Non-Rigid Designators in Modal and Temporal Free Description Logics (Extended Version)

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Abstract

Definite descriptions, such as 'the General Chair of KR 2024', are a semantically transparent device for object identification in knowledge representation. In first-order modal logic, definite descriptions have been widely investigated for their non-rigidity, which allows them to designate different objects (or none at all) at different states. We propose expressive modal description logics with non-rigid definite descriptions and names, and investigate decidability and complexity of the satisfaction problem. We first systematically link satisfiability for the one-variable fragment of first-order modal logic with counting to our modal description logics. Then, we prove a promising NEXPTIME-completeness result for concept satisfiability for the fundamental epistemic multi-agent logic $\mathbf{S5}^n$ and its neighbours, and show that some expressive logics that are undecidable with constant domain become decidable (but Ackermann-hard) with expanding domains. Finally, we conduct a fine-grained analysis of decidability of temporal logics.

1 Introduction

Definite descriptions, like 'the General Chair of KR 2024', are expressions of the form 'the x such that φ.' Together with *individual names* such as 'Pierre,' they are used as *referring expressions* to identify objects in a given domain (Reiter and Dale 2000; Borgida, Toman, and Weddell 2016; Borgida, Toman, and Weddell 2017).

Definite descriptions and individual names can also *fail to designate* any object at all, as in the case of the definite description 'the KR Conference held after KR 2018 and before KR 2020,' or the individual name 'KR 2019.' In order to admit these as genuine terms of the language, while allowing for their possible lack of referents, formalisms based on *free logic* semantics have been developed (Bencivenga 2002; Lehmann 2002; Indrzejczak 2021; Indrzejczak and Zawidzki 2021). In contrast, classical logic approaches assume that individual names always designate, and that definite descriptions can be paraphrased in terms of existence and uniqueness conditions, an approach dating back to Russell (1905). Recently, definite descriptions have been introduced to enrich standard description logics (DLs) with nominals, $\mathcal{ALCO}$ and $\mathcal{ELO}$ (Neuhaus, Kutz, and Righetti 2020; Artale et al. 2020; Artale et al. 2021).

In particular, DLs $\mathcal{ALCO}_u^\iota$ and $\mathcal{ELO}_u^\iota$ (Artale et al. 2020; Artale et al. 2021) include the universal role, u, as well as nominals and definite descriptions of the form $\{\iota C\}$ ('the object in C') as basic concept constructs, while also employing a free logic semantics that allows non-designating terms. Hence, for instance, using nominal $\{\mathsf{kr24}\}$ to designate KR 2024, we can refer to the General Chair of KR 2024 by means of the definite description $\{\iota \exists \mathsf{isGenChair}.\{\mathsf{kr24}\}\}$. Then, in $\mathcal{ALCO}_u^\iota$, we can say that Pierre is the General Chair of KR 2024 with the following concept:

$$\exists u.(\{\mathsf{pierre}\} \sqcap \{\iota \exists \mathsf{isGenChair}.\{\mathsf{kr24}\}\}).$$

The free DLs $\mathcal{ALCO}_u^\iota$ and $\mathcal{ELO}_u^\iota$ have, respectively, EXPTIME-complete ontology satisfiability and PTIME-complete entailment problems, thus matching the complexity of the classical DL counterparts.

Names and descriptions display interesting behaviours also in *modal* (epistemic, temporal) settings. These are indeed *referentially opaque* contexts, where the *intension* (i.e., the meaning) of a term might not coincide with its *extension* (that is, its referent) (Fitting 2004). Here, referring expressions can behave as *non-rigid designators*, meaning that they can designate different individuals across different states (epistemic alternatives, instants of time, etc.).

For example, in an epistemic scenario, even if everybody is aware that Pierre is the General Chair of KR 2024, not everyone thereby knows that he is also the General Chair of the (only, so far, and excluding virtual location) KR Conference held in Southeast Asia, despite the fact that 'the KR Conference held in Southeast Asia' and 'KR 2024' refer (to this day) to the same object. Indeed, 'the KR Conference held in Southeast Asia' can be conceived to designate another event by someone unaware of its actual reference to KR 2024. Similarly, in a temporal setting, 'the General Chair of KR' refers to Pierre in 2024, but will designate someone else over the years. So, for instance, if we assume that Pierre works and will continue working in Europe, we can conclude that the General Chair of KR currently works in Europe, but we should not infer that it will always remain the case.

Due to this fundamental and challenging interplay between designation and modalities, non-rigid descriptions and names have been widely investigated in first-order modal and temporal logics as individual concepts or flexible terms capable of taking different values across states (Coc-

chiarella 1984; Garson 2001; Braüner and Ghilardi 2007; Kröger and Merz 2008; Fitting and Mendelsohn 2012; Corsi and Orlandelli 2013; Indrzejczak 2020; Orlandelli 2021; Indrzejczak and Zawidzki 2023).

The aim of this contribution is to introduce modal DLs that have sufficient expressive power to represent the phenomena discussed above, explore their relationship to standard modal DLs without definite descriptions and non-designating names, and investigate decidability and complexity of reasoning in these languages.

In detail, we propose the language $\mathcal{ML}^n_{\mathcal{ALCO}^\iota_u}$, which is a modalised extension of the free DL $\mathcal{ALCO}^\iota_u$. For instance, under an epistemic reading of the modal operator $\Box$, we can express with an $\mathcal{ML}^n_{\mathcal{ALCO}^\iota_u}$ concept that *of* Pierre it is *known* that he is the General Chair of KR 2024:

$$\exists u.(\{\mathsf{pierre}\} \sqcap \Box\{\iota \exists \mathsf{isGenChair}.\{\mathsf{kr24}\}\}),$$

while *of* Pierre it is *not* known that he is the General Chair of the KR Conference held in Southeast Asia:

$$\exists u.(\{\mathsf{pierre}\} \sqcap \neg\Box\{\iota \exists \mathsf{isGenChair}. \{\iota(\mathsf{KRConf} \sqcap \exists \mathsf{hasLoc}.\mathsf{SEAsiaLoc})\}\}).$$

In a temporal setting, where $\Box$ is read as 'always' and its dual $\Diamond$ as 'at some point in the future', KR 2024 can be made a rigid designator, which refers to the same object at all time instants. This can be achieved, for example, by means of an ontology that holds globally, at all time instants, and consists of the following concept inclusion (CI):

$$\{\mathsf{kr24}\} \sqsubseteq \Box\{\mathsf{kr24}\}.$$

In contrast, we can use the nominal $\{\mathsf{kr}\}$ to refer to the current edition of the KR Conference, for instance stating that KR 2024 is the current KR Conference with the concept $\exists u.(\{\mathsf{kr24}\} \sqcap \{\mathsf{kr}\})$. Moreover, by reading $\bigcirc$ as 'next year', we can exploit its lack of rigidity to express, e.g., that there will be *other* KR Conferences in the future, with the CIs:

$$\top \sqsubseteq \Diamond \exists u.\{\mathsf{kr}\}, \qquad \{\mathsf{kr}\} \sqsubseteq \neg\bigcirc\{\mathsf{kr}\}.$$

Compared to first-order modal logics with non-rigid designators (Stalnaker and Thomason 1968; Fitting and Mendelsohn 2012), both the definition of the language and the scope distinctions for modal operators are simplified in modalised DLs, as first-order variables are replaced by a class-based DL syntax that leaves quantification and predicate abstraction implicit. It is, however, possible to translate our modal DL back to a natural fragment of first-order modal logic with definite descriptions and predicate abstraction.

In this work, we consider interpretations with both constant domains (in which the first-order domain is fixed across all worlds) and expanding domains (in which the domain can grow when moving to accessible worlds). We first establish, for any class of Kripke frames and both constant and expanding domains, polytime satisfiability-preserving reductions (with and without ontology) to the language $\mathcal{ML}^n_{\mathcal{ALCO}_u}$ without definite descriptions. We will show that, in addition, we can assume that each nominal designates in every world, but importantly is still non-rigid.

We then study the satisfiability problem for various fundamental modal logics with epistemic and temporal interpretations. While for first-order modal logics with only rigid designators and no counting the restriction to *monodic* formulas (in which modal operators are only applied to formulas with a single free variable) very often ensures decidability, this is no longer the case if non-rigid designators and/or some counting are admitted (Gabbay et al. 2003). For our modal DLs, this implies that the standard recipe for designing decidable languages — apply modal operators only to concepts — does not always work anymore. Here, we explore in detail when this recipe still works, and when it does not.

First, we closely link the two main sources of bad computational behaviour, non-rigid designators and counting, enabling us to use the results and machinery introduced for first-order modal logics with counting (Hampson and Kurucz 2012; Hampson and Kurucz 2015; Hampson 2016). Then, we prove that, rather surprisingly, for some fundamental modal epistemic logic, non-rigid designators come for free: concept satisfiability for the modal logics of all Kripke frames with n accessibility relation, $\mathbf{K}^n$, and of all Kripke frames with n equivalence relations, $\mathbf{S5}^n$, is in NEXPTIME and thus no harder than without names at all. This holds under both constant and expanding domains, and the proof is by showing the exponential finite model property. This answers an open problem discussed in (Hampson 2016). With ontologies, however, concept satisfiability becomes undecidable under constant domains. While for $\mathbf{S5}^n$ (because of symmetric accessibility relations) constant domains coincide with expanding domains, for $\mathbf{K}^n$ decidability under expanding domains remains open with ontologies. As a fundamental example of an expressive modal logic, we investigate the extension $\mathbf{K}^{*n}$ of $\mathbf{K}^n$ with a modal operator interpreted by the transitive closure of the union of the n accessibility relations, which can be interpreted as common knowledge (Fagin et al. 1995) but also as a fragment of propositional dynamic logic, PDL (Harel, Kozen, and Tiuryn 2001). In this case concept satisfiability is undecidable under expanding and constant domains, but becomes decidable, though Ackermann-hard, for the corresponding logic, $\mathbf{K}\boldsymbol{f}^{*n}$, on finite acyclic models with expanding domains. This answers an open problem posed in (Wolter and Zakharyaschev 2001). Note that Ackermann-hardness means that the time required to establish (un)satisfiability is not bounded by any primitive recursive (or computable) function. We refer the reader to Table 1 for an overview of our results. Recall that with rigid designators all these problems are known to be in 2NEXPTIME (Wolter and Zakharyaschev 2001; Gabbay et al. 2003).

Finally, in the temporal setting, we show that undecidability is a widespread phenomenon: concept satisfiability under global ontology with constant domain is undecidable for all our $\mathcal{ALCO}$-based fragments; the same applies to concept satisfiability in the languages with the universal modality and the temporal $\Diamond$ operator. Reasoning becomes decidable only when considering concept satisfiability (under global ontology) with expanding domains over finite flows of time (though the problem is Ackermann-hard), or in fragments with the $\bigcirc$ operator only, for which we prove EXPTIME-

modal logic L	domain	concept sat.	concept sat. under global ont.
$\mathbf{K}^n$, $n \geq 1$	const.	NEXP-c.	undecidable
	exp.	NEXP-c.	?
$\mathbf{S5}$	—	NEXP-complete	
$\mathbf{S5}^n$, $n \geq 2$	—	NEXP-c.	undecidable
$\mathbf{K}^{*n}$, $n \geq 1$	const.	Σ^1_1-complete	
	exp.	undecidable	
$\mathbf{Kf}^{*n}$, $n \geq 1$	const.	undecidable	
	exp.	decidable, Ackermann-hard	

Table 1: Concept satisfiability (under global ontology) for $L_{\mathcal{ALCO}^{\iota}_{u}}$

membership of concept satisfiability (without ontologies).

It is to be emphasised that the non-rigidity of symbol interpretation by itself is *not* the cause for the satisfiability problem to become harder. For instance, rigid roles are known to often cause harder satisfaction problems than non-rigid roles (Gabbay et al. 2003). What makes non-rigid designators computationally much harder than rigid designators is their ability to count in an unbounded way across worlds.

Related Work Other than in non-modal DL languages (Neuhaus, Kutz, and Righetti 2020; Artale et al. 2020; Artale et al. 2021), and in the already mentioned first-order modal, temporal and hybrid logic settings, definite descriptions have been recently investigated in the context of *propositional* modal, temporal and hybrid logics with nominals and the @ operator (Walega and Zawidzki 2023). Here, the additional ι operator allows one to refer to the (one and only) state of a model that satisfies a certain condition.

Non-rigid rigid designators have received, to the best of our knowledge, little attention in modal DLs, despite the wide body of research both on temporal (Wolter and Zakharyaschev 1998; Artale and Franconi 2005; Lutz, Wolter, and Zakharyaschev 2008; Artale et al. 2014) and epistemic (Donini et al. 1998; Artale, Lutz, and Toman 2007; Calvanese et al. 2008; Console and Lenzerini 2020) extensions. As a notable exception, in an epistemic DL context, non-rigid individual names appear in (Mehdi and Rudolph 2011), where abstract individual names are interpreted on an infinite common domain, but without definite descriptions.

2 Preliminaries

The $\mathcal{ML}^n_{\mathcal{ALCO}^{\iota}_{u}}$ language is a modalised extension of the free description logic (DL) $\mathcal{ALCO}^{\iota}_{u}$ (Artale et al. 2020; Artale et al. 2021). Let $\mathsf{N_C}$, $\mathsf{N_R}$ and $\mathsf{N_I}$ be countably infinite and pairwise disjoint sets of *concept names*, *role names* and *individual names*, respectively, and let $I = \{1, \ldots, n\}$ be a finite set of *modalities*. $\mathcal{ML}^n_{\mathcal{ALCO}^{\iota}_{u}}$ *terms* and *concepts* are defined by the following grammar:

$$\tau ::= a \mid \iota C,$$
$$C ::= A \mid \{\tau\} \mid \neg C \mid (C \sqcap C) \mid \exists r.C \mid \exists u.C \mid \Diamond_i C,$$

where $a \in \mathsf{N_I}$, $A \in \mathsf{N_C}$, $r \in \mathsf{N_R}$, $u \notin \mathsf{N_R}$ is the *universal role*, and $\Diamond_i$, with $i \in I$, is a *diamond* operator. A term of the form ιC is called a *definite description* and C its *body*; a concept $\{\tau\}$ is called a (*term*) *nominal*. All the usual syntactic abbreviations are assumed: $\bot = A \sqcap \neg A$, $\top = \neg\bot$, $C \sqcup D = \neg(\neg C \sqcap \neg D)$, $C \Rightarrow D = \neg C \sqcup D$, $\forall s.C = \neg\exists s.\neg C$, for $s \in \mathsf{N_R} \cup \{u\}$, and *box* operator $\Box_i C = \neg\Diamond_i\neg C$. A *concept inclusion* (*CI*) is of the form $C \sqsubseteq D$, for concepts C, D. We use $C \equiv D$ to abbreviate $C \sqsubseteq D$ and $D \sqsubseteq C$. An *ontology* $\mathcal{O}$ is a finite set of CIs.

Fragments $\mathcal{ML}^n_{\mathcal{ALCO}_u}$, $\mathcal{ML}^n_{\mathcal{ALCO}^{\iota}}$, and $\mathcal{ML}^n_{\mathcal{ALCO}}$ of the full language are defined by restricting the available DL constructors: they do not contain, respectively, definite descriptions, the universal role, and both definite descriptions and the universal role.

Given a concept C, the set of *subconcepts* of C, denoted by $\mathsf{sub}(C)$, is defined as usual: we only note that $\mathsf{sub}(\{\iota C\})$ contains C along with its own subconcepts. The *signature* of C, denoted by Σ_C, is the set of concept, role and individual names in C. The signature of a CI or an ontology is defined similarly. The *modal depth* of terms and concepts is the maximum number of nested modal operators: $md(A) = 0$, $md(\iota C) = md(C)$ and $md(\Diamond_i C) = md(C) + 1$, for example. The *modal depth* of a CI or an ontology is the maximum modal depth of their concepts.

A *frame* is a pair $\mathfrak{F} = (W, \{R_i\}_{i\in I})$, where W is a non-empty set of *worlds* (or *states*) and each $R_i \subseteq W \times W$, for $i \in I$, is a binary *accessibility relation* on W. A *partial interpretation with expanding domains* based on a frame $\mathfrak{F} = (W, \{R_i\}_{i\in I})$ is a triple $\mathfrak{M} = (\mathfrak{F}, \Delta, \mathcal{I})$, where Δ is a function associating with every $w \in W$ a non-empty set, Δ^w, called the *domain of* w *in* $\mathfrak{M}$, such that $\Delta^w \subseteq \Delta^v$, whenever wR_iv, for some $i \in I$; and $\mathcal{I}$ is a function associating with every $w \in W$ a *partial* DL interpretation $\mathcal{I}_w = (\Delta^w, \cdot^{\mathcal{I}_w})$ that maps every $A \in \mathsf{N_C}$ to a subset $A^{\mathcal{I}_w}$ of Δ^w, every $r \in \mathsf{N_R}$ to a subset $r^{\mathcal{I}_w}$ of $\Delta^w \times \Delta^w$, the universal role u to the set $\Delta^w \times \Delta^w$, and every a in *some subset* of $\mathsf{N_I}$ to an element $a^{\mathcal{I}_w}$ in Δ^w. In particular, every $\cdot^{\mathcal{I}_w}$ is a total function on $\mathsf{N_C} \cup \mathsf{N_R}$ but a *partial* function on $\mathsf{N_I}$. If $\mathcal{I}_w$ is defined on $a \in \mathsf{N_I}$, then we say that a *designates at* w. If every $a \in \mathsf{N_I}$ designates at $w \in W$, then $\mathcal{I}_w$ is called *total*. We say that $\mathfrak{M} = (\mathfrak{F}, \Delta, \mathcal{I})$ is a *total* interpretation if every $\mathcal{I}_w$, $w \in W$, is a *total* interpretation. In the sequel, we refer to partial interpretations as interpretations, and add the adjective 'total' explicitly whenever this is the case.

An *interpretation with constant domains* is defined as a special case, where the function Δ is such that $\Delta^w = \Delta^v$, for every $w, v \in W$. With an abuse of notation, we denote the common domain by Δ and call it the *domain of* $\mathfrak{M}$.

Given $\mathfrak{M} = (\mathfrak{F}, \Delta, \mathcal{I})$, with $\mathfrak{F} = (W, \{R_i\}_{i\in I})$, we define the *value* $\tau^{\mathcal{I}_w}$ of a term τ in world $w \in W$ as $a^{\mathcal{I}_w}$, for $\tau = a$, and as follows, for $\tau = \iota C$:

$$(\iota C)^{\mathcal{I}_w} = \begin{cases} d, & \text{if } C^{\mathcal{I}_w} = \{d\}, \text{ for some } d \in \Delta^w; \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

A term τ is said to *designate at* w if $\tau^{\mathcal{I}_w}$ is defined. The *extension* $C^{\mathcal{I}_w}$ of a concept C in $w \in W$ is defined as follows, where $s \in \mathsf{N_R} \cup \{u\}$:

$$(\neg C)^{\mathcal{I}_w} = \Delta^w \setminus C^{\mathcal{I}_w},$$
$$(C \sqcap D)^{\mathcal{I}_w} = C^{\mathcal{I}_w} \cap D^{\mathcal{I}_w},$$

$$(\exists s.C)^{\mathcal{I}_w} = \{d \in \Delta^w \mid \exists e \in C^{\mathcal{I}_w} : (d,e) \in s^{\mathcal{I}_w}\},$$
$$(\Diamond_i C)^{\mathcal{I}_w} = \left\{d \in \Delta^w \mid \exists v \in W : wR_iv \text{ and } d \in C^{\mathcal{I}_v}\right\},$$
$$\{\tau\}^{\mathcal{I}_w} = \begin{cases} \{\tau^{\mathcal{I}_w}\}, & \text{if } \tau \text{ designates at } w, \\ \emptyset, & \text{otherwise.} \end{cases}$$

A concept C is *satisfied at* $w \in W$ *in* $\mathfrak{M}$ if $C^{\mathcal{I}_w} \neq \emptyset$; C is *satisfied in* $\mathfrak{M}$ if it is satisfied at some $w \in W$ in $\mathfrak{M}$. A CI $C \sqsubseteq D$ is *satisfied in* $\mathfrak{M}$, written $\mathfrak{M} \models C \sqsubseteq D$, if $C^{\mathcal{I}_w} \subseteq D^{\mathcal{I}_w}$, for every $w \in W$. An ontology $\mathcal{O}$ is *satisfied in* $\mathfrak{M}$, written $\mathfrak{M} \models \mathcal{O}$, if every CI in $\mathcal{O}$ is satisfied in $\mathfrak{M}$; we also say a concept C is *satisfied in* $\mathfrak{M}$ *under an ontology* $\mathcal{O}$ if $\mathfrak{M} \models \mathcal{O}$ and $C^{\mathcal{I}_w} \neq \emptyset$, for some $w \in W$.

An ontology $\mathcal{O}'$ is called a *model conservative extension* of an ontology $\mathcal{O}$ if every interpretation that satisfies $\mathcal{O}'$ also satisfies $\mathcal{O}$, and every interpretation that satisfies $\mathcal{O}$ can be turned to satisfy $\mathcal{O}'$ by modifying the interpretation of symbols in $\Sigma_{\mathcal{O}'} \setminus \Sigma_{\mathcal{O}}$, while keeping fixed the interpretation of symbols in $\Sigma_{\mathcal{O}}$. Similarly, a concept C' is said to be a *model conservative extension* of a concept C if every interpretation that satisfies C' also satisfies C, and every interpretation satisfying C can be turned into an interpretation that satisfies C', by modifying the interpretation of symbols in $\Sigma_{C'} \setminus \Sigma_C$, while keeping fixed the interpretation of symbols in Σ_C.

Remark 1 (Encoding of assertions)**.** Assertions can be introduced as syntactic sugar using the universal role, with $C(\tau)$ and $r(\tau_1, \tau_2)$ abbreviations for, respectively, concepts

$$\exists u.(\{\tau\} \sqcap C) \text{ and } \exists u.(\{\tau_1\} \sqcap \exists r.\{\tau_2\}).$$

To avoid ambiguities, we use square brackets when applying negation and modal operators to assertions, as in $\neg[C(\tau)]$ and $\Diamond_i[C(\tau)]$. Observe that, in an assertion of the form $\Diamond_i C(\tau)$, the diamond acts as a *de re* operator, since the concept $\Diamond_i C$ applies to the object, if any, designated by the term τ at the current world w, and the assertion is false at a world whenever τ fails to designate at w. On the other hand, in an expression of the form $\Diamond_i[C(\tau)]$, the diamond plays the role of a *de dicto* modality, as it refers to the world of evaluation for the whole assertion $C(\tau)$. Using the lambda abstraction notation for first-order modal logic (Fitting and Mendelsohn 2012), assertion $\Diamond_i A(a)$ corresponds to $\exists x.(\langle \lambda y.x = y \rangle(a) \wedge \Diamond_i A(x))$, whereas $\Diamond_i[A(a)]$ stands for $\Diamond_i \exists x.(\langle \lambda y.x = y \rangle(a) \wedge A(x))$; see Appendix A for details.

3 Reasoning Problems and Reductions

Let $\mathcal{C}$ be a class of frames (e.g., frames with n equivalence relations) and $\mathcal{ML}^n_{\mathcal{DL}}$ a language. We consider the following two main reasoning problems.

Concept $\mathcal{C}$-Satisfiability: Given an $\mathcal{ML}^n_{\mathcal{DL}}$-concept C, is there an interpretation $\mathfrak{M}$ based on a frame in $\mathcal{C}$ such that C is satisfied in $\mathfrak{M}$?

Concept $\mathcal{C}$-Satisfiability under Global Ontology: Given an $\mathcal{ML}^n_{\mathcal{DL}}$-concept C and an $\mathcal{ML}^n_{\mathcal{DL}}$-ontology $\mathcal{O}$, is there an interpretation $\mathfrak{M}$ based on a frame in $\mathcal{C}$ such that C is satisfied in $\mathfrak{M}$ under $\mathcal{O}$?

In the sequel, for the case of concept $\mathcal{C}$-satisfiability under global ontology, we will assume without loss of generality that C is a concept name. Indeed, we can extend $\mathcal{O}$ with $A \equiv C$, for a fresh concept name A, and consider satisfiability of A under the extended ontology, which is a model conservative extension of $\mathcal{O}$.

We begin with a few observations on polytime reductions between the concept satisfiability problems (under global ontology) for different languages and semantic conditions, including (non-)rigid designators, total and partial interpretations, definite description and nominals, the universal role, and expanding and constant domains. These observations will be useful in our constructions below. All proofs are available in the appendix.

No-RDA Subsumes RDA An interpretation $\mathfrak{M}$ satisfies the *rigid designator assumption* (*RDA*) if every individual name $a \in \mathsf{N_I}$ is a *rigid designator* in $\mathfrak{M}$, in the sense that, for every $w, v \in W$ such that wR_iv, if a designates at w, then it designates at v and $a^{\mathcal{I}_w} = a^{\mathcal{I}_v}$. For instance, concept

$$\exists u.(\{a\} \sqcap \Box C) \sqcap \Diamond \exists u.(\{a\} \sqcap \neg C),$$

is unsatisfiable in interpretations *with the RDA*, but is satisfiable otherwise as a can designate differently at different worlds. Note that an individual that fails to designate at all worlds is vacuously rigid. The following proposition shows the RDA can be enforced in interpretations by an ontology.

Proposition 1. *In $\mathcal{ML}^n_{\mathcal{ALCO}_u}$ and $\mathcal{ML}^n_{\mathcal{ALCO}^\iota_u}$, concept $\mathcal{C}$-satisfiability under global ontology with the RDA is polytime-reducible to concept $\mathcal{C}$-satisfiability under global ontology, with both constant and expanding domains.*

The proof is based on the observation that CIs of the form $\{a\} \sqsubseteq \Box_i\{a\}$, for $i \in I$, ensure that a can be made a rigid designator in any interpretation. This provides a reduction for the case of global ontology, where a given $\mathcal{O}$ is extended with these CIs for every a. A similar reduction is provided in Appendix B for concept satisfiability in total interpretations.

In the sequel, we assume implicitly that interpretations do *not* satisfy the RDA, and explicitly write 'with the RDA' where necessary.

From Total to Partial Satisfiability and Back Partial interpretations are a generalisation of the classical, total, interpretations, where all nominals designate at all possible worlds. It turns out that satisfiability in partial and total interpretations are polytime-reducible to each other.

Proposition 2. *In $\mathcal{ML}^n_{\mathcal{ALCO}_u}$ and $\mathcal{ML}^n_{\mathcal{ALCO}^\iota_u}$, total concept $\mathcal{C}$-satisfiability* (*under global ontology*) *is polytime-reducible to concept $\mathcal{C}$-satisfiability* (*under global ontology, respectively*)*, with both constant and expanding domains.*

We sketch the proof for the case of global ontology. Let $\mathcal{O}$ be an ontology. Consider the extension $\mathcal{O}'$ of $\mathcal{O}$ with CIs $\top \sqsubseteq \exists u.\{a\}$, for all individual names a in $\mathcal{O}$. Clearly, these CIs ensure that each a designates in every accessible world. The case of concept satisfiability is shown in the appendix.

Next, we provide the converse reduction.

Proposition 3. *In $\mathcal{ML}^n_{\mathcal{ALCO}_u}$ and $\mathcal{ML}^n_{\mathcal{ALCO}^\iota_u}$, concept $\mathcal{C}$-satisfiability (under global ontology) is polytime-reducible to total concept $\mathcal{C}$-satisfiability (under global ontology, respectively), with both constant and expanding domains.*

We again sketch the proof for the case of global ontology. Let $\mathcal{O}$ be an ontology. Consider $\mathcal{O}'$ obtained from $\mathcal{O}$ by replacing every nominal $\{a\}$ in $\mathcal{O}$ with a fresh concept name N_a and by extending the result with all CIs $N_a \sqsubseteq \{a\}$. It follows that every a in $\mathcal{O}'$ can designate in all worlds, but the corresponding concept N_a may still be interpreted by the empty set in some worlds, thus reflecting the fact that a in $\mathcal{O}$ could have failed to designate in those worlds. The case of concept satisfiability is dealt with in the appendix.

Normal Form for Ontologies and Concepts Next, we define normal form that will help us prove further polytime-reductions, e.g., Lemma 6 and Proposition 8, and then complexity upper bounds. Let $\mathcal{O}$ be an ontology and C a subconcept in $\mathcal{O}$. Denote by $\mathcal{O}[C/A]$ the result of replacing every occurrence of C in $\mathcal{O}$ with a fresh concept name A, called the *surrogate* of C. Clearly, $\mathcal{O}[C/A] \cup \{C \equiv A\}$ is a model conservative extension of $\mathcal{O}$. We can systematically apply this procedure to obtain an ontology in *normal form* where connectives are applied only to concept names: e.g., definite descriptions occur only in the form of ιB, for a concept name B. If surrogates are introduced for innermost connectives first, then the transformation runs in polytime.

Lemma 4. *For any $\mathcal{ML}^n_{\mathcal{ALCO}^\iota_u}$ ontology $\mathcal{O}$, we can construct in polytime an $\mathcal{ML}^n_{\mathcal{ALCO}^\iota_u}$ ontology $\mathcal{O}'$ in normal form that is a model conservative extension of $\mathcal{O}$. Moreover, $\mathcal{O}'$ uses the same set of connectives as $\mathcal{O}$.*

If the language contains the universal role, then a similar construction transforms concepts into normal form. For a single modality ($n = 1$), we can use $\Box_1^k \forall u.(C \Leftrightarrow A)$, for all $k \leq md(\mathcal{O})$. If $n > 1$, then we need to carefully select sequences of boxes to avoid an exponential blowup. So, for an $\mathcal{ML}^n_{\mathcal{ALCO}^\iota_u}$ concept D and its subconcept C, we define the set of *C-relevant paths in D* as the sequences $(i_1, \dots, i_n)$ of the $\Diamond_{i_j}$ operators under which C occurs in D. For example, for $D = \Diamond_1 \neg A \sqcap \Diamond_2 \Diamond_3 A$, we have $\mathsf{rp}(D, A) = \{(1), (2, 3)\}$ and $\mathsf{rp}(D, \neg A) = \{(1)\}$. Note that the maximum length of a path in $\mathsf{rp}(D, C)$ is $md(D)$. We also define the '$\Box$-modality' for each path: for a concept E, we recursively define

$$\Box^\epsilon E = E \text{ and } \Box^{i \cdot \pi} E = \Box_i \Box^\pi E, \text{ for any path } \pi.$$

As before, the *surrogate* of C is a fresh concept name A, and $D[C/A]$ denotes the result of replacing C with A in D.

Lemma 5. *Let D be an $\mathcal{ML}^n_{\mathcal{ALCO}^\iota_u}$ concept and C its subconcept. Denote by D' the conjunction of $D[C/A]$ and*

$$\Box^\pi \forall u.(C \Leftrightarrow A), \text{ for all } \pi \in \mathsf{rp}(D, C). \qquad (1)$$

Then D' is a model conservative extension of D. Moreover, $\mathsf{rp}(D', A) = \mathsf{rp}(D, C)$ and $\mathsf{rp}(D', E) = \mathsf{rp}(D, E)$, for any subconcept E of C.

For any concept D, by repeatedly replacing non-atomic subconcepts with their surrogates, one can obtain a concept D^* in *normal form*, which is a conjunction of a concept name and concepts of the form (1). By Lemma 5, D^* is a model conservative extension of D. Moreover, if surrogates are introduced for innermost non-atomic concepts first, the procedure runs in polytime in the size of D.

Spy Points: Eliminating the Universal Role Our next observation allows us to eliminate occurrences of the universal role from ontologies.

Lemma 6. *Let $\mathcal{O}$ be an $\mathcal{ML}^n_{\mathcal{ALCO}^\iota_u}$ ontology in normal form. Denote by $\mathcal{O}'$ the $\mathcal{ML}^n_{\mathcal{ALCO}^\iota}$ ontology obtained from $\mathcal{O}'$ by replacing*

- *each CI of the form $B \sqsubseteq \exists u.B'$ with $B \sqsubseteq \exists r.B'$, and*
- *each CI of the form $\exists u.B \sqsubseteq B'$ with the following:*

$$\top \sqsubseteq \exists r.\{e\}, \;\; A \sqsubseteq \{e\}, \;\; \neg B' \sqsubseteq \exists r.A, \;\; \exists r.A \sqsubseteq \neg B,$$

where r, e and A are fresh role, nominal and concept names, respectively. Then $\mathcal{O}'$ is a model conservative extension of $\mathcal{O}$, and the size of $\mathcal{O}'$ is linear in the size of $\mathcal{O}$.

Intuitively, positive occurrences of $\exists u.B'$ ensure that B' is non-empty, which can also be achieved with a fresh role r. For negative occurrences of $\exists u.B'$, we use a spy-point e, which is accessible, via a fresh r, from every domain element and belongs to A whenever $\neg B'$ is non-empty (that is, whenever B' does not coincide with the domain). If this is the case, then no domain element can be in B, which, by contraposition, implies $\exists u.B \sqsubseteq B'$. Note that e can be rigid.

From Nominals to Definite Descriptions and Back We first observe that nominals can be easily encoded with definite descriptions. For instance, given an ontology $\mathcal{O}$, take a fresh concept name N_a for each individual name a in $\mathcal{O}$, and let $\mathcal{O}'$ be the result of replacing every occurrence of $\{a\}$ in $\mathcal{O}$ with $\{\iota N_a\}$. Clearly, $\mathcal{O}'$ is a model conservative extension of $\mathcal{O}$, and vice versa. Note, however, that $\mathcal{O}'$ contains no nominals, and so there is neither a distinction between partial and total interpretations, nor between the RDA and no-RDA cases. In general, we have the following result.

Proposition 7. *In $\mathcal{ML}^n_{\mathcal{ALCO}_u}$ and $\mathcal{ML}^n_{\mathcal{ALCO}^\iota_u}$, concept $\mathcal{C}$-satisfiability (under global ontology) is polytime-reducible to $\mathcal{ML}^n_{\mathcal{ALC}^\iota_u}$ concept $\mathcal{C}$-satisfiability (under global ontology, respectively), with both constant and expanding domains.*

Conversely, we now show how to replace definite descriptions ιC with nominals using the universal role.

Proposition 8. *$\mathcal{ML}^n_{\mathcal{ALCO}^\iota_u}$ concept $\mathcal{C}$-satisfiability (under global ontology) is polytime-reducible to $\mathcal{ML}^n_{\mathcal{ALCO}_u}$ concept $\mathcal{C}$-satisfiability (under global ontology, respectively), with both constant and expanding domains.*

The proof reduces the total $\mathcal{C}$-satisfiability problems in $\mathcal{ML}^n_{\mathcal{ALCO}^\iota_u}$ to total $\mathcal{C}$-satisfiability in $\mathcal{ML}^n_{\mathcal{ALCO}_u}$, which, by Propositions 3 and 2 gives us the required result. We sketch the case of the global ontology. By Lemma 4, we can assume that the given $\mathcal{O}$ is in normal form. Let $\mathcal{O}^*$ be the result of replacing each $A_{\{\iota B\}} \equiv \{\iota B\}$ in $\mathcal{O}$ with CIs

$$A_{\{\iota B\}} \sqsubseteq B \sqcap \{a_B\} \text{ and } B \sqcap \forall u.(B \Rightarrow \{a_B\}) \sqsubseteq A_{\{\iota B\}},$$

where a_B is a fresh nominal. Intuitively, the first CI ensures that the surrogate for $\{\iota B\}$ belongs to B and is never interpreted by more than one element. The second CI ensures that if B is a singleton, then that element belongs to the surrogate for $\{\iota B\}$. Formally, we show that $\mathcal{O}^*$ is a model conservative extension of $\mathcal{O}$. The case of concept satisfiability relies on normal form of concepts (Lemma 5) and is treated in the appendix.

Expanding to Constant Domains It is known that, for the satisfiability problems, the interpretations with expanding domains can be simulated by constant domain interpretations using a fresh concept name representing the domain to relativise concepts and CIs; see e.g., (Gabbay et al. 2003, Proposition 3.32 (ii), (iv)). We restate this standard result in our setting for completeness:

Proposition 9. *In $\mathcal{ML}^n_{\mathcal{ALCO}_u}$, concept $\mathcal{C}$-satisfiability* (*under global ontology*) *with expanding domains is polytime-reducible to concept $\mathcal{C}$-satisfiability* (*under global ontology, respectively*) *with constant domain.*

In the sequel, we implicitly adopt the *constant domain assumption* (Gabbay et al. 2003), and explicitly write when we consider interpretations with expanding domains instead.

4 Non-Rigid Designators and Counting

We prove a strong link between non-rigid designators and the first-order one-variable modal logic enriched with the 'elsewhere' quantifier, $\mathcal{ML}^n_{\text{Diff}}$, introduced and investigated in (Hampson and Kurucz 2012; Hampson and Kurucz 2015; Hampson 2016). We define $\mathcal{ML}^n_{\text{Diff}}$ using DL-style syntax: concepts in $\mathcal{ML}^n_{\text{Diff}}$ are of the form

$$C ::= A \mid \neg C \mid (C \sqcap C) \mid \exists u.C \mid \exists^{\neq} u.C \mid \Diamond_i C,$$

where $i \in I$. Observe that the language has no terms and no roles apart from the universal role u. All constructs are interpreted as before, with the addition of

$$(\exists^{\neq} u.C)^{\mathcal{I}_w} = \big\{d \in \Delta^w \mid C^{\mathcal{I}_w} \setminus \{d\} \neq \emptyset\big\}.$$

Note that our language contains existential quantification, which in (Hampson and Kurucz 2015) is introduced as an abbreviation for $C \sqcup \exists^{\neq} u.C$. $\mathcal{ML}^n_{\text{Diff}}$ can be regarded as a basic first-order modal logic with counting because the counting quantifier $\exists^{=1} u.C$ with

$$(\exists^{=1} u.C)^{\mathcal{I}_w} = \big\{d \in \Delta^w \mid |C^{\mathcal{I}_w}| = 1\big\}$$

is logically equivalent to the $\mathcal{ML}^n_{\text{Diff}}$-concept $\exists u.(C \sqcap \neg \exists^{\neq} u.C)$. Conversely, $\exists^{\neq} u.C$ is logically equivalent to $\exists u.C \sqcap (C \Rightarrow \neg \exists^{=1} u.C)$. So, one could replace $\exists^{\neq}$ by $\exists^{=1}$ in the definition $\mathcal{ML}^n_{\text{Diff}}$ and obtain a logic with exactly the same expressive power.

Theorem 10. *(1) $\mathcal{C}$-satisfiability of $\mathcal{ML}^n_{\mathcal{ALCO}_u}$-concepts* (*under global ontology*) *can be reduced in double exponential time to $\mathcal{C}$-satisfiability of $\mathcal{ML}^n_{\text{Diff}}$-concepts* (*under global ontology, respectively*)*, both with constant and expanding domains.*

(2) Conversely, $\mathcal{C}$-satisfiability of $\mathcal{ML}^n_{\text{Diff}}$-concepts (*under global ontology*) *is polytime-reducible to $\mathcal{C}$-satisfiability of $\mathcal{ML}^n_{\mathcal{ALCO}_u}$-concepts* (*under global ontology, respectively*)*, with both constant and expanding domains.*

The proof of (1) and proofs in Sections 5 and 6 use *types* $\boldsymbol{t}$, sets of (negated) subconcepts of the input concept (and ontology) such that, for any subconcept C, either $C \in \boldsymbol{t}$ or $\neg C \in \boldsymbol{t}$. A *quasistate* T is a set of types and stands for a world satisfying exactly the types in T (Gabbay et al. 2003). Given a quasistate T, one can check in polytime whether its *non-modal abstraction*, obtained by replacing all concepts starting with a modal operator by a concept name, is satisfiable in a DL interpretation. On the other hand, assuming that $\mathcal{C}$-satisfiability of $\mathcal{ML}^n_{\text{Diff}}$-concepts is decidable, one can check whether its *non-DL abstraction*, in which existential restrictions and nominals are replaced by concept names, is satisfiable in a $\mathcal{ML}^n_{\text{Diff}}$ model in which the concept names replacing nominals are interpreted as singletons. As observed above, to enforce interpretation of a concept name A as a singleton in all worlds one can use the $\mathcal{ML}^n_{\text{Diff}}$ CI $\top \sqsubseteq \exists^{=1} u.A$. For concept satisfiability (without ontology) one makes the same statement for all worlds reachable along relevant paths. The reduction is now implemented using both abstractions. It requires double exponential time because the number of quasistates is double exponential in the size of the input. Item (2) is proved by lifting to the modal description logic setting the observation of Gargov and Goranko (1993) that the difference modality and nominals are mutually interpretable by each other; see the equivalences that show the same expressive power of $\exists^{\neq}$ and $\exists^{=1}$.

Using Theorem 10, one can transfer a large number of (un)decidability results and lower complexity bounds from first-order modal logics with 'elsewhere' to modal DLs with non-rigid designators. Conversely, the results proved below entail new results for first-order modal logics with 'elsewhere' and/or counting.

5 Reasoning in Modal Free Description Logics

The aim of this section is to show the results presented in Table 1 and also discuss a few related results. For some basic frame classes, it will be convenient to use standard modal logic notation when discussing the satisfiability problem. So, given a propositional modal logic L with n operators, the class $\mathcal{C}_L$ of frames comprising all frames validating L, and a DL fragment $\mathcal{DL}$, then $L_{\mathcal{DL}}$ *concept satisfiability* (*under global ontology*) is the problem to decide $\mathcal{C}_L$-satisfiability of $\mathcal{ML}^n_{\mathcal{DL}}$-concepts (under global ontology, respectively). We focus on the following logics L characterised by classes $\mathcal{C}_L$ of frames, with $n \geq 1$:

$\mathbf{K}^n$: $\mathcal{C}_L$ is the class of all frames $(W, R_1, \dots, R_n)$;

$\mathbf{S5}^n$: $\mathcal{C}_L$ is the class of all frames $(W, R_1, \dots, R_n)$ such that the R_i are equivalence relations;

$\mathbf{K}^{*n}$: $\mathcal{C}_L$ is the class of all frames $(W, R_1, \dots, R_n, R)$ such that R is the transitive closure of $R_1 \cup \dots \cup R_n$;

$\mathbf{Kf}^{*n}$: $\mathcal{C}_L$ is as for $\mathbf{K}^{*n}$ and, in addition, W is finite and R is irreflexive (in other words, there is no chain $w_0 R_{i_1} w_1 \cdots R_{i_n} w_n$ with $w_0 = w_n$).

To illustrate the language $\mathbf{S5}^2_{\mathcal{ALCO}^\iota_u}$, we express that Agent 2 *knows that* Agent 1 does *not know of* the General

Chair of KR 2024 that they are the General Chair of the KR Conference held in Southeast Asia:

$$\Box_2 \exists u.(\{\iota \exists \mathsf{isGenChair}.\{\mathsf{kr24}\}\} \sqcap \neg \Box_1 \{\iota \exists \mathsf{isGenChair}.\{\iota(\mathsf{KRConf} \sqcap \exists \mathsf{hasLoc}.\mathsf{SEAsiaLoc})\}\}).$$

The two main new results in Table 1 are the NEXPTIME upper bound for $\mathbf{K}^n$ and $\mathbf{S5}^n$ and decidability of $\mathbf{Kf}^{*n}$. The remaining results are by (sometimes non-trivial) reductions to known results.

Theorem 11. *For $L \in \{\mathbf{K}^n, \mathbf{S5}^n\}$ with $n \geq 1$, $L_{\mathcal{ALCO}_u^\iota}$ concept satisfiability is in* NEXPTIME *with both expanding and constant domains.*

We provide a sketch of the main ideas of the proof for $\mathbf{S5}^n$. First, observe (using an unfolding argument) that any satisfiable concept C is satisfied in world w_0 in a model based on a frame $\mathfrak{F} = (W, R_1, \dots, R_n)$ such that the domain W of $\mathfrak{F}$ is a prefix-closed set of words of the form

$$\vec{w} = w_0 i_0 w_1 \cdots i_{m-1} w_m$$

where $1 \leq i_j \leq n$, $i_j \neq i_{j+1}$, and each R_i is the smallest equivalence relation containing all pairs of the form $(\vec{w}, \vec{w}iw) \in W \times W$. One can assume that m is smaller than the modal depth of C. Next, one can substitute the first-order domain by quasistates (as introduced above) and work with *quasimodels* $\mathfrak{Q} = (\mathfrak{F}, \boldsymbol{q}, \mathfrak{R})$, in which $\boldsymbol{q}$ associates with any world a quasistate, and a set of *runs* $\mathfrak{R}$ represents first-order domain elements as functions mapping every world to a type in $\boldsymbol{q}(w)$. If $\boldsymbol{t} \in \boldsymbol{q}(w)$ contains a nominal, there is only one $r \in \mathfrak{R}$ with $r(w) = \boldsymbol{t}$. Now, one can apply selective filtration and some surgery to such a quasimodel to obtain a quasimodel of the above form with at most exponential outdegree (and so of at most exponential size), from which one can then reconstruct a model of exponential size.

Theorem 12. *$\mathbf{Kf}^{*n}_{\mathcal{ALCO}_u^\iota}$ concept satisfiability under global ontology is decidable with expanding domains, for $n \geq 1$.*

The proof is again based on appropriate quasimodels, which are now based on expanding domain models using finite frames $\mathfrak{F} = (W, R_1, \dots, R_n)$ such that $(R_1 \cup \cdots R_n)^*$ contains no cycles. Unfolding shows that we may assume that (W, R) is a forest. We show decidability by proving a recursive bound on the size of these models. We fix an ordering $\boldsymbol{t}_1, \dots, \boldsymbol{t}_k$ of the types and represent a quasistate as a vector $(x_1, \dots, x_k) \in (\mathbb{N} \cup \{\infty\})^k$, where x_i represents the number of nodes that satisfy type $\boldsymbol{t}_i$ in a world (equivalently, the number of runs through $\boldsymbol{t}_i$). Let $|\vec{x}| = x_1 + \cdots + x_k$ for $\vec{x} = (x_1, \dots, x_k)$. Observe that expanding domains correspond to the condition that wR_iv implies $|\boldsymbol{q}(w)| \leq |\boldsymbol{q}(v)|$. To obtain a recursive bound on the size of a finite model satisfying a concept we then apply Dickson's Lemma to the quasistates. Define the product ordering $\leq$ on $(\mathbb{N} \cup \{\infty\})^k$ by setting, for $\vec{x} = (x_1, \dots, x_k)$ and $\vec{y} = (y_1, \dots, y_k)$, $\vec{x} \leq \vec{y}$ if $x_i \leq y_i$ for $1 \leq i \leq k$. A pair $\vec{x}, \vec{y}$ with $\vec{x} \leq \vec{y}$ is called an *increasing pair*. Dickson's Lemma states every infinite sequence $\vec{x}_1, \vec{x}_2, \dots \in (\mathbb{N} \cup \{\infty\})^k$ contains an increasing pair $\vec{x}_{i_1}, \vec{x}_{i_2}$ with $i_1 < i_2$. In fact, assuming $|\vec{x}_i| \leq |\vec{x}_{i+1}|$ for all $i \geq 0$ and given recursive bounds on $|\vec{x}_1|$ and $|\vec{x}_{i+1}| - |\vec{x}_i|$ one can compute a recursive bound on the length of the longest sequence without any increasing pair (Figueira et al. 2011). Now, the proof of a recursive bound on the size of a satisfying model consists in manipulating the quasimodel so that the outdegree of the forest is recursively bounded and Dickson's Lemma becomes applicable. The expanding domain assumption is crucial for this.

We comment on the remaining results in Table 1. The NEXPTIME-hardness results already hold without nominals (Gabbay et al. 2003, Theorem 14.14) (the proof goes through also with expanding domain). The lower bounds for $\mathbf{Kf}^{*n}$ and $\mathbf{K}^{*n}$ follow from the following lemma and the corresponding lower bounds proved in the next section for temporal DLs (Table 2).

Lemma 13. *Concept satisfiability for $\mathbf{LTLf}_{\mathcal{ALCO}_u}$ and $\mathbf{LTL}_{\mathcal{ALCO}_u}$ are polytime-reducible to concept satisfiability for $\mathbf{Kf}^{*n}_{\mathcal{ALCO}_u}$ and $\mathbf{K}^{*n}_{\mathcal{ALCO}_u}$, respectively, with and without ontology and under constant and expanding domains.*

The proof of this reduction from logics of linear frames with transitive closure to logics of branching frames with transitive closure is not trivial but can be done by adapting the reduction given in the proof of Theorem 6.24 in (Gabbay et al. 2003) for product modal logics. Finally, the undecidability of concept satisfiability for $\mathbf{K}^n$ and $\mathbf{S5}^n$ under ontologies with constant domain can be proved in the same way as the undecidability of $\mathcal{ML}^n_{\text{Diff}}$ extended with the universal modality over arbitrary frames shown in (Hampson and Kurucz 2012) using the reduction of Theorem 10.

For many important modal logics the decidability status of modal DLs with non-rigid designators remains open. Most prominently, for the modal logics of (reflexive) transitive frames $\mathbf{K4}$ (and $\mathbf{S4}$, respectively), decidability of concept satisfiability with or without ontologies and with expanding or constant domains is open. As a "finitary" approximation and a first step to understand $\mathbf{K4}$ and $\mathbf{S4}$, we prove, as a specialisation of the proof of Theorem 12, decidability of concept satisfiability for the Gödel-Löb logic $\mathbf{GL}_{\mathcal{ALCO}_u^\iota}$ ($\mathbf{GL}$ is the logic of all transitive and Noetherian[1] frames (Boolos 1995)) and Grzegorczyk's logic $\mathbf{Grz}_{\mathcal{ALCO}_u^\iota}$ ($\mathbf{Grz}$ is the logic of all reflexive and transitive Noetherian frames (Grzegorczyk 1967)) in expanding domain models with and without ontologies. Alternatively, the decidability of concept satisfiability for $\mathbf{GL}_{\mathcal{ALCO}_u^\iota}$ and $\mathbf{Grz}_{\mathcal{ALCO}_u^\iota}$ can be proved by a double exponential time reduction (similar to Theorem 10) to satisfiability in expanding domain products of transitive Noetherian frames, which is known to be decidable (Gabelaia et al. 2006).

6 Reasoning in Temporal Free Description Logics

In this section, we consider the temporal setting. For the temporal DL language, we build $\mathcal{TL}_{\mathcal{ALCO}_u^\iota}$ *terms*, *concepts*, *concept inclusions* and *ontologies* similarly to the $\mathcal{ML}^n_{\mathcal{ALCO}_u^\iota}$ case, with $n = 2$: the language has two modalities — temporal operators 'sometime in the future', $\Diamond$, and

[1] (W, R) is *Noetherian* if there is no infinite chain $w_0 R w_1 R \cdots$ with $w_i \neq w_{i+1}$.

'at the next moment', $\bigcirc$. In particular, the $\mathcal{TL}_{\mathcal{ALCO}^{\iota}_{u}}$ *concepts* are defined by the grammar

$$C ::= A \mid \{\tau\} \mid \neg C \mid (C \sqcap C) \mid \exists r.C \mid \exists u.C \mid \Diamond C \mid \bigcirc C,$$

where τ is a $\mathcal{TL}_{\mathcal{ALCO}^{\iota}_{u}}$ *term*, defined as in Section 2.

A *flow of time* $\mathfrak{F}$ is a pair $(T, <)$, where T is either the set $\mathbb{N}$ of non-negative integers or a subset of $\mathbb{N}$ of the form $[0, n]$, for $n \in \mathbb{N}$, and $<$ is the strict linear order on T. Elements of T are called *instants* (rather than worlds). A flow of time $(T, <)$ naturally gives rise to a frame $(T, <, succ)$ for $\mathcal{TL}_{\mathcal{ALCO}^{\iota}_{u}}$, where *succ* is the successor relation: $succ = \{(t, t+1) \mid t, t+1 \in T\}$. So, we will often say that an interpretation $\mathfrak{M}$ is based on a flow $\mathfrak{F}$ if its frame is induced by $\mathfrak{F}$. If $\mathfrak{M}$ is based on $(\mathbb{N}, <)$, then we call it an $\mathbf{LTL}_{\mathcal{ALCO}^{\iota}_{u}}$ *interpretation* and often denote (with an abuse of notation) by $\mathfrak{M} = (\Delta, (\mathcal{I}_t)_{t\in\mathbb{N}})$; whereas, if it is based on $([0, n], <)$, for some $n \in \mathbb{N}$, it is called an $\mathbf{LTL}\boldsymbol{f}_{\mathcal{ALCO}^{\iota}_{u}}$ *interpretation* and often denoted by $\mathfrak{M} = (\Delta, (\mathcal{I}_t)_{t\in[0,n]})$.

Given $\mathfrak{M} = (\Delta, (\mathcal{I}_t)_{t\in T})$, the *value* of a $\mathcal{TL}_{\mathcal{ALCO}^{\iota}_{u}}$ term τ at $t \in T$ and the *extension* of a $\mathcal{TL}_{\mathcal{ALCO}^{\iota}_{u}}$ concept C at $t \in T$, are defined as in the modal case for $n = 2$: in particular, we have

$$(\bigcirc D)^{\mathcal{I}_t} = \begin{cases} D^{\mathcal{I}_{t+1}}, & \text{if } t+1 \in T, \\ \emptyset, & \text{otherwise,} \end{cases} \quad \text{and} \quad (\Diamond D)^{\mathcal{I}_t} = \bigcup_{t' \in T \text{ with } t<t'} D^{\mathcal{I}_{t'}}.$$

Note that $\Diamond$ is interpreted by $<$ and thus does not include the current instant.

We will consider restrictions of the base language $\mathcal{TL}_{\mathcal{ALCO}^{\iota}_{u}}$ along both the DL and temporal dimensions. First, $\mathcal{TL}_{\mathcal{ALCO}^{\iota}}$, $\mathcal{TL}_{\mathcal{ALCO}_{u}}$ and $\mathcal{TL}_{\mathcal{ALCO}}$ stand for the fragments of $\mathcal{TL}_{\mathcal{ALCO}^{\iota}_{u}}$ without the universal role, definite descriptions and both constructs, respectively. In addition to the basic free description logic $\mathcal{ALCO}^{\iota}_{u}$, we define temporal extensions of the light-weight free DL $\mathcal{ELO}^{\iota}_{u}$, which does not contain negation (and so disjunction). More precisely, the language $\mathcal{TL}_{\mathcal{ELO}^{\iota}_{u}}$ is obtained from $\mathcal{TL}_{\mathcal{ALCO}^{\iota}_{u}}$ by allowing only $\top$ (considered as a primitive logical symbol), concept names, term nominals, conjunctions and existential restrictions in the construction of concepts. Then, by removing the universal role or/and definite descriptions, we define $\mathcal{TL}_{\mathcal{ELO}^{\iota}}$, $\mathcal{TL}_{\mathcal{ELO}_{u}}$ and $\mathcal{TL}_{\mathcal{ELO}}$ in the obvious way.

In the temporal dimension, given a DL $\mathcal{DL}$, the fragments $\mathcal{TL}^{\Diamond}_{\mathcal{DL}}$ and $\mathcal{TL}^{\bigcirc}_{\mathcal{DL}}$ are obtained from $\mathcal{TL}_{\mathcal{DL}}$ by disallowing the $\bigcirc$ and the $\Diamond$ operators, respectively. Both the $\Diamond\Box$- and the $\bigcirc$-fragment corresponds to the unimodal language $\mathcal{ML}^{1}_{\mathcal{DL}}$, but with different accessibility relations.

In the following we will combine the syntactic restrictions (fragments) with the semantic restrictions on interpretations and refer, for example, to the satisfiability problem for $\mathcal{TL}^{\Diamond}_{\mathcal{ALCO}_{u}}$ concepts in $\mathbf{LTL}\boldsymbol{f}_{\mathcal{ALCO}^{\iota}_{u}}$ interpretations simply as $\mathbf{LTL}\boldsymbol{f}^{\Diamond}_{\mathcal{ALCO}_{u}}$ concept satisfiability.

As an example, in $\mathbf{LTL}\boldsymbol{f}_{\mathcal{ALCO}^{\iota}_{u}}$, we express that whoever is a Program Chair of KR will not be Program Chair of KR again, but is always appointed as either the General Chair or a PC member of next year's KR, by means of the CI:

$$\exists \mathsf{isProgChair}.\{\mathsf{kr}\} \sqsubseteq \neg\Diamond\exists \mathsf{isProgChair}.\{\mathsf{kr}\} \sqcap (\{\iota \exists \mathsf{isGenChair}.\bigcirc\{\mathsf{kr}\}\} \sqcup \exists \mathsf{isPCMember}.\bigcirc\{\mathsf{kr}\}).$$

We begin the study of the satisfiability problems for temporal DLs based on $\mathcal{ALCO}$ by showing that concept satisfiability in constant domains is undecidable or even Σ^1_1-complete over the infinite flow of time $(\mathbb{N}, <)$. This is very different from the classical case with RDA, which was shown to be decidable in the absence of definite descriptions (Gabbay et al. 2003, Theorem 14.12).

Theorem 14. *With constant domains, concept satisfiability is Σ^1_1-complete for $\mathbf{LTL}^{\Diamond}_{\mathcal{ALCO}_{u}}$ and $\mathbf{LTL}_{\mathcal{ALCO}_{u}}$, and undecidable for $\mathbf{LTL}\boldsymbol{f}^{\Diamond}_{\mathcal{ALCO}_{u}}$; also, concept satisfiability under global ontology is Σ^1_1-complete for $\mathbf{LTL}^{\Diamond}_{\mathcal{ALCO}}$ and $\mathbf{LTL}_{\mathcal{ALCO}}$, and undecidable for $\mathbf{LTL}\boldsymbol{f}^{\Diamond}_{\mathcal{ALCO}}$.*

The lower bounds follow, using Theorem 10 (2), from the respective undecidability results for the first-order one-variable temporal logic with counting (Hampson and Kurucz 2015). They could also be proven more directly by encoding the Σ^1_1-complete recurrence and undecidable reachability problems for the Minsky counter machines (Minsky 1961; Alur and Henzinger 1994). Intuitively, the value of each counter of the Minsky machine can be represented as the cardinality of a certain concept. Then, non-rigid nominals (or indeed the counting quantifier) can be used to ensure that the value of the counter is incremented/decremented (depending the command) by the transition: for instance, CI of the form

$$Q_i \sqsubseteq \bigcirc R_k \Leftrightarrow (R_k \sqcup \{a_k\})$$

could be used to say that from state Q_i, the value of counter k is increased by one. Note, however, that this CI uses the $\bigcirc$ operator. Without it, the proof is considerably more elaborate and represents each counter as a pair of concepts: R_k is used to increment the counter, while S_k to decrement it, so that the counter value is the cardinality of $R_k \sqcap \neg S_k$. Both concepts are made 'monotone': $R_k \sqsubseteq \Box R_k$ and $S_k \sqsubseteq \Box S_k$, and for each transition of the Minsky machine, the non-rigid nominals pick an element that, for example, has never been in R_k before but will remain in R_k from the next instant on: $\neg R_k \sqcap \Box R_k$. A sequence of these elements allows us to linearly order the domain and construct a 'diagonal' in the two-dimensional interpretation necessary for the encoding of the computation using only the $\Diamond$ operator.

Reasoning in expanding domains turns out to be less complex, and we obtain the following:

Theorem 15. *(1) With expanding domains, concept satisfiability is undecidable for $\mathbf{LTL}^{\Diamond}_{\mathcal{ALCO}_{u}}$, and concept satisfiability under global ontology is undecidable for $\mathbf{LTL}^{\Diamond}_{\mathcal{ALCO}}$.*

(2) With expanding domains, concept satisfiability (under global ontology) is decidable for $\mathbf{LTL}\boldsymbol{f}_{\mathcal{ALCO}^{\iota}_{u}}$. However, both problems are Ackermann-hard for $\mathbf{LTL}\boldsymbol{f}^{\Diamond}_{\mathcal{ALCO}_{u}}$; moreover, concept satisfiability under global ontology is Ackermann-hard for $\mathbf{LTL}\boldsymbol{f}^{\Diamond}_{\mathcal{ALCO}}$.

Undecidability and Ackermann-hardness are proven similarly to Theorem 14. In this case, however, the master problems are, respectively, the ω-reachability and reachability problems for lossy Minsky machines (Konev, Wolter, and Zakharyaschev 2005; Schnoebelen 2010), which in addition to normal transitions can also arbitrarily decrease the counter values. Such computations can be naturally encoded *backwards* in interpretations with expanding domains: the

temporal logic	concept satisfiability		concept sat. under global ontology	
	const. domain	exp. domains	const. domain	exp. domains
$\mathbf{LTL}^{\Diamond}_{\mathcal{ALCO}_u}$ and $\mathbf{LTL}_{\mathcal{ALCO}_u}$	Σ^1_1-complete	undecidable	Σ^1_1-complete	undecidable
$\mathbf{LTL}\boldsymbol{f}^{\Diamond}_{\mathcal{ALCO}_u}$ and $\mathbf{LTL}\boldsymbol{f}_{\mathcal{ALCO}_u}$	undecidable	decidable, Ackermann-hard	undecidable	decidable, Ackermann-hard
$\mathbf{LTL}^{\Diamond}_{\mathcal{ALCO}}$ and $\mathbf{LTL}_{\mathcal{ALCO}}$	?	?	Σ^1_1-complete	undecidable
$\mathbf{LTL}\boldsymbol{f}^{\Diamond}_{\mathcal{ALCO}}$ and $\mathbf{LTL}\boldsymbol{f}_{\mathcal{ALCO}}$	?	?	undecidable	decidable, Ackermann-hard
$\mathbf{LTL}^{\bigcirc}_{\mathcal{ALCO}_u}$ / $\mathbf{LTL}^{\bigcirc}_{\mathcal{ALCO}}$	EXP-complete / in EXP	EXP-complete / in EXP	undecidable	?
$\mathbf{LTL}\boldsymbol{f}^{\bigcirc}_{\mathcal{ALCO}_u}$ / $\mathbf{LTL}\boldsymbol{f}^{\bigcirc}_{\mathcal{ALCO}}$	EXP-complete / in EXP	EXP-complete / in EXP	undecidable	decidable

Table 2: Concept satisfiability (under global ontology) for temporal DLs

arbitrary decreases of counter values correspond to the extension of the interpretation domain with fresh elements.

The positive decidability results over the finite flows of time follows from Theorem 12 by Lemma 13 (together with the reduction in Proposition 8).

Temporal Free DLs Based on $\mathcal{ELO}$. Next, we transfer the above results to the $\mathcal{ELO}$ family. As $\mathcal{TL}_{\mathcal{ELO}}$ concepts do not contain negation and the empty concept ($\bot$), they are trivially satisfiable. Thus, our main reasoning problem is based on the notion of entailment (rather than satisfiability).

CI Entailment (over Finite Flows): Given a $\mathcal{TL}_{\mathcal{DL}}$-CI $C_1 \sqsubseteq C_2$ and a $\mathcal{TL}_{\mathcal{DL}}$-ontology $\mathcal{O}$, is it the case that $C_1^{\mathcal{I}_t} \subseteq C_2^{\mathcal{I}_t}$, for every $t \in T$ in every interpretation $\mathfrak{M}$ satisfying $\mathcal{O}$ and based on $(\mathbb{N}, <)$ (every finite flow, respectively)?

It turns out that disjunction can be modelled in the temporal extension of $\mathcal{ELO}$ with the help of the $\Diamond$ modality (Artale et al. 2007): intuitively, any CI of the form $\top \sqsubseteq B_1 \sqcup B_2$ is replaced with $\top \sqsubseteq \exists q.(\Diamond X_1 \sqcap \Diamond X_2)$, which says that X_1 and X_2 occur in some order in the future (possibly on another domain element). It then remains to check the order of X_1 and X_2 and, if, say, X_1 precedes X_2, then B_1 is chosen, otherwise B_2 is chosen. So, this reduction shows that the entailment problem for the fragments of $\mathcal{TL}^{\Diamond}_{\mathcal{ELO}}$ essentially has the same complexity as the complement of the satisfiability problem for the corresponding $\mathcal{TL}^{\Diamond}_{\mathcal{ALCO}}$ fragment:

Theorem 16. *(1) With constant domains, $\mathbf{LTL}^{\Diamond}_{\mathcal{ELO}}$ CI entailment is Π^1_1-complete and $\mathbf{LTL}\boldsymbol{f}^{\Diamond}_{\mathcal{ELO}}$ CI entailment is undecidable.*

(2) With expanding domains, $\mathbf{LTL}^{\Diamond}_{\mathcal{ELO}}$ CI entailment is undecidable.

(3) With expanding domains, $\mathbf{LTL}\boldsymbol{f}^{\Diamond}_{\mathcal{ELO}}$ CI entailment is decidable but Ackermann-hard.

Next-Only Temporal Free DLs. As we have seen above, the $\Diamond$-only fragments normally exhibit the same bad computational behaviour as the full logics with both $\Diamond$ and $\bigcirc$. We now provide some results for the fragments that contain only $\bigcirc$. We begin with some positive results for the satisfiability problem (without global ontology):

Theorem 17. *With constant and with expanding domains, concept satisfiability is* EXPTIME*-complete for $\mathbf{LTL}^{\bigcirc}_{\mathcal{ALCO}_u}$ and $\mathbf{LTL}\boldsymbol{f}^{\bigcirc}_{\mathcal{ALCO}_u}$ and in* EXPTIME *for $\mathbf{LTL}^{\bigcirc}_{\mathcal{ALCO}}$ and $\mathbf{LTL}\boldsymbol{f}^{\bigcirc}_{\mathcal{ALCO}}$.*

The EXPTIME upper complexity bound can be shown by a type elimination procedure, similarly to the case of the product $\mathbf{Alt} \times \mathbf{K}_n$ of modal logics $\mathbf{Alt}$, whose accessibility relation is a partial function, and multi-modal $\mathbf{K}_n$, which is a notational variant of $\mathcal{ALC}$; see (Gabbay et al. 2003, Theorem 6.6). One has to, in addition, take care of nominals and the universal role, but that can be done in exponential time. The matching lower bound is inherited from $\mathcal{ALCO}_u$, but for the fragment without the universal role the exact complexity remains an open problem.

Our final result indicates that with the global ontology, the $\bigcirc$-fragments behaves nearly as badly as the full language:

Theorem 18. *With constant domains, concept satisfiability under global ontology is undecidable for $\mathbf{LTL}^{\bigcirc}_{\mathcal{ALCO}}$ and $\mathbf{LTL}\boldsymbol{f}^{\bigcirc}_{\mathcal{ALCO}}$.*

The proof is by a direct reduction of the reachability problem for Minsky machines, similarly to the simplified sketch for Theorem 14; note the proof makes use of the spy-point universal role elimination in Lemma 6.

7 Discussion and Future Work

We have introduced and investigated novel fragments of first-order modal logic with non-rigid (and possibly non-referring) individual names and definite descriptions. Potential applications that remain to be explored include business process management, where formalisms for representing the dynamic behaviour of data and information are crucial (Delgrande et al. 2023; Deutsch et al. 2018), and context, knowledge or standpoint-dependent reasoning for which possible worlds semantics is needed (Ghidini and Serafini 2017; Gómez Álvarez, Rudolph, and Strass 2023).

Besides the open decidability problems discussed above, future research directions include the extension of our results to more expressive monodic fragments (Gabbay et al. 2003; Hodkinson, Wolter, and Zakharyaschev 2002), automated support for the construction of definite descriptions and referring expressions (Artale et al. 2021; Kurucz, Wolter, and Zakharyaschev 2023), the design of

'practical' reasoning algorithms for the languages considered here, and the extension of our results to modal DLs with hybrid (Braüner 2014; Indrzejczak and Zawidzki 2023), branching-time (Hodkinson, Wolter, and Zakharyaschev 2002; Gutiérrez-Basulto, Jung, and Lutz 2012), dynamic (Harel 1979), or non-normal operators (Dalmonte et al. 2023).

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A Details on Introduction and Preliminaries

We define the *reflexive diamond* operator as $\Diamond_i^+ C = C \sqcup \Diamond_i C$, and the *reflexive box* operator as $\Box_i^+ C = \neg\Diamond_i^+\neg C$.

The set of *subconcepts* of a concept C, denoted by $\mathsf{sub}(C)$, is defined inductively as follows:

$$\begin{aligned}
\mathsf{sub}(A) &= \{A\},\\
\mathsf{sub}(\{a\}) &= \{\{a\}\},\\
\mathsf{sub}(\{\iota C\}) &= \{\{\iota C\}\} \cup \mathsf{sub}(C),\\
\mathsf{sub}(\neg C) &= \{\neg C\} \cup \mathsf{sub}(C),\\
\mathsf{sub}(C \sqcap D) &= \{C \sqcap D\} \cup \mathsf{sub}(C) \cup \mathsf{sub}(D),\\
\mathsf{sub}(\exists s.C) &= \{\exists s.C\} \cup \mathsf{sub}(C), \text{ with } s \in \mathsf{N_R} \cup \{u\},\\
\mathsf{sub}(\Diamond_i C) &= \{\Diamond_i C\} \cup \mathsf{sub}(C).
\end{aligned}$$

The *modal depth* of terms and concepts is defined by mutual induction:

$$\begin{aligned}
md(a) &= 0,\\
md(\iota C) &= md(C),\\
md(A) &= 0,\\
md(\{\tau\}) &= md(\tau),\\
md(\neg C) &= md(C),\\
md(C \sqcap D) &= \max\{md(C), md(D)\},\\
md(\exists s.C) &= md(C), \text{ with } s \in \mathsf{N_R} \cup \{u\},\\
md(\Diamond_i C) &= md(C) + 1.
\end{aligned}$$

The *modal depth* of a CI or an ontology is the maximum modal depth of concepts that occur in them.

Standard Translation to First-Order Modal Logic The following definitions are adjusted from (Fitting and Mendelsohn 2012); see, e.g., Definitions 11.1.1-4. The alphabet of the *quantified modal language*, $\mathcal{QML}^{\iota}_{\lambda}$, consists of: countably infinite and pairwise disjoint sets of *predicates* $\mathsf{N_P}$ (of fixed arities ≥ 0, with a distinguished *equality* binary predicate, $=$), *individual names* $\mathsf{N_I}$ and *variables* Var; the *Boolean operators* $\neg, \wedge$; the *existential quantifier* $\exists$; the *predicate abstraction operator* λ; the *definite description operator* ι; and the *modal operators* $\Diamond_i$ (*diamond*), for each *modality* $i \in I$. *Terms* τ and *formulas* φ of $\mathcal{QML}^{\iota}_{\lambda}$ are defined by mutual induction:

$$\begin{aligned}
&\tau ::= x \mid a \mid \iota x.\varphi,\\
&\varphi ::= P(x_1, \ldots, x_n) \mid x = y \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \exists x\, \varphi \mid \Diamond_i\varphi\\
&\qquad\qquad \mid \langle\lambda x.\varphi\rangle(\tau),
\end{aligned}$$

where $a \in \mathsf{N_I}$, $P \in \mathsf{N_P}$ (n-ary), and $x, x_1, \ldots, x_n \in \mathsf{Var}$. Standard abbreviations are assumed, and *free variables* are defined as in (Fitting and Mendelsohn 2012).

A *partial interpretation with expanding domains* is a structure $\mathfrak{M} = (\mathfrak{F}, \Delta, \mathcal{I})$, where $\mathfrak{F} = (W, \{R_i\}_{i\in I})$ is a *frame*, with W being a non-empty set of *worlds* and $R_i \subseteq W \times W$ being an *accessibility relation* on W, for each modality $i \in I$; Δ is a function associating with every $w \in W$ a non-empty set, Δ^w, called the *domain of w in $\mathfrak{M}$*, such that $\Delta^w \subseteq \Delta^v$, whenever wR_iv, for some $i \in I$; $\mathcal{I}$ is a function associating with each $w \in W$ a *partial* first-order interpretation $\mathcal{I}_w$ with domain Δ so that $P^{\mathcal{I}_w} \subseteq \Delta^n$, for each predicate $P \in \mathsf{N_P}$ of arity n, and $a^{\mathcal{I}_w} \in \Delta$, for *some* subset of constants $a \in \mathsf{N_I}$, with no additional requirement (in particular, there is no assumption that all constants are rigid designators in $\mathfrak{M}$). An *assignment in* $\mathfrak{M}$ is a function σ from Var to Δ. An *x-variant* of an assignment σ is an assignment that can differ from σ only on x. The definitions of *value* $\tau^{\mathfrak{M},w}_{\sigma}$ of term τ under assignment σ at world w of $\mathfrak{M}$, and *satisfaction* $\mathfrak{M}, w \models^{\sigma} \varphi$ of formula φ under assignment σ at world w in $\mathfrak{M}$ are defined by mutual induction. First, we have

$$\tau^{\mathfrak{M},w}_{\sigma} = \begin{cases} \sigma(x), & \text{if } \tau \text{ is } x \in \mathsf{Var};\\ a^{\mathcal{I}_w}, & \text{if } \tau \text{ is } a \in \mathsf{N_I} \text{ and } a^{\mathcal{I}_w} \text{ is defined};\\ \sigma'(x), & \text{if } \tau \text{ is } \iota x.\varphi \text{ and } \mathfrak{M}, w \models^{\sigma'} \varphi, \text{ for}\\ & \text{\textit{exactly one} } x\text{-variant } \sigma' \text{ of } \sigma,\\ \text{undefined}, & \text{otherwise.} \end{cases}$$

If $\tau^{\mathfrak{M},w}_{\sigma}$ is defined, then we say that *τ designates under σ at w of $\mathfrak{M}$*. Next,

$$\begin{aligned}
&\mathfrak{M}, w \models^{\sigma} P(x_1, \ldots, x_n) && \text{iff } (\sigma(x_1), \ldots, \sigma(x_n)) \in P^{\mathcal{I}_w};\\
&\mathfrak{M}, w \models^{\sigma} x = y && \text{iff } \sigma(x) = \sigma(y);\\
&\mathfrak{M}, w \models^{\sigma} \neg\varphi && \text{iff } \mathfrak{M}, w \not\models^{\sigma} \varphi;\\
&\mathfrak{M}, w \models^{\sigma} \varphi \wedge \psi && \text{iff } \mathfrak{M}, w \models^{\sigma} \varphi \text{ and } \mathfrak{M}, w \models^{\sigma} \psi;\\
&\mathfrak{M}, w \models^{\sigma} \exists x\, \varphi && \text{iff } \mathfrak{M}, w \models^{\sigma'} \varphi, \text{ for some } x\text{-variant } \sigma' \text{ of } \sigma;\\
&\mathfrak{M}, w \models^{\sigma} \Diamond_i\varphi && \text{iff } \mathfrak{M}, v \models^{\sigma} \varphi, \text{ for some } v \in W \text{ such that } wR_iv;\\
&\mathfrak{M}, w \models^{\sigma} \langle\lambda x.\varphi\rangle(\tau) && \text{iff } \tau \text{ designates under } \sigma \text{ at } w \text{ of } \mathfrak{M} \text{ and } \mathfrak{M}, w \models^{\sigma'} \varphi, \text{ where } \sigma' \text{ is the } x\text{-variant of } \sigma \text{ with } \sigma'(x) = \tau^{\mathfrak{M},w}_{\sigma}.
\end{aligned}$$

We now introduce the *standard translation* of an $\mathcal{ML}^n_{\mathcal{ALCO}^{\iota}_u}$ concept C into a $\mathcal{QML}^{\iota}_{\lambda}$ formula $\pi_x(C)$ with at most one free variable x, defined as follows:

$$\begin{aligned}
\pi_x(A) &= A(x),\\
\pi_x(\{a\}) &= \langle\lambda y.x = y\rangle(a),\\
\pi_x(\{\iota C\}) &= \langle\lambda y.x = y\rangle(\iota z.\pi_z(C)),\\
\pi_x(\neg C) &= \neg\pi_x(C),\\
\pi_x(C \sqcap D) &= (\pi_x(C) \wedge \pi_x(D)),\\
\pi_x(\exists r.C) &= \exists y\, (r(x, y) \wedge \pi_y(C)),\\
\pi_x(\exists u.C) &= \exists y\, \pi_y(C),\\
\pi_x(\Diamond_i C) &= \Diamond_i\pi_x(C).
\end{aligned}$$

The following proposition shows that $\mathcal{ML}^n_{\mathcal{ALCO}^{\iota}_u}$ concepts can indeed be seen as a fragment of $\mathcal{QML}^{\iota}_{\lambda}$, via the standard translation above.

Proposition 19. *For every $\mathcal{ML}^n_{\mathcal{ALCO}^{\iota}_u}$ concept C, partial interpretation $\mathfrak{M} = (\mathfrak{F}, \Delta, \mathcal{I})$, world w of $\mathfrak{M}$, and $d \in \Delta^w$, we have $d \in C^{\mathcal{I}_w}$ iff $\mathfrak{M}, w \models^{\sigma} \pi_x(C)$, where σ is an assignment in $\mathfrak{M}$ such that $\sigma(x) = d$.*

Proof. By induction on the structure of C.

$C = A$. We have $d \in A^{\mathcal{I}_w}$ iff $\sigma(x) \in A^{\mathcal{I}_w}$, where σ is an assignment such that $\sigma(x) = d$. That is, $\mathfrak{M}, w \models^{\sigma} A(x)$.

$C = \{a\}$. We have $d \in \{a\}^{\mathcal{I}_w}$ iff a designates at w and $d = a^{\mathcal{I}_w}$, meaning that a designates under σ at w and $\sigma(x) = a^{\mathcal{I}_w}$, where σ is an assignment with $\sigma(x) = d$. Equivalently, a designates under σ at w and $\sigma'(x) = \sigma'(y)$, where σ' is a y-variant of σ such that $\sigma'(y) = a^{\mathcal{I}_w}$. The previous step means that a designates under σ at w and $\mathfrak{M}, w \models^{\sigma'} x = y$, where σ' is a y-variant of σ such that $\sigma'(y) = a^{\mathcal{I}_w}$. That is, $\mathfrak{M}, w \models^{\sigma} \langle \lambda y.x = y \rangle(a)$.

$C = \{\iota C\}$. We have $d \in \{\iota C\}^{\mathcal{I}_w}$ iff ιC designates at w and $d = (\iota C)^{\mathcal{I}_w}$, i.e., $C^{\mathcal{I}_w} = \{e\}$, for some $e \in \Delta^w$, and $d = e$. This means that $C^{\mathcal{I}_w} = \{e\}$, for some $e \in \Delta^w$, and $\sigma(x) = e$, where σ is an assignment in $\mathfrak{M}$ such that $\sigma(x) = d$. By induction hypothesis, we have equivalently that $\mathfrak{M}, w \models^{\sigma'} \pi_z(C)$, where σ' is the (unique) z-variant of σ such that $\sigma'(z) = e$, and $\sigma(x) = e$. In other words, $\iota z.\pi_z(C)$ designates under σ at w of $\mathfrak{M}$, and $\sigma(x) = \iota z.\pi_z(C)^{\mathfrak{M},w}_{\sigma}$. The previous step can then be rewritten as: $\iota z.\pi_z(C)$ designates under σ at w of $\mathfrak{M}$, and $\sigma(x) = \sigma''(y)$, where σ'' is a y-variant of σ such that $\sigma''(y) = \iota z.\pi_z(C)^{\mathfrak{M},w}_{\sigma}$. Given that $\sigma(x) = \sigma''(x)$ (as y-variants, their values coincide on x), we have equivalently that $\iota z.\pi_z(C)$ designates under σ at w of $\mathfrak{M}$ and $\sigma''(x) = \sigma''(y)$, i.e., $\mathfrak{M}, w \models^{\sigma''} x = y$. Since σ'' is a y-variant of σ such that $\sigma''(y) = \iota z.\pi_z(C)^{\mathfrak{M},w}_{\sigma}$, the last step is equivalent by definition to $\mathfrak{M}, w \models^{\sigma} \langle \lambda y.x = y \rangle(\iota z.\pi_z(C))$.

$C = \neg D$. We have $d \in (\neg D)^{\mathcal{I}_w}$ iff $d \notin D^{\mathcal{I}_w}$. By induction hypothesis, we have equivalently $\mathfrak{M}, w \not\models^{\sigma} \pi_x(D)$, where σ is an assignment in $\mathfrak{M}$ such that $\sigma(x) = d$. That is, $\mathfrak{M}, w \models^{\sigma} \neg\pi_x(D)$.

$C = D \sqcap E$. We have $d \in (D \sqcap E)^{\mathcal{I}_w}$ iff $d \in D^{\mathcal{I}_w}$ and $d \in E^{\mathcal{I}_w}$. By induction hypothesis, we have equivalently $\mathfrak{M}, w \models^{\sigma'} \pi_x(D)$ and $\mathfrak{M}, w \models^{\sigma''} \pi_x(E)$, where σ', σ'' are assignments in $\mathfrak{M}$ such that $\sigma'(x) = \sigma''(x) = d$. Hence, since the truth of $\pi_x(D)$ and $\pi_x(E)$ depends only on the values assigned to x, we have equivalently that $\mathfrak{M}, w \models^{\sigma} \pi_x(D)$ and $\mathfrak{M}, w \models^{\sigma} \pi_x(E)$, i.e., $\mathfrak{M}, w \models^{\sigma} \pi_x(D) \wedge \pi_x(E)$, for an assignment σ in $\mathfrak{M}$ such that $\sigma(x) = d$.

$C = \exists r.D$. We have $d \in (\exists r.D)^{\mathcal{I}_w}$ iff there exists $e \in D^{\mathcal{I}_w}$ such that $(d, e) \in r^{\mathcal{I}_w}$ By induction hypothesis, this means that $(d, \sigma''(y)) \in r^{\mathcal{I}_w}$ and $\mathfrak{M}, w \models^{\sigma''} \pi_y(D)$, for some assignment σ'' in $\mathfrak{M}$ such that $\sigma''(y) = e$. Equivalently, we have $(\sigma'(x), \sigma'(y)) \in r^{\mathcal{I}_w}$ and $\mathfrak{M}, w \models^{\sigma'} \pi_y(D)$, for some assignment σ' in $\mathfrak{M}$ such that $\sigma'(x) = d$ and $\sigma'(y) = e$, i.e., $\mathfrak{M}, w \models^{\sigma'} r(x, y) \wedge \pi_y(D)$. By considering the assignment σ defined as σ', except possibly on y, we have equivalently that σ' is a y-variant of σ such that $\mathfrak{M}, w \models^{\sigma'} r(x, y) \wedge \pi_y(D)$. Hence, the previous step means that $\mathfrak{M}, w \models^{\sigma} \exists y(r(x, y) \wedge \pi_y(D))$, where σ is an assignment in $\mathfrak{M}$ such that $\sigma(x) = d$.

$C = \exists u.D$. We have $d \in (\exists u.D)^{\mathcal{I}_w}$ iff there exists $e \in D^{\mathcal{I}_w}$. By induction hypothesis, this is equivalent to $\mathfrak{M}, w \models^{\sigma'} \pi_y(D)$, for some assignment σ' in $\mathfrak{M}$ such that $\sigma'(y) = e$. By considering the assignment σ defined as σ', except possibly on y, we have equivalently that σ' is a y-variant of σ such that $\mathfrak{M}, w \models^{\sigma'} \pi_y(D)$, that is, $\mathfrak{M}, w \models^{\sigma} \exists y \pi_y(D)$.

$C = \Diamond_i D$. We have $d \in (\Diamond_i D)^{\mathcal{I}_w}$ iff there exists $v \in W$ such that wR_iv and $d \in D^{\mathcal{I}_v}$. By induction hypothesis, the previous step means that there exists $v \in W$ such that wR_iv and $\mathfrak{M}, v \models^{\sigma} \pi_x(D)$, with σ assignment in $\mathfrak{M}$ such that $\sigma(x) = d$. Equivalently, by definition, $\mathfrak{M}, w \models^{\sigma} \Diamond_i \pi_x(D)$. □

Extended Examples The following examples extend the formalisations of the scenarios discussed in the Introduction.

Example 1. *Of* Pierre, Agent 1 *knows* that he is the General Chair of KR 2024:

$$\exists u.(\{\mathsf{pierre}\} \sqcap \Box_1\{\iota\exists \mathsf{isGenChair}.\{\mathsf{kr24}\}\}),$$

Agent 1 does not know that the General Chair of KR 2024 is the General Chair of the KR Conference held in Southeast Asia:

$$\neg\Box_1\exists u.(\{\iota\exists \mathsf{isGenChair}.\{\mathsf{kr24}\}\} \sqcap \{\iota\exists \mathsf{isGenChair}.\{\iota(\mathsf{KRConf} \sqcap \exists \mathsf{hasLoc}.\mathsf{SEAsiaLoc})\}\}),$$

Therefore, Agent 1 does not know of Pierre that he is the General Chair of the KR Conference held in Southeast Asia:

$$\exists u.(\{\mathsf{pierre}\} \sqcap \neg\Box_1\{\iota\exists \mathsf{isGenChair}.\{\iota(\mathsf{KRConf} \sqcap \exists \mathsf{hasLoc}.\mathsf{SEAsiaLoc})\}\})$$

Example 2. Agent 2 knows *that* Agent 1 knows *of* the General Chair of KR 2024 that they are busy:

$$\Box_2\exists u.(\{\iota\exists \mathsf{isGenChair}.\{\mathsf{kr24}\}\} \sqcap \Box_1\mathsf{Busy}),$$

abbreviated as $\Box_2[\Box_1\mathsf{Busy}(\iota\exists \mathsf{isGenChair}.\{\mathsf{kr24}\})]$.

However, Agent 2 also knows that Agent 1 does *not* know *that* the General Chair of the KR Conference held in Southeast Asia is busy:

$$\Box_2\neg\Box_1\exists u.(\{\iota\exists \mathsf{isGenChair}.\{\iota(\mathsf{KRConf} \sqcap \exists \mathsf{hasLoc}.\mathsf{SEAsiaLoc})\}\} \sqcap \mathsf{Busy}),$$

abbreviated as

$$\Box_2\neg\Box_1[\mathsf{Busy}(\{\iota\exists \mathsf{isGenChair}.\{\iota(\mathsf{KRConf} \sqcap \exists \mathsf{hasLoc}.\mathsf{SEAsiaLoc})\}\})].$$

Hence, Agent 2 knows that Agent 1 does not know of the General Chair of KR 2024 that they are the General Chair of the KR Conference held in Southeast Asia:

$$\Box_2\exists u.(\{\iota\exists \mathsf{isGenChair}.\{\mathsf{kr24}\}\} \sqcap \neg\Box_1\{\iota\exists \mathsf{isGenChair}.\{\iota(\mathsf{KRConf} \sqcap \exists \mathsf{hasLoc}.\mathsf{SEAsiaLoc})\}\}),$$

abbreviated as

$$\Box_2[\neg\Box_1\{\iota\exists \mathsf{isGenChair}.\{\iota(\mathsf{KRConf} \sqcap \exists \mathsf{hasLoc}.\mathsf{SEAsiaLoc})\}\}(\{\iota\exists \mathsf{isGenChair}.\{\mathsf{kr24}\}\})].$$

Example 3. KR24 is a rigid designator:

$$\{\mathsf{kr24}\} \sqsubseteq \Box\{\mathsf{kr24}\}, \quad \Diamond\{\mathsf{kr24}\} \sqsubseteq \{\mathsf{kr24}\}.$$

KR24 is the current KR Conference, but from next year there will be more:

$$\exists u.(\{\mathsf{kr24}\} \sqcap \{\mathsf{kr}\}),$$
$$\top \sqsubseteq \Diamond^{+} \exists u.\{\mathsf{kr}\}$$
$$\{\mathsf{kr}\} \sqsubseteq \neg \bigcirc \{\mathsf{kr}\}$$

Whoever is a Program Chair of KR will not be Program Chair again, but always becomes either the General Chair or a PC member of next year's KR:

$$\exists \mathsf{isProgChair}.\{\mathsf{kr}\} \sqsubseteq \neg \Diamond \exists \mathsf{isProgChair}.\{\mathsf{kr}\} \sqcap (\{\iota \exists \mathsf{isGenChair}.\bigcirc\{\mathsf{kr}\}\} \sqcup \exists \mathsf{isPCMember}.\bigcirc\{\mathsf{kr}\}).$$

B Proofs for Section 3

Given an $\mathcal{ML}^n_{\mathcal{ALCO}^{\iota}_u}$ concept C and its subconcept B, we define the set of *B-relevant paths in C* by induction on the structure of C as follows (D is a subconcept of C possibly containing B as its own subconcept):

$$\mathsf{rp}(D, B) = \begin{cases} \{\epsilon\}, & \text{if } D = B, \\ \emptyset, & \text{otherwise,} \end{cases}$$
$$\mathsf{rp}(\{\iota D\}, B) = \mathsf{rp}(D, B),$$
$$\mathsf{rp}(\neg D, B) = \mathsf{rp}(D, B),$$
$$\mathsf{rp}(D_1 \sqcap D_2, B) = \mathsf{rp}(D_1, B) \cup \mathsf{rp}(D_2, B),$$
$$\mathsf{rp}(\exists s.D, B) = \mathsf{rp}(D, B), \text{ for } s \in \mathsf{N_R} \cup \{u\},$$
$$\mathsf{rp}(\Diamond_i D, B) = \{i \cdot \pi \mid \pi \in \mathsf{rp}(D, B)\}.$$

It can be seen that we have the following inclusions, which will be helpful in proofs by induction on the structure of concepts:

$$\mathsf{rp}(C, \{\iota B\}) \subseteq \mathsf{rp}(C, B),$$
$$\mathsf{rp}(C, \neg B) \subseteq \mathsf{rp}(C, B),$$
$$\mathsf{rp}(C, B_1 \sqcap B_2) \subseteq \mathsf{rp}(C, B_1) \cap \mathsf{rp}(C, B_2),$$
$$\mathsf{rp}(C, \exists s.B) \subseteq \mathsf{rp}(C, B), \text{ for } s \in \mathsf{N_R} \cup \{u\},$$
$$\mathsf{rp}(C, \Diamond_i B) \subseteq \{\pi \mid \pi \cdot i \in \mathsf{rp}(C, B)\}.$$

Note that the sets on the left are included in the sets on the right because, for example, concept C may contain fewer occurrences of $\neg B$ than of B, and so, not every B-relevant path in C is a $\neg B$-relevant path in C. The set of B-relevant paths in C induces the set of worlds that are reachable via these sequences of $\Diamond_{i_j}$ operators: given a world $w \in W$, we denote by $\mathsf{rw}(w, C, B)$ the set of *B-relevant worlds for C and w*, consisting of worlds $v \in W$ such that $w_0 R_{i_1} w_1 R_{i_2} \cdots R_{i_n} w_n$ for $(i_1, i_2, \ldots, i_n) \in \mathsf{rp}(C, B)$ and $w_0 = w$ and $w_n = v$. Note that $\mathsf{rw}(w, C, C) = \{w\}$. Moreover, the inclusions between $\mathsf{rp}(C, B)$ given above naturally translate into the following for $\mathsf{rw}(w, C, B)$:

$$\mathsf{rw}(w, C, \{\iota B\}) \subseteq \mathsf{rw}(w, C, B),$$
$$\mathsf{rw}(w, C, \neg B) \subseteq \mathsf{rw}(w, C, B),$$
$$\mathsf{rw}(w, C, B_1 \sqcap B_2) \subseteq \mathsf{rw}(w, C, B_1) \cap \mathsf{rw}(w, C, B_2),$$
$$\mathsf{rw}(w, C, \exists s.B) \subseteq \mathsf{rw}(w, C, B), \text{ for } s \in \mathsf{N_R} \cup \{u\},$$
$$\mathsf{rw}(w, C, \Diamond_i B) \subseteq \{v \mid v R_i u \text{ and } u \in \mathsf{rw}(w, C, B)\}.$$

Proposition 1. *In $\mathcal{ML}^n_{\mathcal{ALCO}_u}$ and $\mathcal{ML}^n_{\mathcal{ALCO}^{\iota}_u}$, concept $\mathcal{C}$-satisfiability under global ontology with the RDA is polytime-reducible to concept $\mathcal{C}$-satisfiability under global ontology, with both constant and expanding domains.*

Proof. Let A be a concept name and $\mathcal{O}$ an ontology. Define $\mathcal{O}'$ by adding to $\mathcal{O}$ the CIs

$$\{a\} \sqsubseteq \Box_i \{a\}, \tag{2}$$

for all $i \in I$ and all individual names a occurring in $\mathcal{O}$. It can be seen that A is $\mathcal{C}$-satisfiable under $\mathcal{O}$ with the RDA iff A is $\mathcal{C}$-satisfiable under $\mathcal{O}'$. Indeed, the ($\Rightarrow$) direction is immediate, since an interpretation with the RDA that satisfies $\mathcal{O}$ satisfies the CIs of the form (2), and hence $\mathcal{O}'$. For the ($\Leftarrow$) direction, suppose that $\mathfrak{M}'$ is an interpretation based on a frame from $\mathcal{C}$, with either constant or expanding domains, such that $\mathfrak{M}' \models \mathcal{O}'$ and $A^{\mathcal{I}'_w} \neq \emptyset$, for some $w \in W$. Due to the CIs of the form (2), for every $w, v \in W$ with $w R_i v$, if a occurs in $\mathcal{O}$ and $a^{\mathcal{I}'_w}$ is defined, then $a^{\mathcal{I}'_v}$ is defined and $a^{\mathcal{I}'_w} = a^{\mathcal{I}'_v}$. We can then define $\mathfrak{M}$ as $\mathfrak{M}'$, except that all individual names a that do not occur in $\mathcal{O}$ now fail to designate in $\mathfrak{M}$ in every world. it can be seen that $\mathfrak{M}$ is based on the same frame and the same domains, satisfies the RDA and A under $\mathcal{O}$. □

Proposition 20. *In $\mathcal{ML}^n_{\mathcal{ALCO}_u}$ and $\mathcal{ML}^n_{\mathcal{ALCO}^{\iota}_u}$, total concept $\mathcal{C}$-satisfiability (under global ontology) with the RDA is polytime-reducible to total concept $\mathcal{C}$-satisfiability (under global ontology, respectively), with both constant and expanding domains.*

Proof. An argument similar to the proof of Proposition 1 can be used to show that a concept name A is satisfied in a total interpretation with the RDA based on a frame from $\mathcal{C}$ under $\mathcal{O}$ iff A is satisfied in a total interpretation based on a frame from $\mathcal{C}$ under $\mathcal{O}$ extended with CIs of the form (2).

For the concept satisfiability problem, without global ontology, the reduction above needs to be modified. Let C be a $\mathcal{ML}^n_{\mathcal{ALCO}^{\iota}_u}$ concept. Denote by C' the concept obtained from C by adding to it the conjuncts of the following form, for individual names a occurring in C:

$$\Box^{\pi} \forall u.(\{a\} \Rightarrow \Box_i \{a\}), \text{ for all } \pi \cdot i \in \overline{\mathsf{rp}}(C, \{a\}), \tag{3}$$

where $\overline{\mathsf{rp}}(C, \{a\})$ is the closure under taking prefixes of $\mathsf{rp}(C, \{a\})$. We show that C is $\mathcal{C}$-satisfiable with the RDA iff C' is $\mathcal{C}$-satisfiable.

For the ($\Rightarrow$) direction, observe that any interpretation $\mathfrak{M}$ with the RDA that satisfies C also satisfies C': all the conjuncts of the form (3) hold by the definition of the RDA.

For the ($\Leftarrow$) direction, suppose that $\mathfrak{M}'$ is a total interpretation based on a frame from $\mathcal{C}$ that satisfies C' at world w. We define an interpretation $\mathfrak{M}$ that coincides with $\mathfrak{M}'$, except that, for every $v \in W$, we set $a^{\mathcal{I}_v} = a^{\mathcal{I}'_w}$ (observe that, since $\mathfrak{M}'$ is total, $a^{\mathcal{I}'_w}$ is defined for every a). Thus, $\mathfrak{M}$ is a total interpretation satisfying the RDA.

It remains to show the following:

Claim 1. $C^{\mathcal{I}'_w} = C^{\mathcal{I}_w}$.

Proof. By induction on the structure of C: for each subconcept B of C, we show that

$$B^{\mathcal{I}'_v} = B^{\mathcal{I}_v}, \text{ for every } v \in \mathsf{rw}(w, C, B). \tag{4}$$

The base case of $B = A$, for a concept name A, is straightforward.

For the base case of $B = \{a\}$, where a is an individual name, both $\{a\}^{\mathcal{I}'_v}$ and $\{a\}^{\mathcal{I}_v}$ are defined as the interpretations are total. In addition, by construction, $\{a\}^{\mathcal{I}'_v}$ coincides with $\{a\}^{\mathcal{I}'_w} = \{a\}^{\mathcal{I}_w}$, both of which are also defined. On the other hand, $\{a\}^{\mathcal{I}_v}$ is equal to $\{a\}^{\mathcal{I}_w}$ due to conjuncts (3) applied along the path connecting w to v. Thus, we obtain $a^{\mathcal{I}_v} = \{a\}^{\mathcal{I}_v}$, for every $v \in \mathsf{rw}(w, C, \{a\})$, as required.

For the induction step, we need to consider the following cases.

If B is of the form $\{\iota B'\}$ or $\neg B'$ or $\exists s.B'$, then (4) is immediate from the induction hypothesis as $\mathsf{rw}(w, C, B) \subseteq \mathsf{rw}(w, C, B')$.

If B is of the form $B_1 \sqcap B_2$, then (4) is also immediate from the induction hypothesis as $\mathsf{rw}(w, C, B) \subseteq \mathsf{rw}(w, C, B_1) \cap \mathsf{rw}(w, C, B_2)$.

Finally, if $B = \Diamond_i B'$, then, by the induction hypothesis, we have $(B')^{\mathcal{I}'_v} = (B')^{\mathcal{I}_v}$, for all $v \in \mathsf{rw}(w, C, B')$. Since $\mathsf{rw}(w, C, \Diamond_i B') \subseteq \{v \mid vR_iu \text{ and } u \in \mathsf{rw}(w, C, B')\}$, we obtain $(\Diamond_i B')^{\mathcal{I}'_v} = (\Diamond_i B')^{\mathcal{I}_v}$, for all $v \in \mathsf{rw}(w, C, \Diamond_i B')$, as required.

This completes the proof of Claim 1. □

It follows that C is satisfied at w in a total interpretation $\mathfrak{M}$ with the RDA based on a frame in $\mathcal{C}$. □

Observe that, for the reduction to hold in the global ontology case, an individual name a does not have to designate at every world. For instance, consider an interpretation $\mathfrak{M}$ with $W = \{w, v\}$, wRv, $\Delta = \{d\}$, and such that a does not designate in w but $a^{\mathcal{I}_v} = d$. In this example, a is rigid and the CI of the form (2) is satisfied in both w and v: a does not designate at w, and v has no R-successors.

However, in the concept satisfiability case, we cannot enforce that any *partial* interpretation satisfying at some world C', i.e., C and the additional conjuncts of the form (3), can be transformed into a partial interpretation with the RDA that satisfies C, as witnessed by the following counterexample. Let C be the concept

$$\forall u.\neg\{a\} \sqcap \Box\exists u.\{a\},$$

and let C' be the conjunction of C with the concept $\forall u.(\{a\} \Rightarrow \Box\{a\})$. Consider a partial interpretation $\mathfrak{M}$ with constant domain such that wRv_i and v_iRu, for $i = 1, 2$, and $\Delta = \{d, e\}$. Moreover, let a be non-designating in w, while $a^{\mathcal{I}_{v_1}} = d$, $a^{\mathcal{I}_{v_2}} = e$, and $a^{\mathcal{I}_u} = d$. It can be seen that $\mathfrak{M}$ is a partial interpretation satisfying C' in w. However, we cannot turn $\mathfrak{M}$ into a partial interpretation with the RDA that satisfies C. This is due to the fact that C' is bounded on the levels of successors of w that it can reach, whereas, under global ontology, the corresponding CI $\{a\} \sqsubseteq \Box\{a\}$ would hold at all worlds of the interpretation.

Proposition 2. *In $\mathcal{ML}^n_{\mathcal{ALCO}_u}$ and $\mathcal{ML}^n_{\mathcal{ALCO}^\iota_u}$, total concept $\mathcal{C}$-satisfiability (under global ontology) is polytime-reducible to concept $\mathcal{C}$-satisfiability (under global ontology, respectively), with both constant and expanding domains.*

Proof. First, consider the case of satisfiability under global ontology: let A be a concept name and $\mathcal{O}$ an ontology. Define $\mathcal{O}'$ by adding to $\mathcal{O}$ the CIs

$$\top \sqsubseteq \exists u.\{a\} \tag{5}$$

for every a occurring in $\mathcal{O}$. Trivially, every total interpretation $\mathfrak{M}$ satisfying A under $\mathcal{O}$ also satisfies A under $\mathcal{O}'$. Conversely, every interpretation $\mathfrak{M}$ satisfying A under $\mathcal{O}'$ can be easily extended to a total interpretation satisfying A under $\mathcal{O}'$ (and so under $\mathcal{O}$), as it only remains to choose the interpretation of individual names not occurring in $\mathcal{O}$, which can be done in an arbitrary way.

Now, consider the case of satisfiability: let C be a concept. For the conjunction C' of C with concepts of the form

$$\Box^\pi\exists u.\{a\}, \text{ for all } \pi \in \mathsf{rp}(C, \{a\}),$$

for each a occurring in C, it can be seen that any total interpretation satisfying C satisfies C' too; conversely, any interpretation $\mathfrak{M}'$ satisfying C' can be turned into a total interpretation $\mathfrak{M}$ satisfying C, by ensuring that every individual name designates at each world. The proof that $C^{\mathcal{I}'_w} = C^{\mathcal{I}_w}$ is inductive and relies on the fact that $\mathfrak{M}'$ and $\mathfrak{M}$ coincide on the interpretation of every individual name a from $\mathcal{O}$ in all $v \in \mathsf{rw}(w, C, \{a\})$; see Claim 1 for a similar proof. □

Proposition 3. *In $\mathcal{ML}^n_{\mathcal{ALCO}_u}$ and $\mathcal{ML}^n_{\mathcal{ALCO}^\iota_u}$, concept $\mathcal{C}$-satisfiability (under global ontology) is polytime-reducible to total concept $\mathcal{C}$-satisfiability (under global ontology, respectively), with both constant and expanding domains.*

Proof. For satisfiability under global ontology, let A be a concept name and $\mathcal{O}$ an ontology. Let $\mathcal{O}'$ be the ontology obtained from $\mathcal{O}$ by replacing every nominal $\{a\}$ in $\mathcal{O}$ with a fresh concept name N_a, and by adding the CIs $N_a \sqsubseteq \{a\}$, for all individual names a occurring in $\mathcal{O}$. It can be seen that A is $\mathcal{C}$-satisfiable under $\mathcal{O}$ iff A is $\mathcal{C}$-satisfiable under $\mathcal{O}'$ in total interpretations.

($\Rightarrow$) Given an interpretation $\mathfrak{M}$ such that $\mathfrak{M} \models \mathcal{O}$ and $A^{\mathcal{I}_w} \neq \emptyset$, for some world $w \in W$, define the total interpretation $\mathfrak{M}'$ defined as $\mathfrak{M}$, except the following, for every $w \in W$:

- $N_a^{\mathcal{I}'_w} = \{a\}^{\mathcal{I}_w}$, for a that occurs in $\mathcal{O}$;
- $a^{\mathcal{I}'_w} = a^{\mathcal{I}_w}$, if a designates at w in $\mathfrak{M}$; and $a^{\mathcal{I}'_w}$ is arbitrary, otherwise, for any individual name a.

It can be seen that $\mathfrak{M}' \models \mathcal{O}'$ and $A^{\mathcal{I}'_w} \neq \emptyset$.

($\Leftarrow$) Given a total interpretation $\mathfrak{M}'$ such that $\mathfrak{M}' \models \mathcal{O}'$ and $A^{\mathcal{I}'_w} \neq \emptyset$, for some world $w \in W$, we define an interpretation $\mathfrak{M}$ that coincides with $\mathfrak{M}'$, except for the following, for every $w \in W$:

- $a^{\mathcal{I}_w} = d$, if a occurs in $\mathcal{O}$ and $N_a^{\mathcal{I}'_w} = \{d\}$, for some $d \in \Delta^w$; and a fails to designate at w in $\mathfrak{M}$, otherwise.

It can be seen that $\mathfrak{M} \models \mathcal{O}$ and $A^{\mathcal{I}_w} \neq \emptyset$.

For the concept satisfiability problem, the reduction above is modified similarly to the proof of Proposition 2: let C' be the conjunction result of replacing each $\{a\}$ with N_a in C and the following additional conjuncts

$$\Box^{\pi} \forall u.(N_a \Rightarrow \{a\}), \text{ for all } \pi \in \mathsf{rp}(C, \{a\}).$$

It can be seen that any interpretation $\mathfrak{M}$ satisfying C gives rise to a total interpretation $\mathfrak{M}'$ satisfying C' by using the extension of $\{a\}^{\mathcal{I}_v}$ to interpret both a and N_a at any v in $\mathfrak{M}'$ and arbitrarily choosing the value of a at v in $\mathfrak{M}'$ if it is undefined in $\mathfrak{M}$. The result clearly satisfies C'. Conversely, the additional conjuncts guarantee that N_a behaves like a nominal term (its extension contains at most one element at every world) and so C is satisfied in an interpretation $\mathfrak{M}$ obtained from any interpretation $\mathfrak{M}$ by redefining each a as the corresponding N_a (which may give a partial interpretation, of course). Again, the proof that $C^{\mathcal{I}'_w} = C^{\mathcal{I}_w}$ is inductive and relies on the fact that $\mathfrak{M}'$ and $\mathfrak{M}$ coincide on the interpretation of every individual name a from $\mathcal{O}$ in all $v \in \mathsf{rw}(w, C, \{a\})$; see Claim 1 for a similar proof. □

An ontology is in *normal form* if it consists of CIs of the form

$$\begin{array}{ll}
A \sqsubseteq \{a\}, & \{a\} \sqsubseteq A, \\
A \sqsubseteq \{\iota A_1\}, & \{\iota A_1\} \sqsubseteq A, \\
A \sqsubseteq \neg A_1, & \neg A_1 \sqsubseteq A, \\
A \sqsubseteq A_1 \sqcap A_2, & A_1 \sqcap A_2 \sqsubseteq A, \\
A \sqsubseteq \exists s.A_1, & \exists s.A_1 \sqsubseteq A, \\
A \sqsubseteq \Diamond_i A_1, & \Diamond_i A_1 \sqsubseteq A,
\end{array}$$

where A, A_1 and A_2 are concept names, $s \in \mathsf{N_R} \cup \{u\}$ and $i \in I$.

Lemma 4. *For any $\mathcal{ML}^n_{\mathcal{ALCO}^{\iota}_u}$ ontology $\mathcal{O}$, we can construct in polytime an $\mathcal{ML}^n_{\mathcal{ALCO}^{\iota}_u}$ ontology $\mathcal{O}'$ in normal form that is a model conservative extension of $\mathcal{O}$. Moreover, $\mathcal{O}'$ uses the same set of connectives as $\mathcal{O}$.*

Proof. Let C be a subconcept in $\mathcal{O}$, and let $\mathcal{O}[C/A]$ be the result of replacing every occurrence of C in $\mathcal{O}$ with a fresh concept name A. Clearly, $\mathcal{O}[C/A] \cup \{C \equiv A\}$ is a model conservative extension of $\mathcal{O}$. By repeated application of this procedure, starting from innermost connective first, we obtain in polynomial time an ontology $\mathcal{O}'$ in normal form having the same set of connectives as $\mathcal{O}$. □

Lemma 5. *Let D be an $\mathcal{ML}^n_{\mathcal{ALCO}^{\iota}_u}$ concept and C its subconcept. Denote by D' the conjunction of $D[C/A]$ and*

$$\Box^{\pi} \forall u.(C \Leftrightarrow A), \text{ for all } \pi \in \mathsf{rp}(D, C). \tag{1}$$

Then D' is a model conservative extension of D. Moreover, $\mathsf{rp}(D', A) = \mathsf{rp}(D, C)$ and $\mathsf{rp}(D', E) = \mathsf{rp}(D, E)$, for any subconcept E of C.

Proof. Let $\mathfrak{M}$ be an interpretation satisfying D. We define an interpretation $\mathfrak{M}'$ that extends $\mathfrak{M}$ with the interpretation of A by taking $A^{\mathcal{I}_w} = C^{\mathcal{I}_w}$. Clearly, D' is satisfied in $\mathfrak{M}'$. Conversely, suppose D' is satisfied at w in an interpretation $\mathfrak{M}'$. The additional conjuncts ensure that $A^{\mathcal{I}'_v} = C^{\mathcal{I}'_v}$, for all $v \in \mathsf{rw}(w, D, C)$. In other words, we have $C[C/A]^{\mathcal{I}'_v} = C^{\mathcal{I}'_v}$, for all $v \in \mathsf{rw}(w, D, C)$. We show

Claim 2. $D[C/A]^{\mathcal{I}'_w} = D^{\mathcal{I}'_w}$.

Proof. We prove this by induction on the structure of D that, for all subconcepts B of $D[C/A]$, we have

$$B[C/A]^{\mathcal{I}'_v} = B^{\mathcal{I}'_v}, \text{ for all } v \in \mathsf{rw}(w, D, B). \tag{6}$$

For the basis of induction, we need to consider two cases. If B is either a concept name different from A or a term nominal $\{a\}$, then (6) is immediate as $B[C/A]$ coincides with B, If B is A, then (6) is by construction.

For the induction step, we need to consider the following cases. If B is of the form $\{\iota B'\}$ or $\neg B'$ or $\exists s.B'$, then (6) is immediate from the induction hypothesis as $\mathsf{rw}(w, D, B) \subseteq \mathsf{rw}(w, D, B')$. If B is of the form $B_1 \sqcap B_2$, then (6) is also immediate from the induction hypothesis as $\mathsf{rw}(w, D, B) \subseteq \mathsf{rw}(w, D, B_1) \cap \mathsf{rw}(w, D, B_2)$. Finally, if $B = \Diamond_i B'$, then, by the induction hypothesis, we have $B'[C/A]^{\mathcal{I}'_v} = (B')^{\mathcal{I}'_v}$, for all $v \in \mathsf{rw}(w, D, B')$. Since $\mathsf{rw}(w, D, \Diamond_i B) \subseteq \{v \mid vR_i u \text{ and } u \in \mathsf{rw}(w, D, B)\}$, we obtain $(\Diamond_i B')[C/A]^{\mathcal{I}'_v} = (\Diamond_i B')^{\mathcal{I}'_v}$, for all $v \in \mathsf{rw}(w, D, \Diamond_i B')$, as required. □

Also by induction on the structure of D, it can be seen that $\mathsf{rp}(D', A) = \mathsf{rp}(D, C)$, for any subconcept C of D, and $\mathsf{rp}(D', E) = \mathsf{rp}(D, E)$, for any subconcept E of C. □

Lemma 6. *Let $\mathcal{O}$ be an $\mathcal{ML}^n_{\mathcal{ALCO}^{\iota}_u}$ ontology in normal form. Denote by $\mathcal{O}'$ the $\mathcal{ML}^n_{\mathcal{ALCO}^{\iota}}$ ontology obtained from $\mathcal{O}'$ by replacing*

- *each CI of the form $B \sqsubseteq \exists u.B'$ with $B \sqsubseteq \exists r.B'$, and*
- *each CI of the form $\exists u.B \sqsubseteq B'$ with the following:*

$$\top \sqsubseteq \exists r.\{e\}, \;\; A \sqsubseteq \{e\}, \;\; \neg B' \sqsubseteq \exists r.A, \;\; \exists r.A \sqsubseteq \neg B,$$

where r, e and A are fresh role, nominal and concept names, respectively. Then $\mathcal{O}'$ is a model conservative extension of $\mathcal{O}$, and the size of $\mathcal{O}'$ is linear in the size of $\mathcal{O}$.

Proof. First, we consider the positive occurrences of the universal role. Suppose $B \sqsubseteq \exists u.B'$ is satisfied at all w in some interpretation $\mathfrak{M}$. Then we extend $\mathfrak{M}$ to $\mathfrak{M}'$ by interpreting the fresh role r as the universal role at all w. Then $B \sqsubseteq \exists r.B'$ is satisfied at all w in $\mathfrak{M}'$.

Conversely, suppose $B \sqsubseteq \exists r.B'$ is satisfied at all w in some $\mathfrak{M}'$. Then, clearly, $B \sqsubseteq \exists u.B'$ is satisfied in $\mathfrak{M}'$ as well.

Second, we consider the negative occurrences of the universal role. Suppose that $\exists u.B \sqsubseteq B'$ is satisfied at all w in some interpretation $\mathfrak{M}$. Then we extend $\mathfrak{M}$ to $\mathfrak{M}'$ by interpreting the fresh role r as the universal role at all w, choosing one arbitrary element as the interpretation of the fresh nominal s, and, for every $w \in W$, including $e^{\mathcal{I}'_w}$ in $A^{\mathcal{I}'_w}$ if $B^{\mathcal{I}_w} = \emptyset$ and leaving $A^{\mathcal{I}'_w}$ empty otherwise. We show that the four CIs are satisfied at all w in $\mathfrak{M}'$. The first and second CIs are satisfied by consruction. If $(\neg B')^{\mathcal{I}'_w} = \emptyset$, then the third CI is trivially satisfied at w in $\mathfrak{M}'$. So, assume that

$(\neg B')^{\mathcal{I}'_w} \neq \emptyset$. As $\exists u.B \sqsubseteq B'$ is satisfied at w in $\mathfrak{M}$, we have $B^{\mathcal{I}'_w} = \emptyset$ and so the third CI is also satisfied in $\mathfrak{M}'$ because, by construction, $e^{\mathcal{I}'_w} \in A^{\mathcal{I}'_w}$. To show that the last CI is also satisfied at w in $\mathfrak{M}'$, suppose that $e^{\mathcal{I}'_w} \in A^{\mathcal{I}'_w}$. Then, by construction, $B^{\mathcal{I}_w} = \emptyset$ and so every domain element is in $(\neg B)^{\mathcal{I}'_w}$, as required by the fourth CI.

Conversely, suppose the four CIs are satisfied at all w in some $\mathfrak{M}'$. By the first CI, $(d, e^{\mathcal{I}'_w}) \in r^{\mathcal{I}'_w}$, for every element d of the domain. Assume first that $B^{\mathcal{I}'_w} \neq \emptyset$. Take any $d \in B^{\mathcal{I}'_w}$. By the fourth CI, $e^{\mathcal{I}'_w} \notin A^{\mathcal{I}'_w}$. Therefore, by the third CI, every element of the domain is in $(B')^{\mathcal{I}'_w}$ for otherwise $e^{\mathcal{I}'_w} \in A^{\mathcal{I}'_w}$. Thus, $\exists u.B \sqsubseteq B'$ is satisfied at all w in $\mathfrak{M}'$. □

Proposition 7. *In $\mathcal{ML}^n_{\mathcal{ALCO}_u}$ and $\mathcal{ML}^n_{\mathcal{ALCO}^\iota_u}$, concept $\mathcal{C}$-satisfiability (under global ontology) is polytime-reducible to $\mathcal{ML}^n_{\mathcal{ALC}^\iota_u}$ concept $\mathcal{C}$-satisfiability (under global ontology, respectively), with both constant and expanding domains.*

Proof. We first consider the case of satisfiability under global ontology. Given a concept name A and an $\mathcal{ML}^n_{\mathcal{ALCO}}$ ontology $\mathcal{O}$, take a fresh concept name N_a for each individual name a in $\mathcal{O}$, and let $\mathcal{O}'$ be the result of replacing every occurrence of $\{a\}$ in $\mathcal{O}$ with $\{\iota N_a\}$. It can be seen that, if A is satisfied in an interpretation $\mathfrak{M}$ under $\mathcal{O}$, then A is satisfied in $\mathfrak{M}'$ under $\mathcal{O}'$, where $\mathfrak{M}'$ is obtained from $\mathfrak{M}$ by setting, for every w in $\mathfrak{M}$, $N_a^{\mathcal{I}'_w} = \{a^{\mathcal{I}_w}\}$, if a designates at w in $\mathfrak{M}$, and $N_a^{\mathcal{I}'_w} = \emptyset$, otherwise; note that the interpretation of nominals can be chosen arbitrarily as $\mathcal{O}'$ does not contain them. Conversely, if A is satisfied in an interpretation $\mathfrak{M}'$ under $\mathcal{O}'$, then A is satisfied in $\mathfrak{M}$ under $\mathcal{O}$, where $\mathfrak{M}$ is obtained from $\mathfrak{M}'$ by setting $a^{\mathcal{I}_w} = (\iota N_a)^{\mathcal{I}'_w}$, for every w in $\mathfrak{M}$ and every a occurring in $\mathcal{O}$; note that the resulting interpretation is not necessarily total and does not necessarily satisfy the RDA.

The same construction and argument work for the satisfiability problem too. □

Proposition 8. *$\mathcal{ML}^n_{\mathcal{ALCO}^\iota_u}$ concept $\mathcal{C}$-satisfiability (under global ontology) is polytime-reducible to $\mathcal{ML}^n_{\mathcal{ALCO}_u}$ concept $\mathcal{C}$-satisfiability (under global ontology, respectively), with both constant and expanding domains.*

Proof. By Propositions 3 and 2, it is sufficient to reduce $\mathcal{ML}^n_{\mathcal{ALCO}^\iota_u}$ *total* concept $\mathcal{C}$-satisfiability (under global ontology) to *total* $\mathcal{ML}^n_{\mathcal{ALCO}_u}$ concept $\mathcal{C}$-satisfiability (under global ontology, respectively).

First, consider the case of satisfiability under global ontology: let A be a concept name and $\mathcal{O}$ an $\mathcal{ML}^n_{\mathcal{ALCO}^\iota_u}$ ontology. By Lemma 4, we assume that $\mathcal{O}$ is in normal form and, in particular, (*a*) all definite descriptions are applied to concept names only and (b) term nominals occur only in CIs of the form $A_{\{\tau\}} \equiv \{\tau\}$, for $A_{\{\tau\}} \in \mathsf{N_C}$ and terms τ that are either $b \in \mathsf{N_I}$ or ιB, for $B \in \mathsf{N_C}$. We then define a translation of an $\mathcal{ML}^n_{\mathcal{ALCO}^\iota_u}$ ontology $\mathcal{O}$ in normal form into an $\mathcal{ML}^n_{\mathcal{ALCO}_u}$ ontology $\mathcal{O}^*$. Let $\mathcal{O}^*$ be the result of replacing each $A_{\{\iota B\}} \equiv \{\iota B\}$ in $\mathcal{O}$ with CIs of the form

$$A_{\{\iota B\}} \sqsubseteq B \sqcap \{a_B\} \text{ and } B \sqcap \forall u.(B \Rightarrow \{a_B\}) \sqsubseteq A_{\{\iota B\}},$$

where a_B is a fresh nominal. We then show the following.

Claim 3. *$\mathcal{O}^*$ is a model conservative extension of $\mathcal{O}$.*

Proof. Clearly, for any interpretation $\mathfrak{M}^*$ such that $\mathfrak{M}^* \models \mathcal{O}^*$, we have $\mathfrak{M}^* \models \mathcal{O}$ as well.

Conversely, given a total interpretation $\mathfrak{M}$ based on a frame in $\mathcal{C}$ such that $\mathfrak{M} \models \mathcal{O}$, we define the total interpretation $\mathfrak{M}^*$ by extending $\mathfrak{M}$, for every $w \in W$, with

- $a_B^{\mathcal{I}^*_w} = d$, if $B^{\mathcal{I}_w} = \{d\}$, for some $d \in \Delta^w$; and $a_B^{\mathcal{I}^*_w}$ is arbitrary, otherwise.

It can be seen that $\mathfrak{M}^* \models \mathcal{O}^*$. □

Hence, A is satisfied under $\mathcal{O}$ in a total interpretation based on a frame from $\mathcal{C}$ iff A is satisfied under $\mathcal{O}^*$ in a total interpretation based on a frame from $\mathcal{C}$.

Second, we consider the case of concept satisfiability. Let C be an $\mathcal{ML}^n_{\mathcal{ALCO}^\iota_u}$ concept. By repeatedly applying Lemma 5, we transform C into normal form, where, in particular, each subconcept of C of the form $\{\iota B\}$, where B is a concept name, has a surrogate $A_{\{\iota B\}}$ and corresponding conjuncts of the form (1). Consider now the $\mathcal{ML}^n_{\mathcal{ALCO}_u}$ concept C^* obtained from the normal form of C by replacing $\Box^\pi \forall u.(A_{\{\iota B\}} \equiv \{\iota B\})$ with the following

$$\Box^\pi \forall u.(A_{\{\iota B\}} \Rightarrow B \sqcap \{a_B\}) \sqcap$$
$$\Box^\pi \forall u.(B \sqcap \forall u.(B \Rightarrow \{a_B\}) \Rightarrow A_{\{\iota B\}})$$

where a_B is a fresh nominal for B. Recall that π range over all paths in $\mathsf{rp}(C, \{\iota B\})$. We now show the following.

Claim 4. *C^* is a model-conservative extension of C.*

Proof. It can be seen that any interpretation $\mathfrak{M}^*$ that satisfies C^* also satisfies C.

Conversely, given a total interpretation $\mathfrak{M}$ based on a frame from $\mathcal{C}$ that satisfies C, we define the total interpretation $\mathfrak{M}^*$ extending $\mathfrak{M}$, for every $w \in W$, with

- $a_B^{\mathcal{I}^*_w} = d$, if $B^{\mathcal{I}_w} = \{d\}$, for some $d \in \Delta^w$; and $a_B^{\mathcal{I}^*_w}$ is arbitrary, otherwise.

It can be seen that $\mathfrak{M}^*$ is a total interpretation based on a frame from $\mathcal{C}$ that satisfies C^*. □

Hence, C is total $\mathcal{C}$-satisfiable iff C^* is total $\mathcal{C}$-satisfiable. This concludes the proof of the proposition. □

Proposition 9. *In $\mathcal{ML}^n_{\mathcal{ALCO}_u}$, concept $\mathcal{C}$-satisfiability (under global ontology) with expanding domains is polytime-reducible to concept $\mathcal{C}$-satisfiability (under global ontology, respectively) with constant domain.*

Proof. By Propositions 3 and 2, it is sufficient to reduce $\mathcal{ML}^n_{\mathcal{ALCO}_u}$ *total* concept $\mathcal{C}$-satisfiability (under global ontology) with expanding domains to *total* $\mathcal{ML}^n_{\mathcal{ALCO}_u}$ concept $\mathcal{C}$-satisfiability (under global ontology, respectively) with constant domain.

Let Ex be a fresh concept name, used to represent the objects that actually *exist* at a given world domain. We introduce the $\cdot^{\downarrow\mathsf{Ex}}$ *relativisation*, mapping an $\mathcal{ML}^n_{\mathcal{ALCO}_u}$ concept C to an $\mathcal{ML}^n_{\mathcal{ALCO}_u}$ concept $C^{\downarrow\mathsf{Ex}}$ as follows, where $s \in \mathsf{N_R} \cup \{u\}$:

$$A^{\downarrow\mathsf{Ex}} = A, \quad \{a\}^{\downarrow\mathsf{Ex}} = \mathsf{Ex} \sqcap \{a\}, \quad (\neg D)^{\downarrow\mathsf{Ex}} = \neg D^{\downarrow\mathsf{Ex}},$$
$$(\exists s.D)^{\downarrow\mathsf{Ex}} = \exists s.(\mathsf{Ex} \sqcap D^{\downarrow\mathsf{Ex}}),\ (D \sqcap E)^{\downarrow\mathsf{Ex}} = D^{\downarrow\mathsf{Ex}} \sqcap E^{\downarrow\mathsf{Ex}},$$
$$(\Diamond_i D)^{\downarrow\mathsf{Ex}} = \Diamond_i (D)^{\downarrow\mathsf{Ex}}.$$

The $\cdot^{\downarrow\mathsf{Ex}}$ relativisation of CIs and ontologies is then obtained by replacing every concept C with $C^{\downarrow\mathsf{Ex}}$.

For the case of satisfiability under global ontology, it can be seen that A is $\mathcal{C}$-satisfiable in total interpretations with expanding domains under global ontology $\mathcal{O}$ iff $\mathsf{Ex} \sqcap A$ is $\mathcal{C}$-satisfiable in total interpretations with constant domains under ontology that consists of $\mathcal{O}^{\downarrow\mathsf{Ex}}$ along with the following CIs:

$$\mathsf{Ex} \sqsubseteq \Box_i \mathsf{Ex}, \text{ for all } i \in I;$$

cf. the translation of φ in (Gabbay et al. 2003, Proposition 3.32 (ii), (iv)).

Similarly, for concept satisfiability, it can be seen that C is $\mathcal{C}$-satisfiable in total interpretations with expanding domains iff the conjunction C' of $\mathsf{Ex} \sqcap C^{\downarrow\mathsf{Ex}}$ with the following concepts:

$$\Box^\pi \forall u.(\mathsf{Ex} \Rightarrow \Box_i \mathsf{Ex}), \text{ for all } i \in I \text{ and } \pi \cdot i \in \mathsf{rp}(C, B) \text{ with a subconcept } B \text{ of } C.$$

is $\mathcal{C}$-satisfiable in total interpretations with constant domains; cf. C' in (Gabbay et al. 2003, Proposition 3.32 (ii)). □

C Proofs for Section 4

Theorem 10. *(1) $\mathcal{C}$-satisfiability of $\mathcal{ML}^n_{\mathcal{ALCO}_u}$-concepts (under global ontology) can be reduced in double exponential time to $\mathcal{C}$-satisfiability of $\mathcal{ML}^n_{\text{Diff}}$-concepts (under global ontology, respectively), both with constant and expanding domains.*

(2) Conversely, $\mathcal{C}$-satisfiability of $\mathcal{ML}^n_{\text{Diff}}$-concepts (under global ontology) is polytime-reducible to $\mathcal{C}$-satisfiability of $\mathcal{ML}^n_{\mathcal{ALCO}_u}$-concepts (under global ontology, respectively), with both constant and expanding domains.

Proof. (1) We first consider the case of satisfiability under global ontology. Let A be a concept name and $\mathcal{O}$ an $\mathcal{ML}^n_{\mathcal{ALCO}_u}$ ontology. We assume that A occurs in $\mathcal{O}$. Let $\mathsf{con}(\mathcal{O})$ be the closure under single negation of the set of concepts occurring in $\mathcal{O}$. A *type* for $\mathcal{O}$ is a subset $\boldsymbol{t}$ of $\mathsf{con}(\mathcal{O})$ such that $\neg C \in \boldsymbol{t}$ iff $C \notin \boldsymbol{t}$, for all $\neg C \in \mathsf{con}(\mathcal{O})$. A *quasistate for* $\mathcal{O}$ is a non-empty set $\boldsymbol{T}$ of types for $\mathcal{O}$. The *description of* $\boldsymbol{T}$ is the $\mathcal{ML}^n_{\mathcal{ALCO}_u}$-concept

$$\Xi_{\boldsymbol{T}} = \forall u.(\bigsqcup_{\boldsymbol{t} \in \boldsymbol{T}} \boldsymbol{t}) \sqcap \bigsqcap_{\boldsymbol{t} \in \boldsymbol{T}} \exists u.\boldsymbol{t}.$$

Let $\mathcal{S}_\mathcal{O}$ denote the set of all quasistates $\boldsymbol{T}$ for $\mathcal{O}$ that are *$\mathcal{ALCO}_u$-satisfiable*, that is, such that $\Xi^t_{\boldsymbol{T}}$ is satisfiable, where $\Xi^t_{\boldsymbol{T}}$ denotes the result of replacing every outermost occurrence of $\Diamond_i C$ by $A_{\Diamond_i C}$, for a fresh concept name $A_{\Diamond_i C}$. It should be clear that $\mathcal{S}_\mathcal{O}$ can be computed in double exponential time as satisfiabilty of $\mathcal{ALCO}_u$-concepts under $\mathcal{ALCO}_u$ ontologies is in ExpTime.

We now reserve for any $C \in \mathsf{con}(\mathcal{O})$ of the form $\{a\}$ or $\exists r.C'$ a fresh concept name A_C. Define a mapping $\cdot^\sharp$ that associates with every $\mathcal{ML}^n_{\mathcal{ALCO}_u}$-concept an $\mathcal{ML}^n_{\text{Diff}}$-concept by replacing outermost occurrences of concepts of the form $\exists r.C$ or concepts of the form $\{a\}$ by the respective fresh concept name. Let $\mathcal{O}'$ be the extension of $\mathcal{O}^\sharp$ with the following CIs:

$$\top \sqsubseteq \exists u.(\{a\}^\sharp \sqcap \neg \exists^{\neq} u.\{a\}^\sharp), \text{ for every } a \in \mathsf{N_I} \text{ in } \mathcal{O},$$
$$\top \sqsubseteq \bigsqcup_{\boldsymbol{T} \in \mathcal{S}_\mathcal{O}} \Xi^\sharp_{\boldsymbol{T}}.$$

Then we have the following:

Lemma 21. *A is $\mathcal{C}$-satisfiable under $\mathcal{O}$ iff there is a type $\boldsymbol{t}$ for $\mathcal{O}$ such that $A \in \boldsymbol{t}$ and $\boldsymbol{t}^\sharp$ is $\mathcal{C}$-satisfiable under $\mathcal{O}'$.*

Proof. Suppose A is satisfied at $w \in W$ in an $\mathcal{ML}^n_{\mathcal{ALCO}_u}$ interpretation $\mathfrak{M}$ based on a frame from $\mathcal{C}$ and $\mathcal{O}$ is satisfied in $\mathfrak{M}$. Then we can define the type $\boldsymbol{t}$ by reading it off from $e \in A^{\mathcal{I}_w}$:

$$\boldsymbol{t} = \big\{C \in \mathsf{con}(\mathcal{O}) \mid e \in C^{\mathcal{I}_w}\big\}.$$

Then, we define an $\mathcal{ML}^n_{\text{Diff}}$ interpretation $\mathfrak{M}'$ from $\mathfrak{M}$ by providing the interpretation of the additional concept names A_C for $\mathcal{ML}^n_{\mathcal{ALCO}_u}$ concepts C of the form $\{a\}$ and $\exists r.C'$: we simply set $A_C^{\mathcal{I}'_v} = C^{\mathcal{I}_v}$, for all $v \in W$. It should be clear that $\mathfrak{M}'$ satisfies $\mathcal{O}'$ and that $e \in (\boldsymbol{t}^\sharp)^{\mathcal{I}'_w}$.

Conversely, suppose there is a type $\boldsymbol{t}$ for $\mathcal{O}$ such that $A \in \boldsymbol{t}$ and $\boldsymbol{t}^\sharp$ is satisfied in an $\mathcal{ML}^n_{\text{Diff}}$ interpretation $\mathfrak{M}'$ based on a frame from $\mathcal{C}$ such that $\mathcal{O}'$ is also satisfied in $\mathfrak{M}'$. We define $\mathfrak{M}$ by extending $\mathfrak{M}'$ by suitable interpretations of nominals and role names. Let $v \in W'$. Observe that for any nominal a in $\mathcal{O}$, $(a^\sharp)^{\mathcal{I}'_v}$ is a singleton, by definition of $\mathcal{O}^\sharp$. Hence we define $a^{\mathcal{I}_v}$ as its single element. For any role name r, we define $r^{\mathcal{I}_v}$ as the maximal relation that is consistent with existential restrictions in the types we aim to satisfy: for $d \in (\boldsymbol{t}^\sharp)^{\mathcal{I}'_v}$ and $d' \in (\boldsymbol{t}'^\sharp)^{\mathcal{I}'_v}$, let $(d, d') \in r^{\mathcal{I}_v}$ if, for all $\exists r.D \in \mathsf{con}(\mathcal{O})$, if $D \in \boldsymbol{t}'$, then $\exists r.D \in \boldsymbol{t}$. It is easy to see that $\mathfrak{M}$ is as required. □

For the concept $\mathcal{C}$-satisfiability problem we adapt the proof in the same way as in Proposition 2: the additional CIs of $\mathcal{O}'$ need to be replaced by conjuncts with suitable prefixes of the box operators and the universal role restrictions.

(2) Gargov and Goranko (1993) observed that the logic of the difference modality and nominals have the same expressive power. Their technical result can be presented in our setting in the following way. Let C be an $\mathcal{ML}^n_{\text{Diff}}$-concept and $\exists^{\neq} u.B$ its subconcept. Denote by $C^{\ddagger_B}$ the result of replacing every occurrence of $\exists^{\neq} u.B$ in D with

$$\exists u.B \sqcap (\{a_B\} \Rightarrow \exists u.(\neg\{a_B\} \sqcap B)),$$

where a_B is a fresh nominal associated with concept B (note that we replace only a single subconcept here). Let $\mathcal{O}_B^{\mathsf{wt}}$ consist of the following CI:

$$B \sqsubseteq \exists u.(\{a_B\} \sqcap B).$$

Then we can extend the proof of Lemmas 4.2 and 4.3 in (Gargov and Goranko 1993) to the modal setting (provided that interpretations do not satisfy the RDA, and so nominals can be interpreted differently in different worlds) to obtain the following:

Lemma 22. *For any interpretation $\mathfrak{M}$ satisfying $\mathcal{O}_B^{\mathsf{wt}}$, we have $C^{\mathcal{I}(w)} = (C^{\ddagger_B})^{\mathcal{I}(w)}$, for all w in $\mathfrak{M}$.*

This result gives us the reduction for the case of satisfiability under global ontology. Let $\mathcal{O}$ be an $\mathcal{ML}^n_{\text{Diff}}$-ontology and A a concept name. By Lemma 4, we can assume that $\mathcal{O}$ is in normal form. Denote by $\mathcal{O}^\ddagger$ the result of replacing every CI $C_1 \sqsubseteq C_2$ in $\mathcal{O}$ with $C_1^\ddagger \sqsubseteq C_2^\ddagger$, where $C_i^\ddagger$ is the result of applying $\cdot^{\ddagger_B}$ to C_i, for every subconcept of the form $\exists^{\neq} u.B$; note that the order of applications of the $\cdot^{\ddagger_B}$ does not matter as the ontology is in normal form. Let $\mathcal{O}' = \mathcal{O}^\ddagger \cup \mathcal{O}^{\mathsf{wt}}$, where $\mathcal{O}^{\mathsf{wt}}$ is the union of all $\mathcal{O}_B^{\mathsf{wt}}$, for concepts $\exists^{\neq} u.B$ in $\mathcal{O}$. It should be clear that, if A is satisfied under $\mathcal{O}$ at w in an interpretation $\mathfrak{M}$ based on frame from $\mathcal{C}$, then we can extend $\mathfrak{M}$ to $\mathfrak{M}'$ by interpreting the nominals a_B for concepts of the form $\exists^{\neq} u.B$: for each such nominal, if $B^{\mathcal{I}_v} \neq \emptyset$, then we pick any $e \in B^{\mathcal{I}_v}$ and assign $a_B^{\mathcal{I}_v} = e$; otherwise, we leave $a_B^{\mathcal{I}_v}$ undefined. Clearly, $\mathfrak{M}'$ satisfies $\mathcal{O}^{\mathsf{wt}}$. Thus, by Lemma 22, $\mathfrak{M}'$ satisfies $\mathcal{O}^\ddagger$, and also A is satisfied at w in $\mathfrak{M}'$.

Conversely, if A is satisfied under $\mathcal{O}'$ at w in an interpretation $\mathfrak{M}'$ based on a frame from $\mathcal{C}$, then, by Lemma 22, A is satisfied under $\mathcal{O}$ at w in $\mathfrak{M}'$. We can now take the reduct $\mathfrak{M}$ of $\mathfrak{M}'$ by removing the interpretation of nominals: as $\mathcal{O}$ does not contain nominals, A is satisfied under $\mathcal{O}$ at w in $\mathfrak{M}$.

Next, we consider the case of concept satisfiability. Let D be an $\mathcal{ML}^n_{\text{Diff}}$-concept. We repeatedly apply Lemma 5, starting from the innermost occurrences of subconcepts of the form $\exists^{\neq} u.C$, to transform D into normal form D', where each 'elsewhere' quantifier is applied only to concept names. We then construct $D^\ddagger$ by applying the $\cdot^{\ddagger_B}$ to D', for each B such that $\exists^{\neq} u.B$ occurs in D'; again, the order of applications does not matter as D' is in normal form. Consider the conjunction of $D^\ddagger$ with all concepts of the form

$$\Box^\pi \forall u.(B \Rightarrow \exists u.(\{a_B\} \sqcap B)),$$

for $\pi \in \mathsf{rp}(D', \exists^{\neq} u.B)$ and concepts of the form $\exists^{\neq} u.B$ in D' (note that, by Lemma 5, $\mathsf{rp}(D', \exists^{\neq} u.B)$ coincides with $\mathsf{rp}(D, \exists^{\neq} u.C)$ for the respective C in D). We can easily modify the argument presented above (see the proof of Proposition 2) to show that D is satisfied in an interpretation $\mathfrak{M}$ iff D' is satisfied in an interpretation $\mathfrak{M}'$ that additionally interprets nominals a_B as 'witnesses' for concepts of the form $\exists^{\neq} u.B$. □

D Proofs for Section 5

We first observe that for $\mathbf{S5}$, concept satisfiability under global ontology can be reduced to concept satisfiability without ontology by using the concept $C_{\mathcal{O}} \sqcap \Box C_{\mathcal{O}}$ with $C_{\mathcal{O}}$ the conjunction of all $C \Rightarrow D$ with $C \sqsubseteq D \in \mathcal{O}$.

Theorem 11. *For $L \in \{\mathbf{K}^n, \mathbf{S5}^n\}$ with $n \geq 1$, $L_{\mathcal{ALCO}_u^\iota}$ concept satisfiability is in* NExpTime *with both expanding and constant domains.*

Proof. By Proposition 8, we can drop definite descriptions (hence consider $L_{\mathcal{ALCO}_u}$) and by Proposition 3, it is sufficient to consider total satisfiability. By Proposition 9, satisfiability with expanding domains can be reduced in polytime to satisfiability with constant domain, so we consider constant domain models. We give the proof for $\mathbf{S5}^n_{\mathcal{ALCO}_u}$. The proof for $\mathbf{K}^n_{\mathcal{ALCO}_u}$ can then be derived in a straightforward manner.

We show the exponential finite model property (every $\mathbf{S5}^n_{\mathcal{ALCO}_u}$-concept that is satisfiable, is satisfiable in a model of exponential size). The NExpTime upper bound then follows directly. We proceed in a number of steps. First, we argue that tree-shaped models are sufficient. Then, we introduce quasimodels as a surrogate for models that is easier to manipulate and show the exponential finite model property for tree-shaped quasimodels. Finite exponential-size models are then easily constructed.

Recall that a frame $\mathfrak{F} = (W, R_1, \dots, R_n)$ is an $\mathbf{S5}^n$-frame if all R_i are equivalence relations. Then such an $\mathbf{S5}^n$-frame $\mathfrak{F} = (W, R_1, \dots, R_n)$ is called a *tree-shaped* $\mathbf{S5}^n$*-frames* if there exists a $w_0 \in W$ such that the domain W of $\mathfrak{F}$ is a prefix-closed set of words of the form

$$\vec{w} = w_0 i_0 w_1 \cdots i_{m-1} w_m, \tag{7}$$

where $1 \leq i_j \leq n$, $i_j \neq i_{j+1}$, and each R_i is the smallest equivalence relation containing all pairs of the form $(\vec{w}, \vec{w}iw) \in W \times W$. We call w_0 the root of $\mathfrak{F}$. We define the *depth* of $\mathfrak{F}$ as the maximal m such that W contains a word of the form (7). A *tree-shaped* $\mathbf{S5}^n_{\mathcal{ALCO}_u}$*-model* takes the form $\mathfrak{M} = (\mathfrak{F}, \Delta, \mathcal{I})$ with $\mathfrak{F}$ a tree-shaped $\mathbf{S5}^n$-frame.

Lemma 23. *$\mathbf{S5}^n_{\mathcal{ALCO}_u}$ is determined by tree-shaped $\mathbf{S5}^n_{\mathcal{ALCO}_u}$-models: an $\mathcal{ML}^n_{\mathcal{ALCO}_u}$-concept C is satisfiable in an $\mathbf{S5}^n_{\mathcal{ALCO}_u}$-model iff it is satisfiable in the root of a tree-shaped $\mathbf{S5}^n_{\mathcal{ALCO}_u}$-model of depth bounded by the modal depth of C.*

Proof. Assume that C has modal depth K and assume $\mathfrak{M} = (\mathfrak{F}, \Delta, \mathcal{I})$ with $\mathfrak{F} = (W, R_1, \dots, R_n)$ and $w_0 \in W$ with $C^{\mathcal{I}_{w_0}} \neq \emptyset$ are given. Unfold $\mathfrak{F}$ into

$$\mathfrak{F}^* = (W^*, R_1^*, \dots, R_n^*)$$

where W^* is the set of words $\vec{w}$ of the form (7) with $(w_j, w_{j+1}) \in R_{i_j}$, $w_j \neq w_{j+1}$, $i_j \neq i_{j+1}$, and $m \leq K$, and R_i^* is the smallest equivalence relation containing all pairs $(\vec{w}, \vec{w}iw) \in W^* \times W^*$. Define a model $\mathfrak{M}^* = (\mathfrak{F}^*, \Delta^*, \mathcal{I}^*)$ by setting $\Delta^* = \Delta$ and $\mathcal{I}^*_{\vec{w}} = \mathcal{I}_{w_m}$ for $\vec{w} \in W^*$ of the form (7). Clearly $\mathfrak{M}^*$ is tree-shaped. Moreover, for all $\mathcal{ML}^n_{\mathcal{ALCO}_u}$-concepts D of modal depth not exceeding K, one can show by induction that $D^{\mathcal{I}^*_{w_0}} = D^{\mathcal{I}_{w_0}}$. In particular, $C^{\mathcal{I}^*_{w_0}} \neq \emptyset$. □

Given an $\mathcal{ML}^n_{\mathcal{ALCO}_u}$ concept C_0, let $\mathsf{con}(C_0)$ be the closure under single negation of the set of concepts occurring in C_0. A *type* for C_0 is a subset $\boldsymbol{t}$ of $\mathsf{con}(C_0)$ such that

C1 $\neg C \in \boldsymbol{t}$ iff $C \notin \boldsymbol{t}$, for all $\neg C \in \mathsf{con}(C_0)$;

C2 $C \sqcap D \in \boldsymbol{t}$ iff $C, D \in \boldsymbol{t}$, for all $C \sqcap D \in \mathsf{con}(C_0)$.

Note that there are at most $2^{|\mathsf{con}(C_0)|}$ types for C_0.

A *quasistate for* C_0 is a non-empty set $\boldsymbol{T}$ of types for C_0 satisfying the following conditions:

Q1 for every $\{a\} \in \mathsf{con}(C_0)$, there exists exactly one $\boldsymbol{t} \in \boldsymbol{T}$ such that $\{a\} \in \boldsymbol{t}$;

Q2 for every $\boldsymbol{t} \in \boldsymbol{T}$ and every $\exists r.C \in \boldsymbol{t}$, there exists $\boldsymbol{t}' \in \boldsymbol{T}$ such that $\{\neg D \mid \neg\exists r.D \in \boldsymbol{t}\} \cup \{C\} \subseteq \boldsymbol{t}'$;

Q3 for every $\boldsymbol{t} \in \boldsymbol{T}$, $\exists u.C \in \boldsymbol{t}$ iff there exists $\boldsymbol{t}' \in \boldsymbol{T}$ such that $C \in \boldsymbol{t}'$.

A *basic structure for* C_0 is a pair $(\mathfrak{F}, \boldsymbol{q})$, where $\mathfrak{F} = (W, R_1, \ldots, R_n)$ is a tree-shaped $\mathbf{S5}^n$-frame with root w_0 and $\boldsymbol{q}$ is a function associating with every $w \in W$ a quasistate $\boldsymbol{q}(w)$ for C_0, satisfying

B1 there exists a type $\boldsymbol{t} \in \boldsymbol{q}(w_0)$ such that $C_0 \in \boldsymbol{t}$.

A *run through* $(\mathfrak{F}, \boldsymbol{q})$ is a function ρ mapping each world $w \in W$ into a type $\rho(w) \in \boldsymbol{q}(w)$ and satisfying the following condition for every $\Diamond_i C \in \mathsf{con}(C_0)$:

R1 $\Diamond_i C \in \rho(w)$ if there exists $v \in W$ such that wR_iv and $C \in \rho(v)$.

R2 if $\Diamond_i C \in \rho(w)$ then there exists $v \in W$ such that wR_iv and $C \in \rho(v)$.

An $\mathbf{S5}^n_{\mathcal{ALCO}_u}$ *quasimodel for* C_0 is a triple $\mathfrak{Q} = (\mathfrak{F}, \boldsymbol{q}, \mathfrak{R})$, where $(\mathfrak{F}, \boldsymbol{q})$ is a basic structure for C_0 and $\mathfrak{R}$ is a set of runs through $(\mathfrak{M}, \boldsymbol{q})$ such that the following condition holds:

M1 for every $w \in W$ and every $\boldsymbol{t} \in \boldsymbol{q}(w)$, there exists $\rho \in \mathfrak{R}$ with $\rho(w) = \boldsymbol{t}$;

M2 for every $w \in W$ and for every $\boldsymbol{t} \in \boldsymbol{q}(w)$, with $\{a\} \in \boldsymbol{t}$, there exists exactly one $\rho \in \mathfrak{R}$ such that $\{a\} \in \rho(w)$.

The next lemma provides a link between quasimodels and standard models.

Lemma 24. *An $\mathcal{ML}^n_{\mathcal{ALCO}_u}$-concept C_0 is satisfiable in the root of a tree-shaped $\mathbf{S5}^n_{\mathcal{ALCO}_u}$-model based on a frame $\mathfrak{F}$ iff there exists a $\mathbf{S5}^n_{\mathcal{ALCO}_u}$-quasimodel for C_0 based on the same frame $\mathfrak{F}$.*

Proof. ($\Rightarrow$) Let $\mathfrak{M} = (\mathfrak{F}, \Delta, \mathcal{I})$, with $\mathfrak{F} = (W, R_1, \ldots, R_n)$ be a tree-shaped $\mathbf{S5}^n_{\mathcal{ALCO}_u}$-model of modal depth at most $\mathrm{md}(C_0)$ with $C^{\mathcal{I}_{w_0}} \neq \emptyset$ for the root w_0 of $\mathfrak{F}$. Let $\boldsymbol{t}^{\mathcal{I}_w}(d) = \{C \in \mathsf{con}(C_0) \mid d \in C^{\mathcal{I}_w}\}$, for every $d \in \Delta$ and $w \in W$. Clearly, $\boldsymbol{t}^{\mathcal{I}_w}(d)$ is a concept type for C_0 since it satisfies (**C1**)–(**C2**). We now define a triple $\mathfrak{Q} = (\mathfrak{F}, \boldsymbol{q}, \mathfrak{R})$, where

- $\boldsymbol{q}$ is the function from W to the set of quasistates for C_0 defined by setting $\boldsymbol{q}(w) = \{\boldsymbol{t}^{\mathcal{I}_w}(d) \mid d \in \Delta\}$, for every $w \in W$;
- $\mathfrak{R}$ is the set of functions ρ_d from W to the set of types for C_0 defined by setting $\rho_d(w) = \boldsymbol{t}^{\mathcal{I}_w}(d)$, for every $d \in \Delta$ and $w \in W$.

It is easy to show that $\mathfrak{Q}$ is a quasimodel for C_0. Indeed, $\boldsymbol{q}$ is well-defined, as $\boldsymbol{q}(w)$ is a set of types for C_0 satisfying (**Q1**)–(**Q3**), for every $w \in W$. Moreover, $(\mathfrak{F}, \boldsymbol{q})$ is a basic structure for C_0 since $\mathfrak{M}$ satisfies C_0 in the root w_0 of $\mathfrak{F}$ and thus $(\mathfrak{F}, \boldsymbol{q})$ satisfies (**B1**). The set of runs, $\mathfrak{R}$, by construction, satisfies (**R1**) and (**R2**). Finally, by definition of $\boldsymbol{q}(w)$ and of ρ, $\mathfrak{Q}$ satisfies (**M1**), and, since for every $w \in W$, if $\{a\} \in \boldsymbol{t}$, there is exactly one $d \in \Delta$ such that $d = \{a\}^{\mathcal{I}_w}$, thus $\mathfrak{Q}$ satisfies (**M2**).

($\Leftarrow$) Suppose there is a quasimodel $\mathfrak{Q} = (\mathfrak{F}, \boldsymbol{q}, \mathfrak{R})$ for C_0. Define a model $\mathfrak{M} = (\mathfrak{F}, \Delta, \mathcal{I})$, by setting $\Delta = \mathfrak{R}$, and, for any $A \in \mathsf{N_C}$, $r \in \mathsf{N_R}$ and $a \in \mathsf{N_I}$:

- $A^{\mathcal{I}_w} = \{\rho \in \Delta \mid A \in \rho(w)\}$;
- $r^{\mathcal{I}_w} = \{(\rho, \rho') \in \Delta \times \Delta \mid \{\neg C \mid \neg\exists r.C \in \rho(w)\} \subseteq \rho'(w)\}$;
- $u^{\mathcal{I}_w} = \Delta \times \Delta$;
- $a^{\mathcal{I}_w} = \rho$, for the unique $\rho \in \mathfrak{R}$ such that $\{a\} \in \rho(w)$.

Observe that Δ is well-defined since W is a non-empty set and $\boldsymbol{q}(w) \neq \emptyset$, for all $w \in W$. Thus, by (**M1**), $\mathfrak{R} \neq \emptyset$. Also, by (**Q1**) and (**M2**), $a^{\mathcal{I}_w}$ is well-defined. We now require the following claim.

Claim 5. *For every $C \in \mathsf{con}(C_0)$, $w \in W$ and $\rho \in \Delta$, $\rho \in C^{\mathcal{I}_w}$ iff $C \in \rho(w)$.*

Proof. The proof is by induction on C. The base cases, $C = A$ and $C = \{a\}$, follow immediately from the definition of $\mathfrak{M}$. We then consider the inductive cases.

Let $C = \neg D$. $\neg D \in \rho(w)$ iff, by (**C1**), $D \notin \rho(w)$. By induction, $D \notin \rho(w)$ iff $\rho \notin D^{\mathcal{I}_w}$ iff $\rho \in (\neg D)^{\mathcal{I}_w}$.

Let $C = D \sqcap E$. Similar to the previous case, now by using (**C2**).

Let $C = \exists u.D$. $\rho \in (\exists u.D)^{\mathcal{I}_w}$ iff there exists $\rho' \in D^{\mathcal{I}_w}$. By inductive hypothesis, $\rho' \in D^{\mathcal{I}_w}$ iff $D \in \rho'(w)$. By (**Q3**), the previous step holds iff $\exists u.D \in \rho(w)$.

Let $C = \exists r.D$. ($\Rightarrow$) Suppose that $\rho \in (\exists r.D)^{\mathcal{I}_w}$. Then, there exists $\rho' \in \Delta$ such that $(\rho, \rho') \in r^{\mathcal{I}_w}$ and $\rho' \in D^{\mathcal{I}_w}$. By inductive hypothesis, $\rho' \in D^{\mathcal{I}_w}$ iff $D \in \rho'(w)$. By contradiction, assume that $\exists r.D \notin \rho(w)$, then, by (**C1**), $\neg\exists r.D \in \rho(w)$. By definition of $r^{\mathcal{I}_w}$, since $(\rho, \rho') \in r^{\mathcal{I}_w}$, then, $\neg D \in \rho'(w)$, thus contradicting, by (**C1**), that $D \in \rho'(w)$. ($\Leftarrow$) Conversely, suppose that $\exists r.D \in \rho(w)$. By (**Q2**) and (**M1**), there exists a $\rho' \in \Delta$ such that $\{\neg E \mid \neg\exists r.E \in \rho(w)\} \cup \{D\} \subseteq \rho'(w)$. By inductive hypothesis and the definition of $r^{\mathcal{I}_w}$, $\rho' \in D^{\mathcal{I}_w}$ and $(\rho, \rho') \in r^{\mathcal{I}_w}$. Thus, $\rho \in (\exists r.D)^{\mathcal{I}_w}$.

Let $C = \Diamond_i D$. $\rho \in (\Diamond_i D)^{\mathcal{I}_w}$ iff there exists $v \in W$ such that $(w, v) \in R_i$ and $\rho \in D^{\mathcal{I}_v}$. By inductive hypothesis, $\rho \in D^{\mathcal{I}_v}$ iff $D \in \rho(v)$. Hence, by (**R1**) and (**R2**), such a v exists iff $\Diamond_i D \in \rho(w)$. □

Now we can easily finish the proof of Lemma 24 by observing that, by (**B1**), there exists a world $w' \in W$ and a type $\boldsymbol{t} \in \boldsymbol{q}(w')$ such that $C_0 \in \boldsymbol{t}$. Thus, by (**M1**), there exists $\rho \in \mathfrak{R}$ with $\rho(w') = \boldsymbol{t}$ and, by Claim 5, $\rho \in C_0^{\mathcal{I}_{w'}}$. □

We now show the exponential finite model property in terms of quasimodels.

Lemma 25. *There exists an $\mathbf{S5}^n_{\mathcal{ALCO}_u}$ quasimodel for C_0 iff there exists an $\mathbf{S5}^n_{\mathcal{ALCO}_u}$ quasimodel for C_0 of exponential size in the length of C_0.*

Proof. Assume $\mathfrak{Q} = (\mathfrak{F}, \boldsymbol{q}, \mathfrak{R})$ with $\mathfrak{F} = (W, R_1, \dots, R_n)$ is given such that W is a prefix-closed set of words of the form (7), where $1 \leq i_j \leq m$, $i_j \neq i_{j+1}$, and m is bounded by $\mathrm{md}(C_0)$.

For each $\vec{w} \in W$ and $\boldsymbol{t} \in \boldsymbol{q}(\vec{w})$ we fix a *proto-run,* $\rho_{\vec{w},\boldsymbol{t}} \in \mathfrak{R}$, *for $\vec{w}$ and $\boldsymbol{t}$* with $\rho_{\vec{w},\boldsymbol{t}}(\vec{w}) = \boldsymbol{t}$. Let $\mathcal{F}(\vec{w})$ denote the set of selected proto-runs through $\boldsymbol{q}(\vec{w})$, that is $\mathcal{F}(\vec{w}) = \{\rho_{\vec{w},\boldsymbol{t}} \mid \boldsymbol{t} \in \boldsymbol{q}(\vec{w})\}$. We also fix for any $\Diamond_i C \in \boldsymbol{t} \in \boldsymbol{q}(\vec{w})$ a *proto-witness* $\vec{w}' = v_{\vec{w},i,C,\boldsymbol{t}} \in W$ such that $\vec{w}'$ takes the form $\vec{w}iv$ (and so $(\vec{w}, \vec{w}') \in R_i$) and $C \in \rho_{\vec{w},\boldsymbol{t}}(\vec{w}')$, whenever such a $\vec{w}'$ exists. Note that such a $\vec{w}'$ might not exist since by the definition of R_i as equivalence closure in tree-shaped models the witness for $\Diamond_i C$ might also be $\boldsymbol{t}$ itself if $C \in \boldsymbol{t}$ or $\vec{w}$ could take the form $\vec{w}_0 iw$ for some $\vec{w}_0$ and either $C \in \rho_{\vec{w},\boldsymbol{t}}(\vec{w}_0)$ or there is w' with $\vec{w}_0 iw' \in W$ and $C \in \rho_{\vec{w},\boldsymbol{t}}(\vec{w}_0 iw')$.

We next define inductively sets $W_0, W_1, \dots, W_K \subseteq W$ with K the depth of $\mathfrak{F}$ by setting $W_0 = \{w_0\}$ and $W_{j+1} = W^1_{j+1} \cup \dots \cup W^n_{j+1}$, where for $1 \leq i \leq n$:

$$W^i_{j+1} = \{v_{\vec{w},i,C,\boldsymbol{t}} \mid \Diamond_i C \in \boldsymbol{t} \in \boldsymbol{q}(\vec{w}), \vec{w} \in W_j\}.$$

Let $W' = W_0 \cup \dots \cup W_K$ and let $R'_1, \dots, R'_n$ and $\boldsymbol{q}'$ be the restrictions of $R_1, \dots, R_n$ and $\boldsymbol{q}$ to W':

$$R'_i = R_i \cap (W' \times W'), \quad \boldsymbol{q}'(\vec{w}) = \boldsymbol{q}(\vec{w}) \text{ for } \vec{w} \in W'$$

Also let $\mathfrak{R}'$ denote the set of restrictions of the proto-runs $\rho_{\vec{w},\boldsymbol{t}}$ with $\vec{w} \in W'$ to W'. Let $\mathfrak{F}' = (W', R_1, ' \dots, R'_n)$ and $\mathfrak{Q}' = (\mathfrak{F}', \boldsymbol{q}', \mathfrak{R}')$. Then $\mathfrak{Q}'$ is a quasimodel except that the proto-runs in $\mathfrak{R}'$ might not all satisfy (**R2**). Note that (**R2**) holds for every proto-run $\rho_{\vec{w},\boldsymbol{t}}$ at the world $\vec{w}$ for those $\Diamond_i C$ for which $i_{m-1} \neq i$ (assuming that $\vec{w}$ takes the form (7)) but that there might not be witnesses for $\Diamond_i C$ on $\vec{w}' \neq \vec{w}$ and if $i_{m-1} = i$. To ensure (**R2**) also in these cases we introduce, in a careful way, copies of existing worlds that provide the witnesses for such $\Diamond_i C$.

Denote the set of all bijections on $\mathfrak{R}'$ by $B(\mathfrak{R}')$. The new domain is constructed using copies of the original domain elements indexed by bijections in $B(\mathfrak{R}')$.

Take for any $\vec{w} \in W'$ and $\rho \in \mathfrak{R}'$ the bijection $\sigma_{\vec{w},\rho} \in B(\mathfrak{R}')$ that swaps ρ with the proto-run $\rho_{\vec{w},\boldsymbol{t}}$ with $\boldsymbol{t} = \rho(\vec{w})$ and maps the remaining runs to themselves. The *$\vec{w}$-repair set*, $\mathrm{Rep}(\vec{w})$, is the set of all $\sigma_{\vec{w},\rho}$ with $\rho \in \mathfrak{R}'$. Note that the identity function, id, on $\mathfrak{R}'$ is an element of $\mathrm{Rep}(\vec{w})$.

For any $(\vec{w}, \sigma) \in W' \times B(\mathfrak{R}')$, define

$$Suc_i(\vec{w}, \sigma) = \{(\vec{w}', \sigma') \in W' \times B \mid \vec{w}' = \vec{w}iw \in W', \\ \sigma' = \tau \circ \sigma \text{ for some } \tau \in \mathrm{Rep}(\vec{w})\}$$

and let W'' denote the set of all words

$$\vec{u} = (\vec{w}_0, \sigma_0) i_0 (\vec{w}_1, \sigma_1) i_1 \cdots i_{m-1} (\vec{w}_m, \sigma_m), \qquad (8)$$

where $(\vec{w}_0, \sigma_0) := (w_0, id)$ and $(\vec{w}_{i+1}, \sigma_{i+1}) \in Suc_i(\vec{w}_i, \sigma_i)$ for $0 \leq i < m$.

Define $\boldsymbol{q}''$ by setting $\boldsymbol{q}''(\vec{u}) = \boldsymbol{q}'(\vec{w}_m)$ for any $\vec{u} \in W''$ of the form (8). Associate with any run $\rho \in \mathfrak{R}'$ a unique run $\rho^\uparrow$ on W'' by setting $\rho^\uparrow(\vec{u}) := \sigma_m(\rho)(\vec{w}_m)$ for any $\vec{u} \in W''$ of the form (8) and let $\mathfrak{R}'' = \{\rho^\uparrow \mid \rho \in \mathfrak{R}'\}$.

We show that $(\mathfrak{F}'', \boldsymbol{q}'', \mathfrak{R}'')$ with $\mathfrak{F}'' = (W'', R''_1, \dots, R''_n)$ the tree-shaped $\mathbf{S5}^n_{\mathcal{ALCO}_u}$ frame defined by W'', is as required.

We show that conditions (**B1**), (**R1**), (**R2**), (**M1**), and (**M2**) are satisfied.

Observe first that $\rho \mapsto \rho^\uparrow$ is a bijection from $\mathfrak{R}'$ onto $\mathfrak{R}''$. To show this observe that the mapping is surjective by definition. To show that it is injective, assume that $\rho_1 \neq \rho_2$. Take $\vec{w}_m \in W'$ of the form (7) with $\rho_1(\vec{w}_m) \neq \rho_2(\vec{w}_m)$. Then let

$$\vec{v} = (\vec{w}_0, id) i_0 (\vec{w}_1, id) i_1 \cdots i_{m-1} (\vec{w}_m, id) \in W'',$$

where $\vec{w}_{i+1} = \vec{w}_i iw_{i+1}$ for $i < m$. We have $\rho^\uparrow_1(\vec{v}) = id(\rho_1)(\vec{w}_m) \neq id(\rho_2)(\vec{w}_m)$, as required.

(**B1**) is straightforward.

We prove (**M1**) and (**M2**) in one go by showing that for any $\vec{u} \in W''$ of the form (8) and $\boldsymbol{t} \in \boldsymbol{q}''(\vec{u})$ there is a bijection between $V_0 = \{\rho \in \mathfrak{R}'' \mid \rho(\vec{u}) = \boldsymbol{t}\}$ and $V_1 = \{\rho \in \mathfrak{R}' \mid \rho(\vec{w}_m) = \boldsymbol{t}\}$ (recall that $\boldsymbol{q}''(\vec{u}) = \boldsymbol{q}'(\vec{w}_m)$).

Indeed, we claim that $F\colon V_0 \to V_1$ defined by setting $F(\rho^\uparrow) = \sigma_m(\rho)$ is a bijection. As $\rho \mapsto \rho^\uparrow$ is bijective and $\rho^\uparrow(\vec{u}) = \sigma_m(\rho)(\vec{w}_m)$, F is a well defined function into V_1. It is injective since σ_m is also bijective, by definition. It is surjective since for any $\rho \in \mathfrak{R}'$ with $\rho(\vec{w}_m) = \boldsymbol{t}$ we have $F((\sigma_m^{-1}(\rho))^\uparrow) = \sigma_m \sigma_m^-(\rho) = \rho$ and $\sigma_m^{-1}(\rho) \in \mathfrak{R}'$.

We next show (**R1**) and (**R2**). Assume that $\Diamond_i C \in \mathsf{con}(C_0)$ and $\rho^\uparrow \in \mathfrak{R}''$. Consider $\vec{u} \in W''$ of the form (8).

For (**R1**), assume there exists $\vec{v} \in W''$ such that $\vec{u} R''_i \vec{v}$ and $C \in \rho^\uparrow(\vec{v})$. We have to show that $\Diamond_i C \in \rho^\uparrow(\vec{u})$. Observe that the runs in $\mathfrak{R}'$ satisfy (**R1**) in $\mathfrak{Q}'$ and that, moreover,

R3 if $\Diamond_i C \in \rho(\vec{w})$ with $\rho \in \mathfrak{R}'$ and $(\vec{w}, \vec{w}') \in R'_i$, then $\Diamond_i C \in \rho(\vec{w}')$.

We distinguish the following four cases.

1. $\vec{v} = \vec{u}$. Then $\Diamond_i C \in \rho^\uparrow(\vec{u})$ since we have $\Diamond_i C \in \boldsymbol{t}$ whenever $C \in \boldsymbol{t}$ for $\Diamond_i C \in \mathsf{con}(C_0)$ for any $\boldsymbol{t}$.
2. $\vec{v} = \vec{u}i(\vec{u}iw_{m+1}, \sigma_{m+1})$ for some (w_{m+1}, σ_{m+1}) with $\vec{w}_m iw_{m+1} \in W'$ with $\sigma_{m+1} = \tau \circ \sigma_n$ and $\tau \in \mathrm{Rep}(\vec{w}_m)$. We have $\rho^\uparrow(\vec{u}) = \sigma_m(\rho)(\vec{w}_m)$ and $\rho^\uparrow(\vec{v}) = \tau \circ \sigma_m(\rho)(\vec{w}_m iw_{m+1})$. From $C \in \tau \circ \sigma_m(\rho)(\vec{w}_m iw_{m+1})$ it follows that $\Diamond_i C \in \tau \circ \sigma_m(\rho)(\vec{w}_m)$ since $\tau \circ \sigma_m(\rho)$ satisfies condition (**R1**) for runs in $\mathfrak{Q}'$. Now $\tau \circ \sigma_m(\rho)$ coincides with $\sigma_m(\rho)$ on $\vec{w}_m$. Hence $\Diamond_i C \in \sigma_m(\rho)(\vec{w}_m)$, as required.
3. $\vec{v} = (\vec{w}_0, \sigma_0) i_0 (\vec{w}_1, \sigma_1) i_1 \cdots i_{m-2} (\vec{w}_{m-1}, \sigma_{m-1})$ and $i_{m-1} = i$.
   We have $\sigma_m = \tau' \circ \sigma_{m-1}$ for some $\tau' \in \mathrm{Rep}(\vec{w}_{m-1})$. Note that $\rho^\uparrow(\vec{u}) = \tau' \circ \sigma_{m-1}(\rho)(\vec{w}_{m-1} iw_m)$ and $\rho^\uparrow(\vec{v}) = \sigma_{m-1}(\rho)(\vec{w}_{m-1})$. Now $\tau' \circ \sigma_{m-1}(\rho)$ coincides with $\sigma_{m-1}(\rho)$ on $\vec{w}_{m-1}$. Hence $C \in \tau' \circ \sigma_{m-1}(\rho)(\vec{w}_{m-1})$. Hence $\Diamond_i C \in \tau' \circ \sigma_{m-1}(\rho)(\vec{w}_{m-1} iw_m)$ since $\tau' \circ \sigma_{m-1}$ satisfies condition (**R1**) for runs in $\mathfrak{Q}'$, as required.

4. for $\vec{v}' = (\vec{w}_0, \sigma_0) i_0 (\vec{w}_1, \sigma_1) i_1 \cdots i_{m-2} (\vec{w}_{m-1}, \sigma_{m-1})$ and $i_{m-1} = i$ we have $\vec{v} = \vec{v}' i (\vec{w}_{m-1} i w'_m, \sigma'_m)$ for some (w'_m, σ'_m) with $\vec{w}_{m-1} i w'_m \in W'$ and $\sigma'_m = \tau \circ \sigma_{m-1}$ for some $\tau \in \mathrm{Rep}(\vec{w}_{m-1})$. We can then first show that $\Diamond_i C \in \rho^\uparrow(\vec{v}')$ in exactly the same way as in the proof of Point 2. Now we can argue in the same way as in the proof of Point 3 and using (**R3**) that $\Diamond_i C \in \rho^\uparrow(\vec{u})$.

For (**R2**), assume $\Diamond_i C \in \rho^\uparrow(\vec{u})$. We have to show that there exists $\vec{v} \in W''$ such that $\vec{u} R''_i \vec{v}$ and $C \in \rho^\uparrow(\vec{v})$. We have $\rho^\uparrow(\vec{u}) = \sigma_m(\rho)(\vec{w}_m)$ and distinguish the following three cases:

1. $C \in \sigma_m(\rho)(\vec{w}_m)$. Then we are done since then for $\vec{v} = \vec{u}$ we have $C \in \rho^\uparrow(\vec{v})$ and we have $\vec{u} R''_i \vec{v}$.
2. for $\boldsymbol{t} = \sigma_m(\rho)(\vec{w}_m)$ we have for the proto-run $\rho_{\vec{w}_m, \boldsymbol{t}} \in \mathfrak{R}$ that there is a proto-witness $\vec{w}_{m+1} = v_{\vec{w}_m, i, C, \boldsymbol{t}} \in W'$ of the form $\vec{w}_m i w_{m+1}$. Then $C \in \rho_{\vec{w}_m, \boldsymbol{t}}(\vec{w}_{m+1})$.
   Then let $\vec{v} = \vec{u} i (\vec{w}_{m+1}, \tau \circ \sigma_m)$ where τ swaps $\sigma_m(\rho)$ with $\rho_{\vec{w}_m, \boldsymbol{t}}$. We have $\vec{u} R''_i \vec{v}$ and $\rho^\uparrow(\vec{v}) = \tau \circ \sigma_m(\rho)(\vec{w}_{m+1}) = \rho_{\vec{w}_m, \boldsymbol{t}}(\vec{w}_{m+1})$. So $C \in \rho^\uparrow(\vec{v})$, as required.
3. $i_{m-1} = i$ and $\Diamond_i C \in \sigma_{m-1}(\rho)(\vec{w}_{m-1})$. If $C \in \sigma_{m-1}(\rho)(\vec{w}_{m-1})$, then $C \in \rho^\uparrow(\vec{v})$ for $\vec{v} = (\vec{w}_0, \sigma_0) i_0 (\vec{w}_1, \sigma_1) i_1 \cdots i_{m-2} (\vec{w}_{m-1}, \sigma_{m-1})$ and we are done.
   Otherwise we can proceed with $\sigma_{m-1}(\rho)(\vec{w}_{m-1})$ in the same way as we did with $\sigma_m(\rho)(\vec{w}_m)$ in Point 2 and find for $\vec{v}' = (\vec{w}_0, \sigma_0) i_0 (\vec{w}_1, \sigma_1) i_1 \cdots i_{m-2} (\vec{w}_{m-1}, \sigma_{m-1})$ a $\vec{v} = \vec{v}' i (\vec{w}_{m-1} i w'_m, \sigma'_m)$ for some (w'_m, σ'_m) with $\vec{w}_{m-1} i w'_m \in W'$ and $\sigma'_m = \tau \circ \sigma_{m-1}$ for some $\tau \in \mathrm{Rep}(\vec{w}_{m-1})$ such that $C \in \rho^\uparrow(\vec{v})$.

Clearly, the quasimodel $(\mathfrak{F}'', \boldsymbol{q}'', \mathfrak{R}'')$ is of at most exponential size, as required. □

The exponential finite model property now follows from Lemma 25 using Lemma 24. □

Theorem 12. *$\mathbf{K}\boldsymbol{f}^{*n}_{\mathcal{ALCO}^\iota_u}$ concept satisfiability under global ontology is decidable with expanding domains, for $n \geq 1$.*

As before, it suffices to consider total concept satisfiability under ontologies without definite descriptions. For $\mathbf{K}\boldsymbol{f}^{*n}_{\mathcal{ALCO}_u}$, the decidability proof for concept satisfiability without ontologies is easily extended to a proof with ontologies, so we consider concept satisfiability without ontology. Call a finite frame $\mathfrak{F} = (W, R_1, \ldots, R_n, R)$ a *finite tree-shaped* frame if R is the transitive closure of $R_1 \cup \cdots \cup R_n$ and there exists a $w_0 \in W$ such that the domain W of $\mathfrak{F}$ is a prefix closed set of words of the form

$$\vec{w} = w_0 i_0 w_1 \cdots i_{n-1} w_m \tag{9}$$

where $1 \leq i_j \leq n$, and

$$R_i = \{(\vec{w}, \vec{w} i w) \mid \vec{w} i w \in W\}.$$

We call w_0 the root of $\mathfrak{F}$. A *finite tree-shaped model* takes the form $\mathfrak{M} = (\mathfrak{F}, \Delta, \mathcal{I})$ with $\mathfrak{F}$ a finite tree-shaped frame. A straightforward unfolding argument shows that any $\mathcal{ML}^n_{\mathcal{ALCO}_u}$-concept that is satisfiable in a $\mathbf{K}\boldsymbol{f}^{*n}_{\mathcal{ALCO}_u}$-model is also satisfiable in a finite tree-shaped model. So we start with finite tree-shaped models.

Given an $\mathcal{ML}^n_{\mathcal{ALCO}_u}$-concept C_0, let $\mathsf{con}(C_0)$ be the closure under single negation of the set of concepts occurring in C_0. A *type* for C_0 is a subset $\boldsymbol{t}$ of $\mathsf{con}(C_0)$ such that

C1 $\neg C \in \boldsymbol{t}$ iff $C \notin \boldsymbol{t}$, for all $\neg C \in \mathsf{con}(C_0)$;

C2 $C \sqcap D \in \boldsymbol{t}$ iff $C, D \in \boldsymbol{t}$, for all $C \sqcap D \in \mathsf{con}(C_0)$.

Note that there are at most $2^{|\mathsf{con}(C_0)|}$ types for C_0. We fix an ordering $\boldsymbol{t}_1, \ldots, \boldsymbol{t}_k$ of the set of types for C_0. We use vectors $\vec{x} = (x_1, \ldots, x_k) \in (\mathbb{N} \cup \{\infty\})^k$ to represent that the type $\boldsymbol{t}_i$ is satisfied by x_i-many elements in a world w, for $1 \leq i \leq n$. Let $|\vec{x}| = \sum_{i=1}^k x_i$. A *quasistate* for C_0 is a vector $\vec{x} = (x_1, \ldots, x_k) \in (\mathbb{N} \cup \{\infty\})^k$ with $|\vec{x}| > 0$ satisfying the following conditions:

Q1 for every $\{a\} \in \mathsf{con}(C_0)$,

$$\sum_{\{a\} \in \boldsymbol{t}_i} x_i = 1;$$

Q2 for every $x_i > 0$ and every $\exists r.C \in \boldsymbol{t}_i$, there exists $x_j > 0$ such that $\{\neg D \mid \neg \exists r.D \in \boldsymbol{t}_i\} \cup \{C\} \subseteq \boldsymbol{t}_j$;

Q3 for every $x_i > 0$ and $\exists u.C \in \boldsymbol{t}_i$ iff there exists $x_j > 0$ with such that $C \in \boldsymbol{t}_j$.

A *basic structure for* C_0 is a pair $(\mathfrak{F}, \boldsymbol{q})$, where $\mathfrak{F} = (W, R_1, \ldots, R_n)$ is a finite tree-shaped frame with root w_0 and $\boldsymbol{q}$ is a function associating with every $w \in W$ a quasistate $\boldsymbol{q}(w)$ for C_0, satisfying

B1 there exists $i \leq k$ with $\boldsymbol{q}(w_0)_i > 0$ such that $C_0 \in \boldsymbol{t}_i$.

Call a subset V of W *R_i-closed* if $v \in V$ whenever $w \in V$ and $w R_i v$. Call V *R-closed* if V is R_i-closed for $1 \leq i \leq n$. A *run through* $(\mathfrak{F}, \boldsymbol{q})$ is a partial function ρ mapping worlds $w \in W$ to natural numbers $\rho(w) \leq k$ with $\boldsymbol{q}(w)_{\rho(w)} > 0$ such that the domain of ρ is R-closed and satisfies the following condition for every $\Diamond_i C \in \mathsf{con}(C_0)$:

R1 $\Diamond_i C \in \boldsymbol{t}_{\rho(w)}$ if there exists $v \in W$ such that $w R_i v$ and $C \in \boldsymbol{t}_{\rho(v)}$;

R2 if $\Diamond_i C \in \boldsymbol{t}_{\rho(w)}$ then there exists $v \in W$ such that $w R_i v$ and $C \in \boldsymbol{t}_{\rho(v)}$.

A *quasimodel for* C_0 is a triple $\mathfrak{Q} = (\mathfrak{F}, \boldsymbol{q}, \mathfrak{R})$, where $(\mathfrak{F}, \boldsymbol{q})$ is a basic structure for C_0, and $\mathfrak{R}$ is a set of runs through $(\mathfrak{M}, \boldsymbol{q})$ such that the following condition holds:

M1 for every $w \in W$ and every $i \leq k$,

$$|\{\rho \in \mathfrak{R} \mid \rho(w) = i\}| = \boldsymbol{q}(w)_i.$$

The following lemma provides a link between quasimodels for C_0 and models satisfying C_0.

Lemma 26. *An $\mathcal{ML}^n_{\mathcal{ALCO}_u}$-concept C_0 is satisfiable in the root of a finite tree-shaped model iff there exists a quasimodel for C_0 based on the same frame.*

Proof. ($\Rightarrow$) Let $\mathfrak{M} = (\mathfrak{F}, \Delta, \mathcal{I})$, with $\mathfrak{F} = (W, R_1, \ldots, R_n)$ be a finite tree-shaped model with $C^{\mathcal{I}_{w_0}} \neq \emptyset$ for the root w_0 of $\mathfrak{F}$. Let $\boldsymbol{t}^{\mathcal{I}_w}(d) = \{C \in \mathsf{con}(C_0) \mid d \in C^{\mathcal{I}_w}\}$, for every $d \in \Delta$ and $w \in W$. Clearly, $\boldsymbol{t}^{\mathcal{I}_w}(d)$ is a type for C_0 since it satisfies (**C1**)–(**C2**). We now define a triple $\mathfrak{Q} = (\mathfrak{F}, \boldsymbol{q}, \mathfrak{R})$, where

- $\boldsymbol{q}$ is the function from W to the set of quasistates for C_0 defined by setting $\boldsymbol{q}(w) = (x_1, \ldots, x_k)$ with
$$x_i = |\{d \in \Delta^w \mid \boldsymbol{t}^{\mathcal{I}_w}(d) = \boldsymbol{t}_i\}|$$
for $1 \leq i \leq k$ and every $w \in W$;
- $\mathfrak{R}$ is the set of functions ρ_d from W to $\{1, \ldots, k\}$ defined by setting $\rho_d(w) = i$ if $\boldsymbol{t}^{\mathcal{I}_w}(d) = \boldsymbol{t}_i$, for every $d \in \Delta$ and $w \in W$.

It is easy to show that $\mathfrak{Q}$ is a quasimodel for C_0.
($\Leftarrow$) Suppose there is a quasimodel $\mathfrak{Q} = (\mathfrak{F}, \boldsymbol{q}, \mathfrak{R})$ for C_0. Define a model $\mathfrak{M} = (\mathfrak{F}, \Delta, \mathcal{I})$, by setting $\Delta^w = \{\rho \in \mathfrak{R} \mid w \in \mathrm{dom}(\rho)\}$, and, for any $A \in \mathsf{N_C}$, $r \in \mathsf{N_R}$ and $a \in \mathsf{N_I}$:

- $A^{\mathcal{I}_w} = \{\rho \in \Delta^w \mid A \in \rho(w)\}$;
- $r^{\mathcal{I}_w} = \{(\rho, \rho') \in \Delta^w \times \Delta^w \mid \{\neg C \mid \neg\exists r.C \in \rho(w)\} \subseteq \rho'(w)\}$;
- $u^{\mathcal{I}_w} = \Delta^w \times \Delta^w$;
- $a^{\mathcal{I}_w} = \rho$, for the unique $\rho \in \Delta^w$ such that $\{a\} \in \rho(w)$.

Observe that Δ is well-defined and expanding since $|\vec{x}| > 0$ for every quasistate $\vec{x}$ and since the domain of each $\rho \in \mathfrak{R}$ is R-closed. The following claim is now straightforward from the definition of quasimodels for C_0.

Claim 6. *For every $C \in \mathsf{con}(C_0)$, $w \in W$ and $\rho \in \Delta$, $\rho \in C^{\mathcal{I}_w}$ iff $C \in \boldsymbol{t}_{\rho(w)}$.*

Now we can easily finish the proof of the proposition by observing that, by (**B1**), there exists $i \leq k$ with $\boldsymbol{q}(w_0)_i > 0$ such that $C_0 \in \boldsymbol{t}_i$. Then there exists a $\rho \in \mathfrak{R}$ with $\rho(w_0) = i$. Hence, by the above claim, $\rho \in C_0^{\mathcal{I}_{w_0}}$. □

The *size* of a quasimodel $\mathfrak{Q} = (\mathfrak{F}, \boldsymbol{q}, \mathfrak{R})$ with $\mathfrak{F} = (W, R_1, \ldots, R_n)$ is defined as
$$|W| \times \max\{|\boldsymbol{q}(w)| \mid w \in W\}.$$
Our next aim is to show that there exists a quasimodel for C_0 of size bounded by a recursive function in $|C_0|$ (in particular with $|\boldsymbol{q}(w)| < \infty$ for all w). We are going to apply Dickson's Lemma. For some $k > 0$, let $(\mathbb{N}^k, \leq)$ be the set of k-tuples of natural numbers ordered by the natural product ordering: for $\vec{x} = (x_1, \ldots, x_k)$ and $\vec{y} = (y_1, \ldots, y_k)$ we set $\vec{x} \leq \vec{y}$ if $x_i \leq y_i$ for $1 \leq i \leq k$. A pair $\vec{x}, \vec{y}$ with $\vec{x} \leq \vec{y}$ is called an *increasing pair*. Dickson's Lemma states every infinite sequence $\vec{x}_1, \vec{x}_2, \ldots \in \mathbb{N}^k$ contains an increasing pair $\vec{x}_{i_1}, \vec{x}_{i_2}$ with $i_1 < i_2$. In fact, assuming $|\vec{x}_i| \leq |\vec{x}_{i+1}|$ for all $i \geq 0$ and given recursive bounds on $|\vec{x}_1|$ and $|\vec{x}_{i+1}| - |\vec{x}_i|$ one can compute a recursive bound on the length of the longest sequence without any increasing pair (Figueira et al. 2011).

Assume a finite tree-shaped frame $\mathfrak{F} = (W, R_1, \ldots, R_n)$ with root w_0 and worlds of the form (9) is given. Then we say that $\vec{w}$ is the *predecessor* of $\vec{w}iw \in W$ in $\mathfrak{F}$ and that $\vec{w}iw \in W$ is a *successor* of $\vec{w}$ in $\mathfrak{F}$. Worlds reachable along a path of predecessors from a world are called *ancestors* and worlds reachable along a path of successors from a world are called *descendants*. We set $\vec{w} < \vec{w}'$ if $\vec{w}$ is a predecessor of $\vec{w}'$ and $W_{\vec{w}} = \{\vec{w}\} \cup \{\vec{v} \in W \mid \vec{w} < \vec{v}\}$.

We now show that one can reduce quasimodels for C_0 to quasimodels for C_0 not having increasing pairs $|\boldsymbol{q}(\vec{w})|, |\boldsymbol{q}(\vec{w}')|$ with $\vec{w} < \vec{w}'$ and satisfying recursive bounds for $|\boldsymbol{q}(w_0)|$, $|\boldsymbol{q}(\vec{w}iw)| - |\boldsymbol{q}(\vec{w})|$, and the outdegree of each $\vec{w}$. We then obtain a recursive bound on the size of such quasimodels from Dickson's Lemma.

Lemma 27. *There exists a quasimodel for C_0 iff there exists a quasimodel $\mathfrak{Q} = (\mathfrak{F}, \boldsymbol{q}, \mathfrak{R})$ for C_0 with*

1. *$|\boldsymbol{q}(w_0)| \leq 2^{|C_0|}$, for the root w_0 of $\mathfrak{Q}$;*
2. *if v is a successor of w, then $|\boldsymbol{q}(v)| \leq |\boldsymbol{q}(w)| + 2^{|C_0|}$;*
3. *if $w < v$, then $\boldsymbol{q}(w) \not\leq \boldsymbol{q}(v)$;*
4. *the outdgree of w in $\mathfrak{F}$ is bounded by $|\boldsymbol{q}(w)| \times |C_0|$.*

Proof. Assume that a quasimodel $\mathfrak{Q}$ for C_0 with $\mathfrak{Q} = (\mathfrak{F}, \boldsymbol{q}, \mathfrak{R})$ and $\mathfrak{F} = (W, R_1, \ldots, R_n)$ with root w_0 is given. We may assume that every run $\rho \in \mathfrak{R}$ has a *root* $w \in W$ in the sense that $W_w = \mathrm{dom}(\rho)$. We let $\mathfrak{R}_w$ denote the set of runs with root w. We introduce a few rules that allow us to modify $\mathfrak{Q}$:

Minimize root. Assume $\boldsymbol{q}(w_0) = (x_1, \ldots, x_k)$. Pick for any $x_i > 0$ a single $\rho_i \in \mathfrak{R}$ with $\rho_i(w_0) = i$. Now construct an updated $\mathfrak{Q}' = (\mathfrak{F}', \boldsymbol{q}', \mathfrak{R}')$ by defining

- $\mathfrak{F}' = \mathfrak{F}$,
- $\boldsymbol{q}'(w_0) = (\min\{x_1, 1\}, \ldots, \min\{x_k, 1\})$ and $\boldsymbol{q}'(w) = \boldsymbol{q}(w)$ for all $w \neq w_0$,
- $\mathfrak{R}'$ as the set of selected runs ρ_i, all runs in $\mathfrak{R} \setminus \mathfrak{R}_{w_0}$, and the restrictions of all remaining $\rho \in \mathfrak{R}_{w_0}$ to W_w for w a successor of w_0.

It is easy to see that $\mathfrak{Q}'$ is a quasimodel for C_0 satisfying Condition 1.

Minimize non-root: Assume v is a successor of w and $\boldsymbol{q}(v) = (x_1, \ldots, x_k)$. Call $i \leq k$ increasing at v if there exists $\rho_i \in \mathfrak{R}_v$ such that $\rho_i(v) = i$. Select for any such i such a $\rho_i \in \mathfrak{R}_v$. We construct an updated $\mathfrak{Q}' = (\mathfrak{F}', \boldsymbol{q}', \mathfrak{R}')$ by defining

- $\mathfrak{F}' = \mathfrak{F}$,
- $\boldsymbol{q}'(v) = (x_i', \ldots, x_k')$ with
$$x_i' = |\{\rho \in \mathfrak{R} \mid w \in \mathrm{dom}(\rho), \rho(v) = i\}| + 1$$
if i is increasing at v and $x_i' = x_i$ otherwise. Set $\boldsymbol{q}'(w') = \boldsymbol{q}(w')$ for all $w' \neq v$.
- $\mathfrak{R}'$ as the set of selected runs ρ_i, all runs in $\mathfrak{R} \setminus \mathfrak{R}_v$, and the restrictions of all remaining $\rho \in \mathfrak{R}_v$ to $W_{v'}$ for v' a successor of v.

It is easy to see that $\mathfrak{Q}'$ is a quasimodel for C_0 satisfying Condition 2 for w and v.

Drop interval 1: Consider $\vec{w} = \vec{w}'iw \in W$ and $\vec{v} = \vec{v}'jv \in W$ with $\vec{w} < \vec{v}$ and

- $\boldsymbol{q}(\vec{w}) = (x_1, \dots, x_k)$;
- $\boldsymbol{q}(\vec{v}) = (y_1, \dots, y_k)$;
- $x_i \leq y_i$ for all $i \leq k$.

Now construct an updated $\mathfrak{Q}' = (\mathfrak{F}', \boldsymbol{q}', \mathfrak{R}')$ by defining

- $\mathfrak{F}'$ is the frame determined by W' with
$$W' = (W \setminus W_{\vec{w}}) \cup \{\vec{w}' i \vec{r} \mid \vec{v}' j \vec{r} \in W_{\vec{v}}\}.$$
- Set $\boldsymbol{q}'(\vec{u}) = \boldsymbol{q}(\vec{u})$ for all $\vec{u} \in W \setminus W_{\vec{w}}$ and $\boldsymbol{q}'(\vec{w}' i \vec{r}) = \boldsymbol{q}(\vec{v}' j \vec{r})$ for all $\vec{v}' j \vec{r} \in W_{\vec{v}}$.
- Take for any $\rho \in \mathfrak{R}$ with $\vec{w}' \in \mathrm{dom}(\rho)$ a $f(\rho) \in \mathfrak{R}$ with $\vec{v} \in \mathrm{dom}(f(\rho))$ such that $f(\rho)(\vec{v}) = \rho(\vec{w})$. We can ensure that all $f(\rho)$ are distinct since $x_i \leq y_i$ for all $i \leq k$. Then let $\rho' \in \mathfrak{R}'$ with ρ' defined by setting $\rho'(\vec{u}) = \rho(\vec{u})$ for $\vec{u} \in W \setminus W_{\vec{w}}$ and $\rho'(\vec{w}' i \vec{r}) = f(\rho)(\vec{v}' j \vec{r})$ for all $\vec{v}' j \vec{r} \in W_{\vec{v}}$.
  Also add all $\rho \in \mathfrak{R}$ with $\mathrm{dom}(\rho) \cap W_{\vec{w}} = \emptyset$ to $\mathfrak{R}'$. Finally, for all ρ with $\mathrm{dom}(\rho) \subseteq W_{\vec{w}}$ which were not selected as an $f(\rho)$, add the ρ' to $\mathfrak{R}'$ defined by setting $\rho'(\vec{w}' i \vec{r}) = \rho(\vec{v}' j \vec{r})$ for all $\vec{v}' j \vec{r} \in W_{\vec{v}}$.

It is easy to see that $\mathfrak{Q}'$ is a quasimodel for C_0.

Drop interval 2: Consider the root w_0 and $v \in W$ with

- $\boldsymbol{q}(w_0) = (x_1, \dots, x_k)$;
- $\boldsymbol{q}(v) = (y_1, \dots, y_k)$;
- $x_i \leq y_i$ for all $i \leq k$.

Then construct an updated $\mathfrak{Q}' = (\mathfrak{F}', \boldsymbol{q}', \mathfrak{R}')$ by restricting $\mathfrak{Q}$ to W_v in the obvious way.

Restrict outdegree. Consider $\vec{w} \in W$ and assume $\boldsymbol{q}(\vec{w}) = (x_1, \dots, x_k)$. To bound the number of successors of $\vec{w}$ to $|\boldsymbol{q}(\vec{w})| \times |C_0|$ pick for every $j \leq k$ with $x_j > 0$ and every $\Diamond_i C \in \boldsymbol{t}_j$ for every run $\rho \in \mathfrak{R}$ with $\rho(\vec{w}) = j$ a successor $\vec{w} i w \in W$ with $C \in \boldsymbol{t}_{\rho(\vec{w} i w)}$ or $\Diamond_i C \in \boldsymbol{t}_{\rho(\vec{w} i w)}$. Let V be the set of successors of W picked in this way. Then construct an updated $\mathfrak{Q}' = (\mathfrak{F}', \boldsymbol{q}', \mathfrak{R}')$ by restricting $\mathfrak{Q}$ to

$$W' = (W \setminus W_{\vec{w}}) \cup \{\vec{w}\} \cup \bigcup_{\vec{v} \in V} W_{\vec{v}}.$$

It is easy to see that $\mathfrak{Q}'$ is a quasimodel for C_0 such that the outdegree of $\vec{w}$ is bounded by $|\boldsymbol{q}(\vec{w})| \times |C_0|$.

By applying the rules above exhaustively, we obtain a quasimodel satisfying the conditions of the lemma. □

As argued above, it follows from Lemma 27 that we have a recursive bound on the size of a quasimodel for C_0. By Lemma 26, we then also have a recursive bound on the size of a finite tree-shaped model satisfying C_0. This shows the decidability of concept-satisfiability for $\mathbf{Kf}^{*n}_{\mathcal{ALCO}_u}$ in expanding domain models.

We now discuss concept satisfiability for $\mathbf{GL}_{\mathcal{ALCO}^\iota_u}$ and $\mathbf{Grz}_{\mathcal{ALCO}^\iota_u}$ under expanding domains. We focus on $\mathbf{GL}_{\mathcal{ALCO}^\iota_u}$, $\mathbf{Grz}_{\mathcal{ALCO}^\iota_u}$ can be treated similarly. Note that a straightforward selective filtration argument applied to appropriate quasimodels shows that in $\mathbf{GL}_{\mathcal{ALCO}^\iota_u}$ over expanding domain models every satisfiable concept is satisfied in a finite model. This observation implies that a $\mathcal{ML}^n_{\mathcal{ALCO}^\iota_u}$-concept C with a single modal operator is satisfiable in an expanding domain model for $\mathbf{GL}_{\mathcal{ALCO}^\iota_u}$ iff it is satisfiable in a model for $\mathbf{Kf}^{*n}_{\mathcal{ALCO}^\iota_u}$ with expanding domains, where the modal operator in C is interpreted by the transitive closure of $R_1 \cup \dots \cup R_n$. Hence the decidability of concept-satisfiability for $\mathbf{GL}_{\mathcal{ALCO}^\iota_u}$ over expanding domain models is a consequence of the decidability of concept satisfiability for $\mathbf{Kf}^{*n}_{\mathcal{ALCO}^\iota_u}$ over expanding domain models.

The decidability of concept-satisfiability for $\mathbf{GL}_{\mathcal{ALCO}^\iota_u}$ over expanding domain models can also be derived from decidability results for expanding domain product modal logics. We define the relevant modal logics using DL syntax. Consider the modal DL with a single role, o, interpreted as a Noetherian strict linear order. In DL syntax, concepts in $\mathcal{ML}^n_{lin}$ are of the form

$$C ::= A \mid \neg C \mid (C \sqcap C) \mid \exists o.C \mid \Diamond_i C,$$

where $i \in I$.

Theorem 28. *Let $\mathcal{C}$ be a class of frames. Then $\mathcal{C}$-satisfiability of $\mathcal{ML}^n_{\mathcal{ALCO}_u}$-concepts can be reduced in double exponential time to $\mathcal{C}$-satisfiability of $\mathcal{ML}^n_{lin}$-concepts, both with constant and expanding domains.*

Proof. The proof is similar to the proof of Theorem 10 (1). Assume an $\mathcal{ML}^n_{\mathcal{ALCO}_u}$-concept C is given. Assume for simplicity that C contains a single modal operator, denoted by $\Box$. Let m be the modal depth of C. Let $\mathsf{con}(C)$ be the closure under single negation of the set of concepts occurring in C. A *type* for C is a subset $\boldsymbol{t}$ of $\mathsf{con}(C)$ such that $\neg C \in \boldsymbol{t}$ iff $C \notin \boldsymbol{t}$, for all $\neg C \in \mathsf{con}(\mathcal{O})$. A *quasistate for D* is a non-empty set $\boldsymbol{T}$ of types for C. The *description of $\boldsymbol{T}$* is the $\mathcal{ML}^n_{\mathcal{ALCO}_u}$-concept

$$\Xi_{\boldsymbol{T}} = \forall u.(\bigsqcup_{\boldsymbol{t} \in \boldsymbol{T}} \boldsymbol{t}) \sqcap \bigsqcap_{\boldsymbol{t} \in \boldsymbol{T}} \exists u.\boldsymbol{t}.$$

Let $\mathcal{S}_C$ denote the set of all quasistates $\boldsymbol{T}$ for C that are *$\mathcal{ALCO}_u$-satisfiable*, that is, such that $\Xi^t_{\boldsymbol{T}}$ is satisfiable, where $\Xi^t_{\boldsymbol{T}}$ denotes the result of replacing every outermost occurrence of $\Diamond_i D$ by $A_{\Diamond_i D}$, for a fresh concept name $A_{\Diamond_i D}$. We know that $\mathcal{S}_C$ can be computed in double exponential time.

Let $\exists o^+.D = D \sqcup \exists o.D$ and $\forall o^+.D = D \sqcap \forall o.D$. We reserve for any $D \in \mathsf{con}(C)$ of the form $\{a\}$ or $\exists r.D'$ or $\exists u.D'$ a fresh concept name A_D. Define a mapping $.^\sharp$ that associates with every $\mathcal{ML}^n_{\mathcal{ALCO}_u}$-concept an $\mathcal{ML}^n_{lin}$-concept by replacing outermost occurrences of concepts of the form $\{a\}$ or $\exists r.D'$ or $\exists u.D'$ by the respective fresh concept name.

Let C^* denote the conjunction of:

1. $C^\sharp$;
2. $\Box^{\leq m} \bigsqcup_{\boldsymbol{T} \in \mathcal{S}_C} (\forall o^+.(\bigsqcup_{\boldsymbol{t} \in \boldsymbol{T}} \boldsymbol{t}^\sharp) \sqcap \bigsqcap_{\boldsymbol{t} \in \boldsymbol{T}} \exists o^+.\boldsymbol{t}^\sharp)$;
3. $\Box^{\leq m} \bigsqcap_{\exists u.D \in \mathsf{con}(C)} (A_{\exists u.D} \Leftrightarrow \exists o^+.D^\sharp)$;
4. $\Box^{\leq m} \exists o^+.\{a\}^\sharp$, for every a in C;
5. $\Box^{\leq m} \forall o^+.(\{a\}^\sharp \Rightarrow \forall o.\neg\{a\}^\sharp)$, for every a in C.

The following lemma completes the proof of the reduction.

Lemma 29. *C is $\mathcal{C}$-satisfiable iff C^* is $\mathcal{C}$-satisfiable in a $\mathcal{ML}^n_{lin}$-model, with constant and expanding domains.*

□

Now decidability of concept satisfiability for $\mathbf{GL}_{\mathcal{ALCO}^{\iota}_{u}}$ over expanding domain models follows from decidability of satisfiability of $\mathcal{ML}^1_{lin}$-concepts in models with expanding domains and transitive Noetherian frames (Gabelaia et al. 2006).

E Proofs for Section 6

The following two theorems share the same proof below.

Theorem 14. *With constant domains, concept satisfiability is Σ^1_1-complete for $\mathbf{LTL}^{\Diamond}_{\mathcal{ALCO}_u}$ and $\mathbf{LTL}_{\mathcal{ALCO}_u}$, and undecidable for $\mathbf{LTL}\boldsymbol{f}^{\Diamond}_{\mathcal{ALCO}_u}$; also, concept satisfiability under global ontology is Σ^1_1-complete for $\mathbf{LTL}^{\Diamond}_{\mathcal{ALCO}}$ and $\mathbf{LTL}_{\mathcal{ALCO}}$, and undecidable for $\mathbf{LTL}\boldsymbol{f}^{\Diamond}_{\mathcal{ALCO}}$.*

Theorem 15. *(1) With expanding domains, concept satisfiability is undecidable for $\mathbf{LTL}^{\Diamond}_{\mathcal{ALCO}_u}$, and concept satisfiability under global ontology is undecidable for $\mathbf{LTL}^{\Diamond}_{\mathcal{ALCO}}$.*

(2) With expanding domains, concept satisfiability (under global ontology) is decidable for $\mathbf{LTL}\boldsymbol{f}_{\mathcal{ALCO}^{\iota}}$. However, both problems are Ackermann-hard for $\mathbf{LTL}\boldsymbol{f}^{\Diamond}_{\mathcal{ALCO}_u}$; moreover, concept satisfiability under global ontology is Ackermann-hard for $\mathbf{LTL}\boldsymbol{f}^{\Diamond}_{\mathcal{ALCO}}$.

Proof. The proofs of these two theorems rely on the following results: the problem of satisfiability of $\mathcal{TL}^{\Diamond}_{\text{Diff}}$ concepts is known (Hampson and Kurucz 2015) to be

1. Σ^1_1-complete in constant-domain interpretations over $(\mathbb{N}, <)$;
2. recursively enumerable, but undecidable in constant-domain interpretations over finite flows of time;
3. undecidable (co-r.e.) in expanding-domains interpretations over $(\mathbb{N}, <)$;
4. decidable, but Ackermann-hard in expanding-domains interpretations over finite flows of time.

As a consequence of Theorem 10 (2) and results 1–3 above, we obtain the following: in constant-domain interpretations, concept satisfiability is Σ^1_1-hard for $\mathbf{LTL}^{\Diamond}_{\mathcal{ALCO}_u}$ and undecidable for $\mathbf{LTL}\boldsymbol{f}^{\Diamond}_{\mathcal{ALCO}_u}$, while in expanding-domains interpretations, the problem is undecidable for $\mathbf{LTL}^{\Diamond}_{\mathcal{ALCO}_u}$. As a consequence of result 4, concept satisfiability in expanding-domains interpretations is Ackermann-hard for $\mathbf{LTL}\boldsymbol{f}^{\Diamond}_{\mathcal{ALCO}_u}$.

To prove the results for the languages without the universal modality, we first replace the satisfiability problem for a given concept, say, C, with the satisfiability problem for a concept name A under the global ontology $\mathcal{O} = \{A \equiv C\}$. Next, we transform global ontology $\mathcal{O}$ into normal form; see Lemma 4. Next, the reduction in Lemma 6 allows us to obtain the undecidability (including Σ^1_1-hardness) and Ackermann-hardness results for the problem of concept satisfiability under global ontology in languages without the universal role.

The membership in Σ^1_1 follows from the fact that the problem can be encoded in second-order arithmetic with existential quantifiers over binary predicate symbols for concept names and ternary predicate symbols for role names.

To show decidability of concept satisfiability (under global ontology) for $\mathbf{LTL}\boldsymbol{f}_{\mathcal{ALCO}^{\iota}}$, we first eliminate definite descriptions; see Proposition 8. The result then is a consequence of Theorem 28 and the decidability results proved for concept satisfiability in the language with both 'next' and 'eventually' (Gabelaia et al. 2006, Theorem 3); note that for $\mathbf{LTL}\boldsymbol{f}^{\Diamond}_{\mathcal{ALCO}_u}$ this follows more directly from Theorem 10 (1) and the result 4 above. □

Theorem 16. *(1) With constant domains, $\mathbf{LTL}^{\Diamond}_{\mathcal{ELO}}$ CI entailment is Π^1_1-complete and $\mathbf{LTL}\boldsymbol{f}^{\Diamond}_{\mathcal{ELO}}$ CI entailment is undecidable.*

(2) With expanding domains, $\mathbf{LTL}^{\Diamond}_{\mathcal{ELO}}$ CI entailment is undecidable.

(3) With expanding domains, $\mathbf{LTL}\boldsymbol{f}^{\Diamond}_{\mathcal{ELO}}$ CI entailment is decidable but Ackermann-hard.

Proof. We prove these results by reduction from the satisfiability under global ontology in the languages based on $\mathcal{ALC}$. Let A be a concept name and $\mathcal{O}$ a $\mathcal{TL}_{\mathcal{ALCO}^{\iota}_{u}}$ ontology. By Lemma 4, we assume that $\mathcal{O}$ is in normal form. First, we eliminate occurrences of $\neg$: every CI of the form $B_1 \sqsubseteq \neg B_2$ is replaced by $B_1 \sqcap B_2 \sqsubseteq \bot$ and every CI of the form $\neg B_1 \sqsubseteq B_2$ is first replaced by $\top \sqsubseteq B_1 \sqcup B_2$. Then, we eliminate the disjunction from the C by using the method suggested in (Artale et al. 2007): each CI of the form $\top \sqsubseteq B_1 \sqcup B_2$ is replaced with the following:

$$\begin{aligned} \top &\sqsubseteq \exists q.(\Diamond X_1 \sqcap \Diamond X_2),\\ \exists q.\Diamond(X_1 \sqcap \Diamond X_2) &\sqsubseteq B_1,\\ \exists q.\Diamond(X_1 \sqcap X_2) &\sqsubseteq B_1,\\ \exists q.\Diamond(X_2 \sqcap \Diamond X_1) &\sqsubseteq B_2, \end{aligned}$$

where q is a fresh role name, and X_1 and X_2 are fresh concept names. One can clearly see that every interpretation $\mathfrak{M}'$ satisfying the four CI above also satisfies $\top \sqsubseteq B_1 \sqcup B_2$. Indeed, for every $t \in \mathbb{N}$ and every $d \in \Delta^t$, there is a $d' \in \Delta^t$ such that $d' \in (\Diamond X_i)^{\mathcal{I}_t}$, for $i = 1, 2$. Thus, $d' \in X_i^{\mathcal{I}_{t_i}}$, for $t_i > t$ and $i = 1, 2$. Depending on whether $t_1 \geq t_2$ or not, we have $d \in B_i^{\mathcal{I}_t}$ for either $i = 1$ or $i = 2$, thus satisfying $\top \sqsubseteq B_1 \sqcup B_2$.

Conversely, if $\top \sqsubseteq B_1 \sqcup B_2$ is satisfied at all $w \in W$ in some $\mathfrak{M}$ over $(\mathbb{N}, <)$ such that the Δ^t is countably infinite, for each $t \in \mathbb{N}$, then $\mathfrak{M}$ can be extended by interpreting the fresh symbols to satisfy the four CIs above: for every $t \in \mathbb{N}$ and every $d \in \Delta^t$, we pick a unique element $d' \in \Delta^t$ and make it the $q^{\mathcal{I}_t}$-successor of d; since the domains are countably infinite, we can choose these $q^{\mathcal{I}_t}$-successor in such a way that no d' is the $q^{\mathcal{I}_{t_1}}$-successor of d_1 and the $q^{\mathcal{I}_{t_2}}$-successor of d_2 for distinct pairs (t_1, d_1) and (t_2, d_2). It now remains to define the interpretation of X_1 and X_2: for every $t \in \mathbb{N}$ and every $d \in \Delta^t$, let d' be the $q^{\mathcal{I}_t}$-successor of d; if $d \in B_1^{\mathcal{I}_t}$, then we include d' in $X_1^{\mathcal{I}_{t+1}}$ and in $X_2^{\mathcal{I}_{t+2}}$; otherwise, we include d' in $X_1^{\mathcal{I}_{t+2}}$ and in $X_2^{\mathcal{I}_{t+1}}$. The interpretation of X_1 and X_2 is well-defined due to our assumption on the interpretation of q.

A similar transformation works for the finite flows, but we need to reserve two last instants for the interpretation of X_1 and X_2. To achieve this, we 'relativise' all CIs in a

given ontology $\mathcal{O}$: every CI of the form $C_1 \sqsubseteq C_2$ is replaced by $\Diamond\Diamond\top \sqcap C_1 \sqsubseteq C_2$ (alternatively, $\bigcirc\bigcirc\top$ can be used). It should be clear that $\mathcal{O}$ is satisfied in an interpretation based on a flow $([0,n],<)$ iff the relativised ontology $\mathcal{O}'$ is satisfied in an interpretation based on a flow $([0,n+2],<)$; the two interpretations coincide on the first n instants, but the additional two instants are simply ignored by the relativised CIs and so, all concept and role names can be assumed to be empty there. Now, we can apply the procedure for eliminating the negation described above (note that the two additional time instants can now be used to interpret X_1 and X_2 in any suitable way).

So, we now have a $\mathcal{TL}_{\mathcal{ALCO}_u^\iota}$ ontology $\mathcal{O}'$ in normal form without any occurrences of the $\neg$ operator. We have also shown that A is satisfied under $\mathcal{O}$ in an interpretation based on an infinite (respectively, finite) flow and having countably infinite domains iff A is satisfied under $\mathcal{O}'$ in an interpretation based on an infinite (respectively, finite) flow and having countably infinite domains. It remains to eliminate $\bot$. To this end, we again use the method suggested in (Artale et al. 2007) and take a fresh concept name L and replace all occurrences of $\bot$ with L. In addition, we extend $\mathcal{O}'$ with the following CIs:

$$\exists s.L \sqsubseteq L, \text{ for all role names } s \text{ in } \mathcal{O}',$$
$$\Diamond L \sqsubseteq L.$$

Denote the resulting ontology by $\mathcal{O}''$. We can easily show that A is satisfied under $\mathcal{O}'$ iff $A \sqsubseteq L$ is *not* entailed by $\mathcal{O}''$. □

Theorem 17. *With constant and with expanding domains, concept satisfiability is* EXPTIME*-complete for* $\mathbf{LTL}^{\circ}_{\mathcal{ALCO}_u}$ *and* $\mathbf{LTL}\boldsymbol{f}^{\circ}_{\mathcal{ALCO}_u}$ *and in* EXPTIME *for* $\mathbf{LTL}^{\circ}_{\mathcal{ALCO}}$ *and* $\mathbf{LTL}\boldsymbol{f}^{\circ}_{\mathcal{ALCO}}$.

Proof. Let C_0 be an $\mathcal{TL}^{\circ}_{\mathcal{ALCO}_u}$ concept. By Proposition 3, it is sufficient to consider total satisfiability. Set $d = md(C_0)$. Observe that C_0 is satisfiable in an interpretation of the form $(\Delta, (\mathcal{I}_t)_{t\in[0,n]})$ with $n \in \mathbb{N}$ iff it is satisfiable in an interpretation of the form $(\Delta, (\mathcal{I}_t)_{t\in[0,n]})$ with $n \leq d$. Fix $m_0 \leq d$. Below, we give an exponential-time subroutine that checks satisfiability in an interpretation of the form $(\Delta, (\mathcal{I}_t)_{t\in[0,m_0]})$. By going through all $m_0 \leq d$, we obtain an exponential-time algorithm that checks satisfiability for $\mathbf{LTL}\boldsymbol{f}^{\circ}_{\mathcal{ALCO}_u}$ in constant domains, which, by Proposition 9, also gives the same upper bound for the problem in expanding domains. An algorithm for $\mathbf{LTL}^{\circ}_{\mathcal{ALCO}_u}$ in constant domains simply checks satisfiability in an interpretation of the form $(\Delta, (\mathcal{I}_t)_{t\in[0,m_0]})$ for $m_0 = d$; the case of expanding domains is again due to Proposition 9.

As before, a *type* for C_0 is a subset $\boldsymbol{t}$ of $\mathsf{con}(C_0)$ such that

C1 $\neg C \in \boldsymbol{t}$ iff $C \notin \boldsymbol{t}$, for all $\neg C \in \mathsf{con}(C_0)$;
C2 $C \sqcap D \in \boldsymbol{t}$ iff $C, D \in \boldsymbol{t}$, for all $C \sqcap D \in \mathsf{con}(C_0)$.

Denote by $\boldsymbol{T}$ the set of types for C_0. We clearly have $|\boldsymbol{T}| \leq 2^{|\mathsf{con}(C_0)|}$. Given an interpretation $\mathfrak{M} = (\Delta, (\mathcal{I}_t)_{t\in[0,m_0]})$, denote by $\boldsymbol{t}^{\mathcal{I}_i}(d)$ the type realised by $d \in \Delta$ in $\mathcal{I}_i$, for $i \leq m_0$.

The subroutine is an elimination procedure on the set of all (m_0+1)-tuples ρ of types $(\boldsymbol{t}_0, \cdots, \boldsymbol{t}_{m_0})$, called *runs*. Runs can be thought of as elements in $\boldsymbol{T}^{m_0+1}$ and so the number of runs is bounded by $2^{|\mathsf{con}(C_0)|(m_0+1)}$. The components of a run ρ are denoted $\rho(0), \ldots, \rho(m_0)$. We write

$$\rho \to_{r,i} \rho'$$

if $C \in \rho'(i)$ implies $\exists r.C \in \rho(i)$ for all $\exists r.C \in \mathsf{con}(C_0)$. Given an interpretation $\mathfrak{M} = (\Delta, (\mathcal{I}_t)_{t\in[0,m_0]})$, denote by $\rho^{\mathfrak{M}}_d = (\boldsymbol{t}^{\mathcal{I}_0}(d), \ldots, \boldsymbol{t}^{\mathcal{I}_{m_0}}(d))$ the run realised by $d \in \Delta$ in $\mathfrak{M}$.

Let $a_1, \ldots, a_n$ be the nominals in C_0. To construct the initial set $\mathfrak{R}_0$ of runs, we fix two parameters:

- an (m_0+1)-tuple $(U^0, \ldots, U^{m_0})$ of *u-types*, where each U^i is a set containing either $\exists u.C$ or $\neg\exists u.C$, for every $\exists u.C \in \mathsf{con}(C_0)$;
- a set $\mathfrak{N}$ of at most $n(m_0+1)$ *nominal runs* $\rho \in \boldsymbol{T}^{m_0+1}$ such that
  - $U^i \subseteq \rho(i)$, for all $\rho \in \mathfrak{N}$ and $i \leq m_0$, and
  - for every $\rho \in \mathfrak{N}$, there is $i \leq m_0$ such that $\{a_j\} \in \rho(i)$, for some $1 \leq j \leq n$;
  - for every $i \leq m_0$ and every $1 \leq j \leq n$, there is exactly one nominal run $\rho \in \mathfrak{N}$ such that $\{a_j\} \in \rho(i)$.

Then we take $\mathfrak{R}_0 = \mathfrak{N} \cup \mathfrak{O}$, where $\mathfrak{O}$ is the set of *all* runs without nominals consistent with the fixed u-types, that is, all runs $\rho \in \boldsymbol{T}^{m_0+1}$ such that

- $U^i \subseteq \rho(i)$, for all $i \leq m_0$, and
- $\{a_j\} \notin \rho(i)$, for all $i \leq m_0$ and $1 \leq j \leq n$.

We now exhaustively apply the run elimination procedure and to remove each run ρ from $\mathfrak{R}_0$ that satisfies one of the following:

- there is $i \leq m_0$ and $\bigcirc C \in \rho(i)$ and either $i = m_0$ or $i < m_0$ but $C \notin \rho(i+1)$;
- there is $i < m_0$ and $C \in \rho(i+1)$ but $\bigcirc C \notin \rho(i)$, for $\bigcirc C \in \mathsf{con}(C_0)$;
- there is $i \leq m_0$ and $\exists u.C \in \rho(i)$ but there is no ρ' with $C \in \rho'(i)$;
- there is $i \leq m_0$ and $\exists r.C \in \rho(i)$ but there is no ρ' with $C \in \rho'(i)$ and $\rho \to_{r,i} \rho'$.

Assume that $\mathfrak{R}$ is the remaining set of runs.

Lemma 30. *The following conditions are equivalent:*

(1) C_0 *is satisfiable in a total interpretation* $\mathfrak{M} = (\Delta, (\mathcal{I}_t)_{t\in[0,m_0]})$ *such that*

- $U^i = \{\exists u.C \in \mathsf{con}(C_0) \mid (\exists u.C)^{\mathcal{I}_i} = \Delta\} \cup \{\neg\exists u.C \in \mathsf{con}(C_0) \mid (\exists u.C)^{\mathcal{I}_i} = \emptyset\}$, *for* $i \leq m_0$;
- $\mathfrak{N} = \{\rho^{\mathfrak{M}}_{a_j^{\mathcal{I}_i}} \mid i \leq m_0, 1 \leq j \leq n\}$.

(2) the set $\mathfrak{R}$ *of runs satisfies the following conditions*:

- $\mathfrak{N} \subseteq \mathfrak{R}$;
- $C_0 \in \rho(0)$ *for some* $\rho \in \mathfrak{R}$.

Proof. If (1) holds, then take a witness total interpretation $\mathfrak{M}$. Let $\mathfrak{R}'$ be the set of runs $\rho_d^{\mathfrak{M}}$ for $d \in \Delta$. It is easy to see that $\mathfrak{R} \supseteq \mathfrak{R}' \supseteq \mathfrak{R}$. Moreover, $\rho_{d_0}^{\mathfrak{M}}$, for some $d_0 \in C_0^{\mathcal{I}_0}$, provides the requited witness run for (2).

Conversely, assume that (2) holds. Construct a total interpretation $\mathfrak{M}$ by setting:

- $\Delta = \mathfrak{R}$;
- $a^{\mathcal{I}_i} = \rho$ iff $\{a\} \in \rho(i)$, for $i \leq m_0$;
- $\rho \in A^{\mathcal{I}_i}$ iff $A \in \rho(i)$, for $i \leq m_0$;
- $(\rho, \rho') \in r^{\mathcal{I}_i}$ iff $\rho \to_{r,i} \rho'$, for $i \leq m_0$.

It is now easy to show that $\mathfrak{M}$ is well-defined and as required for (1). □

The exponential-time subroutine for satisfiability in interpretations of length $m_0 + 1$ is obtained by going through all u-types and sets of nominal runs (of which there are only exponentially many). □

Theorem 18. *With constant domains, concept satisfiability under global ontology is undecidable for* $\mathbf{LTL}^{\circ}_{\mathcal{ALCO}}$ *and* $\mathbf{LTL}\boldsymbol{f}^{\circ}_{\mathcal{ALCO}}$.

Proof. The result is obtained by a reduction of the (undecidable) halting problem for (two-counter) Minsky machines starting with 0 as initial counter values (Degtyarev, Fisher, and Lisitsa 2002; Baader et al. 2017).

A *(two-counter) Minsky machine* is a pair $M = (Q, P)$, where $Q = \{q_0, \ldots, q_L\}$ is a set of *states* and $P = (I_0, \ldots, I_{L\text{-}1})$ is a sequence of *instructions*. We assume q_0 to be the *initial state*, and q_L to be the *halting state*. Moreover, the instruction I_i is executed at state q_i, for $0 \leq i < L$. Each instruction I can have one of the following forms, where $r \in \{r_1, r_2\}$ is a *register* that stores non-negative integers as values, and q, q' are states:

1. $I = +(r, q)$ – add 1 to the value of register r and go to state q;
2. $I = -(r, q, q')$ – if the value of register r is (strictly) greater than 0, subtract 1 from the value of r and go to state q; otherwise, go to state q'.

Given a Minsky machine M, a *configuration of* M is a triple (q, v_1, v_2), where q is a state of M and $v_1, v_2 \in \mathbb{N}$ are the *values* of registers r_1, r_2, respectively. In the following, we denote $\overline{k} = 3 - k$, for $k \in \{1, 2\}$. Given $i, j \geq 0$, we write $(q_i, v_1, v_2) \Rightarrow_M (q_j, v_1', v_2')$ iff one of the following conditions hold:

- $I_i = +(r_k, q_j)$, $v_k' = v_k + 1$ and $v_{\overline{k}}' = v_{\overline{k}}$;
- $I_i = -(r_k, q_j, q')$, $v_k > 0$, $v_k' = v_k - 1$, and $v_{\overline{k}}' = v_{\overline{k}}$;
- $I_i = -(r_k, q', q_j)$, $v_k = v_k' = 0$, and $v_{\overline{k}}' = v_{\overline{k}}$.

Given an *input* $(v_1, v_2) \in \mathbb{N} \times \mathbb{N}$, the *computation of* M *on input* (v_1, v_2) is the (unique) longest sequence of configurations $(q^0, v_1^0, v_2^0) \Rightarrow_M (q^1, v_1^1, v_2^1) \Rightarrow_M \ldots$, such that $q^0 = q_0$, $v_1^0 = v_1$ and $v_2^0 = v_2$. We say that M *halts on input* $(0, 0)$ if the computation of M on input $(0, 0)$ is finite: thus, its initial configuration takes the form $(q_0, 0, 0)$, while its last configuration has the form (q_L, v_1, v_2), for some $v_1, v_2 \in \mathbb{N}$. The *halting problem for Minsky machines* is the problem of deciding, given a Minsky machine M, whether M halts on input $(0, 0)$. This problem is known to be undecidable; see, e.g., (Degtyarev, Fisher, and Lisitsa 2002; Baader et al. 2017).

To represent the computation of a Minsky machine M on input $(0, 0)$, we use the temporal dimension to model successive configurations in the computation. We introduce a concept name Q_i, for each state q_i of M. Concept names R_1, R_2 are used to represent registers r_1, r_2, respectively: the value of register r_k at any computation step will coincide with the cardinality of the extension of R_k at the corresponding moment of time. The incrementation of the value of register r_k is modelled by requiring that the extension of concept $\neg R_k \sqcap \bigcirc R_k$ is included in that of nominal $\{a_k\}$; dually, the decrementation of register r_k is modelled by including the extension of $R_k \sqcap \bigcirc \neg R_k$ in the nominal $\{b_k\}$. Since the individual names a_k and b_k are interpreted *non-rigidly*, the (unique) element added or subtracted from the extension of R_k will vary over time.

We first construct a $\mathcal{TL}^{\circ}_{\mathcal{ALCO}_u}$ ontology $\mathcal{O}_M$ such that

- Q_0 is satisfiable under $\mathcal{O}_M$ over $(\mathbb{N}, <)$ iff machine M never halts on $(0, 0)$;
- Q_0 is satisfiable under $\mathcal{O}_M$ over finite flows of time iff machine M halts on $(0, 0)$;

Then, Lemmas 4 and 6 can be used to replace the ontology $\mathcal{O}_M$ with a $\mathcal{TL}^{\circ}_{\mathcal{ALCO}}$ ontology $\mathcal{O}_M$ with the same property, thus establishing the required results.

We use the following CIs to ensure that concepts Q_i representing states are either empty or coincide with the whole domain at any moment of time:

(**S1**) $\exists u.Q_i \sqsubseteq Q_i$, for all $0 \leq i \leq L$;
(**S2**) $Q_i \sqcap Q_j \sqsubseteq \bot$, for all $0 \leq i < j \leq L$.

Next, we encode instructions. We begin with incrementation. An instruction of the form $I_i = +(r_k, q_j)$ is represented by the following CIs:

(**I1**) $Q_i \sqsubseteq \exists u.(\{a_k\} \sqcap \neg R_k)$;
(**I2**) $Q_i \sqsubseteq \bigcirc R_k \Leftrightarrow (R_k \sqcup \{a_k\})$;
(**I3**) $Q_i \sqsubseteq \bigcirc R_{\overline{k}} \Leftrightarrow R_{\overline{k}}$;
(**I4**) $Q_i \sqsubseteq \bigcirc Q_j$.

An instructions of the form $I_i = -(r_k, q_j, q_h)$ is represented by the following CIs:

(**D1**) $Q_i \sqcap \exists u.R_k \sqsubseteq \exists u.(\{b_k\} \sqcap R_k)$;
(**D2**) $Q_i \sqsubseteq \bigcirc R_k \Leftrightarrow (R_k \sqcap \neg\{b_k\})$;
(**D3**) $Q_i \sqsubseteq \bigcirc R_{\overline{k}} \Leftrightarrow R_{\overline{k}}$;
(**D4**) $Q_i \sqcap \exists u.R_k \sqsubseteq \bigcirc Q_j$;
(**D5**) $Q_i \sqcap \neg\exists u.R_k \sqsubseteq \bigcirc Q_h$.

Given $\mathfrak{M} = (\Delta, (\mathcal{I}_t)_{t \in \mathfrak{T}})$, we say that an instant $t \in \mathfrak{T}$ *encodes* a configuration (q_i, v_1, v_2) whenever $Q_i^{\mathcal{I}_t} = \Delta$ and $|R_k^{\mathcal{I}_t}| = v_k$, for $k = 1, 2$. We show the following:

Claim 7. *For any $\mathfrak{M} = (\Delta, (\mathcal{I}_t)_{t\in\mathfrak{T}})$ that satisfies (**S1**)–(**D5**), if $t \in \mathfrak{T}$ encodes (q_i, v_1, v_2) and $(q_i, v_1, v_2) \Rightarrow_M (q', v_1', v_2')$, then $t+1$ encodes (q', v_1', v_2').*

Proof of Claim. We consider the following cases.

- Suppose $I_i = +(r_k, q_j)$. By (**I4**) with (**S1**), we have $Q_j^{\mathcal{I}_{t+1}} = \Delta$, and thus $q' = q_j$. Note that, by (**S2**), $Q_{j'}^{\mathcal{I}_{t+1}} = \emptyset$ for any $j' \neq j$. Then, by (**I1**), $a_k^{\mathcal{I}_t}$ is defined and $a_k^{\mathcal{I}_t} \notin R_k^{\mathcal{I}_t}$. By (**I2**), $R_k^{\mathcal{I}_{t+1}} = R_k^{\mathcal{I}_t} \cup \{a_k^{\mathcal{I}_t}\}$, whence $|R_k^{\mathcal{I}_{t+1}}| = |R_k^{\mathcal{I}_t}| + 1$, and so $v_k' = v_k + 1$. For the other counter, by (**I3**), $R_{\overline{k}}^{\mathcal{I}_{t+1}} = R_{\overline{k}}^{\mathcal{I}_t}$, whence $v_{\overline{k}}' = v_{\overline{k}}$.
- Suppose $I_i = -(r_k, q_j, q_h)$. By (**D4**) with (**S1**), if $R_k^{\mathcal{I}_t} \neq \emptyset$, then $Q_j^{\mathcal{I}_{t+1}} = \Delta$, and thus $q' = q_j$. Otherwise, if $R_k^{\mathcal{I}_t} = \emptyset$, then, by (**D5**) with (**S1**), $Q_h^{\mathcal{I}_{t+1}} = \Delta$, and thus $q' = q_h$. Note that in either case, by (**S2**), $Q_{j'}^{\mathcal{I}_{t+1}} = \emptyset$ for any other j'. We next deal with the values of counters. If $R_k^{\mathcal{I}_t} \neq \emptyset$, then, by (**D1**), $b_k^{\mathcal{I}_t}$ is defined and $b_k^{\mathcal{I}_t} \in R_k^{\mathcal{I}_t}$. Then, by (**D2**), we have $R_k^{\mathcal{I}_{t+1}} = R_k^{\mathcal{I}_t} \setminus \{b_k^{\mathcal{I}_t}\}$. It follows that $|R_k^{\mathcal{I}_{t+1}}| = |R_k^{\mathcal{I}_t}| - 1$ and so $v_k' = v_k - 1$. On the other hand, if $R_k^{\mathcal{I}_t} = \emptyset$, then, by (**D2**), $R_k^{\mathcal{I}_{t+1}} = \emptyset$ and so $v_k' = v_k = 0$. For the other counter, by (**D3**), $R_{\overline{k}}^{\mathcal{I}_{t+1}} = R_{\overline{k}}^{\mathcal{I}_t}$, whence $v_{\overline{k}}' = v_{\overline{k}}$.

So, in each case, $t+1$ encodes (q', v_1', v_2'). □

It remains to encode the initial configuration and the acceptance condition. For the initial configuration, we two CIs

(**O1**) $Q_0 \sqcap R_k \sqsubseteq \bot$, for $k = 1, 2$;

Clearly, these two CIs ensure that the value of both counters is 0 whenever the state is the initial state q_0. For the non-termination condition over $(\mathbb{N}, <)$ and the termination condition over finite flows of time, we use the following CIs, respectively:

(**A1**) $Q_L \sqsubseteq \bot$.
(**A2**) $\neg\bigcirc\top \sqsubseteq Q_L$.

We claim that Q_0 is satisfiable under $\mathcal{O}_M$ over $(\mathbb{N}, <)$ iff machine M never halts on $(0, 0)$. Indeed, if Q_0 is satisfiable under $\mathcal{O}_M$, then by (**O1**), moment 0 encodes configuration $(q_0, 0, 0)$. By Claim 7, moments 0, 1, 2, etc. encode configurations (q^t, v_1^t, v_2^t) such that $(q^t, v_1^t, v_2^t) \Rightarrow_M (q^{t+1}, v_1^{t+1}, v_2^{t+1})$, for each t. By (**A1**), the halting state q_L is never reached, and so machine M does not halt on $(0, 0)$.

Conversely, $(q^0, v_1^0, v_2^0) \Rightarrow_M \ldots \Rightarrow_M (q^t, v_1^t, v_2^t) \Rightarrow_M \ldots$ be the computation of M on input $(0, 0)$, where $q^0 = q_0$ and $v_k^0 = 0$, for $k = 1, 2$. We construct an interpretation $\mathfrak{M} = (\Delta, (\mathcal{I}_t)_{t\in\mathbb{N}})$ that satisfies Q_0 under $\mathcal{O}_M$. Let Δ be a fixed countable set. Given $t \in \mathbb{N}$ and $q^t = q_j$, for some $0 \leq i \leq L$, we set $Q_i^{\mathcal{I}_t} = \Delta$ and $Q_j^{\mathcal{I}_t} = \emptyset$, for all $j \neq i$. In addition, we set $R_1^{\mathcal{I}_0} = R_2^{\mathcal{I}_0} = \emptyset$. Then, for every instant $t \in \mathbb{N}$, we define the interpretations $R_1^{\mathcal{I}_{t+1}}$, $R_2^{\mathcal{I}_{t+1}}$, $a_k^{\mathcal{I}_t}$, and $b_k^{\mathcal{I}_t}$ inductively as follows.

- If $q^t = q_i$ and $I_i = +(r_k, q_j)$, then we choose some $d \notin R_k^{\mathcal{I}_t}$ and set $a_k^{\mathcal{I}_t} = d$ and $R_k^{\mathcal{I}_{t+1}} = R_k^{\mathcal{I}_t} \cup \{d\}$; we also choose an arbitrary $b_k^{\mathcal{I}_t}$ and set $R_{\overline{k}}^{\mathcal{I}_{t+1}} = R_{\overline{k}}^{\mathcal{I}_t}$.
- If $q^t = q_i$ and $I_i = -(r_k, q_j, q_h)$, then we have two options:
  - if $R_k^{\mathcal{I}_t} \neq \emptyset$, then we choose an arbitrary $e \in R_k^{\mathcal{I}_t}$ and set $b_k^{\mathcal{I}_t} = e$ and $R_k^{\mathcal{I}_{t+1}} = R_k^{\mathcal{I}_t} \setminus \{e\}$;
  - if $R_k^{\mathcal{I}_t} = \emptyset$, then we choose an arbitrary $b_k^{\mathcal{I}_t}$ and set $R_k^{\mathcal{I}_{t+1}} = R_1^{\mathcal{I}_t}$.

  In either case, we also choose an arbitrary $a_k^{\mathcal{I}_t}$ and set $R_{\overline{k}}^{\mathcal{I}_{t+1}} = R_{\overline{k}}^{\mathcal{I}_t}$.

It can be seen that $\mathfrak{M}$ satisfies Q_0 under $\mathcal{O}_M$.

Next, we claim that Q_0 is satisfiable under $\mathcal{O}_M$ over finite flows of time iff machine M halts on $(0, 0)$. Indeed, if Q_0 is satisfiable under $\mathcal{O}_M$ over some $([0, n], <)$, then by (**O1**), moment 0 encodes configuration $(q_0, 0, 0)$. By Claim 7, moments $0, 1, 2, \ldots, n$ etc. encode configurations (q^t, v_1^t, v_2^t) such that $(q^t, v_1^t, v_2^t) \Rightarrow_M (q^{t+1}, v_1^{t+1}, v_2^{t+1})$, for each $0 \leq t < n$. By (**A2**), the state q^n is the halting state, q_L, as required. The converse direction is similar to the case of $(\mathbb{N}, <)$. □