}}$ with $\rho_0 \notin \mathfrak{p}(\boldsymbol{w})$ violates the claim at $\boldsymbol{w}$: we have $\rho_0 \notin \mathfrak{fin}(\boldsymbol{w})$ but $\rho_0 \in \mathfrak{fin}(\boldsymbol{v}_0)$ for some $\boldsymbol{v}_0 \in W_{\downarrow\boldsymbol{w}}$. It follows, in particular, that the multiset $\mathfrak{R}_{\boldsymbol{w},\rho_0(\boldsymbol{w})}$ is infinite. Observe that, since W is finite, we have $\sum_{\boldsymbol{v}\in W}\{\mathfrak{R}(\rho) \mid \rho \in \mathfrak{fin}(\boldsymbol{v})\} < \aleph_0$. So, the multiset of all $\rho \in \mathfrak{R}$ such that $\rho \in \mathfrak{fin}(\boldsymbol{v})$, for some $\boldsymbol{v} \in \mathsf{dom}\rho$, is finite. It follows that there is a weak run $\rho' \in \mathfrak{R}_{\boldsymbol{w},\rho_0(\boldsymbol{w})}$ such that $\rho' \notin \mathfrak{fin}(\boldsymbol{v})$ for all $\boldsymbol{v} \in \mathsf{dom}\rho'$ (in fact, the total multiplicity of such weak runs is $\aleph_0$). Now we obtain $\mathfrak{R}'$ from $\mathfrak{R}$ as follows. Define a weak run ρ'_0 by setting, for all $\boldsymbol{v} \in \mathsf{dom}\rho_0$,

$$\rho'_0(\boldsymbol{v}) = \begin{cases} \rho'(\boldsymbol{v}), & \text{if } \boldsymbol{v} \in W_{\downarrow\boldsymbol{w}}, \\ \rho_0(\boldsymbol{v}), & \text{otherwise.} \end{cases}$$

We replace ρ_0 with ρ'_0 and the $\rho_{0\downarrow\boldsymbol{u}}$, for R_a-successors $\boldsymbol{u}$ of $\boldsymbol{w}$: formally, $\mathfrak{R}'$ coincides with $\mathfrak{R}$ except that

$$\begin{aligned} \mathfrak{R}'(\rho_0) &= 0, \\ \mathfrak{R}'(\rho'_0) &= \mathfrak{R}(\rho'_0) + \mathfrak{R}(\rho_0), \\ \mathfrak{R}'(\rho_{0\downarrow\boldsymbol{u}}) &= \mathfrak{R}(\rho_{0\downarrow\boldsymbol{u}}) + \mathfrak{R}(\rho_0), \text{ for all } \boldsymbol{u} \text{ with } \boldsymbol{w}R_a\boldsymbol{u},\ a \in A_0; \end{aligned}$$

see Fig. 5. Note that we have to include $\mathfrak{R}(\rho'_0)$ and $\mathfrak{R}(\rho_{0\downarrow\boldsymbol{u}})$ as summands because $\mathfrak{R}$ can already contain ρ'_0 and $\rho_{0\downarrow\boldsymbol{u}}$. Also, the types in $\rho'_{\downarrow\boldsymbol{w}}$ are effectively counted twice, in ρ' as well as in ρ'_0, but that has no effect on **(card)** as they have infinite multiplicity. Finally, we have to update $\mathfrak{p}$ if $\rho_0 \in \mathfrak{p}(\boldsymbol{v})$, for some $\boldsymbol{v}$. Observe that, by our assumption, $\boldsymbol{v} \neq \boldsymbol{w}$. So, we can replace ρ_0 in $\mathfrak{p}(\boldsymbol{v})$ by $\rho_{0\downarrow\boldsymbol{u}}$ if $\boldsymbol{v} \in W_{\downarrow\boldsymbol{u}}$ for some $\boldsymbol{u}$ with $\boldsymbol{w}R_a\boldsymbol{u}$, $a \in A_0$, and by ρ'_0, otherwise (if $\boldsymbol{v} \notin W_{\downarrow\boldsymbol{w}}$).

We repeat this construction until no weak run violating $(\dagger_\infty)$ at $\boldsymbol{w}$ is left. This concludes the induction step and the proof of the claim.

We now introduce four rules that manipulate weak quasimodels. By starting from any $\mathfrak{Q}_0$ satisfying $(\dagger_\infty)$ and applying the rules exhaustively, we construct a sequence $\mathfrak{Q}_0, \dots, \mathfrak{Q}_N$ of weak quasimodels satisfying $(\dagger_\infty)$ and obtain $\mathfrak{Q}_N$ that satisfies the conditions of the lemma. To avoid notational clutter, in the

description of the rules we assume that a given $\mathfrak{Q} = (\mathfrak{F}, \boldsymbol{q}, \mathfrak{R}, \mathfrak{p})$ with a tree-shaped $\mathfrak{F} = (W, \{R_a\}_{a\in A})$ rooted in $\boldsymbol{w}_0$ is transformed into $\mathfrak{Q}'$.

Small Root $\boldsymbol{w}_0$: Define a quasistate candidate $\boldsymbol{n}$ for φ by setting

$$\boldsymbol{n}(t) = \begin{cases} 0, & \text{if } \boldsymbol{q}(\boldsymbol{w}_0, t) < \aleph_0, \\ \aleph_0, & \text{otherwise.} \end{cases}$$

By Lemma 35, there exists a quasistate $\boldsymbol{y}$ for φ with $\boldsymbol{n} \leq \boldsymbol{y} \leq \boldsymbol{q}(\boldsymbol{w}_0)$ and $|\boldsymbol{y}|_f \leq g_0(|\varphi|)$. Consider $\mathfrak{Q}' = (\mathfrak{F}, \boldsymbol{q}', \mathfrak{R}', \mathfrak{p}')$, where $\boldsymbol{q}'$ is defined as $\boldsymbol{q}$ except that $\boldsymbol{q}'(\boldsymbol{w}_0) = \boldsymbol{y}$, function $\mathfrak{p}'$ is obtained from $\mathfrak{p}$ by dropping all $\mathfrak{p}(\boldsymbol{w}_0, t)$ such that $t \notin \boldsymbol{q}'(\boldsymbol{w}_0)$, and $\mathfrak{R}'(\rho)$ is defined by a case distinction as follows.

i. For all $\rho \in \mathfrak{R}_{\boldsymbol{w}_0}$ with $\rho \notin \mathfrak{fin}(\boldsymbol{w}_0)$, set $\mathfrak{R}'(\rho) = \mathfrak{R}(\rho)$;

ii. For all $\rho \in \mathfrak{R}_{\boldsymbol{w}_0}$ with $\rho \in \mathfrak{fin}(\boldsymbol{w}_0)$, using $\boldsymbol{n} \leq \boldsymbol{q}'(\boldsymbol{w}_0) \leq \boldsymbol{q}(\boldsymbol{w}_0)$, one can find $\mathfrak{R}'(\rho) \leq \mathfrak{R}(\rho)$ such that

   (a) $\mathfrak{R}'(\rho) > 0$ if $\rho \in \mathfrak{p}(\boldsymbol{w}_0)$ and $\rho(\boldsymbol{w}_0) \in \boldsymbol{q}'(\boldsymbol{w}_0)$;

   (b) $\sum\{\mathfrak{R}'(\rho) \mid \rho \in \mathfrak{R}_{\boldsymbol{w}_0,t}\} = \boldsymbol{q}'(\boldsymbol{w}_0, t)$, for all t with $\boldsymbol{q}(\boldsymbol{w}_0, t) < \aleph_0$.

iii. If $\rho \notin \mathfrak{R}_{\boldsymbol{w}_0}$ but $\rho = \rho'_{\downarrow \boldsymbol{v}}$ for some $\rho' \in \mathfrak{R}_{\boldsymbol{w}_0}$ and some $\boldsymbol{v}$ with $\boldsymbol{w}_0 R_* \boldsymbol{v}$, then set $\mathfrak{R}'(\rho) = \mathfrak{R}(\rho) - (\mathfrak{R}'(\rho') - \mathfrak{R}(\rho'))$; note that the bracket is non-negative by i) and ii), and either both $\mathfrak{R}'(\rho')$ and $\mathfrak{R}(\rho')$ are finite or both are $\aleph_0$ (due to i), in which case the bracket is assumed to be 0.

iv. If $\rho \notin \mathfrak{R}_{\boldsymbol{w}_0}$ and ρ is not a restriction of $\rho' \in \mathfrak{R}_{\boldsymbol{w}_0}$ (considered in iii), then set $\mathfrak{R}'(\rho) = \mathfrak{R}(\rho)$.

It can be seen that $\mathfrak{Q}'$ is a weak quasimodel for φ with $|\boldsymbol{q}'(\boldsymbol{w}_0)|_f \leq g(|\varphi|)$, thus satisfying Item 1 for $\boldsymbol{w}_0$ and the same frame (so, Item 3 still holds).

Small Non-Root: We generalise the construction used for the root. Consider $\boldsymbol{w}', \boldsymbol{w} \in W$ with $\boldsymbol{w}' R_a \boldsymbol{w}$ and assume $|\boldsymbol{q}(\boldsymbol{w}')|_f \leq g^{d(\boldsymbol{w}')+1}(|\varphi|)$. We aim to retain the multiplicity of all weak runs through $\boldsymbol{w}'$ when we make $\boldsymbol{q}(\boldsymbol{w})$ small. Hence define a quasistate candidate $\boldsymbol{n}$ for φ by taking

$$\boldsymbol{n}(t) = \begin{cases} \sum\{\mathfrak{R}(\rho) \mid \rho \in \mathfrak{R}_{\boldsymbol{w}'} \cap \mathfrak{R}_{\boldsymbol{w},t}\}, & \text{if } \boldsymbol{q}(\boldsymbol{w}, t) < \aleph_0, \\ \aleph_0, & \text{otherwise;} \end{cases}$$

informally, $\boldsymbol{n}(t)$ keeps the number of runs from the predecessor $\boldsymbol{w}'$ of $\boldsymbol{w}$ if the type t has finite multiplicity in $\boldsymbol{q}(\boldsymbol{w})$, but requires infinite multiplicity otherwise. Note that, by $(\dagger_\infty)$, every $\rho \in \mathfrak{R}_{\boldsymbol{w}'}$ with $\rho \in \mathfrak{fin}(\boldsymbol{w}')$ but $\rho \notin \mathfrak{fin}(\boldsymbol{w})$ belongs to $\mathfrak{p}(\boldsymbol{w}')$. Observe that $|\mathfrak{p}(\boldsymbol{w}')| \leq 2^{3|\varphi|}$. Hence, by definition,

$$|\boldsymbol{n}|_f \;\leq\; |\boldsymbol{q}(\boldsymbol{w}')|_f + |\mathfrak{p}(\boldsymbol{w}')| \;\leq\; g^{d(\boldsymbol{w}')+1}(|\varphi|) + 2^{3|\varphi|} \;=\; g^{d(\boldsymbol{w})}(|\varphi|) + 2^{3|\varphi|}.$$

By Lemma 35, there exists a quasistate $\boldsymbol{y}$ for φ with $\boldsymbol{n} \leq \boldsymbol{y} \leq \boldsymbol{q}(\boldsymbol{w})$ such that $|\boldsymbol{y}|_f \leq g_0(|\boldsymbol{n}|_f + |\varphi|)$. Consider $\mathfrak{Q}' = (\mathfrak{F}, \boldsymbol{q}', \mathfrak{R}', \mathfrak{p}')$, where $\boldsymbol{q}'$ is defined as $\boldsymbol{q}$ except that $\boldsymbol{q}'(\boldsymbol{w}) = \boldsymbol{y}$, function $\mathfrak{p}'$ is obtained from $\mathfrak{p}$ by dropping all $\mathfrak{p}(\boldsymbol{w}, t)$ with $t \notin \boldsymbol{q}'(\boldsymbol{w})$, and $\mathfrak{R}'(\rho)$ is defined by a case distinction similar to the previous operation.

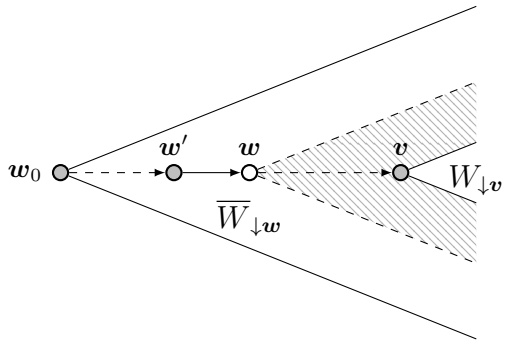

Figure 6: Drop interval operation on quasimodels in the proof of Lemma 36.

i. For all $\rho \in \mathfrak{R}_{\boldsymbol{w}}$ with $\rho \notin \mathfrak{fin}(\boldsymbol{w})$, set $\mathfrak{R}'(\rho) = \mathfrak{R}(\rho)$.

ii. For all $\rho \in \mathfrak{R}_{\boldsymbol{w}}$ with $\rho \in \mathfrak{fin}(\boldsymbol{w})$, using the definition of $\boldsymbol{n}$ and the condition that $\boldsymbol{n} \le \boldsymbol{q}'(\boldsymbol{w}) \le \boldsymbol{q}(\boldsymbol{w})$, one can find $\mathfrak{R}'(\rho) \le \mathfrak{R}(\rho)$ such that

   (a) $\mathfrak{R}'(\rho) > 0$ if $\rho \in \mathfrak{p}(\boldsymbol{w})$ and $\rho(\boldsymbol{w}) \in \boldsymbol{q}(\boldsymbol{w})$;

   (b) $\sum\{\mathfrak{R}'(\rho) \mid \rho \in \mathfrak{R}_{\boldsymbol{w},t}\} = \boldsymbol{q}'(\boldsymbol{w},t)$, for all t with $\boldsymbol{q}(\boldsymbol{w},t) < \aleph_0$;

   (c) $\mathfrak{R}'(\rho) = \mathfrak{R}(\rho)$, for all $\rho \in \mathfrak{R}_{\boldsymbol{w}'}$.

iii. If $\rho \notin \mathfrak{R}_{\boldsymbol{w}}$ but $\rho = \rho'_{\downarrow \boldsymbol{v}}$ for some $\rho' \in \mathfrak{R}_{\boldsymbol{w}}$ and some $\boldsymbol{v}$ with $\boldsymbol{w} R_* \boldsymbol{v}$, then set $\mathfrak{R}'(\rho) = \mathfrak{R}(\rho) - (\mathfrak{R}'(\rho') - \mathfrak{R}(\rho'))$; note that the bracket is again non-negative by i) and ii) and is assumed 0 if both multiplicities of ρ' are $\aleph_0$.

iv. If $\rho \notin \mathfrak{R}_{\boldsymbol{w}}$ and ρ is not a restriction of some $\rho' \in \mathfrak{R}_{\boldsymbol{w}}$ (considered in iii), then set $\mathfrak{R}'(\rho) = \mathfrak{R}(\rho)$.

It can be seen that $\mathfrak{Q}'$ is a weak quasimodel for φ satisfying $|\boldsymbol{q}'(\boldsymbol{w})|_f \le g_0(g^{d(\boldsymbol{w})}(|\varphi|) + 2^{3|\varphi|} + |\varphi|) \le g_0(2^{5g^{d(\boldsymbol{w})}(|\varphi|)}) = g^{d(\boldsymbol{w})+1}(|\varphi|)$. Thus, Item 1 holds for $\boldsymbol{w}$. Since the frame is the same, Item 3 is also satisfied.

Drop Interval: Consider $\boldsymbol{w}', \boldsymbol{w}, \boldsymbol{v} \in W$ with $\boldsymbol{w}' R_a \boldsymbol{w} R_* \boldsymbol{v}$ and $\boldsymbol{q}(\boldsymbol{w}) \le \boldsymbol{q}(\boldsymbol{v})$. Construct $\mathfrak{Q}' = (\mathfrak{F}', \boldsymbol{q}', \mathfrak{R}', \mathfrak{p}')$ as follows. Frame $\mathfrak{F}'$ is obtained from $\mathfrak{F}$ by replacing $W_{\downarrow \boldsymbol{w}}$ by $W_{\downarrow \boldsymbol{v}}$: formally, $\mathfrak{F}'$ is the restriction of $\mathfrak{F}$ to $W' = \overline{W}_{\downarrow \boldsymbol{w}} \cup W_{\downarrow \boldsymbol{v}}$, where $\overline{W}_{\downarrow \boldsymbol{w}} = W \setminus W_{\downarrow \boldsymbol{w}}$, and additionally, $\boldsymbol{w}' R_a \boldsymbol{v}$; see Fig. 6. This clearly preserves the outdegree, and so Item 3 is still satisfied. We set $\boldsymbol{q}'(\boldsymbol{u}) = \boldsymbol{q}(\boldsymbol{u})$ for all $\boldsymbol{u} \in W'$, and so the bound on the norm of quasistates in Item 1 is preserved. It remains to define $\mathfrak{R}'$ and $\mathfrak{p}'$.

The new weak runs in $\mathfrak{R}'$ are obtained by taking, for any $\rho_0 \in \mathfrak{R}_{\boldsymbol{w}'} \cap \mathfrak{R}_{\boldsymbol{w},t}$ and $\rho_1 \in \mathfrak{R}_{\boldsymbol{v},t}$, with $t \in \boldsymbol{q}(\boldsymbol{w})$, the weak run $\rho_0 \oplus_{\boldsymbol{w},\boldsymbol{v}} \rho_1$ defined by setting

$$\rho_0 \oplus_{\boldsymbol{w},\boldsymbol{v}} \rho_1 \colon \boldsymbol{u} \mapsto \begin{cases} \rho_0(\boldsymbol{u}), & \text{if } \boldsymbol{u} \in \overline{W}_{\downarrow \boldsymbol{w}}, \\ \rho_1(\boldsymbol{u}), & \text{if } \boldsymbol{u} \in W_{\downarrow \boldsymbol{v}}. \end{cases}$$

First, we define $\mathfrak{R}'$ on $\rho \in \mathfrak{R}$ with $\rho \notin \mathfrak{R}_{\boldsymbol{w}}$ by setting $\mathfrak{R}'(\rho) = \mathfrak{R}(\rho)$. Then, as follows from $\boldsymbol{q}(\boldsymbol{w}) \leq \boldsymbol{q}(\boldsymbol{v})$, we can define $\mathfrak{R}'$ on weak runs of the form $\rho_0 \oplus_{\boldsymbol{w},\boldsymbol{v}} \rho_1$ and on runs with domain $W_{\downarrow \boldsymbol{v}}$ in such a way that, for all $\rho_0 \in \mathfrak{R}_{\boldsymbol{w}'} \cap \mathfrak{R}_{\boldsymbol{w},t}$ and all $\rho_1 \in \mathfrak{R}_{\boldsymbol{v},t}$ with $t \in \boldsymbol{q}(\boldsymbol{w})$, we have

$$\sum \big\{\mathfrak{R}(\rho) \mid \rho \in \mathfrak{R} \text{ with } \rho \stackrel{\overline{W}_{\downarrow \boldsymbol{w}} \cup \{\boldsymbol{w}\}}{=} \rho_0\big\} = \sum \big\{\mathfrak{R}'(\rho_0 \oplus_{\boldsymbol{w},\boldsymbol{v}} \rho_1') \mid \rho_1' \in \mathfrak{R}_{\boldsymbol{v},t}\big\},$$

$$\sum \big\{\mathfrak{R}(\rho) \mid \rho \in \mathfrak{R} \text{ with } \rho \stackrel{W_{\downarrow \boldsymbol{v}}}{=} \rho_1\big\} = \mathfrak{R}'(\rho_{1 \downarrow \boldsymbol{v}}) + \sum \big\{\mathfrak{R}'(\rho_0' \oplus_{\boldsymbol{w},\boldsymbol{v}} \rho_1) \mid \rho_0' \in \mathfrak{R}_{\boldsymbol{w}'} \cap \mathfrak{R}_{\boldsymbol{w},t}\big\},$$

where $\rho \stackrel{U}{=} \rho'$ means that the two weak runs coincide on U. Define $\mathfrak{p}'$ from $\mathfrak{p}$ and $\mathfrak{R}'$ in the obvious way so that $\mathfrak{Q}'$ is a weak quasimodel for φ.

After applying this rule exhaustively, Item 2 holds for all $\boldsymbol{w} R_* \boldsymbol{v}$ with $\boldsymbol{w} \neq \boldsymbol{w}_0$ and Item 3 still holds. (Observe, however, that we might have to apply the construction of small non-roots again.)

Drop Initial Interval: Consider the root $\boldsymbol{w}_0$ and some $\boldsymbol{v} \in W \setminus \{\boldsymbol{w}_0\}$ such that $\boldsymbol{q}(\boldsymbol{w}_0) \leq \boldsymbol{q}(\boldsymbol{v})$. Then construct an updated $\mathfrak{Q}' = (\mathfrak{F}', \boldsymbol{q}', \mathfrak{R}', \mathfrak{p}')$ by restricting $\mathfrak{Q}$ to $W_{\downarrow \boldsymbol{v}}$ in the obvious way.

By applying the four rules above exhaustively, we obtain a weak quasimodel satisfying Items 1–3 of the lemma.

The proof for $\mathsf{GF}^{=}_{\boxed{1}}\mathsf{ML}_c$ is similar except that we work with finite quasistates only. In particular, we do not require condition $(\dagger_\infty)$ and the definition of $\boldsymbol{n}$ in the *Small Root* and *Small Non-Root* operations becomes simpler as the second option (with $\aleph_0$) is not applicable; also, item i. in both operations becomes irrelevant. ⊣

Theorem 37. *For expanding-domain models,* $\mathbf{K}\boldsymbol{f}_{*n}$*-validity in both* $\mathsf{C}^2_{\boxed{1}}\mathsf{ML}_c$ *and* $\mathsf{GF}^{=}_{\boxed{1}}\mathsf{ML}_c$ *are decidable.*

Proof. We give the proof for $\mathsf{C}^2_{\boxed{1}}\mathsf{ML}_c$, the proof for $\mathsf{GF}^{=}_{\boxed{1}}\mathsf{ML}_c$ is similar. Let g be the recursive function from Lemma 36 and assume a $\mathsf{C}^2_{\boxed{1}}\mathsf{ML}_c$-sentence φ is given. Then φ is satisfiable in a $\mathbf{K}\boldsymbol{f}_{*n}$ frame iff there is weak quasimodel satisfying φ and Items 1–3 of Lemma 36. The existence of such a weak quasimodel $\mathfrak{Q} = (\mathfrak{F}, \boldsymbol{q}, \mathfrak{R}, \mathfrak{p})$ is decidable: by Theorem 34 and Items 1 and 2 of Lemma 36, the length n of paths $\boldsymbol{w}_0 R_{a_0} \cdots R_{a_n} \boldsymbol{w}_n$ in $\mathfrak{F}$ is recursively bounded in the size $|\varphi|$ of φ. Hence, by Item 3 of Lemma 36, the number of worlds of $\mathfrak{F}$ is recursively bounded in $|\varphi|$. Then also $|\boldsymbol{q}(\boldsymbol{w})|_f$ and the number of distinct $\rho \in \mathfrak{R}$ are recursively bounded. ⊣

It is worth pointing out that the size of weak quasimodels constructed in Lemma 36 is bounded only by a primitive recursive function in $|\varphi|$, but as we shall see below (Theorem 42 and Lemma 43), one cannot do better: the problem is Ackermann-hard even for the one-variable fragment.

8.2. One-Variable Fragment

We consider the one-variable fragment $\mathsf{Q}^{1=}\mathsf{ML}_c$ and show that, for expanding domains, $\mathbf{K}_n$-validity in this fragment is PSPACE-complete; recall that the problem for constant domains is coNEXPTIME-complete (see Theorem 19). We begin by proving a variant of Lemma 35 for $\mathsf{Q}^{1=}\mathsf{ML}_c$.

Lemma 38. *For each $\mathsf{Q}^{1=}\mathsf{ML}_c$-sentence φ, each quasistate $\boldsymbol{m}$ and each finite quasistate candidate $\boldsymbol{n} \leq \boldsymbol{m}$ for φ, there exists a finite quasistate $\boldsymbol{m}'$ for φ such that $\boldsymbol{n} \leq \boldsymbol{m}' \leq \boldsymbol{m}$ and $|\boldsymbol{m}'| \leq |\boldsymbol{n}| + |\varphi|$.*

Proof. Let B the set of types for φ that contains t_c with $x = c \in t_c$, for each constant c in φ, and t_ψ with $\psi \in t_\psi$, for $\exists y\, \psi \in \mathit{sub}_x(\varphi)$ such that every (some) type $t \in \boldsymbol{m}$ contains $\exists y\, \psi$. It is clear that $|B| \leq |\varphi|$. We define a quasistate candidate $\boldsymbol{m}'$ for φ by taking

$$\boldsymbol{m}'(t) = \begin{cases} \min(\boldsymbol{m}(t), \boldsymbol{n}(t) + 1), & \text{if } t \in B, \\ \boldsymbol{n}(t), & \text{otherwise.} \end{cases}$$

It follows that $\boldsymbol{n} \leq \boldsymbol{m}' \leq \boldsymbol{m}$ and $|\boldsymbol{m}'| \leq |\boldsymbol{n}| + |B| \leq |\boldsymbol{n}| + |\varphi|$, and so it remains to show that $\boldsymbol{m}'$ is realisable.

To this end, let $\mathfrak{B}$ be the first-order interpretation with domain $\boldsymbol{m}'$ and such that $c^{\mathfrak{B}} = t_c$, where $t_c \in B$ is the (unique) type with $(x = c) \in t_c$ and $P^{\mathfrak{B}} = \{t \mid P(x) \in \overline{t}\}$, for predicate names P. For each $t \in \boldsymbol{m}'$, it then follows from a routine induction on ψ that $\mathfrak{B} \models \overline{\psi}[t]$ iff $\psi \in t$, for all $\psi \in \mathit{sub}_x(\varphi)$. In particular, for each $t \in \boldsymbol{m}'$, we have $\mathfrak{B} \models \overline{t}[t]$, whence $\boldsymbol{m}'(t) = |\{t \in \mathfrak{B} \mid \mathfrak{B} \models \overline{t}[t]\}|$, which is to say that $\boldsymbol{m}'$ is realisable. ⊣

We next show a variant of Lemma 36 for $\mathsf{Q}^{1=}\mathsf{ML}_c$.

Lemma 39. *For expanding domains, for each $\mathsf{Q}^{1=}\mathsf{ML}_c$-sentence φ, there exists a quasimodel satisfying φ based on a $\mathbf{Tree}^d_n$ frame iff there is a weak quasimodel $\mathfrak{Q} = (\mathfrak{F}, \boldsymbol{q}, \mathfrak{R}, \mathfrak{p})$ satisfying φ based on a (finite) $\mathbf{Tree}^d_n$ frame $\mathfrak{F}$ such that, for all worlds $\boldsymbol{w} \in W$,*

$$|\boldsymbol{q}(\boldsymbol{w})| \leq 1 + (d(\boldsymbol{w}) + 1) \cdot |\varphi|.$$

Proof. The ($\Leftarrow$) direction is by Lemma 14. For the ($\Rightarrow$) direction, by Lemma 13, we obtain a weak quasimodel $\mathfrak{Q}_0 = (\mathfrak{F}, \boldsymbol{q}_0, \mathfrak{R}_0, \mathfrak{p}_0)$ satisfying φ based on a finite $\mathbf{Tree}^d_n$ frame $\mathfrak{F}$ with root $\boldsymbol{w}_0$. We now follow a similar approach to that taken in the proof of Lemma 36, by exhaustively applying transformation rules to $\mathfrak{Q}_0$ to reduce the size of each quasistate: we construct a sequence $\mathfrak{Q}_0, \ldots, \mathfrak{Q}_N$ of weak quasimodels sastifying φ such that $\mathfrak{Q}_N$ fully meets the size restrictions in the claim. The difference here is that we only require rules for *Small Root* and *Small Non-Root* because the frame remains the same and is in fact of depth bounded by d. In both rules, we utilise Lemma 38 in place of Lemma 35, which allows us to bound the quasistate size by a polynomial in $|\varphi|$.

Small Root $\boldsymbol{w}_0$: We start from the quasistate candidate $\boldsymbol{n}$ defined by taking $\boldsymbol{n}(t_0) = 1$ and $\boldsymbol{n}(t) = 0$ for all $t \neq t_0$, where $t_0 \in \boldsymbol{q}(\boldsymbol{w}_0)$ contains φ, to obtain a

new quasistate at $\boldsymbol{w}_0$ so that $|\boldsymbol{q}'(\boldsymbol{w_0})| \leq 1+|\varphi|$. We can then construct the runs of $\mathfrak{R}'$ as in the proof of Lemma 36, by taking one copy $\mathfrak{p}(\boldsymbol{w}_0, t)$ for all $t \in \boldsymbol{q}'(\boldsymbol{w}_0)$ and restricting all other runs of $\mathfrak{R}$ to the domain $W \setminus \{\boldsymbol{w}_0\}$.

Small Non-Root: Given $\boldsymbol{w}$ and $\boldsymbol{w'}$ with $\boldsymbol{w}'R_a\boldsymbol{w}$, we start from the quasistate candidate $\boldsymbol{n}$ defined as follows:

$$\boldsymbol{n}(t) = \begin{cases} \sum\{\mathfrak{R}(\rho) \mid \rho \in \mathfrak{R}_{\boldsymbol{w}'} \cap \mathfrak{R}_{\boldsymbol{w},t}\}, & \text{if } t \in \boldsymbol{q}(\boldsymbol{w}'), \\ 0, & \text{otherwise,} \end{cases}$$

to obtain a new quasistate at $\boldsymbol{w}$ so that $|\boldsymbol{q}'(\boldsymbol{w})| \leq |\boldsymbol{q}(\boldsymbol{w}')| + |\varphi|$. We can again construct the runs of $\mathfrak{R}'$ as in the proof of Lemma 36, by taking one copy $\mathfrak{p}(\boldsymbol{w}, t)$ for all $t \in \boldsymbol{q}'(\boldsymbol{w})$ and restricting all other runs of $\mathfrak{R}$ to the domain $W \setminus \{\boldsymbol{w}\}$. We thereby obtain a new weak quasimodel $\mathfrak{Q}'$ satisfying φ and such that $|\boldsymbol{q}'(\boldsymbol{w})| \leq |\boldsymbol{q}(\boldsymbol{w}')| + |\varphi| \leq 1 + (d(\boldsymbol{w}) + 1) \cdot |\varphi|$, provided that $\boldsymbol{q}(\boldsymbol{w}')$ has already been reduced.

By repeated application of the two rules from the root to the leaves, we construct the required weak quasimodel. ⊣

In fact, the construction in Lemma 14 (which preserves the bound on the size of quasistates at a given depth) implies the following stronger result:

Lemma 40. *For expanding domains, for each $\mathsf{Q}^{1=}\mathsf{ML}_c$-sentence φ, if there is a quasimodel satisfying φ based on a $\mathbf{Tree}^d_n$ frame, then there is a quasimodel $\mathfrak{Q} = (\mathfrak{F}, \boldsymbol{q}, \mathfrak{R})$ satisfying φ based on a (finite) $\mathbf{Tree}^d_n$ frame $\mathfrak{F}$ such that, for all $\boldsymbol{w} \in W$,*

$$|\boldsymbol{q}(\boldsymbol{w})| \leq 1 + (d(\boldsymbol{w}) + 1) \cdot |\varphi|.$$

We are now in a position to show the PSPACE complexity bound. We generalise the standard PSPACE upper bound proof for propositional $\mathbf{K}_n$ (see, e.g., [52, 53]) to $\mathsf{Q}^{1=}\mathsf{ML}_c$ using the lemma above.

In what follows it will be convenient to associate each tuple $\boldsymbol{t} = (t_0, \ldots, t_\ell)$ of types with a multiset $\boldsymbol{n_t}(t) = |\{0 \leq i \leq \ell : t_i = t\}|$, by discarding the inherent ordering.

Theorem 41. *For expanding-domain models, $\mathbf{K}_n$-validity in $\mathsf{Q}^{1=}\mathsf{ML}_c$ is* PSPACE*-complete.*

Proof. The PSPACE lower bound follows from PSPACE-hardness of the underlying (propositional) modal logic $\mathbf{K}_n$. For the upper bound, we define a recursive function $\mathrm{QKSAT}_{\varphi,N}(k, \boldsymbol{t})$ that takes as input an integer $k \in \mathbb{N}$ and a tuple $\boldsymbol{t} = (t_0, \ldots, t_\ell)$ of types for φ of size $\ell < N$ and returns **True** iff the following conditions are met:

(tab1) the multiset $\boldsymbol{n_t}$ associated with $\boldsymbol{t}$ is a quasistate for φ;

(tab2) for each type t_i in $\boldsymbol{t}$ and $\Diamond_a\psi \in t_i$, there exists a tuple $\boldsymbol{t}' = (t'_0, \ldots, t'_m)$ of types for φ, with $\ell \leq m < N$, such that

(i) $\psi \in t'_i$,

(ii) $t_j \to_a t'_j$ for all $0 \leq j \leq \ell$,

(iii) $k > 0$ and $\mathrm{QKSAT}_{\varphi,N}(k-1, \boldsymbol{t}')$ returns **True**.

Note that $\mathrm{QKSAT}_{\varphi,N}(k, \boldsymbol{t})$ is inherently non-deterministic as it must explore all possible choices for $\boldsymbol{t}'$ in condition **(tab2)**. However, since the size of $\boldsymbol{t}'$ is bounded by N, by appealing to Savitch's theorem we require only that $\mathrm{QKSAT}_{\varphi,N}$ uses at most a polynomial amount of space. We note that **(tab1)** can be checked in polynomial space since the one-variable fragment FO^1 is NP-complete. Checking **(tab2)**(i)–(ii) can be done 'in-place,' while the depth of the recursion required for **(tab2)**(iii) is bounded by k, and so it too can be performed in polynomial space. The fixed parameter N places a bound on the maximum length of the tuple of types that can ever appear as arguments to $\mathrm{QKSAT}_{\varphi,N}$, which, by Lemma 40, will be chosen to be polynomial in $|\varphi|$.

For soundness and correctness, we claim that, for all $k \in \mathbb{N}$ and tuples $\boldsymbol{t} = (t_0, \ldots, t_\ell)$, the call $\mathrm{QKSAT}_{\varphi,N}(k, \boldsymbol{t})$ returns **True** if and only if there is quasimodel $\mathfrak{Q} = (\mathfrak{F}, \boldsymbol{q}, \mathfrak{R})$ for φ (not necessarily satisfying φ) such that

(QK1) $\mathfrak{F} = (W, \{R_a\}_{a \in A})$ is a $\mathbf{Tree}_n^k$ frame rooted in $\boldsymbol{w}_0$,

(QK2) $\boldsymbol{q}(\boldsymbol{w}_0) = \boldsymbol{n}_{\boldsymbol{t}}$,

(QK3) $|\boldsymbol{q}(\boldsymbol{w})| \leq N$ for all $\boldsymbol{w} \in W$.

We prove this by induction on k, so let $k \geq 0$ be fixed and suppose that the claim holds for all $m < k$.

($\Rightarrow$) Suppose that $\mathrm{QKSAT}_{\varphi,N}(k, \boldsymbol{t})$ returns **True**, for $\boldsymbol{t} = (t_0, \ldots, t_\ell)$. Let S be the set of all pairs $(i, \Diamond_a \psi)$ with $\Diamond_a \psi \in t_i$. For each $\sigma = (i, \Diamond_a \psi) \in S$, by **(tab2)**, there is some $\boldsymbol{t}^\sigma = (t_0^\sigma, \ldots, t_m^\sigma)$, for $\ell \leq m < N$, such that (i) $\psi \in t_i^\sigma$, (ii) $t_j \to_a t_j^\sigma$, for all $j \leq \ell$, and (iii) $k > 0$ and $\mathrm{QKSAT}_{\varphi,N}(k-1, \boldsymbol{t}^\sigma)$ returns **True**. By the induction hypothesis, there is some quasimodel $\mathfrak{Q}^\sigma = (\mathfrak{F}^\sigma, \boldsymbol{q}^\sigma, \mathfrak{R}^\sigma)$ for φ based on a $\mathbf{Tree}_n^{k-1}$ frame $\mathfrak{F}^\sigma$ rooted in w_0^σ with $\boldsymbol{q}^\sigma(w_0^\sigma) = \boldsymbol{n}_{\boldsymbol{t}^\sigma}$ and $|\boldsymbol{q}^\sigma(\boldsymbol{w})| \leq N$ for all $\boldsymbol{w} \in W^\sigma$. From this (possibly empty) collection of quasimodels, we define a new quasimodel $\mathfrak{Q} = (\mathfrak{F}, \boldsymbol{q}, \mathfrak{R})$ by taking $\mathfrak{F}$ to be the disjoint union of the frames $\mathfrak{F}^\sigma$, for $\sigma \in S$, conjoined by introducing fresh root node w_0:

$$W = \{w_0\} \cup \{w_0 a \boldsymbol{w} \mid \boldsymbol{w} \in W^\sigma,\ \sigma = (i, \Diamond_a \psi) \in S\}.$$

(Note that when $k = 0$, it follows from **(tab2)**(iii) that S must be empty, and so $\mathfrak{F}$ is a $\mathbf{Tree}_n^0$ frame comprising just the root node w_0.) Over this new frame, we define $\boldsymbol{q}$ such that $\boldsymbol{q}(w_0) = \boldsymbol{n}_{\boldsymbol{t}}$, which is guaranteed to be a quasistate by **(tab1)**, and $\boldsymbol{q}(w_0 a \boldsymbol{w}) = \boldsymbol{q}^\sigma(\boldsymbol{w})$ for all $\boldsymbol{w} \in W^\sigma$.

By **(card)**, for each $0 \leq j \leq \ell$ and $\sigma \in S$, there is some unique $\rho_j^\sigma \in \mathfrak{R}^\sigma$ such that $\rho_j^\sigma(w_0^\sigma) = t_j^\sigma$. Hence we may define a new run ρ_j by taking $\rho_j(w_0) = t_j$ and $\rho_j(w_0 a \boldsymbol{w}) = \rho_j^\sigma(\boldsymbol{w})$ for all $\boldsymbol{w} \in W^\sigma$. We then take $\mathfrak{R}$ to be the set of all such ρ_j for $0 \leq j \leq \ell$, together with all $\rho \in \mathfrak{R}^\sigma$ that are not the restriction any ρ_j to W^σ.

It is clear from the construction that $\mathfrak{Q}$ satisfies (QK1)–(QK3), and so it remains to show that $\mathfrak{Q}$ satisfies conditions **(r-coh)** and **(r-sat)**. For **(r-coh)**, we note that each run is coherent across each of the subframes $\mathfrak{F}^\sigma$, since the restriction of each run to W^σ is an original run of $\mathfrak{Q}^\sigma$, which are coherent. For the root, suppose that $\psi \in \rho_j(w_0 a w_0^\sigma)$ for some $0 \leq j \leq \ell$ and $\sigma \in S$. By construction, $\rho_j(w_0 a w_0^\sigma) = \rho_j^\sigma(w_0^\sigma) = t_j^\sigma$. Hence, by (ii), $\Diamond_a \psi \in t_j = \rho_j(w_0)$, as required. Similarly, for **(r-sat)**, we note that each run is saturated across each $\mathfrak{F}^\sigma$. For the root, if $\Diamond_a \psi \in \rho_i(w_0) = t_i$ then $(i, \Diamond_a \psi) \in S$, and so, by (i), $\psi \in t_i^\sigma$. Hence, by construction, $\psi \in \rho_j^\sigma(w_0^\sigma)$, which is to say that $\psi \in \rho_j(w_0 a w_0^\sigma)$, as required.

($\Leftarrow$) Conversely, suppose that $\mathfrak{Q} = (\mathfrak{F}, \boldsymbol{q}, \mathfrak{R})$ is a quasimodel for φ satisfying (QK1)–(QK3). By (QK2), $\boldsymbol{q}(\boldsymbol{w}_0)$ is a quasistate for φ, as required by **(tab1)**. By **(card)**, $\boldsymbol{q}(\boldsymbol{w}, t) = |\mathfrak{R}_{\boldsymbol{w},t}|$, for every $\boldsymbol{w} \in W$ and type t for φ. Hence, without loss of generality we may enumerate the runs of $\mathfrak{R}$ as $\rho_0, \rho_1, \ldots,$ such that $\rho_j(\boldsymbol{w}_0) = t_j$, for all $0 \leq j \leq \ell$. For **(tab2)**, let $i \leq \ell$ and $\Diamond_a \psi \in t_i$, which is to say that $\Diamond_a \psi \in \rho_i(\boldsymbol{w}_0)$. Hence, by **(r-sat)**, there is some $\boldsymbol{v} \in W$ such that $\boldsymbol{w}_0 R_a \boldsymbol{v}$ and $\psi \in \rho_i(\boldsymbol{v})$, which also implies that $k > 0$. So let $\boldsymbol{t}' = (\rho_j(\boldsymbol{v}) : \rho_j \in \mathfrak{R}_{\boldsymbol{v}})$ be a tuple of types for φ of size $|\mathfrak{R}_{\boldsymbol{v}}|$, so that **(tab2)**(i) holds trivially. For **(tab2)**(ii), suppose that $\psi \in \rho_j(\boldsymbol{v})$ for $t_j \in \boldsymbol{t}$, which is to say $\rho_j \in \mathfrak{R}_{\boldsymbol{w}_0}$ and $\rho_j(\boldsymbol{w}_0) = t_j$. Hence, by **(r-coh)**, $\Diamond_a \psi \in t_j$, as required. For **(tab2)**(iii), consider the restriction $\mathfrak{Q}^v$ of $\mathfrak{Q}$ to $W_{\downarrow v}$, which clearly satisfies (QK1)–(QK3): by the induction hypothesis, $\mathrm{QKSAT}_{\varphi,N}(k-1, \boldsymbol{t}')$ returns **True**, as required.

It then follows from Lemmas 11 and 40 that φ is $\mathbf{K}_n$-satisfiable iff there is a tuple $\boldsymbol{t}$ of types with $\varphi \in t_0$, such that $\mathrm{QKSAT}_{\varphi,N}(d(\varphi), \boldsymbol{t})$ returns **True**, where $N = 1 + (d(\varphi)+1) \cdot |\varphi| = O(|\varphi|^2)$, which can be decided in PSPACE. $\dashv$

9. First-Order Temporal Logic

We consider standard temporal logics with the temporal operators for 'eventually' and 'next' defined over the natural numbers and finite strict linear orders. The decidability and complexity of monodic fragments of temporal logics without NRDC features has been investigated extensively [10, 54, 11, 55, 56]. Here we discuss what happens with NRDC feature. We exploit known negative results for the one-variable fragment by giving a polytime reduction to modal logics over the frame classes $\mathbf{K}_{*n}$ and $\mathbf{K}\boldsymbol{f}_{*n}$. This allows us to prove matching lower bounds for $\mathbf{K}_{*n}$ and $\mathbf{K}\boldsymbol{f}_{*n}$. Conversely, our decidability results for $\mathbf{K}\boldsymbol{f}_{*n}$ with expanding domains translate to decidability results for temporal logics over finite strict linear orders with expanding domains.

We introduce the relevant frame classes and notation for languages. By $\mathbf{LTL}^\Diamond$ we denote the frame class containing single frame $(\mathbb{N}, <)$ with $\mathbb{N}$ the natural numbers and $<$ its standard strict ordering. The class of finite strict orders can then be defined as $\mathbf{LTL}\boldsymbol{f}^\Diamond = \{(\{0, \ldots, n\}, <_{|\{0,\ldots,n\}}) \mid n \geq 0\}$. In these frames we interpret the modal language with a single temporal operator, $\Diamond$ ('eventually'). For convenience, we denote the languages $\mathsf{Q}^{1=}\mathsf{ML}_\iota$, $\mathsf{C}^2_{[1]}\mathsf{ML}_\iota$,

and $\mathsf{GF}^{=}_{\boxed{1}}\mathsf{ML}_\iota$ with a single modality interpreted over these frames by $\mathsf{Q}^{1=}\mathsf{LTL}^{\Diamond}_\iota$, $\mathsf{C}^2_{\boxed{1}}\mathsf{LTL}^{\Diamond}_\iota$, and $\mathsf{GF}^{=}_{\boxed{1}}\mathsf{LTL}^{\Diamond}_\iota$, respectively.

We also consider the classes of frames extended with the successor relation $S = \{\,(i, i+1) \mid i \in \mathbb{N}\,\}$ interpreting operator $\bigcirc$ ('next'): we denote $\mathbf{LTL} = \{(\mathbb{N}, <, S)\}$ and $\mathbf{LTL}\boldsymbol{f} = \{(\{0, \dots, n\}, <_{|\{0,\dots,n\}}, S_{|\{0,\dots,n\}}) \mid n \geq 0\}$. The bimodal languages with $\Diamond$ and $\bigcirc$ interpreted over these frame classes are denoted $\mathsf{Q}^{1=}\mathsf{LTL}_\iota$, $\mathsf{C}^2_{\boxed{1}}\mathsf{LTL}_\iota$, and $\mathsf{GF}^{=}_{\boxed{1}}\mathsf{LTL}_\iota$, respectively.

The proof of Lemma 29 shows that, for all temporal frame classes $\mathcal{C}$ introduced above and temporal fragments $\mathcal{L}$, global $\mathcal{C}$-consequence in $\mathcal{L}$ is polytime-reducible to $\mathcal{C}$-validity in $\mathcal{L}$. In what follows we therefore consider validity only.

The computational behaviour of $\mathsf{Q}^{1=}\mathsf{LTL}^{\Diamond}_\iota$ and $\mathsf{Q}^{1=}\mathsf{LTL}_\iota$ is well understood. We use the facts that, by Theorem 6, we can always eliminate definite descriptions and partial designators and, by Theorem 3, we can replace constants by the "elsewhere" quantifier. Then the following result follows from [13, Table 1], which is partly based on ideas first developed in [57, 45].

Theorem 42. *In $\mathsf{Q}^{1=}\mathsf{LTL}_\iota$ and its fragment $\mathsf{Q}^{1=}\mathsf{LTL}^{\Diamond}_\iota$ with the $\Diamond$-operator only:*
(1) *for constant domains, $\mathbf{LTL}$-validity is Σ^1_1-complete, while $\mathbf{LTL}\boldsymbol{f}$-validity is undecidable and co-r.e.;*
(2) *for expanding domains, $\mathbf{LTL}$-validity is undecidable and r.e., while $\mathbf{LTL}\boldsymbol{f}$-validity is decidable but Ackermann-hard.*

We note that monodic fragments beyond the one-variable fragment have not yet been considered in the temporal context. The following result allows us to transfer decidability and complexity results between the temporal and modal domain.

Lemma 43. *For both constant and expanding domains, $\mathbf{LTL}$-validity in $\mathsf{Q}^{1=}\mathsf{LTL}_\iota$, $\mathsf{C}^2_{\boxed{1}}\mathsf{LTL}_\iota$ and $\mathsf{GF}^{=}_{\boxed{1}}\mathsf{LTL}_\iota$ are polytime-reducible to $\mathbf{K}_{*n}$-validity in $\mathsf{Q}^{1=}\mathsf{ML}_\iota$, $\mathsf{C}^2_{\boxed{1}}\mathsf{ML}_\iota$ and $\mathsf{GF}^{=}_{\boxed{1}}\mathsf{ML}_\iota$, respectively. This also holds if $\mathbf{LTL}$ and $\mathbf{K}_{*n}$ are replaced by $\mathbf{LTL}\boldsymbol{f}$ and $\mathbf{K}\boldsymbol{f}_{*n}$, respectively.*

Proof. The proof of this reduction from logics of linear frames with transitive closure to logics of branching frames with transitive closure is not trivial but can be done by adapting in a straightforward way the reduction given in the proof of [4, Theorem 6.24] for product modal logics. ⊣

Using Lemma 43 and Theorem 42, we obtain all lower bounds for the frame classes $\mathbf{K}_{*n}$ and $\mathbf{K}\boldsymbol{f}_{*n}$ stated in Table 1. Conversely, from Theorem 37 and Lemma 43, we obtain the following decidability result for temporal monodic fragments over strict finite orders and expanding domains.

Theorem 44. *For expanding-domain models, $\mathbf{LTL}\boldsymbol{f}$-validity in $\mathsf{C}^2_{\boxed{1}}\mathsf{LTL}_\iota$ and $\mathsf{GF}^{=}_{\boxed{1}}\mathsf{LTL}_\iota$ are decidable.*

10. Discussion

We have shown that monodic fragments of important first-order modal logics are decidable even if non-rigid constants, definite descriptions, or counting (i.e.,

NRDC features) are present in the language. The proof method is illustrated for the guarded fragment with constants and equality, and for the two-variable fragment with counting. We conjecture that our technique can be extended to other decidable fragments of first-order logic, such as variants of the guarded negation fragment [58], fluted fragments [59, 60], or extensions of the two-variable fragment with semantically-constrained relations, e.g., transitive or equivalence relations [8].

Our results can also be applied to many modal and temporal descriptions logics (DLs). These are monodic fragments of modal and temporal first-order logics, where the explicit quantification of variables is replaced by the implicit quantification in the DL constructs, and which have been investigated and applied extensively [61, 62, 63, 64]. Very powerful positive results directly follow from what is shown in this article: for instance, decidability and upper complexity bounds for monodic two-variable fragments with counting transfer to modal and temporal DLs based on $\mathcal{ALCQHIO}^u$ simply because the latter can be regarded as a fragment of C^2. It would be of interest to explore whether our techniques can be applied to analyse temporal ontology-mediated query answering [65] with NRDC features or very expressive DLs with NRDC features not yet considered in a modal or temporal context [66].

An alternative approach to defining decidable fragments of first-order modal logic is to restrict the relative position of modal operators and quantifiers to certain patterns, called *bundles* (such as $\exists x\Diamond$ and $\Diamond\forall x$). *Bundled fragments* of first-order modal logic have been investigated extensively [67, 68]. However, to the best of our knowledge, so far this approach has only been applied to fragments without NRDC features. Thus, exploring how bundled fragments behave with NRDC features remains a topic for future research.

Finally, it would be of interest to systematically consider the role of monodicity and NRDC features within *term modal logics*, a large family of languages in which modal operators are indexed by typically non-rigid names for agents [69, 70, 71]. While decidability has been extensively studied already [72, 73, 74], we conjecture that our techniques can be applied to variations of term modal logics as well.

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