A NON-UNIFORM VIEW OF CRAIG INTERPOLATION IN MODAL LOGICS WITH LINEAR FRAMES

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Abstract. Normal modal logics extending the logic K4.3 of linear transitive frames are known to lack the Craig interpolation property, except some logics of bounded depth such as S5. We turn this 'negative' fact into a research question and pursue a non-uniform approach to Craig interpolation by investigating the following interpolant existence problem: decide whether there exists a Craig interpolant between two given formulas in any fixed logic above K4.3. Using a bisimulation-based characterisation of interpolant existence for descriptive frames, we show that this problem is decidable and coNP-complete for all finitely axiomatisable normal modal logics containing K4.3. It is thus not harder than entailment in these logics, which is in sharp contrast to other recent non-uniform interpolation results. We also extend our approach to Priorean temporal logics (with both past and future modalities) over the standard time flows—the integers, rationals, reals, and finite strict linear orders—none of which is blessed with the Craig interpolation property.

§1. Introduction. Unlike classical and intuitionistic first-order and propositional logics, numerous modal logics, L, do not enjoy the Craig interpolation property (CIP): they contain valid implications $\varphi \to \psi$ without an interpolant in L—a formula ι in the shared signature of φ and ψ such that both $\varphi \to \iota$ and $\iota \to \psi$ are also valid in L. Typical examples of such L are first-order modal logics with constant domains between K and S5 [13] and propositional modal logics with linear transitive Kripke frames of unbounded depth [14, 39]. There have been various attempts to classify propositional modal logics with the CIP, successful for extensions of S4 [14, Section 8] and unsuccessful for extensions of K4 or GL, where the CIP turned out to be undecidable [9, Sections 14, 17].

While establishing the CIP of a logic L typically gives rise to further research problems—develop proof systems that admit efficient/elegant interpolant computation [29, 3], investigate the complexity of computing interpolants from proofs [25, Sections 17, 18], consider restrictions on the shape of interpolants such as in, say, Lyndon's interpolation [31], or employ the CIP to investigate related properties such as Beth definability [10, 11]—a counterexample to the CIP has usually terminated further research of Craig interpolants and their applications for the unfortunate logic in question.

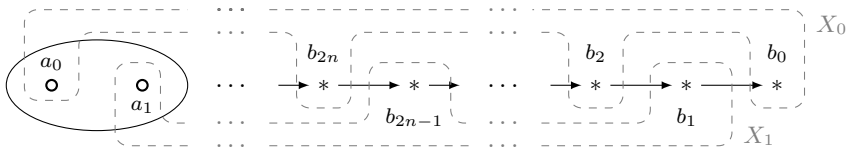
In this article, we take a different, non-uniform view of Craig interpolation and aim to understand interpolants also for logics L without the CIP. We consider

the following *interpolant existence problem* (IEP) for L: given formulas φ and ψ, decide whether $\varphi \to \psi$ has an interpolant in L. For L without the CIP, the existence of an interpolant for φ and ψ does not follow from the validity of $\varphi \to \psi$ in L, and so the IEP does not reduce to validity checking (which is reducible to the IEP). A first question then is whether the former problem is genuinely harder than the latter one. In fact, when the IEP was introduced [24, 1], this was shown to be the case for modal logics with nominals and for the two-variable and guarded fragments of first-order logic. Since then, this has also been confirmed for the one-variable fragment of first-order modal logic $\mathsf{S5}$ and weak $\mathsf{K4}$ [26, 27].

Here, we show that the opposite is true of propositional modal logics containing $\mathsf{K4.3}$, the logic of linear transitive frames: while none of these logics with frames of unbounded depth has the CIP [14, 39], interpolant existence is nevertheless decidable in coNP for finitely axiomatisable logics, and so is as hard as validity [30]. This is the first general result on Craig interpolant existence covering a large family of modal logics and, potentially, a step towards a classification of modal logics according to the complexity of the IEP.

We proceed as follows. To begin with, we give a 'folklore' characterisation of interpolant existence via bisimulations between models based on descriptive frames: $\varphi \to \psi$ does not have an interpolant in L iff φ and $\neg\psi$ can be satisfied in $sig(\varphi) \cap sig(\psi)$-bisimilar models based on descriptive frames for L. If L had the CIP, we could merge these two models into a single one satisfying $\varphi \wedge \neg\psi$ (using, say, bisimulation products [32]) or amalgamate the induced modal algebras [14], which is impossible in our case. Instead, we aim to understand the fine-grained structure of the required bisimilar models and use it to decide their existence. We show that, for some logics (such as first-order definable cofinal subframe logics), any pair of bisimilar models can be transformed into bisimilar models of polynomial size; in other words, such logics enjoy the polysize bisimilar model property. However, for other logics like $\mathsf{GL.3}$, not even models based on infinite Kripke frames are enough despite $\mathsf{GL.3}$ having the finite model property.

We prove, nevertheless, that every pair of bisimilar models satisfying φ and $\neg\psi$ and based on descriptive frames for a finitely axiomatisable $L \supseteq \mathsf{K4.3}$ can be converted to a pair of such models with an understandable structure. In a nutshell, their underlying frames look like a polynomial-size chain of polynomial-size clusters and tadpole-like descriptive frames that comprise a non-degenerate cluster $\{a_0, \dots, a_{k-1}\}$, for some polynomial-size $k > 0$, followed by an infinite descending chain of points b_n, $n < \omega$, which are all irreflexive or all reflexive, with the internal sets (restricting possible valuations) generated as a modal algebra by the singletons $\{b_n\}$ and the k-many pairwise disjoint infinite sets $X_i = \{a_i\} \cup \{b_n \mid n \equiv i \pmod k\}$. The picture below illustrates the underlying Kripke frame and the generators of the tadpole descriptive frame with $k = 2$.



Because of this, we say that all finitely axiomatisable $L \supseteq \mathsf{K4.3}$ have the quasi-polysize bisimilar model property. We show that the existence of such

quasi-polysize bisimilar models can be checked in NP in the size of φ and ψ, for any finitely axiomatisable L.

Finally, we extend the developed techniques to analyse the IEP for a few Priorean temporal logics with past and future modal operators: the logic Lin of all linear frames, the logic $\mathsf{Lin}_{<\omega}$ of all finite strict linear orders, and the logics $\mathsf{Lin}_{\mathbb{Q}}$ of the rationals, $\mathsf{Lin}_{\mathbb{R}}$ of the reals, and $\mathsf{Lin}_{\mathbb{Z}}$ of the integers. We prove that Lin, $\mathsf{Lin}_{\mathbb{Q}}$, and $\mathsf{Lin}_{\mathbb{R}}$ have the polysize bisimilar model property, while $\mathsf{Lin}_{<\omega}$ and $\mathsf{Lin}_{\mathbb{Z}}$ have the quasi-polysize one, with the IEP being coNP-complete. The proofs can be regarded as applications of the general method, which works for all extensions of $\mathsf{K4.3}$, to a few concrete logics with transparent frames. In fact, one could read the Priorean case in parallel with the full general proof, using the former as an illustration of the latter.

The remainder of the article is organised as follows. The introduction is concluded with a brief discussion of related work. §2 contains the necessary modal logic preliminaries. §3 gives the bisimulation-based criterion of interpolant existence and applies it to first-order definable cofinal subframe logics above $\mathsf{K4.3}$. It also provides illustrative examples explaining why the same method does not work in general and what kind of descriptive frames might be needed. §4 establishes the quasi-finite bisimilar model property of all logics above $\mathsf{K4.3}$ and the quasi-polysize bisimilar model property of all finitely axiomatisable ones; for the latter, it gives a coNP-algorithm for deciding the IEP. §5 extends the developed techniques to the Priorean temporal logics mentioned above.

1.1. Related work. The IEP for some logics of linear frames turns out to be closely related to separability of regular languages by first-order definable languages. Formally, the separability problem is to decide whether two input regular languages L_1 and L_2 can be separated by some language L in a given class $\mathcal{L}$ in the sense that $L_1 \subseteq L$ and $L \cap L_2 = \emptyset$. If $\mathcal{L}$ is the class of first-order definable languages over finite words, the separability problem is equivalent to the IEP for the linear temporal logic LTL extending modal logic with the operators 'next' and 'until' over finite strict linear orders. For regular languages of infinite words, the separability problem is equivalent to the IEP for LTL over the natural numbers; see [28] for details. It was shown in [21, 22, 34] that both of these separability problems are decidable in 2ExpTime in the size of NFAs defining L_1 and L_2. It follows that the corresponding IEPs are decidable in 3ExpTime in the size of LTL-formulas. (Separability by other language classes $\mathcal{L}$ are discussed in [33, 35].) These separability results have been obtained using algebraic machinery from semigroup theory, which seems to be orthogonal to our model-theoretic approach to the IEP developed to deal with all modal logics of linear orders. However, for finite strict linear orders and the natural numbers, the algebraic approach also provides an upper bound for the size of interpolants.

It is also worth mentioning that, for these two frame classes, the smallest modal logic with the CIP is LTL extended with fixed-point operators or, equivalently, monadic second-order logic (under very mild conditions on the definition of what a logic is) [15]. Thus, to 'repair' the CIP by extending the expressive power of the logic, we require the addition of second-order features.

§2. Preliminaries. This section provides the basic definitions and facts that will be used later on in the article; consult [16, 17, 9, 5, 6] for more details.

2.1. Descriptive frames for normal modal logics. The formulas, φ, of propositional unimodal logics are built from propositional variables $p_i \in \mathcal{V}$, for some countably-infinite set $\mathcal{V} = \{p_i \mid i < \omega\}$, and constants $\top$, $\bot$ using the Boolean connectives $\neg$, $\wedge$, and the unary possibility operator $\Diamond$. The other Booleans and the necessity operator $\Box$ dual to $\Diamond$ are defined as standard abbreviations. We also use $\Diamond^+\varphi = \varphi \vee \Diamond\varphi$, $\Box^+\varphi = \varphi \wedge \Box\varphi$, and $\Diamond\Gamma = \{\Diamond\varphi \mid \varphi \in \Gamma\}$, for a set Γ of formulas. By a *signature* we mean any set $\sigma \subseteq \mathcal{V}$, denoting by $sig(\varphi)$ the (finite) set of variables in a formula φ. If $sig(\varphi) \subseteq \sigma$, we call φ a *σ-formula*. We denote by $sub(\varphi)$ the set of subformulas of φ together with their negations, and let $|\varphi| = |sub(\varphi)|$.

A (*normal*) *modal logic*, L, is any set of formulas that contains all Boolean tautologies, the modal axiom $\Box(p_0 \to p_1) \to (\Box p_0 \to \Box p_1)$, and is closed under the rules of *modus ponens*, uniform substitution of formulas in place of variables, and necessitation $\varphi/\Box\varphi$. The smallest such logic goes by the moniker K. Given a set Γ of formulas and a modal logic L, the smallest modal logic to contain L and Γ is denoted by $L \oplus \Gamma$. We write $L \oplus \varphi$ for $L \oplus \{\varphi\}$. For example,

$$\begin{aligned}
\mathsf{K4} &= \mathsf{K} \oplus \Box p_0 \to \Box\Box p_0,\\
\mathsf{K4.3} &= \mathsf{K4} \oplus \Box(\Box^+ p_0 \to p_1) \vee \Box(\Box^+ p_1 \to p_0),\\
\mathsf{GL.3} &= \mathsf{K4.3} \oplus \Box(\Box p_0 \to p_0) \to \Box p_0,\\
\mathsf{Log}\{(\mathbb{N}, <)\} &= \mathsf{K4.3} \oplus \Diamond\top \oplus \Box(\Box p_0 \to p_0) \to (\Diamond\Box p_0 \to \Box p_0).
\end{aligned}$$

All logics considered in this article are extensions of $\mathsf{K4.3}$.

We interpret formulas in (*general*) *frames* $\mathfrak{F} = (W, R, \mathcal{P})$, where R is a binary (accessibility) relation on a nonempty set W (of worlds or, more neutrally, points) and $\mathcal{P} \subseteq 2^W$ contains $\emptyset$, W and is closed under $\cap$, $\neg$, and the operator

$$\Diamond^{\mathfrak{F}} X = \{x \in W \mid \exists y \in X\, xRy\}.$$

The structure $\mathfrak{F}^+ = (\mathcal{P}, \cap, \neg, \emptyset, W, \Diamond^{\mathfrak{F}})$ is a Boolean algebra $(\mathcal{P}, \cap, \neg, \emptyset, W)$ with a normal and additive operator $\Diamond^{\mathfrak{F}}$ (BAO, for short). If $\mathfrak{F}^+$ is generated by a set $\mathcal{X} \subseteq \mathcal{P}$ as a BAO, we say that the frame $\mathfrak{F}$ (or the set $\mathcal{P}$) is *generated by* $\mathcal{X}$. If $|\mathcal{X}| = n$, for some $n < \omega$, we call $\mathfrak{F}$ *n-generated* or *finitely generated*. The elements of $\mathcal{P}$ are called *internal sets* in $\mathfrak{F}$. If $\mathcal{P} = 2^W$, $\mathfrak{F}$ is known as a *Kripke frame*; in this case, we drop $\mathcal{P}$ and write $\mathfrak{F} = (W, R)$. A frame $\mathfrak{F} = (W, R, \mathcal{P})$ is *descriptive* if the following conditions hold, for any $x, y \in W$ and any $\mathcal{X} \subseteq \mathcal{P}$:

(dif) $x = y$ iff $\forall X \in \mathcal{P}\,(x \in X \leftrightarrow y \in X)$,
(tig) xRy iff $\forall X \in \mathcal{P}\,(y \in X \to x \in \Diamond^{\mathfrak{F}} X)$,
(com) if $\mathcal{X} \subseteq \mathcal{P}$ has the *finite intersection property* (*fip*, for short)—that is, $\bigcap \mathcal{X}' \neq \emptyset$, for every finite $\mathcal{X}' \subseteq \mathcal{X}$—then $\bigcap \mathcal{X} \neq \emptyset$.

(Frames with **(dif)** are called *differentiated*, with **(tig)** *tight*, and with **(com)** *compact*.) Every BAO is isomorphic to $\mathfrak{F}^+$, for some descriptive frame $\mathfrak{F}$. A finite frame is descriptive iff it is a Kripke frame [9, Section 8].

Given a signature σ, a *σ-model* based on a frame $\mathfrak{F} = (W, R, \mathcal{P})$ is a pair $\mathfrak{M} = (\mathfrak{F}, \mathfrak{v})$ with a *valuation* $\mathfrak{v}\colon \sigma \to \mathcal{P}$. The *atomic σ-type* of $x \in W$ in $\mathfrak{M}$ is

$$at^{\sigma}_{\mathfrak{M}}(x) = \{p_i \mid p_i \in \sigma,\ x \in \mathfrak{v}(p_i)\} \cup \{\neg p_i \mid p_i \in \sigma,\ x \notin \mathfrak{v}(p_i)\}.$$

We omit $\sigma = \mathcal{V}$, saying simply *model* and writing $at_{\mathfrak{M}}(x)$. The *value* of a formula φ in $\mathfrak{M}$ is the set $\mathfrak{v}(\varphi) \in \mathcal{P}$ computed inductively in the obvious way starting from $\mathfrak{v}(p_i)$, $\mathfrak{v}(\top) = W$ and $\mathfrak{v}(\bot) = \emptyset$. A set $X \subseteq W$ is *definable* in $\mathfrak{M}$ if $X = \mathfrak{v}(\varphi)$, for some formula φ, in which case $X \in \mathcal{P}$. If every internal set $X \in \mathcal{P}$ is definable in $\mathfrak{M}$, we say that $\mathfrak{F}$ is $\mathfrak{M}$-*generated*. Every $\mathfrak{F}$ with countable $\mathcal{P}$ is clearly $\mathfrak{M}$-generated, for some model $\mathfrak{M}$.

A formula φ is *true* at x in $\mathfrak{M}$, written $\mathfrak{M}, x \models \varphi$, if $x \in \mathfrak{v}(\varphi)$. The σ-*type of* x *in* $\mathfrak{M}$ is the set $t^{\sigma}_{\mathfrak{M}}(x)$ of all σ-formulas that are true at x in $\mathfrak{M}$. For a set X of points in $\mathfrak{M}$, we let $t^{\sigma}_{\mathfrak{M}}(X) = \{t^{\sigma}_{\mathfrak{M}}(x) \mid x \in X\}$. As before, we drop $\sigma = \mathcal{V}$.

A set Γ of formulas is *finitely satisfiable* in $\mathfrak{M}$ if, for every finite subset $\Gamma' \subseteq \Gamma$, there is $x' \in W$ such that $\Gamma' \subseteq t_{\mathfrak{M}}(x')$; Γ is *satisfiable* in $\mathfrak{M}$ if $\Gamma \subseteq t_{\mathfrak{M}}(x)$, for some $x \in W$. Using these definitions and notations, we can equivalently reformulate conditions **(dif)**, **(tig)**, and **(com)** for $\mathfrak{M}$-generated frames as follows: for any $x, y \in W$ and any set Γ of formulas,

(dif) $x = y$ iff $t_{\mathfrak{M}}(x) = t_{\mathfrak{M}}(y)$,
(tig) xRy iff $\Diamond t_{\mathfrak{M}}(y) \subseteq t_{\mathfrak{M}}(x)$ iff $\{\varphi \mid \Box\varphi \in t_{\mathfrak{M}}(x)\} \subseteq t_{\mathfrak{M}}(y)$,
(com) if Γ is finitely satisfiable in $\mathfrak{M}$, then Γ is satisfiable in $\mathfrak{M}$.

A frame $\mathfrak{F}$ *satisfies* Γ if there is a model $\mathfrak{M}$ based on $\mathfrak{F}$ satisfying Γ. Further, φ is *valid* in $\mathfrak{F}$, written $\mathfrak{F} \models \varphi$, if $\mathfrak{M}, x \models \varphi$, for any model $\mathfrak{M}$ based on $\mathfrak{F}$ and any $x \in W$. We call $\mathfrak{F}$ a *frame for* a logic L and write $\mathfrak{F} \models L$ if $\mathfrak{F} \models \varphi$, for all $\varphi \in L$. Conversely, any class $\mathcal{S}$ of general frames *determines* the modal logic $\mathsf{Log}\,\mathcal{S} = \{\varphi \mid \forall \mathfrak{F} \in \mathcal{S}\ \mathfrak{F} \models \varphi\}$. We write $\mathsf{Log}(\mathfrak{F})$ for $\mathsf{Log}(\{\mathfrak{F}\})$.

A set Γ of formulas is L-*consistent* if $(\bigwedge \Gamma' \to \bot) \notin L$, for any finite $\Gamma' \subseteq \Gamma$. We require the following well-known fact (see, e.g., [9, Section 8.6]):

LEMMA 2.1. *For any modal logic L and any finite signature σ, if Σ is an L-consistent set of σ-formulas, then Σ is satisfiable in a σ-model $\mathfrak{M}$ based on a finitely $\mathfrak{M}$-generated descriptive frame for L.*

By Lemma 2.1, every modal logic L is determined by the class of all descriptive frames for L. A logic L is *Kripke complete* if L is determined by the class of all Kripke frames for L. L is *d-persistent* (aka *canonical*) if $(W, R, \mathcal{P}) \models L$ implies $(W, R) \models L$, for any descriptive frame $(W, R, \mathcal{P})$. L has the *finite model property* (*fmp*) if it is determined by its finite (Kripke) frames.

The smallest logic $\mathsf{K4.3}$ we are interested in is d-persistent; its descriptive and Kripke frames $\mathfrak{F} = (W, R, \mathcal{P})$ are *transitive* and *weakly connected*, that is,

$$\forall x, y, z \in W\ (xRy \land yRz \to xRz),$$
$$\forall x, y, z \in W\ (xRy \land xRz \to y = z \lor yRz \lor zRy).$$

$\mathsf{GL.3}$, on the contrary, is not d-persistent yet has the fmp. In fact, all extensions of $\mathsf{K4.3}$ are Kripke complete [12].

A frame $\mathfrak{F}' = (W', R', \mathcal{P}')$ is a *subframe* of a frame $\mathfrak{F} = (W, R, \mathcal{P})$ if $W' \subseteq W$, $R' = R \restriction_{W'} = R \cap (W' \times W')$, and $\mathcal{P}' \subseteq \mathcal{P}$. For every internal set $V \in \mathcal{P}$, the frame $\mathfrak{F} \restriction_V = \big(V, R \restriction_V, \mathcal{P} \restriction_V\big)$ with $\mathcal{P} \restriction_V = \{V \cap X \mid X \in \mathcal{P}\}$ is a subframe of $\mathfrak{F}$. For a model $\mathfrak{M} = (\mathfrak{F}, \mathfrak{v})$, we let $\mathfrak{M} \restriction_V = (\mathfrak{F} \restriction_V, \mathfrak{v} \restriction_V)$, where $\mathfrak{v} \restriction_V(p) = V \cap \mathfrak{v}(p)$.

Given a frame $\mathfrak{F} = (W, R, \mathcal{P})$ with transitive R and a point $x \in W$, we define the frame $\mathfrak{F}_x = (W_x, R_x, \mathcal{P}_x)$ by taking $W_x = \{y \in W \mid xR^+y\}$, where R^+ is the *reflexive closure* of R (that is, $R^+ = R \cup \{(y, y) \mid y \in W\}$), $R_x = R \restriction_{W_x}$, and $\mathcal{P}_x = \mathcal{P} \restriction_{W_x}$. We call $\mathfrak{F}$ *rooted* if $\mathfrak{F} = \mathfrak{F}_x$, for some $x \in W$, in which case x is called a *root* of $\mathfrak{F}$. Note that $\mathfrak{F}_x$ is not necessarily a subframe of $\mathfrak{F}$, but we have:

(1) if $\mathfrak{F}$ is descriptive and transitive, then $\mathfrak{F}_x$ is descriptive as well.

Indeed, suppose $\mathfrak{F} = (W, R, \mathcal{P})$ is descriptive and $x \in W$. Conditions **(dif)** and **(tig)** for $\mathfrak{F}_x$ are straightforward and left to the reader. To establish **(com)**, consider any $\mathcal{X}_x \subseteq \mathcal{P}_x$ with the fip. Then

$$\mathcal{X} = \{V \in \mathcal{P} \mid V \cap W_x \in \mathcal{X}_x\} \cup \{V \in \mathcal{P} \mid W_x \subseteq V\}$$

also has the fip, and so $\bigcap \mathcal{X} \neq \emptyset$. To prove that $\bigcap \mathcal{X}_x \neq \emptyset$, it suffices to show that $\bigcap\{V \in \mathcal{P} \mid W_x \subseteq V\} \subseteq W_x$. To this end, suppose on the contrary that $y \in \bigcap\{V \in \mathcal{P} \mid W_x \subseteq V\}$ and $y \notin W_x$. Then **(dif)** and **(tig)** give $Z, Y \in \mathcal{P}$ such that $x \in Z$, $y \notin Z$, $y \in Y$, and $x \in \Box\neg Y$. It follows that $Z \cup \neg Y \in \mathcal{P}$, $W_x \subseteq Z \cup \neg Y$, and so $y \in Z \cup \neg Y$, which is a contradiction.

2.2. The structure of linear finitely-generated descriptive frames. From now on, all frames $\mathfrak{F} = (W, R, \mathcal{P})$ are assumed to be rooted frames for $\mathsf{K4.3}$, so their relation R is always transitive and *connected*:

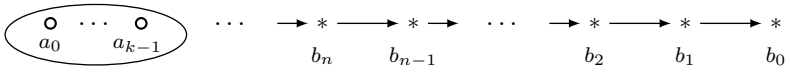
(2) $$\forall x, y \in W \, \big(xRy \vee x = y \vee yRx\big).$$

A *cluster* in $\mathfrak{F}$ is any set of the form $C(x) = \{x\} \cup \{y \in W \mid xRy \wedge yRx\}$ with $x \in W$. If x is *irreflexive*, i.e., xRx does not hold, $C(x)$ is called a *degenerate cluster* and depicted as $\bullet$; a *reflexive* x (for which xRx) is depicted as $\circ$. A non-degenerate cluster with $k \geq 1$ (reflexive) points is depicted as $\textcircled{k}$. The next example will be used many times in what follows.

EXAMPLE 2.2. Consider the frame $\mathfrak{F} = (W_k, R_{k\bullet}, \mathcal{P}_k)$, where $0 < k < \omega$,

$$W_k = A_k \cup \{b_n \mid n < \omega\}, \qquad A_k = \{a_0, \dots, a_{k-1}\},$$
$$xR_{k\bullet}y \text{ iff either } x = a_i \text{ or } x = b_n,\ y = b_m \text{ and } m < n,$$

and $\mathcal{P}_k$ is generated by the sets $X_i = \{a_i\} \cup \{b_n \mid n < \omega,\ n \equiv i \pmod k\}$, for $i < k$, and $\{b_n\}$, for $n < \omega$. (For instance, $\mathcal{P}_1$ consists of all finite subsets of $\{b_n \mid n < \omega\}$ and their complements in W_1.) The underlying Kripke frame $(W_k, R_{k\bullet})$ is shown in the picture below, where all $*$ are $\bullet$.



It is not hard to see that

(3) for any $X \in \mathcal{P}_k$, X is infinite iff $A_k \cap X \neq \emptyset$,

and so $A_k \notin \mathcal{P}_k$. For every nonempty $X \in \mathcal{P}_k$, the set $\Diamond^{\mathfrak{F}} X$ is cofinite in W_k. Using these observations, it is readily checked that $\mathfrak{F}$ is a descriptive frame; we denote it by $\mathfrak{C}(\textcircled{k}, \bullet)$. Clearly, $\mathfrak{C}(\textcircled{k}, \bullet)$ is $\mathfrak{M}$-generated for $\mathfrak{M}$ with $\mathfrak{v}(p_i) = X_i$ if $i < k$, and $\mathfrak{v}(p_i) = \emptyset$ otherwise. The descriptive frame $(W_k, R_{k\circ}, \mathcal{P}_k)$ with $R_{k\circ} = R_{k\bullet} \cup \{(b_n, b_n) \mid n < \omega\}$ is denoted by $\mathfrak{C}(\textcircled{k}, \circ)$; $(W_k, R_{k\circ})$ looks like in the

picture above, with all $* = \circ$. Note that $\mathfrak{C}(\textcircled{k}, \bullet) \models \mathsf{GL.3}$ but $\mathfrak{C}(\textcircled{k}, \circ) \not\models \mathsf{GL.3}$, cf. Example 2.10 (a). $\dashv$

The next lemma, originating in [12], will play a key role in our subsequent constructions. Let $\mathfrak{M}$ be a model based on a rooted frame $\mathfrak{F} = (W, R, \mathcal{P})$ for $\mathsf{K4.3}$, and let Γ be a set of formulas. A point $x \in W$ is called Γ-*maximal in* $\mathfrak{M}$ if $\mathfrak{M}, x \models \Gamma$, and whenever xRy and $\mathfrak{M}, y \models \Gamma$, then yRx. We denote by $\max_{\mathfrak{M}} \Gamma$ the set of all Γ-maximal points in $\mathfrak{M}$.

LEMMA 2.3. *Suppose* Γ *is a set of formulas and* $\mathfrak{M}$ *a model based on a rooted descriptive frame* $\mathfrak{F} = (W, R, \mathcal{P})$ *for* $\mathsf{K4.3}$. *Then the following hold*:
(modal saturation) *if* $\mathfrak{M}, x \models \Diamond \bigwedge \Gamma'$ *for every finite* $\Gamma' \subseteq \Gamma$, *then there is* y *with* xRy *and* $\mathfrak{M}, y \models \Gamma$;
(maximal points) *if there is* x *with* $\mathfrak{M}, x \models \Gamma$, *then* $\max_{\mathfrak{M}} \Gamma \neq \emptyset$.

Given a rooted frame $\mathfrak{F} = (W, R, \mathcal{P})$ for $\mathsf{K4.3}$, let $R^s = \{(x, y) \in R \mid (y, x) \notin R\}$ be the *strict* R-accessibility in $\mathfrak{F}$. Sometimes it will be convenient to view (W, R) as a strict linear order $\mathfrak{F}_c = (W_c, <_R)$ of clusters, where $W_c = \{C(x) \mid x \in W\}$ and $C(x) <_R C(y)$ iff $xR^s y$. A cluster C is *final* in $\mathfrak{F}$ if there is no cluster C' with $C <_R C'$. A cluster C is a *root cluster* if there is no cluster C' with $C' <_R C$, in which case $C <_R C'$ for every $C' \neq C$ in $\mathfrak{F}$; the root cluster in $\mathfrak{F}$ is unique. A cluster C' is an *immediate successor* of a cluster C in $\mathfrak{F}$ if $C <_R C'$ and there is no C'' with $C <_R C'' <_R C'$, in which case C is an *immediate predecessor* of C'. A sequence C_n, $n < \omega$, of clusters in $\mathfrak{F}_c$ is an *infinite ascending chain* if $C_n <_R C_{n+1}$, for all $n < \omega$. $\mathfrak{F}_c$ is *converse well-founded* if it has no infinite ascending chain of clusters.

The next lemma follows from, e.g., the more general [9, Theorems 10.34, 10.35]:

LEMMA 2.4. *If* $\mathfrak{F}$ *is a rooted* n*-generated descriptive frame for* $\mathsf{K4.3}$, *for some* $n < \omega$, *then*

(a) $\mathfrak{F}_c$ *is converse well-founded, and so the strict linear order* $\mathfrak{F}_c^{-1} = (W_c, >_R)$ *is isomorphic to some ordinal*;
(b) *every cluster in* $\mathfrak{F}$ *has at most* 2^n *points.*

PROOF. Let $\mathfrak{F} = (W, R, \mathcal{P})$, let $\leq_R$ be the reflexive closure of $<_R$, and let $\mathcal{G}$ be a finite set generating $\mathcal{P}$ with $|\mathcal{G}| = n$. For $x, y \in W$, we write $x \sim_{\mathcal{G}} y$ in case $x \in G$ iff $y \in G$, for all $G \in \mathcal{G}$, and denote by $[x]_{\mathcal{G}}$ the $\sim_{\mathcal{G}}$-class of x. Clearly, $\big|\{[x]_{\mathcal{G}} \mid x \in W\}\big| = 2^{|\mathcal{G}|} = 2^n$.

(a) Suppose on the contrary that $C(x_i)$, $i < \omega$, is an infinite ascending chain in $\mathfrak{F}_c$. Call $x \in W$ a *middle-point* if $C(x_0) \leq_R C(x) \leq_R C(x_i)$, for some $i < \omega$. Let $V_x = \{[y]_{\mathcal{G}} \mid y \text{ a middle-point with } xRy\}$. Since $V_x \supseteq V_y$ whenever xRy and each V_x is finite, there is $m < \omega$ such that $V_y = V_{x_m}$, for every middle-point y with $C(x_m) \leq_R C(y)$. By induction on the construction of $X \in \mathcal{P}$ from the generators in $\mathcal{G}$, it is readily seen that

(4) if y, z are middle-points, $C(x_m) \leq_R C(y)$, $C(x_m) \leq_R C(z)$, and $y \sim_{\mathcal{G}} z$, then $y \in X$ iff $z \in X$, for all $X \in \mathcal{P}$.

(Indeed, the only non-trivial case is when $X = \Diamond^{\mathfrak{F}} Y$, yRz and $y \in \Diamond^{\mathfrak{F}} Y$. Then there is $x \in Y$ with yRx. If zRx, we are done. Otherwise, x is a middle-point.

As $V_y = V_z$, there is a middle-point x' with zRx' and $x \sim_{\mathcal{G}} x'$. By IH, $x' \in Y$.) As there are finitely many $\sim_{\mathcal{G}}$-classes, there exist $k \neq \ell \geq m$ such that $x_k \sim_{\mathcal{G}} x_\ell$, and so $x_k \in X$ iff $x_\ell \in X$, for all $X \in \mathcal{P}$, by (4). But this contradicts **(dif)**.

(b) It is straightforward to show that if $C(x) = C(y)$ and $x \sim_{\mathcal{G}} y$, then $x \in X$ iff $y \in X$, for all $X \in \mathcal{P}$. So by **(dif)**, every cluster in $\mathfrak{F}$ has $\leq 2^{|\mathcal{G}|}$ points. $\dashv$

Note that the existence of maximal points (Lemma 2.3) in models based on rooted finitely generated descriptive frames for $\mathsf{K4.3}$ also follows from Lemma 2.4. Another consequence is that such a frame $\mathfrak{F}$ contains a unique final cluster, and any non-root cluster in $\mathfrak{F}$ has an immediate predecessor. If $\mathfrak{F}_c^{-1} = (W_c, >_R)$ is isomorphic to an ordinal γ and $\alpha \leq \gamma$, we denote by $C_\alpha^{\mathfrak{F}}$ the cluster that is the image of α under this isomorphism. If α is a non-zero limit ordinal, we call $C_\alpha^{\mathfrak{F}}$ a *limit cluster*. A non-final cluster is a limit cluster iff it does not have an immediate successor. By **(dif)** and Lemma 2.4 (b), we also have:

LEMMA 2.5. *If $\mathfrak{F} = (W, R, \mathcal{P})$ is a rooted finitely generated descriptive frame for $\mathsf{K4.3}$ and $C \in \mathcal{P}$, for some cluster C, then $\{x\} \in \mathcal{P}$, for every $x \in C$.*

Now, suppose $\mathfrak{M}$ is a model based on a rooted finitely $\mathfrak{M}$-generated descriptive frame $\mathfrak{F} = (W, R, \mathcal{P})$ for $L \supseteq \mathsf{K4.3}$. Given a formula μ, a cluster C is called *μ-maximal in $\mathfrak{M}$* if there is a point in C that is $\{\mu\}$-maximal in $\mathfrak{M}$. Further, C is *maximal in $\mathfrak{M}$* if it is μ-maximal in $\mathfrak{M}$, for some μ, and C is *σ-maximal in $\mathfrak{M}$*, for a signature σ, if there is such a σ-formula μ. Every definable in $\mathfrak{M}$ cluster is clearly maximal in $\mathfrak{M}$. The next lemma says that the converse is also true:

LEMMA 2.6. *Suppose $\mathfrak{M}$ is a model based on a rooted finitely $\mathfrak{M}$-generated descriptive frame $\mathfrak{F} = (W, R, \mathcal{P})$ for $\mathsf{K4.3}$. Then*

(a) *every degenerate cluster in $\mathfrak{F}$ is maximal in $\mathfrak{M}$;*
(b) *a cluster is maximal in $\mathfrak{M}$ iff either it is final or has an immediate successor;*
(c) *a cluster is definable in $\mathfrak{M}$ iff it is maximal in $\mathfrak{M}$.*

So limit clusters are not definable and not degenerate, while every other cluster is definable in $\mathfrak{M}$.

PROOF. (a) If $C(x)$ is degenerate, then $\Diamond t_{\mathfrak{M}}(x) \not\subseteq t_{\mathfrak{M}}(x)$ by **(tig)**. So there is a formula μ with $\mathfrak{M}, x \models \mu$ but $\mathfrak{M}, x \not\models \Diamond\mu$.

$(b, \Rightarrow)$ Let $C(x)$ be maximal in $\mathfrak{M}$ with $\mathfrak{M}, x \models \mu$ and $\mathfrak{M}, y \not\models \mu$ whenever $xR^s y$. Suppose $C(x)$ is a limit cluster. Let $S = \{C \in W_c \mid C(x) <_R C\}$ with $y_C \in C$, for $C \in S$. Consider

$$\Gamma = \bigcup_{C \in S} \Diamond t_{\mathfrak{M}}(y_C) \cup \{\psi \mid \Box\psi \in t_{\mathfrak{M}}(x)\} \cup \{\Box\neg\mu\}.$$

Clearly, Γ is finitely satisfiable in $\mathfrak{M}$, and so, by **(com)**, $\Gamma \subseteq t_{\mathfrak{M}}(y)$, for some y. Thus, by **(tig)**, $xRyRy_C$ for all $C \in S$, and so $yR^s y_C$ for all $C \in S$ and yRx. But we also have $\mathfrak{M}, y \models \Box\neg\mu$, contrary to $\mathfrak{M}, x \models \mu$.

$(b, \Leftarrow)$ The (unique) final cluster is maximal in $\mathfrak{M}$ for $\top$. Suppose $C(y)$ is an immediate successor of $C(x)$. If $C(y)$ is degenerate, then $C(y)$ is maximal in $\mathfrak{M}$ by (a), and so there is μ with $\mathfrak{M}, y \models \mu \wedge \neg\Diamond\mu$. It follows that $C(x)$ is $\Diamond(\mu \wedge \neg\Diamond\mu)$-maximal in $\mathfrak{M}$. If $C(y)$ is non-degenerate and $C(x)$ is not maximal in $\mathfrak{M}$, then $\Diamond t_{\mathfrak{M}}(x) \subseteq t_{\mathfrak{M}}(y)$, and so yRx by **(tig)**, contrary to $xR^s y$.

$(c, \Leftarrow)$ Let $C(x)$ be μ-maximal in $\mathfrak{M}$. If $C(x)$ is degenerate, it is defined by $\mu \wedge \neg\Diamond\mu$. Otherwise, take the immediate predecessor $C(y)$ of $C(x)$. By (b), $C(y)$ is τ-maximal in $\mathfrak{M}$, for some τ, so $\Box^+\neg\tau \wedge \Diamond\mu$ defines $C(x)$. $(c, \Rightarrow)$ is obvious. $\dashv$

We require a few important consequences of Lemmas 2.4 and 2.6.

LEMMA 2.7. *If $\mathfrak{F} = (W, R, \mathcal{P})$ is a rooted finitely generated descriptive frame for* K4.3*, then W is countable.*

PROOF. By Lemma 2.4, it suffices to show that the ordinal γ isomorphic to $\mathfrak{F}_c^{-1} = (W_c, >_R)$ is countable. Let $Z = \{\alpha + 1 \mid \alpha < \gamma,\ \alpha + 1 \neq \gamma\}$ be the set of successor ordinals $< \gamma$. Then $|Z| = |\gamma|$ and $C^{\mathfrak{F}}_\beta \in \mathcal{P}$, for any $\beta \in Z$, by Lemma 2.6. As $\mathfrak{F}$ is finitely generated, $\mathcal{P}$ is countable, and so are Z and W. $\dashv$

Given a rooted finitely $\mathfrak{M}$-generated descriptive frame $\mathfrak{F} = (W, R, \mathcal{P})$ for K4.3, let $m_{\mathfrak{F}}$ be the smallest ordinal $\leq \omega$ with degenerate $C^{\mathfrak{F}}_n$ for all $n < m_{\mathfrak{F}}$. We call the (possibly empty) interval $Z = \bigcup_{n<m_{\mathfrak{F}}} C^{\mathfrak{F}}_n$ the *tail of* $\mathfrak{F}$. We may assume that $Z = \{z_n \mid n < m_{\mathfrak{F}}\}$, where all z_n are irreflexive and $z_n R z_{n-1}$, $0 < n < m_{\mathfrak{F}}$. If Z is infinite, then $Z \neq W$ (as $\mathfrak{F}$ is rooted). If $Z \neq W$, we call $C^{\mathfrak{F}}_{m_{\mathfrak{F}}}$ the *head of* Z. In particular, if $Z = \emptyset$, its head is the final (non-degenerate) cluster $C^{\mathfrak{F}}_0$; if $Z \neq W$ and $Z \neq \emptyset$ is finite, its head is the immediate predecessor of $C^{\mathfrak{F}}_{m_{\mathfrak{F}}-1} = \{z_{m_{\mathfrak{F}}-1}\}$; and if Z is infinite, its head is the limit cluster $C^{\mathfrak{F}}_\omega$. Thus, by Lemma 2.6,

(5) the head of a tail is always non-degenerate.

2.3. Building linear models from pieces.

DEFINITION 2.8. The *ordered sum* $\mathfrak{F}_0 \lhd \cdots \lhd \mathfrak{F}_{n-1} = (W, R, \mathcal{P})$ of rooted frames $\mathfrak{F}_i = (W_i, R_i, \mathcal{P}_i)$, $i < n$, for K4.3 with pairwise disjoint W_i is defined by

$$W = \bigcup_{i<n} W_i, \quad R = \bigcup_{i<n} R_i \cup \bigcup_{i<j<n} (W_i \times W_j), \quad \mathcal{P} = \{X_0 \cup \cdots \cup X_{n-1} \mid X_i \in \mathcal{P}_i\}.$$

It is not hard to see that if the $\mathfrak{F}_i$ are descriptive, then $\mathfrak{F}_0 \lhd \cdots \lhd \mathfrak{F}_{n-1}$ is also descriptive. If $\mathfrak{M}_i = (\mathfrak{F}_i, \mathfrak{v}_i)$, then $\mathfrak{M} = \mathfrak{M}_0 \lhd \cdots \lhd \mathfrak{M}_{n-1}$ is the model based on $\mathfrak{F}_0 \lhd \cdots \lhd \mathfrak{F}_{n-1}$ with the valuation $\mathfrak{v}(p) = \bigcup_{i<n} \mathfrak{v}_i(p)$, for any $p \in \mathcal{V}$. We call the $\mathfrak{M}_i$ *$\lhd$-components of* $\mathfrak{M}$.

Now, let $\mathfrak{F} = (W, R, \mathcal{P})$ be a rooted frame for K4.3. An *interval in* $\mathfrak{F}$ is any subset $I \subseteq W$ such that $xRyRz$ and $x, z \in I$ imply $y \in I$, for all $x, y, z \in W$. If $I \cap C \neq \emptyset$, for a cluster C, then clearly $C \subseteq I$. An interval I is *closed* if there are clusters C, C' such that $I = C \cup C' \cup \bigcup\{D \mid C <_R D <_R C'\}$, in which case we write $I = [C, C']$. Given two closed intervals I, I' in $\mathfrak{F}$, we write $I \prec_{\mathfrak{F}} I'$ if I and I' are disjoint and xRx', for all $x \in I$, $x' \in I'$. Notice that if I is a closed internal interval in $\mathfrak{F}$, then $\mathfrak{F}\restriction_I$ is also a rooted frame for K4.3. Also, if $\mathfrak{F}$ is descriptive, then $\mathfrak{F}\restriction_I$ is descriptive as well. And if $\mathfrak{F}$ is finitely $\mathfrak{M}$-generated, for some model $\mathfrak{M}$, then $\mathfrak{F}\restriction_I$ is finitely $\mathfrak{M}\restriction_I$-generated. We clearly have the following:

LEMMA 2.9. *Suppose $\mathfrak{F} = (W, R, \mathcal{P})$ is a rooted frame for* K4.3 *and W is partitioned as $\{I_j \mid j < n\}$, $n < \omega$, with closed intervals $I_j \in \mathcal{P}$ and $I_j \prec_{\mathfrak{F}} I_k$ iff $j < k$. Then*

(a) $\mathfrak{F} = \mathfrak{F}\restriction_{I_0} \lhd \cdots \lhd \mathfrak{F}\restriction_{I_{n-1}}$;

(b) if $\mathfrak{M}$ is a model based on $\mathfrak{F}$, then $\mathfrak{M} = \mathfrak{M}\restriction_{I_0} \lhd \cdots \lhd \mathfrak{M}\restriction_{I_{n-1}}$.

2.4. Canonical formulas. To check whether a frame validates a given finitely axiomatisable logic, we use the canonical formulas of [43, 38, 9, 4] whose basic properties are summarised below in the context of $\mathsf{K4.3}$; for more details consult [9, Section 16.3]. Every logic $L \supseteq \mathsf{K4.3}$ can be represented in the form

$$L = \mathsf{K4.3} \oplus \{\alpha(\mathfrak{G}_j, \mathfrak{D}_j, \bot) \mid j \in J_L\}, \quad \text{for some index set } J_L, \tag{6}$$

where each $\alpha(\mathfrak{G}_j, \mathfrak{D}_j, \bot)$ is a *canonical formula* based on a finite rooted Kripke frame $\mathfrak{G}_j = (V_j, S_j)$ for $\mathsf{K4.3}$ and a (possibly empty) set $\mathfrak{D}_j \subseteq V_j$ of *irreflexive* non-root points in $\mathfrak{G}_j$. If L is finitely axiomatisable, its canonical axiomatisation (6) with finite J_L can be constructed effectively, given any finite set of axioms.

Let $\mathfrak{F} = (W, R, \mathcal{P})$ be any rooted finitely generated descriptive frame for $\mathsf{K4.3}$. By Theorem 2.4, $\mathfrak{F}$ contains a unique final cluster, and any non-root cluster in $\mathfrak{F}$ has an immediate predecessor. The formulas $\alpha(\mathfrak{G}_j, \mathfrak{D}_j, \bot)$ are defined so that $\mathfrak{F} \not\models \alpha(\mathfrak{G}_j, \mathfrak{D}_j, \bot)$ iff there is an injection $f\colon V_j \to W$ such that the following conditions hold, for all $x, y \in V_j$:

($\mathbf{cf}_1$) xS_jy iff $f(x)Rf(y)$ (so x is irreflexive iff $f(x)$ is);

($\mathbf{cf}_2$) if $C(x)$ is the final cluster in $\mathfrak{G}_j$, then $C(f(x))$ is the final cluster in $\mathfrak{F}$;

($\mathbf{cf}_3$) if $x \in \mathfrak{D}_j$ and $C(y)$ is the immediate predecessor of $C(x) = \{x\}$ in $\mathfrak{G}_j$, then $C(f(y))$ is the immediate predecessor of $C(f(x)) = \{f(x)\}$ in $\mathfrak{F}$;

($\mathbf{cf}_4$) $\{f(x)\} \in \mathcal{P}$.

Intuitively, every frame $\mathfrak{F}$ with $\mathfrak{F} \not\models \alpha(\mathfrak{G}_j, \mathfrak{D}_j, \bot)$ can be obtained by inserting certain chains of clusters immediately before some clusters $C(x)$ in $\mathfrak{G}_j$, provided that $x \notin \mathfrak{D}_j$, and by enlarging some non-degenerate clusters in $\mathfrak{G}_j$.

Canonical formulas of the form $\alpha(\mathfrak{G}, \emptyset, \bot)$ axiomatise exactly the *cofinal subframe logics* whose frames are closed under taking cofinal subframes. We remind the reader [9] that a subframe $\mathfrak{F}' = (W', R', \mathcal{P}')$ of a frame $\mathfrak{F} = (W, R, \mathcal{P})$ is called *cofinal* if W' is cofinal in $\mathfrak{F}$ in the sense that, for any $x \in W'$ and $y \in W$, whenever xRy then either $y \in W'$ or there is $z \in W'$ with yRz. Cofinal subframe logics enjoy the fmp, and so are decidable if finitely axiomatisable [42]. Example 2.10 shows the canonical axioms of some extensions of $\mathsf{K4.3}$.

Example 2.10. (a) We prove that

$$\mathsf{GL.3} = \mathsf{K4.3} \oplus \Box(\Box p_0 \to p_0) \to \Box p_0 = \mathsf{K4.3} \oplus \alpha(\circ, \emptyset, \bot) \oplus \alpha(\circ \lhd \bullet, \emptyset, \bot).$$

Let $\mathfrak{F} = (W, R, \mathcal{P})$ be a rooted finitely generated descriptive frame for $\mathsf{K4.3}$. By Lemma 2.7, W is countable, and so $\mathfrak{F}$ is $\mathfrak{M}$-generated, for some model $\mathfrak{M} = (\mathfrak{F}, \mathfrak{v})$. We claim that the following are equivalent:

1. $\mathfrak{M} \not\models \mathsf{GL.3}$;
2. there is a formula ψ with a non-degenerate ψ-maximal cluster in $\mathfrak{M}$;
3. there is a non-degenerate non-limit cluster in $\mathfrak{F}$;
4. $\mathfrak{F} \not\models \alpha(\circ, \emptyset, \bot) \wedge \alpha(\circ \lhd \bullet, \emptyset, \bot)$.

1. $\Rightarrow$ 2. Suppose $\mathfrak{M}, x \not\models \Box(\Box\varphi \to \varphi) \to \Box\varphi$, for some formula φ. Then the $\neg\big(\Box(\Box\varphi \to \varphi) \to \Box\varphi\big)$-maximal cluster C in $\mathfrak{M}$ is non-degenerate.

2. $\Leftrightarrow$ 3. by Lemma 2.6.

2. $\Rightarrow$ 4. Suppose the ψ-maximal cluster C_ψ in $\mathfrak{M}$ is non-degenerate, for some ψ. If C_ψ is the final cluster of $\mathfrak{F}$, then the injection f mapping $\circ$ to a point in C_ψ satisfies **($\mathbf{cf}_1$)**–**($\mathbf{cf}_4$)**, and so $\mathfrak{F} \not\models \alpha(\circ, \emptyset, \bot)$. If C_ψ is not the final cluster, then

the $\neg\psi$-maximal cluster $C_{\neg\psi}$ is the final cluster in $\mathfrak{F}$. If $C_{\neg\psi}$ is non-degenerate, then again $\mathfrak{F} \not\models \alpha(\circ, \emptyset, \bot)$; otherwise $\mathfrak{F} \not\models \alpha(\circ \lhd \bullet, \emptyset, \bot)$ as witnessed by f sending $\bullet$ to the point in the final cluster and $\circ$ to a point in C_ψ.

4. $\Rightarrow$ 1. If $\mathfrak{F} \not\models \alpha(\circ \lhd \bullet, \emptyset, \bot)$, then take an injection f from $\circ \lhd \bullet$ to $\mathfrak{F}$ satisfying $\mathbf{(cf_1)}$–$\mathbf{(cf_4)}$. By $\mathbf{(cf_4)}$ and Lemma 2.6, $\{f(\circ)\} = \mathfrak{v}(\varphi)$, for some φ. As $f(\circ)Rf(\circ)$ by $\mathbf{(cf_1)}$, it is easy to see that $\mathfrak{M}, f(\circ) \not\models \Box(\Box\neg\varphi \to \neg\varphi) \to \Box\neg\varphi$. The case when $\mathfrak{F} \not\models \alpha(\circ, \emptyset, \bot)$ is similar.

(b) Similarly, we can prove that

$$\begin{aligned}\mathsf{Log}\{(\mathbb{N}, <)\} &= \mathsf{K4.3} \oplus \Diamond\top \oplus \Box(\Box p_0 \to p_0) \to (\Diamond\Box p_0 \to \Box p_0)\\ &= \mathsf{K4.3} \oplus \alpha(\bullet, \emptyset, \bot) \oplus \alpha(\circ \lhd \circ, \emptyset, \bot)\end{aligned}$$

by showing that, for every $\mathfrak{M}$ and $\mathfrak{F}$ as above, the following are equivalent:

- $\mathfrak{M} \not\models \mathsf{Log}\{(\mathbb{N}, <)\}$;
- either the final cluster in $\mathfrak{F}$ is degenerate or there is a non-degenerate non-limit cluster different from the final cluster in $\mathfrak{F}$;
- $\mathfrak{F} \not\models \alpha(\bullet, \emptyset, \bot) \wedge \alpha(\circ \lhd \circ, \emptyset, \bot)$.

(c) A prominent example of a non-cofinal subframe logic is $\mathsf{K4.3} \oplus \Diamond p \to \Diamond\Diamond p$ with *dense* frames, whose canonical axioms

$$\mathsf{K4.3} \oplus \alpha(\bullet \lhd \bullet^a, \{a\}, \bot) \oplus \alpha(\bullet \lhd \bullet^a \lhd \circ, \{a\}, \bot) \oplus \alpha(\bullet \lhd \bullet^a \lhd \bullet, \{a\}, \bot)$$

forbid any two consecutive degenerate clusters in finitely generated descriptive frames for the logic (see also Lemma 5.6). $\dashv$

§3. Craig interpolant existence: warming up. In this section, we first give a model-theoretic, bisimulation-based criterion of interpolant non-existence, then apply it to design a coNP-algorithm deciding the IEP in any finitely axiomatisable d-persistent cofinal subframe logic containing $\mathsf{K4.3}$. Finally, we illustrate by examples that a way more involved approach is needed to tackle arbitrary finitely axiomatisable extensions of $\mathsf{K4.3}$.

A formula ι is called a *Craig interpolant* of formulas φ_1 and φ_2 in a logic L if $sig(\iota) \subseteq sig(\varphi_1) \cap sig(\varphi_2)$ and both $\varphi_1 \to \iota$ and $\iota \to \varphi_2$ are in L. We say that L has the *Craig interpolation property* (*CIP*) if an interpolant for φ_1 and φ_2 exists whenever $(\varphi_1 \to \varphi_2) \in L$.

Many standard modal logics have the CIP, including K, $\mathsf{K4}$, $\mathsf{S4}$. In fact, there are a continuum of logics containing $\mathsf{K4}$ with the CIP. However, none of the continuum-many extensions of $\mathsf{K4.3}$ with frames of unbounded depth has the CIP, and very few—not more than 37—out of the continuum-many logics containing $\mathsf{S4}$ enjoy the CIP (deciding whether a finitely axiomatisable logic above $\mathsf{S4}$ has the CIP is in coNExpTime and PSpace-hard). The reader can find proofs of these results and further references in [14, 9]; see also Example 3.6.

We now introduce the model-theoretic notions and tools that are needed in our non-uniform approach to deciding interpolant existence in modal logics.

Given two models $\mathfrak{M}_i$, $i = 1, 2$, based on $\mathfrak{F}_i = (W_i, R_i, \mathcal{P}_i)$ with $x_i \in W_i$, we write $\mathfrak{M}_1, x_1 \equiv_\sigma \mathfrak{M}_2, x_2$, for a signature σ, if $t^\sigma_{\mathfrak{M}_1}(x_1) = t^\sigma_{\mathfrak{M}_2}(x_2)$. The equivalence relation $\equiv_\sigma \subseteq W_1 \times W_2$ can be characterised in terms of bisimulations. Namely, a relation $\boldsymbol{\beta} \subseteq W_1 \times W_2$ is called a *σ-bisimulation* between $\mathfrak{M}_1$ and $\mathfrak{M}_2$ if the following conditions hold whenever $x_1 \boldsymbol{\beta} x_2$:

(atom) $at^{\sigma}_{\mathfrak{M}_1}(x_1) = at^{\sigma}_{\mathfrak{M}_2}(x_2)$;

(move) if $x_1 R_1 y_1$, then there is y_2 such that $x_2 R_2 y_2$ and $y_1 \boldsymbol{\beta} y_2$; and, conversely, if $x_2 R_2 y_2$, then there is y_1 with $x_1 R_1 y_1$ and $y_1 \boldsymbol{\beta} y_2$.

If there is such $\boldsymbol{\beta}$ with $z_1 \boldsymbol{\beta} z_2$, we write $\mathfrak{M}_1, z_1 \sim_\sigma \mathfrak{M}_2, z_2$. We call $\boldsymbol{\beta}$ *global* if, for every $x_1 \in W_1$, there is $x_2 \in W_2$ with $x_1 \boldsymbol{\beta} x_2$, and, for every $x_2 \in W_2$, there is $x_1 \in W_1$ with $x_1 \boldsymbol{\beta} x_2$. In this case, we say that $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are *globally σ-bisimilar* and write $\mathfrak{M}_1 \sim_\sigma \mathfrak{M}_2$.

We employ the following characterisation of $\equiv_\sigma$ (see [20] for a further discussion of the relationship between bisimulations and modal equivalence):

LEMMA 3.1. *For any signature σ, any models $\mathfrak{M}_i$, $i = 1, 2$, based on descriptive frames $\mathfrak{F}_i = (W_i, R_i, \mathcal{P}_i)$, and any $x_i \in W_i$,*

$$\mathfrak{M}_1, x_1 \equiv_\sigma \mathfrak{M}_2, x_2 \quad \textit{iff} \quad \mathfrak{M}_1, x_1 \sim_\sigma \mathfrak{M}_2, x_2.$$

The implication ($\Leftarrow$) holds for arbitrary models.

PROOF. ($\Rightarrow$) We show that $\{(y_1, y_2) \in W_1 \times W_2 \mid t^{\sigma}_{\mathfrak{M}_1}(y_1) = t^{\sigma}_{\mathfrak{M}_2}(y_2)\}$ is a σ-bisimulation between $\mathfrak{M}_1$ and $\mathfrak{M}_2$. Condition **(atom)** is obvious. For **(move)**, suppose $y_1 R_1 z_1$ and $t^{\sigma}_{\mathfrak{M}_1}(y_1) = t^{\sigma}_{\mathfrak{M}_2}(y_2)$. Let $\Gamma = t^{\sigma}_{\mathfrak{M}_1}(z_1)$. Then, for every finite $\Gamma' \subseteq \Gamma$, we have $\mathfrak{M}_1, y_1 \models \Diamond \bigwedge \Gamma'$, and so $\mathfrak{M}_2, y_2 \models \Diamond \bigwedge \Gamma'$ as well. Since $\mathfrak{F}_2$ is descriptive, Lemma 2.3 gives us z_2 with $y_2 R_2 z_2$ and $\mathfrak{M}_2, z_2 \models \Gamma$. It follows that $t^{\sigma}_{\mathfrak{M}_1}(z_1) = t^{\sigma}_{\mathfrak{M}_2}(z_2)$, as required. The implication ($\Leftarrow$) is straightforward. ⊣

Note that if $\mathcal{B}$ is a set of σ-bisimulations between $\mathfrak{M}_1$ and $\mathfrak{M}_2$, then $\bigcup_{\boldsymbol{\beta} \in \mathcal{B}} \boldsymbol{\beta}$ is also a σ-bisimulation between $\mathfrak{M}_1$ and $\mathfrak{M}_2$. It follows that there is always a *largest* σ-bisimulation between $\mathfrak{M}_1$ and $\mathfrak{M}_2$ (which is $\equiv_\sigma$ if both $\mathfrak{M}_i$ are based on descriptive frames).

Variations of the following criterion of interpolant (non-)existence are implicit in various (dis-)proofs of the CIP in modal logics [32, 20].

THEOREM 3.2. *Formulas φ_1 and φ_2 do not have an interpolant in a modal logic L iff there are models $\mathfrak{M}_i$, $i = 1, 2$, based on finitely $\mathfrak{M}_i$-generated descriptive frames $\mathfrak{F}_i = (W_i, R_i, \mathcal{P}_i)$ for L with points $x_i \in W_i$ such that*

$$\mathfrak{M}_1, x_1 \models \varphi_1, \quad \mathfrak{M}_2, x_2 \models \neg\varphi_2, \quad \mathfrak{M}_1, x_1 \sim_\sigma \mathfrak{M}_2, x_2, \text{ for } \sigma = sig(\varphi_1) \cap sig(\varphi_2).$$

If $L \supseteq \mathsf{K4}$, we may assume that x_i is the root of the descriptive frame $\mathfrak{F}_i$, $i = 1, 2$.

PROOF. ($\Leftarrow$) is straightforward (and holds for arbitrary frames for L). For ($\Rightarrow$), consider the signature $\delta = sig(\varphi_1) \cup sig(\varphi_2)$ and the set

$$\Sigma = \{\chi \mid \chi \text{ is a } \sigma\text{-formula and } (\varphi_1 \to \chi) \in L\} \cup \{\neg\varphi_2\}$$

of δ-formulas. As φ_1 and φ_2 have no interpolant in L, Σ is L-consistent, and so, by Lemma 2.1, there exist a δ-model $\mathfrak{M}_2$ based on a finitely $\mathfrak{M}_2$-generated descriptive frame $\mathfrak{F}_2$ and a point x_2 with $\mathfrak{M}_2, x_2 \models \Sigma$. Let $\Sigma' = t^{\sigma}_{\mathfrak{M}_2}(x_2) \cup \{\varphi_1\}$. As Σ' is an L-consistent set of δ-formulas, Lemma 2.1 gives a δ-model $\mathfrak{M}_1$ based on a finitely $\mathfrak{M}_1$-generated descriptive frame $\mathfrak{F}_1$ and an x_1 in $\mathfrak{M}_1$ such that $\mathfrak{M}_1, x_1 \models \Sigma'$. We clearly have $t^{\sigma}_{\mathfrak{M}_1}(x_1) = t^{\sigma}_{\mathfrak{M}_2}(x_2)$, and so $\mathfrak{M}_1, x_1 \sim_\sigma \mathfrak{M}_2, x_2$ by Lemma 3.1. In case $L \supseteq \mathsf{K4}$, (1) allows us to make x_i the root of $\mathfrak{F}_i$. ⊣

The next lemma refines Theorem 3.2; it is used in the proof of Lemma 4.21.

LEMMA 3.3. *If φ_1 and φ_2 do not have an interpolant in a logic $L \supseteq \mathsf{K4.3}$, then there are rooted models $\mathfrak{M}_i, x_i$, $i = 1, 2$, satisfying the criterion Theorem 3.2 such that $C(x_i)$ is not a limit cluster in $\mathfrak{M}_i$, for $i = 1, 2$.*

PROOF. Suppose $\mathfrak{M}, x$ is a rooted δ-model, for some finite signature δ, that is based on a finitely $\mathfrak{M}$-generated descriptive frame $\mathfrak{F} = (W, R, \mathcal{P})$ such that $\mathfrak{M}, x \models \varphi$, for some φ with $sig(\varphi) \subseteq \delta$, and $C(x)$ is a root limit cluster in $\mathfrak{F}$. Pick a fresh variable $q \notin \delta$. For $* \in \{\bullet, \circ\}$, take the frames $\mathfrak{F}^* = * \lhd \mathfrak{F}$, denote the root point of $\mathfrak{F}^*$ by x^*, and consider the $\delta \cup \{q\}$-models $\mathfrak{M}^*$ based on $\mathfrak{F}^*$, which coincide with $\mathfrak{M}$ on $\mathfrak{F}$ and have $\mathfrak{M}, x^* \models p$ iff $\mathfrak{M}, x \models p$, for $p \in \delta$, and $\mathfrak{M}^*, x^* \models q$. To prove the lemma, it suffices to show that there is $* \in \{\bullet, \circ\}$ with (i) $\mathfrak{M}^*, x^* \models \varphi$, (ii) $\mathfrak{M}^*, x^* \sim_\sigma \mathfrak{M}, x$, for any $\sigma \subseteq sig(\varphi)$, and (iii) $\mathsf{Log}(\mathfrak{F}) \subseteq \mathsf{Log}(\mathfrak{F}^*)$.

As the limit cluster $C(x)$ is non-degenerate by Lemma 2.6, we have (i) and (ii). To show (iii), suppose on the contrary that, for each $* \in \{\bullet, \circ\}$, there is a canonical formula $\alpha(\mathfrak{G}^*, \mathfrak{D}^*, \bot)$ with $\mathfrak{F} \models \alpha(\mathfrak{G}^*, \mathfrak{D}^*, \bot)$ and $\mathfrak{F}^* \not\models \alpha(\mathfrak{G}^*, \mathfrak{D}^*, \bot)$. Let f^* be an injection from $\mathfrak{G}^*$ to $\mathfrak{F}^*$ satisfying $\mathbf{(cf_1)}$–$\mathbf{(cf_4)}$ for $\alpha(\mathfrak{G}^*, \mathfrak{D}^*, \bot)$, and let $C(r^*)$ be the root-cluster in $\mathfrak{G}^*$ and $C(y^*)$ its immediate successor in $\mathfrak{G}^*$. By assumption, f^* is not an injection from $\mathfrak{G}^*$ to $\mathfrak{F}$ satisfying $\mathbf{(cf_1)}$–$\mathbf{(cf_4)}$, so $f^*(r^*) = x^*$ and $f^*(y^*) \in W$. As $\{f^*(y^*)\} \in \mathcal{P}$ by $\mathbf{(cf_4)}$ and $C(x)$ is a limit cluster, it follows from Lemma 2.6 that $f^*(y^*) \notin C(x)$, and so $y^* \notin \mathfrak{D}^*$. Suppose, for definiteness, that $f^\circ(y^\circ) R f^\bullet(y^\bullet)$ or $f^\circ(y^\circ) = f^\bullet(y^\bullet)$. Let C be the immediate predecessor of $C(f^\circ(y^\circ))$ in $\mathfrak{F}$. Then C is a non-limit cluster. By Lemma 2.6, $C \in \mathcal{P}$ and, by Lemma 2.5, $\{z\} \in \mathcal{P}$, for every $z \in C$. If C is non-degenerate, then we modify f° by taking $f^\circ(r^\circ) \in C$; otherwise, we modify $f^\bullet$ by taking $f^\bullet(r^\bullet) \in C$. In either case, the modified f^* is an injection from $\mathfrak{G}^*$ to $\mathfrak{F}$ satisfying $\mathbf{(cf_1)}$–$\mathbf{(cf_4)}$, a contradiction. ⊣

We begin our study of the interpolant existence problem (IEP) by showing how the criterion of Theorem 3.2 can be used to decide whether given formulas have an interpolant in any fixed d-persistent cofinal subframe logic $L \supseteq \mathsf{K4.3}$ (defined in §2.4). Suppose that φ_1 and φ_2 do not have an interpolant in L. Let $\sigma = sig(\varphi_1) \cap sig(\varphi_2)$. By Theorem 3.2, there exist models $\mathfrak{M}_i$, $i = 1, 2$, based on descriptive frames $\mathfrak{F}_i = (W_i, R_i, \mathcal{P}_i)$ for L with roots $x_i \in W_i$ such that $\mathfrak{M}_1, x_1 \sim_\sigma \mathfrak{M}_2, x_2$, $\mathfrak{M}_1, x_1 \models \varphi_1$ and $\mathfrak{M}_2, x_2 \models \neg\varphi_2$. We may assume that $\boldsymbol{\beta}$ is the largest σ-bisimulation $\equiv_\sigma$ between $\mathfrak{M}_1$ and $\mathfrak{M}_2$ (for which $x_1 \boldsymbol{\beta} x_2$, of course). We show how to extract from the $\mathfrak{M}_i$ polynomial-size models $\mathfrak{M}'_i$ that still witness that φ_1 and φ_2 lack an interpolant in L. We proceed in two steps.

Step 1: For each $i = 1, 2$ and each $\tau \in sub(\varphi_i)$ satisfied in $\mathfrak{M}_i$, we take a $\{\tau\}$-maximal point $y_\tau \in W_i$ (which exists by Lemma 2.3), and denote the set of all these y_τ by $\boldsymbol{M}_i \subseteq W_i$. Note that $\boldsymbol{M}_i$ is cofinal in $\mathfrak{F}_i$ because each point in $W_i \setminus \boldsymbol{M}_i$ has a $\{\varphi_i\}$- or $\{\neg\varphi_i\}$-maximal R_i-successor. Set

$$T = \big\{t^\sigma_{\mathfrak{M}_1}(x) \mid x \in \{x_1\} \cup \boldsymbol{M}_1\big\} \cup \big\{t^\sigma_{\mathfrak{M}_2}(x) \mid x \in \{x_2\} \cup \boldsymbol{M}_2\big\}. \tag{7}$$

Step 2: As $\mathfrak{M}_1, x_1 \sim_\sigma \mathfrak{M}_2, x_2$ and $\boldsymbol{\beta}$ is the largest σ-bisimulation, each $t \in T$ is satisfied in both $\mathfrak{M}_i$. For $i = 1, 2$, we take a smallest set $\boldsymbol{S}_i \subseteq W_i$ containing a t-maximal point z_t in $\mathfrak{M}_i$ (which exists by Lemma 2.3), for each $t \in T$.

Now, let $W'_i = \{x_i\} \cup \boldsymbol{M}_i \cup \boldsymbol{S}_i$, $R'_i = R_i \restriction_{W'_i}$, $\mathfrak{F}'_i = (W'_i, R'_i)$, and let $\mathfrak{M}'_i$ be the restriction of $\mathfrak{M}_i$ to $\mathfrak{F}'_i$. We let

$$\boldsymbol{k}(\varphi_1, \varphi_2) = 3 + 3\max(|\varphi_1|, |\varphi_2|). \tag{8}$$

Clearly, $|W_i'| \leq \boldsymbol{k}(\varphi_1, \varphi_2)$, so the size of $\mathfrak{M}_i$ is $\mathcal{O}\big(\max(|\varphi_1|, |\varphi_2|)\big)$. As L is d-persistent, $(W_i, R_i) \models L$. By construction, $\mathfrak{F}_i'$ is a cofinal subframe of (W_i, R_i), and so $\mathfrak{F}_i' \models L$ as L is a cofinal subframe logic. Finally, we define $\boldsymbol{\beta}'$ as the restriction of $\boldsymbol{\beta}$ to $W_1' \times W_2'$: $x_1' \boldsymbol{\beta}' x_2'$ iff $t^\sigma_{\mathfrak{M}_1}(x_1') = t^\sigma_{\mathfrak{M}_2}(x_2')$, for all $x_1' \in W_1'$, $x_2' \in W_2'$.

LEMMA 3.4. *(a)* $\mathfrak{M}_1', x_1 \models \varphi_1$, $\mathfrak{M}_2', x_2 \models \neg\varphi_2$, *and (b)* $\boldsymbol{\beta}'$ *is a* σ*-bisimulation between* $\mathfrak{M}_1'$ *and* $\mathfrak{M}_2'$ *with* $x_1 \boldsymbol{\beta}' x_2$.

PROOF. *(a)* follows from the fact that, for any $\tau \in \mathit{sub}(\varphi_i)$ and $x \in W_i'$, $\mathfrak{M}_i, x \models \tau$ iff $\mathfrak{M}_i', x \models \tau$, which can be established by a straightforward induction on the construction of φ_1 and φ_2. We only show ($\Rightarrow$) for $\tau = \Diamond\psi$. If $\mathfrak{M}_i, x \models \Diamond\psi$, then there is $y \in W_i$ with xR_iy and $\mathfrak{M}_i, y \models \psi$. Take $y_\psi \in \boldsymbol{M}_i \subseteq W_i'$. By the $\{\psi\}$-maximality of y_ψ, either $y = y_\psi$ or yR_iy_ψ, and so $xR_i'y_\psi$ and $\mathfrak{M}_i', x \models \Diamond\psi$.

(b) Condition **(atom)** follows from the definition. To establish **(move)**, assume $x\boldsymbol{\beta}'x'$ and $xR_1'y$. Let $t = t^\sigma_{\mathfrak{M}_1}(y)$. Then $t \in T$, and so there is a t-maximal $z_t \in \boldsymbol{S}_2 \subseteq W_2'$ in $\mathfrak{M}_2$. In particular, $t^\sigma_{\mathfrak{M}_2}(z_t) = t$, and so $y\boldsymbol{\beta}'z_t$. As $x\boldsymbol{\beta}x'$ and $\boldsymbol{\beta}$ is the largest σ-bisimulation, there is $z \in W_2$ with $x'R_2z$ and $t^\sigma_{\mathfrak{M}_2}(z) = t$. It follows from the t-maximality of z_t that $z = z_t$ or zR_2z_t, and so $x'R_2'z_t$, as required. ⊣

Thus, the fact that φ_1 and φ_2 have no interpolant in L can always be witnessed (in the sense of Theorem 3.2) by models $\mathfrak{M}_i$ of size polynomial in $\max(|\varphi_1|, |\varphi_2|)$, and so we can say that L has the *polysize bisimilar model property*. This gives the first claim of the following theorem:

THEOREM 3.5. *(a) All d-persistent cofinal subframe logics* $L \supseteq \mathsf{K4.3}$ *have the polysize bisimilar model property. (b) If such an* L *is consistent and finitely axiomatisable, then the IEP for* L *is* coNP*-complete.*

PROOF. We show that $(a) \Rightarrow (b)$; cf. Theorem 4.9 in §4. Indeed, suppose L is given by (6) (with $\mathfrak{G}_j = (V_j, S_j)$ and $\mathfrak{D}_j = \emptyset$, for all j in the finite index set J_L). To decide whether formulas φ_1 and φ_2 do not have an interpolant in L, we guess polynomial-size pointed models $\mathfrak{M}_i, x_i$ based on Kripke frames $\mathfrak{F}_i = (W_i, R_i)$ for $\mathsf{K4.3}$ and restricted to the variables in φ_1 and φ_2. The conditions $\mathfrak{M}_1, x_1 \models \varphi_1$ and $\mathfrak{M}_2, x_2 \models \neg\varphi_2$ are clearly polynomially checkable; that $\mathfrak{M}_1, x_1 \sim_\sigma \mathfrak{M}_2, x_2$, for $\sigma = \mathit{sig}(\varphi_1) \cap \mathit{sig}(\varphi_2)$, can be established in polynomial time using a standard technique from [2, Chapter 7]. Finally, to check whether $\mathfrak{F}_i \models \alpha(\mathfrak{G}_j, \emptyset, \bot)$, for each $j \in J_L$, we simply enumerate all injective functions from $\mathfrak{G}_j$ to $\mathfrak{F}_i$, whose number does not exceed $|W_i|^{|V_j|}$, and verify that at least one of them satisfies $\mathbf{(cf_1)}$–$\mathbf{(cf_2)}$, which can obviously be done in time polynomial in $|W_i|$. (Condition $\mathbf{(cf_3)}$ holds vacuously, and $\mathbf{(cf_4)}$ always holds as $\mathfrak{F}_i$ is a Kripke frame.) ⊣

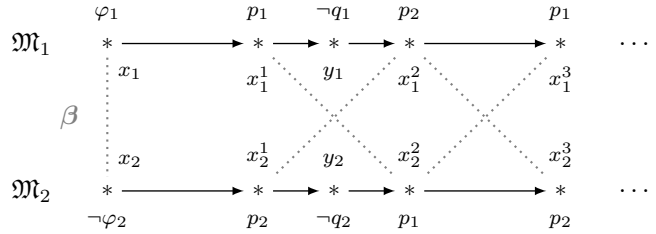
We now give two examples illustrating that the construction above does not work for logics that are not d-persistent, even for logics with the fmp. Prominent examples of such logics are $\mathsf{GL.3}$ and $\mathsf{Log}\{(\mathbb{N}, <)\}$; see Example 2.10. We show that, for these logics, establishing model-theoretically (using Theorem 3.2) that some formulas do not have an interpolant requires a pair of models that are based on infinite descriptive (non-Kripke) frames.

EXAMPLE 3.6. (*a*) Consider the following formulas φ_1 and φ_2:

$$\varphi_1 = \Diamond(p_1 \wedge \Diamond^+\neg q_1) \wedge \Box(p_2 \to \Box^+ q_1),$$
$$\varphi_2 = \neg[\Diamond(p_2 \wedge \Diamond^+\neg q_2) \wedge \Box(p_1 \to \Box^+ q_2)]. \tag{9}$$

To show that $(\varphi_1 \to \varphi_2) \in \mathsf{K4.3} \subseteq \mathsf{GL.3}$, suppose otherwise. Then there exist a model $\mathfrak{M}$ based on a frame $\mathfrak{F} = (W, R)$ for $\mathsf{K4.3}$ and $z \in W$ with $\mathfrak{M}, z \models \varphi_1 \wedge \neg\varphi_2$. So we have $x, x', y, y' \in W$ with $zRxR^+x'$, $zRyR^+y'$, $\mathfrak{M}, x \models p_1$, $\mathfrak{M}, x' \models \neg q_1$, $\mathfrak{M}, y \models p_2$, and $\mathfrak{M}, y' \models \neg q_2$. Since $\mathfrak{F}$ is a frame for $\mathsf{K4.3}$, either $x' = y'$ or $x'Ry'$ or $y'Rx'$. However, none of these is possible because of the boxed conjuncts of φ_1 and $\neg\varphi_2$.

We now use Theorem 3.2 to show that φ_1 and φ_2 do not have an interpolant in $\mathsf{GL.3}$. Let $\sigma = sig(\varphi_1) \cap sig(\varphi_2) = \{p_1, p_2\}$. Observe that any models $\mathfrak{M}_i$ meeting the conditions of Theorem 3.2 cannot be based on a Kripke frame $\mathfrak{F}_i = (W_i, R_i)$ for $\mathsf{GL.3}$. Indeed, let $\boldsymbol{\beta}$ be the corresponding bisimulation. Then $\mathfrak{M}_1, x_1 \models \varphi_1$ implies that there is $x_1^1 \in W_1$ with $x_1 R_1 x_1^1$ and $\mathfrak{M}_1, x_1^1 \models p_1$; we must also have $\mathfrak{M}_1, y_1 \models \neg q_1$, for some y_1 with $x_1^1 R_1^+ y_1$. Similarly, $\mathfrak{M}_2, x_2 \models \neg\varphi_2$ implies that there is $x_2^1 \in W_2$ with $x_2 R_1 x_2^1$ and $\mathfrak{M}_2, x_2^1 \models p_2$, and we also have $\mathfrak{M}_2, y_2 \models \neg q_2$, for some y_2 with $x_2^1 R_2^+ y_2$. As $x_1 \boldsymbol{\beta} x_2$ and $x_1 R_1 x_1^1$, **(move)** gives x_2^2 with $x_2 R_2 x_2^2$ and $x_1^1 \boldsymbol{\beta} x_2^2$. But then $\mathfrak{M}_2, x_2^2 \models p_1$, and so $x_2 R_2 x_2^1 R_2^+ y_2 R_2 x_2^2$ since $\mathfrak{F}_2$ is a frame for $\mathsf{K4.3}$ and in view of $\neg\varphi_2$'s second conjunct. Symmetrically, we find x_1^2 with $x_1 R_1 x_1^1 R_1^+ y_1 R_1 x_1^2$ and $x_1^2 \boldsymbol{\beta} x_2^1$. Using **(move)**, we construct infinite ascending chains of not necessarily distinct points as shown in the picture below.



It follows that the $\mathfrak{F}_i$ are not frames for $\mathsf{GL.3}$; see any of [19, 9, 5] for details.

We now give a descriptive frame for $\mathsf{GL.3}$ that can be used to show that φ_1 and φ_2 do not have an interpolant in $\mathsf{GL.3}$. Take the descriptive frame $\mathfrak{C}(②, \bullet)$ defined in Example 2.2 and construct $\mathfrak{F} = \bullet \lhd \bullet \lhd \mathfrak{C}(②, \bullet)$ (see Definition 2.8), which is a frame for $\mathsf{GL.3}$ by property (*iii*) in Example 2.10 (*a*). Consider the rooted models $\mathfrak{M}_i, x_i$, $i = 1, 2$, shown in Fig. 1, both of which are based on a frame isomorphic to $\mathfrak{F}$. It is readily checked that $\mathfrak{M}_1, x_1 \models \varphi_1$, $\mathfrak{M}_2, x_2 \models \neg\varphi_2$, and the depicted relation $\boldsymbol{\beta}$ is a σ-bisimulation between $\mathfrak{M}_1$ and $\mathfrak{M}_2$ with $x_1 \boldsymbol{\beta} x_2$.

In fact, the argument above shows that none of the logics L in the interval $\mathsf{K4.3} \subseteq L \subseteq \mathsf{GL.3}$ has the CIP.

(*b*) Consider next the logic $\mathsf{Log}\{(\mathbb{N}, <)\}$ and show that the formulas

$$\varphi_1' = \Diamond(p_1 \wedge \Diamond^+\neg q_1) \wedge \Box(p_2 \to \Box^+ q_1) \wedge \Diamond r \wedge \neg\Diamond(r \wedge \Diamond p_1)$$

and φ_2 given by (9) do not have an interpolant in it, though $(\varphi_1 \to \varphi_2) \in \mathsf{K4.3}$, and so $(\varphi_1' \to \varphi_2) \in \mathsf{K4.3} \subseteq \mathsf{Log}\{(\mathbb{N}, <)\}$.

As in (*a*) above, any models $\mathfrak{M}_i$, $i = 1, 2$, satisfying the conditions of Theorem 3.2 for φ_1' and φ_2 cannot be based on Kripke frames, however the reason

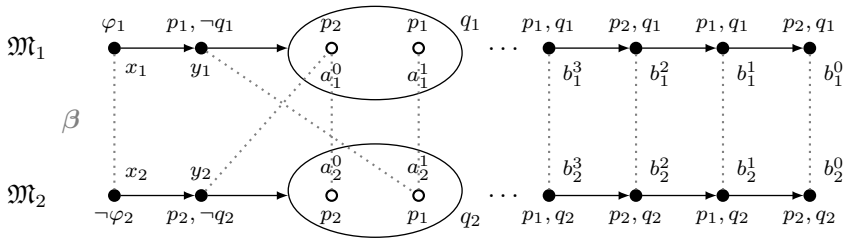
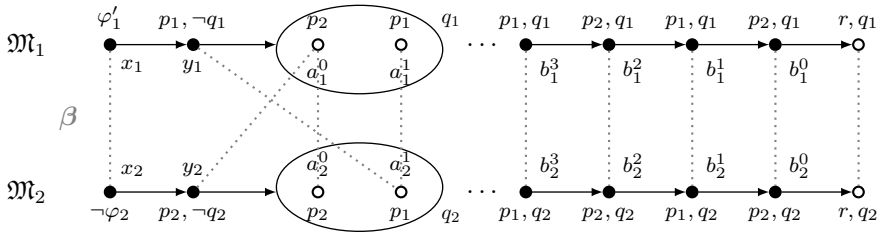


FIGURE 1. σ-bisimilar models based on a descriptive frame for GL.3.

for this is slightly different. Suppose $\boldsymbol{\beta}$ is a bisimulation witnessing these conditions. Then the models $\mathfrak{M}_i$ must contain infinite ascending chains such as those in Example 3.6 (a). Also, the model $\mathfrak{M}_1$ with $\mathfrak{M}_1, x_1 \models \varphi_1'$ must contain a point z such that $x_1 R_1 z$ and $\mathfrak{M}_1, z \models r \wedge \Box\neg p_1$, which means that z is located after all of the x_1^j, $j < \omega$. But then the Kripke frame $\mathfrak{F}_1$ underlying $\mathfrak{M}_1$ is not a frame for $\mathsf{Log}\{(\mathbb{N}, <)\}$, as it refutes its axiom $\Box(\Box p \to p) \to (\Diamond\Box p \to \Box p)$ if we make p true everywhere after the initial ascending chain in $\mathfrak{F}_1$ and false elsewhere.

The picture below shows models $\mathfrak{M}_1$ and $\mathfrak{M}_2$ based on $\bullet \lhd \bullet \lhd \mathfrak{C}(\textcircled{2}, \bullet) \lhd \circ$ and satisfying the conditions of Theorem 3.2 for φ_1' and φ_2. That this frame is a frame for $\mathsf{Log}\{(\mathbb{N}, <)\}$ follows from Example 2.10 (b). $\dashv$



§4. Interpolant existence in logics above K4.3. We now generalise Theorem 3.5 to all finitely axiomatisable logics containing K4.3. It turns out that, even though these logics do not have the polysize bisimilar model property in general, the structure of the models required in Theorem 3.2 is perfectly understandable. We show that one can assemble a pair of bisimilar models witnessing the absence of an interpolant for φ_1 and φ_2 in any $L \supseteq \mathsf{K4.3}$ as the ordered sum of finitely-many 'nice' models, which are either finite or infinite but finitely 'presentable'. Hence, we say that all $L \supseteq \mathsf{K4.3}$ have the 'quasi-finite bisimilar model property'. Moreover, if L is finitely axiomatisable, we can replace 'finite' by 'polynomial in $\boldsymbol{c}_L$ and $\max(|\varphi_1|, |\varphi_2|)$', for some constant $\boldsymbol{c}_L$ depending on L only. In this case, we say that L has the 'quasi-polysize bisimilar model property'.

§4 is organised as follows. In §4.1, we formulate our main results (Theorems 4.5–4.7 and 4.9), and show how Theorem 4.6 implies Theorem 4.9. In §4.2 and §4.3, we prove Theorem 4.5. Then, in §4.4, we show how to fine-tune the proof of Theorem 4.5 and obtain proofs of Theorem 4.6 and 4.7. Finally, in § 4.5, we formulate and prove an interesting consequence of our methods for cofinal subframe logics (Theorem 4.25).

4.1. The quasi-polysize bisimilar model property. Given a finite signature δ, a δ-model $\mathfrak{M} = (\mathfrak{F}, \mathfrak{w})$ is called *simple* if either $\mathfrak{F}$ is finite or $\mathfrak{F} = \mathfrak{C}(\textcircled{k}, *)$, for $0 < k < \omega$ and $* \in \{\bullet, \circ\}$, and, for every $p \in \delta$, there is $A_p \subseteq \{0, \dots, k-1\}$ with $\mathfrak{w}(p) = \bigcup_{i \in A_p} X_i$, where the X_i are the infinite generators of the internal sets in $\mathfrak{C}(\textcircled{k}, *)$ defined in Example 2.2. Thus, even though $\mathfrak{C}(\textcircled{k}, *)$ is infinite, any simple δ-model based on it is fully determined by the *finitary* information provided by the sets A_p, $p \in \delta$, that is, by the atomic δ-types of the points in the $\textcircled{k}$-cluster. A δ-model is called *quasi-finite* if it is the ordered sum of finitely-many simple models.

DEFINITION 4.1. A logic $L \supseteq \mathsf{K4.3}$ is said to have the *quasi-finite bisimilar model property* if, for any formulas φ_1, φ_2 without an interpolant in L, there are rooted quasi-finite δ-models $\mathfrak{N}_1, x_1$ and $\mathfrak{N}_2, x_2$ satisfying conditions (a)–(c) below, for $\delta = sig(\varphi_1) \cup sig(\varphi_2)$ and $\sigma = sig(\varphi_1) \cap sig(\varphi_2)$:

(a) $\mathfrak{N}_1, x_1 \models \varphi_1$ and $\mathfrak{N}_2, x_2 \models \neg\varphi_2$;
(b) $\mathfrak{N}_1$ and $\mathfrak{N}_2$ are based on frames for L;
(c) $\mathfrak{N}_1, x_1 \sim_\sigma \mathfrak{N}_2, x_2$.

Our first result is:

THEOREM 4.2. *All $L \supseteq \mathsf{K4.3}$ have the quasi-finite bisimilar model property.*

We actually prove a stronger Theorem 4.5 that prescribes more structure for the pair of quasi-finite models witnessing the lack of an interpolant, which makes it easy to deduce the existence of a σ-bisimulation between the models. The prescribed structure is easily checkable, which is used in the proof of the main Theorem 4.9. To formulate our 'structural' theorem, we require a few definitions.

For $0 < m < \omega$, let $m^< = \underbrace{\bullet \lhd \cdots \lhd \bullet}_{m}$. An *atomic frame* takes one of the forms

$$(10) \quad m^<, \quad \textcircled{1} \lhd m^<, \quad \textcircled{k}, \quad \mathfrak{C}(\textcircled{k}, \bullet), \quad \mathfrak{C}(\textcircled{k}, \circ), \qquad \text{where } 0 < m < \omega \text{ and } 0 < k \leq 2^{|\delta|}.$$

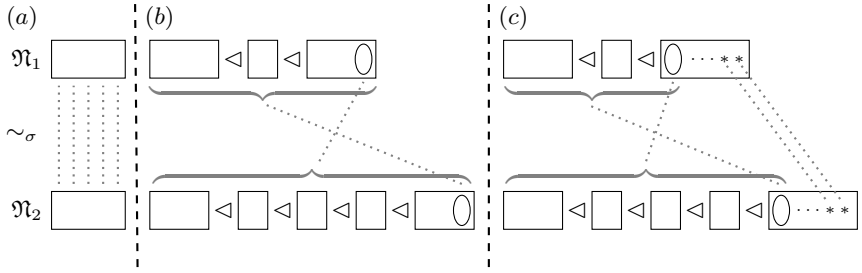
The *size* $\|\mathfrak{F}\|$ of an atomic $\mathfrak{F}$ is defined by taking $\|m^<\| = m$, $\|\textcircled{1} \lhd m^<\| = 1+m$, and $\|\textcircled{k}\| = \|\mathfrak{C}(\textcircled{k}, \bullet)\| = \|\mathfrak{C}(\textcircled{k}, \circ)\| = k$. If $\mathfrak{M} = \mathfrak{M}_0 \lhd \cdots \lhd \mathfrak{M}_{n-1}$, for some $0 < n < \omega$ and simple δ-models $\mathfrak{M}_j$ based on atomic frames $\mathfrak{F}_j$, $j < n$, then we set $\|\mathfrak{M}\| = \|\mathfrak{F}_0 \lhd \cdots \lhd \mathfrak{F}_{n-1}\| = \|\mathfrak{F}_0\| + \cdots + \|\mathfrak{F}_{n-1}\|$.

DEFINITION 4.3. Suppose $\mathfrak{N}_i$, $i = 1, 2$, is the ordered sum of finitely-many simple δ-models based on atomic frames. The pair $(\mathfrak{N}_1, \mathfrak{N}_2)$ is called *σ-matching* if it satisfies one of the following conditions (a)–(c):

(a) $\mathfrak{N}_1$ and $\mathfrak{N}_2$ are simple models based on the same atomic frame $\mathfrak{H}$ with $at^\sigma_{\mathfrak{N}_1}(y) = at^\sigma_{\mathfrak{N}_2}(y)$, for every point y in $\mathfrak{H}$;
(b) the final clusters C_i of $\mathfrak{N}_i$, $i = 1, 2$, are non-degenerate and, for every point y_1 in $\mathfrak{N}_1$, there is $y_2 \in C_2$ with $at^\sigma_{\mathfrak{N}_1}(y_1) = at^\sigma_{\mathfrak{N}_2}(y_2)$, and, for every point y_2 in $\mathfrak{N}_2$, there is $y_1 \in C_1$ with $at^\sigma_{\mathfrak{N}_2}(y_2) = at^\sigma_{\mathfrak{N}_1}(y_1)$;
(c) 1. the last $\lhd$-components of the $\mathfrak{N}_1$ and $\mathfrak{N}_2$ are based on the same atomic frame $\mathfrak{G}$ of the form $\mathfrak{C}(\textcircled{k}, \bullet)$ or $\mathfrak{C}(\textcircled{k}, \circ)$, with $0 < k \leq 2^{|\delta|}$;
2. $at^\sigma_{\mathfrak{N}_1}(y) = at^\sigma_{\mathfrak{N}_2}(y)$, for every point y in the root $\textcircled{k}$-cluster A_k of $\mathfrak{G}$;

3. for every point y_1 in any non-last $\lhd$-component of $\mathfrak{N}_1$, there is $y_2 \in A_k$ with $at^\sigma_{\mathfrak{N}_1}(y_1) = at^\sigma_{\mathfrak{N}_2}(y_2)$ and, for every point y_2 in any non-last $\lhd$-component of $\mathfrak{N}_2$, there is $y_1 \in A_k$ with $at^\sigma_{\mathfrak{N}_2}(y_2) = at^\sigma_{\mathfrak{N}_1}(y_1)$.

If $(\mathfrak{N}_1, \mathfrak{N}_2)$ satisfies condition (x), for $x = a, b, c$, we say that it *is of type* (x).



The following lemma justifies this definition:

LEMMA 4.4. *Suppose* $\mathfrak{N}_i = \mathfrak{N}_i^0 \lhd \cdots \lhd \mathfrak{N}_i^{N-1}$, *for* $i = 1, 2$ *and* $0 < N < \omega$, *and* x_i *is a root of* $\mathfrak{N}_i^0$. *If* $at^\sigma_{\mathfrak{N}_1}(x_1) = at^\sigma_{\mathfrak{N}_2}(x_2)$ *and, for every* $\ell < N$, *the pair* $(\mathfrak{N}_1^\ell, \mathfrak{N}_2^\ell)$ *is* σ*-matching, then* $\mathfrak{N}_1, x_1 \sim_\sigma \mathfrak{N}_2, x_2$.

PROOF. First, we show that, for every $\ell < N$, there is a global σ-bisimulation between $\mathfrak{N}_1^\ell$ and $\mathfrak{N}_2^\ell$. This is clear for $(\mathfrak{N}_1^\ell, \mathfrak{N}_2^\ell)$ of type (a), in which case the identity function on $\mathfrak{H}$ is a σ-bisimulation.

If $(\mathfrak{N}_1^\ell, \mathfrak{N}_2^\ell)$ is of type (b), then the final clusters C_i of $\mathfrak{N}_i^\ell$, $i = 1, 2$, are non-degenerate. Thus, $\boldsymbol{\beta}_1 \cup \boldsymbol{\beta}_2$ is a global σ-bisimulation between $\mathfrak{N}_1^\ell$ and $\mathfrak{N}_2^\ell$,

$$\boldsymbol{\beta}_1 = \big\{(y_1, y_2) \mid y_1 \text{ in } \mathfrak{N}_1^\ell,\ y_2 \text{ in } C_2,\ at^\sigma_{\mathfrak{N}_1^\ell}(y_1) = at^\sigma_{\mathfrak{N}_2^\ell}(y_2)\big\},$$
$$\boldsymbol{\beta}_2 = \big\{(y_1, y_2) \mid y_2 \text{ in } \mathfrak{N}_2^\ell,\ y_1 \text{ in } C_1,\ at^\sigma_{\mathfrak{N}_1^\ell}(y_1) = at^\sigma_{\mathfrak{N}_2^\ell}(y_2)\big\}.$$

If $(\mathfrak{N}_1^\ell, \mathfrak{N}_2^\ell)$ is of type (c), suppose $\mathfrak{N}_i^\ell = \mathfrak{N}_i^{0,\ell} \lhd \cdots \lhd \mathfrak{N}_i^{\boldsymbol{n}_i - 1, \ell}$, for $0 < \boldsymbol{n}_i < \omega$ and $i = 1, 2$. By (c).1, $\mathfrak{N}_1^{\boldsymbol{n}_1 - 1, \ell}$ and $\mathfrak{N}_2^{\boldsymbol{n}_2 - 1, \ell}$ are simple models based on the same atomic frame of the form $\mathfrak{C}(\textcircled{k}, \bullet)$ or $\mathfrak{C}(\textcircled{k}, \circ)$. As in Example 2.2, let $A_k = \{a_s \mid s < k\}$ and $W_k = A_k \cup \{b_n \mid n < \omega\}$ (containing all the points of $\mathfrak{C}(\textcircled{k}, *)$). We claim that

$$at^\sigma_{\mathfrak{N}_1^\ell}(b_n) = at^\sigma_{\mathfrak{N}_2^\ell}(b_n), \quad \text{for all } n < \omega. \tag{11}$$

Indeed, suppose $n < \omega$ and let $s < k$ be such that $n \equiv s \pmod k$. As $\mathfrak{N}_i^{\boldsymbol{n}_i - 1, \ell}$ is a simple model, we have

$$at^\sigma_{\mathfrak{N}_i^\ell}(b_n) = at^\sigma_{\mathfrak{N}_i^{\boldsymbol{n}_i - 1, \ell}}(b_n) = at^\sigma_{\mathfrak{N}_i^{\boldsymbol{n}_i - 1, \ell}}(a_s) = at^\sigma_{\mathfrak{N}_i^\ell}(a_s), \quad \text{for } i = 1, 2,$$

and so (11) follows from (c).2. Now let

$$\boldsymbol{\beta}_1 = \big\{(y_1, y_2) \mid y_1 \text{ in } \mathfrak{N}_1^{0,\ell} \lhd \cdots \lhd \mathfrak{N}_1^{\boldsymbol{n}_1 - 2, \ell},\ y_2 \in A_k,\ at^\sigma_{\mathfrak{N}_1^\ell}(y_1) = at^\sigma_{\mathfrak{N}_2^\ell}(y_2)\big\},$$
$$\boldsymbol{\beta}_2 = \big\{(y_1, y_2) \mid y_2 \text{ in } \mathfrak{N}_2^{0,\ell} \lhd \cdots \lhd \mathfrak{N}_2^{\boldsymbol{n}_2 - 2, \ell},\ y_1 \in A_k,\ at^\sigma_{\mathfrak{N}_1^\ell}(y_1) = at^\sigma_{\mathfrak{N}_2^\ell}(y_2)\big\}.$$

By (11) and (c).2–3, $\boldsymbol{\beta}_1 \cup \boldsymbol{\beta}_2 \cup \big\{(b_n, b_n) \mid n < \omega\big\}$ is a global σ-bisimulation between $\mathfrak{N}_1^\ell$ and $\mathfrak{N}_2^\ell$. Finally, if $\boldsymbol{\beta}^0$ is a global σ-bisimulation between $\mathfrak{N}_1^0$ and $\mathfrak{N}_2^0$, then $\boldsymbol{\beta}^0 \cup \{(x_1, x_2)\}$ is also a global σ-bisimulation between $\mathfrak{N}_1^0$ and $\mathfrak{N}_2^0$

because $at^{\sigma}_{\mathfrak{N}_1}(x_1) = at^{\sigma}_{\mathfrak{N}_2}(x_2)$. The union of the constructed global bisimulations is a (global) bisimulation $\boldsymbol{\beta}$ between $\mathfrak{N}_1$ and $\mathfrak{N}_2$ with $x_1 \boldsymbol{\beta} x_2$, as required. $\dashv$

The following strengthening of Theorem 4.2 will be proved in §4.2–4.3:

THEOREM 4.5. *For any logic $L \supseteq \mathsf{K4.3}$ and formulas φ_1, φ_2 without an interpolant in L, there are rooted δ-models $\mathfrak{N}_1, x_1$ and $\mathfrak{N}_2, x_2$ satisfying (a)–(d) below, for $\delta = sig(\varphi_1) \cup sig(\varphi_2)$ and $\sigma = sig(\varphi_1) \cap sig(\varphi_2)$:*

(a) *$\mathfrak{N}_1, x_1 \models \varphi_1$ and $\mathfrak{N}_2, x_2 \models \neg\varphi_2$;*
(b) *each $\mathfrak{N}_i$, $i = 1, 2$, is based on a frame for L;*
(c) *$at^{\sigma}_{\mathfrak{N}_1}(x_1) = at^{\sigma}_{\mathfrak{N}_2}(x_2)$;*
(d) *there is $N = \mathcal{O}\big(\max(|\varphi_1|, |\varphi_2|)\big)$ such that $\mathfrak{N}_i = \mathfrak{N}_i^0 \lhd \cdots \lhd \mathfrak{N}_i^{N-1}$, $i = 1, 2$, and, for any $\ell < N$,*
   1. *each $\mathfrak{N}_i^\ell$ is the ordered sum of $\mathcal{O}\big(\max(|\varphi_1|, |\varphi_2|)\big)$-many simple δ-models based on atomic frames;*
   2. *the pair $(\mathfrak{N}_1^\ell, \mathfrak{N}_2^\ell)$ is σ-matching.*

Observe that the models provided by Theorem 4.5 are ordered sums of polynomially-many simple models. However, the sizes of these simple models are not necessarily polynomial in $\max(|\varphi_1|, |\varphi_2|)$. Our second main result shows that all *finitely axiomatisable* logics $L \supseteq \mathsf{K4.3}$ have the stronger *quasi-polysize bisimilar model property*: the lack of an interpolant can be witnessed by a pair quasi-finite models of polynomial size. More precisely, suppose L is given by its canonical axioms as $L = \mathsf{K4.3} \oplus \{\alpha(\mathfrak{G}_j, \mathfrak{D}_j, \bot) \mid j \in J_L\}$, for some finite set J_L and $\mathfrak{G}_j = (V_j, S_j)$. Let $\boldsymbol{c}_L = \max_{j \in J_L} |V_j|$. An atomic frame in (10) is called *L-bounded* if it is of the form $m^<$ or $① \lhd m^<$ with $m \leq \boldsymbol{c}_L + 1$, or it has one of the three remaining forms with

$$k \leq \boldsymbol{p}_L(\varphi_1, \varphi_2) := 2(\boldsymbol{k}(\varphi_1, \varphi_2) - 1) \cdot \max\big(\boldsymbol{c}_L + 2, \boldsymbol{k}(\varphi_1, \varphi_2)\big) + \boldsymbol{k}(\varphi_1, \varphi_2),$$

for the polynomial number $\boldsymbol{k}(\varphi_1, \varphi_2)$ defined in (8). In § 4.4, we prove:

THEOREM 4.6. *For any finitely axiomatisable logic $L \supseteq \mathsf{K4.3}$ and formulas φ_1, φ_2 without an interpolant in L, there are rooted δ-models $\mathfrak{N}_1, x_1$ and $\mathfrak{N}_2, x_2$ satisfying (a)–(d) from Theorem 4.5, in which condition (d).1 is strengthened to*

1. *each $\mathfrak{N}_i^\ell$, $i = 1, 2$, is the ordered sum of $\mathcal{O}\big(\max(|\varphi_1|, |\varphi_2|)\big)$-many simple δ-models based on L-bounded atomic frames.*

In §4.4, we also show:

THEOREM 4.7. *All finitely axiomatisable $L \supseteq \mathsf{K4.3}$ have the quasi-polysize bisimilar model property, with the size of witnessing models bounded by*

$$(3\boldsymbol{k}(\varphi_1, \varphi_2) - 1) \cdot \max\big(\boldsymbol{c}_L + 2, \boldsymbol{p}_L(\varphi_1, \varphi_2)\big).$$

REMARK 4.8. As a consequence we obtain that each finitely axiomatisable logic $L \supseteq \mathsf{K4.3}$ has the *quasi-polysize model property*: $\varphi \in L$ iff φ is true in all models $\mathfrak{M}$ that are (i) ordered sums of simple models and (ii) are based on a frame for L of size $\mathcal{O}(|\varphi|^2)$; cf. [43, 30].

In the remainder of §4.1, we show how Theorem 4.6 implies the following:

THEOREM 4.9. *The IEP for any fixed finitely axiomatisable logic $L \supseteq \mathsf{K4.3}$ is* coNP*-complete.*

PROOF. We describe an NP-algorithm deciding the complement of the IEP for L given by its canonical axioms (6). Given φ_1 and φ_2, let $\delta = sig(\varphi_1) \cup sig(\varphi_2)$. We guess polynomial-size N. Then, for each $\ell < N$, we guess $z_\ell \in \{a, b, c\}$, and if $z_\ell = a$, we let $\boldsymbol{n}_1^\ell = \boldsymbol{n}_2^\ell = 1$; otherwise, we guess polynomial-size $\boldsymbol{n}_i^\ell$, for $i = 1, 2$; we also guess simple δ-models $\mathfrak{N}_i^{j,\ell}$, for $\ell < N$, $i = 1, 2$, $j < \boldsymbol{n}_i^\ell$, based on L-bounded atomic frames that are either of the form $\textcircled{k}$, $\mathfrak{C}(\textcircled{k}, \bullet)$, or $\mathfrak{C}(\textcircled{k}, \circ)$, for some $k \leq \boldsymbol{p}_L(\varphi_1, \varphi_2)$, or of the form $m^<$ or $\textcircled{1} \lhd m^<$, for some $m \leq \boldsymbol{c}_L + 1$, and respective roots x_i in $\mathfrak{N}_i^{0,0}$. We then let $\mathfrak{N}_i^\ell = \mathfrak{N}_i^{0,\ell} \lhd \cdots \lhd \mathfrak{N}_i^{\boldsymbol{n}_i^\ell - 1, \ell}$, for $\ell < N$, $i = 1, 2$, and $\mathfrak{N}_i = \mathfrak{N}_i^0 \lhd \cdots \lhd \mathfrak{N}_i^{N-1}$. Checking (c) and $(d).2$ in Theorem 4.6 can clearly be done in time polynomial in $\|\mathfrak{N}_i\|$ (which is polynomial in $\max(|\varphi_1|, |\varphi_2|)$). For (a), we use the following:

LEMMA 4.10. *Checking whether $\mathfrak{M}_0 \lhd \cdots \lhd \mathfrak{M}_{n-1}, x \models \varphi$, for simple $sig(\varphi)$-models $\mathfrak{M}_j$, $j < n$, based on atomic frames with root x in $\mathfrak{M}_0$, can be done in time polynomial in $|\varphi|$ and $\|\mathfrak{M}_0\| + \cdots + \|\mathfrak{M}_{n-1}\|$.*

PROOF. Let $\mathfrak{M} = \mathfrak{M}_0 \lhd \cdots \lhd \mathfrak{M}_{n-1}$. Suppose $\mathfrak{M}_j$ is based on the frame $\mathfrak{C}(\textcircled{k}, *)$ defined in Example 2.2 with points a_s, $s < k$, and b_ℓ, $\ell < \omega$. Using the definition of a simple model, it is readily shown by structural induction that any formula $\psi \in sub(\varphi)$ is satisfiable in $\mathfrak{M}_j$ iff there is $\ell < k + md(\psi)$ with $\mathfrak{M}_j, b_\ell \models \psi$, where $md(\psi)$, the *modal depth* of ψ, is the maximal number of nested modal operators in ψ. The required algorithm is now obvious. $\dashv$

Suppose $L = \mathsf{K4.3} \oplus \{\alpha(\mathfrak{G}_j, \mathfrak{D}_j, \bot) \mid j \in J_L\}$ with finite J_L and $\mathfrak{G}_j = (V_j, S_j)$. To check condition (b) in Theorem 4.6, we require the following:

LEMMA 4.11. *If $\mathfrak{F} = \mathfrak{F}_0 \lhd \cdots \lhd \mathfrak{F}_{n-1}$ with atomic frames $\mathfrak{F}_\ell$, $\ell < n$, then checking whether $\mathfrak{F} \models \alpha(\mathfrak{G}_j, \mathfrak{D}_j, \bot)$, for all $j \in J_L$, can be done in time polynomial in*

$$n_{\mathfrak{F},j} = n \cdot \max\big(\|\mathfrak{F}_0\|, \dots, \|\mathfrak{F}_{n-1}\|, |V_j|\big).$$

PROOF. Let $\mathfrak{F} = (W, R, \mathcal{P})$. Given any $\alpha(\mathfrak{G}_j, \mathfrak{D}_j, \bot)$, we construct the Kripke frame $\mathfrak{H}_j = (W_j, R_j)$, where $R_j = R \restriction_{W_j}$ and $W_j \subseteq W$ comprises

- the underlying sets of all finite $\lhd$-components $\mathfrak{F}_\ell$ of $\mathfrak{F}$;
- the last $|V_j| + 1$-many points $b_{|V_j|}, \dots, b_0$ in $\mathfrak{F}_\ell = \mathfrak{C}(\textcircled{k}, *)$, where $* \in \{\bullet, \circ\}$ and $b_{|V_j|}$ is 'painted' blue (see Example 2.2 for the notation).

Then $\mathfrak{H}_j$ is a subframe of $\mathfrak{F}$ because all finite subsets of $\{b_n \mid n < \omega\}$ are internal in $\mathfrak{C}(\textcircled{k}, *)$. We show below that there is an injection $f\colon V_j \to W$ satisfying $\mathbf{(cf_1)}$–$\mathbf{(cf_4)}$ in $\mathfrak{F}$ iff there is an injection $h\colon V_j \to W_j$ satisfying $\mathbf{(cf_1)}$–$\mathbf{(cf_4)}$ in $\mathfrak{H}_j$ and having no blue points in $h(V_j)$. Note that the latter is checkable in time polynomial in $n_{\mathfrak{F},j}$: just enumerate all (at most $n_{\mathfrak{F},j}^{|V_j|}$-many) injections $V_j \to W_j$ and verify that at least one of them meets the required conditions.

($\Leftarrow$) Suppose $h\colon V_j \to W_j$ is an injection satisfying $\mathbf{(cf_1)}$–$\mathbf{(cf_4)}$ in $\mathfrak{H}_j$ and having no blue points in $h(V_j)$. We claim that h satisfies $\mathbf{(cf_1)}$–$\mathbf{(cf_4)}$ in $\mathfrak{F}$. Indeed, $\mathbf{(cf_1)}$ and $\mathbf{(cf_4)}$ hold since $\mathfrak{H}_j$ is a subframe of $\mathfrak{F}$, and $\mathbf{(cf_2)}$ holds because the final cluster of $\mathfrak{H}_j$ is the final cluster of $\mathfrak{F}$ by definition. To show that h also meets $\mathbf{(cf_3)}$, observe that, as $h(x)$ is not blue for any $x \in V_j$, the immediate predecessor cluster of $C\big(h(x)\big)$ in $\mathfrak{H}_j$ is also the immediate predecessor of $C\big(h(x)\big)$ in $\mathfrak{F}$.

($\Rightarrow$) Let $f\colon V_j \to W$ be an injection satisfying $\mathbf{(cf_1)}$–$\mathbf{(cf_4)}$. To obtain h, we modify those $f(x)$ that belong to infinite $\lhd$-components $\mathfrak{F}_\ell = \mathfrak{C}(\textcircled{k}, *)$. Suppose the intersection of $f(V_j)$ with such an $\mathfrak{F}_\ell$ is not empty. By (3) in Example 2.2 and $\mathbf{(cf_4)}$, $f(V_j) \cap \{a_0, \dots, a_{k-1}\} = \emptyset$, and so the intersection of $f(V_j)$ with $\mathfrak{F}_\ell$ is $\{b_{i_0}, \dots, b_{i_{m_\ell - 1}}\}$, for some $m_\ell \leq |V_j|$. It is readily seen that by taking $h(x) = b_z$ if $f(x) = b_{i_z}$, for $z < m_\ell$, and $h(y) = f(y)$, for $f(y)$ in finite $\lhd$-components, we obtain an injection $h\colon V_j \to W_j$ with $\mathbf{(cf_1)}$–$\mathbf{(cf_4)}$ and no blue points in $h(V_j)$. $\dashv$

If all checks are positive, then, by Lemma 4.4, $\mathfrak{N}_1, x_1$ and $\mathfrak{N}_2, x_2$ satisfy the conditions of Theorem 3.2, and so φ_1 and φ_2 do not have an interpolant in L. $\dashv$

4.2. Partitioning models into globally σ-bisimilar intervals. In this section, we start proving Theorem 4.5. In a nutshell, our plan is as follows. Given φ_1 and φ_2 without an interpolant in $L \supseteq \mathsf{K4.3}$, the criterion of Theorem 3.2 supplies models $\mathfrak{M}_i$, $i = 1, 2$, based on finitely $\mathfrak{M}_i$-generated descriptive frames $\mathfrak{F}_i = (W_i, R_i, \mathcal{P}_i)$ with roots $x_i \in W_i$ such that

- $\mathfrak{M}_1, x_1 \models \varphi_1$ and $\mathfrak{M}_2, x_2 \models \neg\varphi_2$;
- each $\mathfrak{M}_i$, $i = 1, 2$, is based on a frame for L;
- $\mathfrak{M}_1, x_1 \sim_\sigma \mathfrak{M}_2, x_2$, where $\sigma = \mathit{sig}(\varphi_1) \cap \mathit{sig}(\varphi_2)$.

To prove Theorem 4.5, we need to turn the $\mathfrak{M}_i, x_i$ to some $\mathfrak{N}_i, x_i$ with the required structure and still satisfying these three conditions. In view of Example 3.6, extracting the roots x_i and the sets $\boldsymbol{M}_i$, $\boldsymbol{S}_i$ of maximal points from $\mathfrak{M}_i$ (similarly to the proof of Theorem 3.5 (a)) is not enough now, so we need to develop a more involved construction. We proceed in two steps:

- First, we analyse the σ-types in the $\mathfrak{M}_i$ and partition them into *internal* closed intervals $\mathcal{I}_i = \{I_i^\ell \mid \ell < N\}$, for the same $N = \mathcal{O}\big(\max(|\varphi_1|, |\varphi_2|)\big)$, such that $\mathfrak{M}_1 \restriction_{I_1^\ell}$ and $\mathfrak{M}_2 \restriction_{I_2^\ell}$ are globally σ-bisimilar, for every $\ell < N$. By Lemma 2.9, $\mathfrak{M}_i = (\mathfrak{M}_i \restriction_{I_i^0}) \lhd \dots \lhd (\mathfrak{M}_i \restriction_{I_i^{N-1}})$, $i = 1, 2$.
- Then, in §4.3, we complete the proof of Theorem 4.5 by transforming each pair $\big(\mathfrak{M}_1 \restriction_{I_1^\ell}, \mathfrak{M}_2 \restriction_{I_2^\ell}\big)$, $\ell < N$, into a pair $(\mathfrak{N}_1^\ell, \mathfrak{N}_2^\ell)$ of models with the required structure.

We begin with a simple observation on definable closed intervals:

LEMMA 4.12. *Suppose $\mathfrak{M}$ is a model based on a rooted finitely $\mathfrak{M}$-generated descriptive frame $\mathfrak{F} = (W, R, \mathcal{P})$ for $\mathsf{K4.3}$. Then every closed interval $[C, C']$ in $\mathfrak{F}$ with a non-limit cluster C' is definable in $\mathfrak{M}$.*

PROOF. By Lemma 2.6 (a)–(c), the non-limit C' is defined in $\mathfrak{M}$ by some formula γ. Let $\delta = \bot$ if C is the root cluster, and let δ define the immediate predecessor of C in $\mathfrak{M}$ otherwise, which exists by Lemma 2.4 (a) and is definable by Lemma 2.6 (a)–(c). Then $[C, C']$ is defined in $\mathfrak{M}$ by $\neg\Diamond^+\delta \wedge \Diamond^+\gamma$. $\dashv$

Next, we look into the structure of σ-types in any model $\mathfrak{M}$ based on a rooted finitely $\mathfrak{M}$-generated descriptive frame $\mathfrak{F} = (W, R, \mathcal{P})$ for $\mathsf{K4.3}$. Given $x \in W$ and a signature σ, we define the *σ-block* $\boldsymbol{b}^\sigma_\mathfrak{M}(x)$ *of x in $\mathfrak{M}$* by taking

$$\boldsymbol{b}^\sigma_\mathfrak{M}(x) = \begin{cases} \{y \in W \mid \Diamond t^\sigma_\mathfrak{M}(y) \subseteq t^\sigma_\mathfrak{M}(x),\ \Diamond t^\sigma_\mathfrak{M}(x) \subseteq t^\sigma_\mathfrak{M}(y)\}, & \text{if } \Diamond t^\sigma_\mathfrak{M}(x) \subseteq t^\sigma_\mathfrak{M}(x); \\ \{x\}, & \text{otherwise;} \end{cases}$$

in the latter case—when x must be an irreflexive point—the σ-block $\boldsymbol{b}^\sigma_{\mathfrak{M}}(x)$ is called *degenerate*. (It can happen that $\{x\}$ is a degenerate cluster but $\boldsymbol{b}^\sigma_{\mathfrak{M}}(x)$ is not a degenerate σ-block.) We call a set $\boldsymbol{b} \subseteq W$ a *σ-block in $\mathfrak{M}$* if $\boldsymbol{b} = \boldsymbol{b}^\sigma_{\mathfrak{M}}(x)$, for some x. It is readily seen that the relation $x \approx y$ iff $\boldsymbol{b}^\sigma_{\mathfrak{M}}(x) = \boldsymbol{b}^\sigma_{\mathfrak{M}}(y)$ is an equivalence relation on W, and every σ-block $\boldsymbol{b}$ is an interval in $\mathfrak{F}$. (See Example 4.14 below for an illustration.) Observe that

(block) for all σ-blocks $\boldsymbol{b}$ in $\mathfrak{M}$ and $y \in W$, if $y \notin \boldsymbol{b}$, then $t^\sigma_{\mathfrak{M}}(y) \notin t^\sigma_{\mathfrak{M}}(\boldsymbol{b})$.

For degenerate σ-blocks this follows from the definability of degenerate clusters (Lemma 2.6), and for other σ-blocks it is straightforward from the definitions.

LEMMA 4.13. *Suppose $\mathfrak{M}$ is a model based on a rooted finitely $\mathfrak{M}$-generated descriptive frame $\mathfrak{F} = (W, R, \mathcal{P})$ for* K4.3. *For any σ-block $\boldsymbol{b}$ in $\mathfrak{M}$, there exist clusters $C^-_{\boldsymbol{b}}$, $C^+_{\boldsymbol{b}}$ in $\mathfrak{F}$ such that the following hold*:

(*a*) $\boldsymbol{b} = \big[C^-_{\boldsymbol{b}}, C^+_{\boldsymbol{b}}\big]$;
(*b*) *if $C^+_{\boldsymbol{b}}$ is maximal in $\mathfrak{M}$, then it is σ-maximal in $\mathfrak{M}$*;
(*c*) *if $C^+_{\boldsymbol{b}}$ is degenerate, then $\boldsymbol{b} = C^+_{\boldsymbol{b}}$*;
(*d*) *$\boldsymbol{b}$ is definable in $\mathfrak{M}$ iff $C^+_{\boldsymbol{b}}$ is not a limit cluster*;
(*e*) $t^\sigma_{\mathfrak{M}}(\boldsymbol{b}) = t^\sigma_{\mathfrak{M}}\big(C^+_{\boldsymbol{b}}\big)$.

PROOF. (*a*) Let $\mathcal{X}_{\boldsymbol{b}} = \{C \mid C \text{ be a cluster with } \boldsymbol{b} \cap C \neq \emptyset\}$. As $\boldsymbol{b}$ is an interval, $\boldsymbol{b} = \bigcup \mathcal{X}_{\boldsymbol{b}}$. By Lemma 2.4, there is a $<_R$-largest cluster $C^+_{\boldsymbol{b}}$ in $\mathcal{X}_{\boldsymbol{b}}$. Also, there is a $<_R$-largest cluster D with $D <_R C$, for every $C \in \mathcal{X}_{\boldsymbol{b}}$. Suppose there is no $<_R$-smallest cluster in $\mathcal{X}_{\boldsymbol{b}}$. Then D is a limit cluster and if $y \in D$, then $t^\sigma_{\mathfrak{M}}(y) \notin t^\sigma_{\mathfrak{M}}(\boldsymbol{b})$ by **(block)**. So there is a σ-formula μ such that $\mu \in t^\sigma_{\mathfrak{M}}(y)$ and $\Diamond\mu \notin t^\sigma_{\mathfrak{M}}(x)$ for any $x \in \boldsymbol{b}$, and so for any x with $yR^s x$. As D is non-degenerate by Lemma 2.6, it follows that D is $\Diamond\mu$-maximal in $\mathfrak{M}$, contrary to Lemma 2.6 (*b*). Therefore, there is a $<_R$-smallest cluster $C^-_{\boldsymbol{b}}$ in $\mathcal{X}_{\boldsymbol{b}}$, and so $\boldsymbol{b} = \big[C^-_{\boldsymbol{b}}, C^+_{\boldsymbol{b}}\big]$.

(*b*) If $C^+_{\boldsymbol{b}}$ is maximal in $\mathfrak{M}$, then either it is final or has an immediate successor, by Lemma 2.6 (*b*). If $C^+_{\boldsymbol{b}}$ is final, then it is $\top$-maximal in $\mathfrak{M}$. So suppose that $C(y)$ is an immediate successor of $C^+_{\boldsymbol{b}} = C(x)$. If $C^+_{\boldsymbol{b}}$ is not degenerate, then $\Diamond t^\sigma_{\mathfrak{M}}(x) \not\subseteq t^\sigma_{\mathfrak{M}}(y)$ follows from $y \notin \boldsymbol{b}$. So there is a σ-formula μ such that $\mathfrak{M}, x \models \mu$ and $\mathfrak{M}, y \not\models \Diamond\mu$. If $\mathfrak{M}, y \models \mu$, then $C^+_{\boldsymbol{b}}$ is $\Diamond\mu$-maximal in $\mathfrak{M}$. And if $\mathfrak{M}, y \not\models \mu$, then $C^+_{\boldsymbol{b}}$ is μ-maximal in $\mathfrak{M}$. If $C^+_{\boldsymbol{b}}$ is degenerate, we cannot have $\Diamond t^\sigma_{\mathfrak{M}}(x) \subseteq t^\sigma_{\mathfrak{M}}(x)$, for otherwise $t^\sigma_{\mathfrak{M}}(x) \subseteq t^\sigma_{\mathfrak{M}}(y)$, contrary to **(block)**. Thus, $\Diamond t^\sigma_{\mathfrak{M}}(x) \not\subseteq t^\sigma_{\mathfrak{M}}(x)$, and so there is σ-formula μ such that $\mathfrak{M}, x \models \mu$ and $\mathfrak{M}, x \not\models \Diamond\mu$. Therefore, $C^+_{\boldsymbol{b}}$ is μ-maximal in $\mathfrak{M}$.

(*c*) Suppose on the contrary that $C^+_{\boldsymbol{b}} = \{x\} \neq \boldsymbol{b}$. Then $|\boldsymbol{b}| > 1$, and so $\Diamond t^\sigma_{\mathfrak{M}}(x) \subseteq t^\sigma_{\mathfrak{M}}(x)$ follows from $\boldsymbol{b} = \boldsymbol{b}^\sigma_{\mathfrak{M}}(x)$. So, for every σ-formula μ, if $\mathfrak{M}, x \models \mu$ then $\mathfrak{M}, x \models \Diamond\mu$. On the other hand, $C^+_{\boldsymbol{b}}$ is maximal in $\mathfrak{M}$ by Lemma 2.6 (*a*), and so σ-maximal in $\mathfrak{M}$ by (*b*), which is a contradiction.

(*d*, $\Leftarrow$) This is by (*a*) and Lemma 4.12.

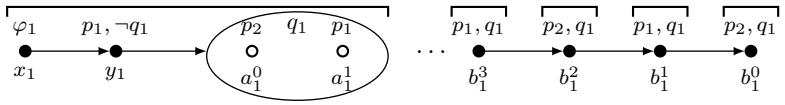
(*d*, $\Rightarrow$) Suppose that $\boldsymbol{b}$ is defined in $\mathfrak{M}$ by some ψ. Then $C^+_{\boldsymbol{b}}$ is ψ-maximal in $\mathfrak{M}$, and so cannot be a limit cluster by Lemma 2.6 (*b*).

(*e*) If $C^+_{\boldsymbol{b}}$ is degenerate, then this is obvious by (*c*). So suppose $C^+_{\boldsymbol{b}} = C(y)$ is non-degenerate and $x \in \boldsymbol{b}$. Then $\Diamond t^\sigma_{\mathfrak{M}}(x) \subseteq t^\sigma_{\mathfrak{M}}(y)$, and so $\Diamond \bigwedge \Gamma \in t^\sigma_{\mathfrak{M}}(y)$

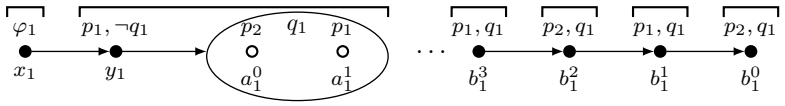
for every finite $\Gamma \subseteq t^\sigma_{\mathfrak{M}}(x)$. By Lemma 2.3 (a), there is z such that yRz and $t^\sigma_{\mathfrak{M}}(z) = t^\sigma_{\mathfrak{M}}(x)$. By **(block)**, we have $z \in \boldsymbol{b}$, and so $z \in C^+_{\boldsymbol{b}}$. $\dashv$

EXAMPLE 4.14. The model $\mathfrak{M}_1$ in Fig. 1 from Example 3.6 (a) is partitioned into the following σ-blocks (indicated by the brackets), for three different σ:

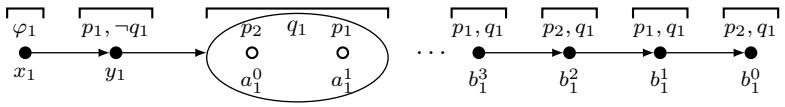
$\sigma = \emptyset$



$\sigma = \{p_1, p_2\}$



$\sigma = \{p_1, p_2, q_1, q_2\}$



To show this for $\sigma = \emptyset$, observe that, for every $n > 0$, we have $\Diamond^n \top \in t^\sigma_{\mathfrak{M}_1}(b_1^n)$, $\Diamond^{n+1}\top \notin t^\sigma_{\mathfrak{M}_1}(b_1^n)$, $\neg\Diamond\top \in t^\sigma_{\mathfrak{M}_1}(b_1^0)$, and $\Diamond^n\top \in t^\sigma_{\mathfrak{M}_1}(a_1^0)$. The cluster $C(a_1^0)$ is not maximal in $\mathfrak{M}_1$ as any formula α that is true at a_1^0 or a_1^1 is also true at b_1^n, for some $n < \omega$ (which is seen by induction on the structure of α). The model $\mathfrak{M}_1$ in Example 3.6 (b) has only one $\emptyset$-block comprising all of its points. $\dashv$

We now return to our models $\mathfrak{M}_i$, $i = 1, 2$, witnessing the lack of interpolants for φ_1 and φ_2. By Lemma 4.13 (a), σ-blocks in each $\mathfrak{M}_i$ are closed intervals that form a partition of W_i (with not all of them being necessarily definable in $\mathfrak{M}_i$). We show that there is a $\prec_{\mathfrak{F}_i}$-respecting bijection between the σ-blocks of the two models. Indeed, suppose that W_1 is partitioned as $\{\boldsymbol{b}^j \mid j \in F\}$ into σ-blocks in $\mathfrak{M}_1$, for some countable set F. For each $j \in F$, we let

$$\boldsymbol{\beta}(\boldsymbol{b}^j) = \{y \in W_2 \mid t^\sigma_{\mathfrak{M}_2}(y) \in t^\sigma_{\mathfrak{M}_1}(\boldsymbol{b}^j)\}.$$

LEMMA 4.15. *For all $j \in F$, the following hold*:

(a) $t^\sigma_{\mathfrak{M}_1}(\boldsymbol{b}^j) = t^\sigma_{\mathfrak{M}_2}\big(\boldsymbol{\beta}(\boldsymbol{b}^j)\big)$;
(b) $\boldsymbol{\beta}(\boldsymbol{b}^j)$ *is a σ-block in $\mathfrak{M}_2$, and $\boldsymbol{b}^j$ is degenerate iff $\boldsymbol{\beta}(\boldsymbol{b}^j)$ is degenerate*;
(c) $\{\boldsymbol{\beta}(\boldsymbol{b}^j) \mid j \in F\}$ *is a partition of W_2*;
(d) $\boldsymbol{b}^j \prec_{\mathfrak{F}_1} \boldsymbol{b}^k$ *iff* $\boldsymbol{\beta}(\boldsymbol{b}^j) \prec_{\mathfrak{F}_2} \boldsymbol{\beta}(\boldsymbol{b}^k)$, *for $j, k \in F$*;
(e) $\boldsymbol{b}^j$ *is definable in $\mathfrak{M}_1$ iff $\boldsymbol{\beta}(\boldsymbol{b}^j)$ is definable in $\mathfrak{M}_2$.*

PROOF. (a) This follows from $\mathfrak{M}_1, x_1 \sim_\sigma \mathfrak{M}_2, x_2$ and Lemma 3.1.

(b) Let $j \in F$. As $\mathfrak{M}_1, x_1 \sim_\sigma \mathfrak{M}_2, x_2$, $\boldsymbol{\beta}(\boldsymbol{b}^j) \neq \emptyset$. Take some $y \in \boldsymbol{\beta}(\boldsymbol{b}^j)$. We show that $\boldsymbol{\beta}(\boldsymbol{b}^j) = \boldsymbol{b}^\sigma_{\mathfrak{M}_2}(y)$. Indeed, this is straightforward from the definitions if $\Diamond t^\sigma_{\mathfrak{M}_2}(y) \subseteq t^\sigma_{\mathfrak{M}_2}(y)$. If $\Diamond t^\sigma_{\mathfrak{M}_2}(y) \not\subseteq t^\sigma_{\mathfrak{M}_2}(y)$, then $\boldsymbol{b}^\sigma_{\mathfrak{M}_2}(y) = \{y\}$. Take some $x \in \boldsymbol{b}^j$ with $t^\sigma_{\mathfrak{M}_1}(x) = t^\sigma_{\mathfrak{M}_2}(y)$. Then $\Diamond t^\sigma_{\mathfrak{M}_1}(x) \not\subseteq t^\sigma_{\mathfrak{M}_1}(x)$, and so $\boldsymbol{b}^j = \{x\}$. Thus, $\boldsymbol{\beta}(\boldsymbol{b}^j) = \{z \in W_2 \mid t^\sigma_{\mathfrak{M}_2}(z) = t^\sigma_{\mathfrak{M}_2}(y)\}$, and so there is a σ-formula μ such that $\mu \in t^\sigma_{\mathfrak{M}_2}(z) = t^\sigma_{\mathfrak{M}_2}(y)$ and $\Diamond\mu \notin t^\sigma_{\mathfrak{M}_2}(z) = t^\sigma_{\mathfrak{M}_2}(y)$. Suppose there is $z \in \boldsymbol{\beta}(\boldsymbol{b}^j)$, $z \neq y$. Then either zR_2y or yR_2z, which is a contradiction.

(*c*) As $\boldsymbol{\beta}(\boldsymbol{b}^j)$ and $\boldsymbol{\beta}(\boldsymbol{b}^k)$ are disjoint for $j \neq k$ by (*a*) and **(block)**, the relation '$y \approx y'$ iff there is $j \in F$ with $y, y' \in \boldsymbol{\beta}(\boldsymbol{b}^j)$' is an equivalence relation on W_2.

(*d*) This follows from $\mathfrak{M}_1, x_1 \sim_\sigma \mathfrak{M}_2, x_2$, (*a*) and **(block)**.

(*e*) This follows from (*b*)–(*d*) and Lemma 4.13 (*a*) and (*d*). ⊣

So, from now on we assume that we have a strict linear order $(F, \prec)$ such that each W_i, $i = 1, 2$, is partitioned as $\{\boldsymbol{b}_i^j \mid j \in F\}$ into σ-blocks in $\mathfrak{M}_i$ with $j \prec k$ iff $\boldsymbol{b}_1^j \prec_{\mathfrak{F}_1} \boldsymbol{b}_1^k$ iff $\boldsymbol{b}_2^j \prec_{\mathfrak{F}_2} \boldsymbol{b}_2^k$, for $j, k \in F$. (We write $j \preceq k$ whenever $j \prec k$ or $j = k$.) Observe that, by Lemmas 2.4 (*a*) and 2.7, $(F, \succ)$ is isomorphic to a countable ordinal. We say that $j \in F$ is a $\succ$-*limit* iff it corresponds to a limit ordinal under this isomorphism. Thus, every $j \in F$ has an immediate $\prec$-predecessor, and if j is not a $\succ$-limit, then it also has an immediate $\prec$-successor. Also, j is a $\succ$-limit iff $C^+_{\boldsymbol{b}_i^j}$ is a limit cluster, for $i = 1, 2$.

Next, we analyse some properties of special σ-blocks. Recall that **Steps 1** and **2** in the proof of Theorem 3.5 (*a*) give us the sets $\boldsymbol{M}_i$ containing the $\{\psi\}$-maximal points in $\mathfrak{M}_i$ that satisfy each formula ψ in $sub(\varphi_i)$ that is satisfiable in $\mathfrak{M}_i$; the set T of the σ-types of points in $\{x_1, x_2\} \cup \boldsymbol{M}_1 \cup \boldsymbol{M}_2$ (cf. (7)); and also the sets $\boldsymbol{S}_i \subseteq W_i$ of t-maximal points in $\mathfrak{M}_i$ satisfying the σ-types t from T. Points in $\{x_i\} \cup \boldsymbol{M}_i \cup \boldsymbol{S}_i$ are called *relevant in* $\mathfrak{M}_i$. A cluster or an interval is *relevant in* $\mathfrak{M}_i$ if it contains a relevant point, and *irrelevant* otherwise. The number of relevant clusters (and of relevant σ-blocks) in $\mathfrak{M}_i$ is clearly bounded by the number of relevant points, that is, by $\boldsymbol{k}(\varphi_1, \varphi_2)$ (defined in (8)). Note that the root and final clusters of $\mathfrak{M}_i$ are always relevant (the latter because $sub(\varphi_i)$ is closed under negation, so the final cluster always intersects with $\boldsymbol{M}_i$).

Example 4.16. For the models $\mathfrak{M}_i$ shown in Fig. 1 from Example 3.6 (*a*) and $\sigma = \{p_1, p_2\}$, we have $\boldsymbol{M}_i = \{x_i, y_i, b_i^1, b_i^0\}$, $\boldsymbol{S}_1 = \{x_1, y_1, a_1^1, b_1^1, b_1^0\}$, and $\boldsymbol{S}_2 = \{x_2, y_2, a_2^0, b_2^1, b_2^0\}$, so only the first two and the last two σ-blocks in the $\mathfrak{M}_i$ are relevant (cf. Example 4.14 for the σ-blocks). ⊣

The next lemma lists a few important properties of relevant σ-blocks:

Lemma 4.17. *For all $j \in F$ and $i = 1, 2$, the following hold*:

(*a*) $\boldsymbol{S}_i \cap \boldsymbol{b}_i^j = \boldsymbol{S}_i \cap C^+_{\boldsymbol{b}_i^j}$;

(*b*) $\big(\{x_i\} \cup \boldsymbol{M}_i \cup \boldsymbol{S}_i\big) \cap C^+_{\boldsymbol{b}_i^j} = \boldsymbol{S}_i \cap C^+_{\boldsymbol{b}_i^j}$;

(*c*) $\boldsymbol{b}_i^j$ *is relevant iff* $\boldsymbol{S}_i \cap C^+_{\boldsymbol{b}_i^j} \neq \emptyset$;

(*d*) *there is a bijection* $f^-\colon \big(\boldsymbol{S}_1 \cap C^+_{\boldsymbol{b}_1^j}\big) \to \big(\boldsymbol{S}_2 \cap C^+_{\boldsymbol{b}_2^j}\big)$ *with* $t^\sigma_{\mathfrak{M}_1}(y) = t^\sigma_{\mathfrak{M}_2}\big(f^-(y)\big)$, *for every* $y \in \boldsymbol{S}_1 \cap C^+_{\boldsymbol{b}_1^j}$;

(*e*) $\boldsymbol{b}_1^j$ *is relevant iff* $\boldsymbol{b}_2^j$ *is relevant.*

Proof. Recall the following properties of the $\boldsymbol{S}_i$ defined in **Step 2** of the proof of Theorem 3.5:

1. if $x \in \boldsymbol{S}_i$, then x is $t^\sigma_{\mathfrak{M}_i}(x)$-maximal in $\mathfrak{M}_i$;
2. if $x \in \{x_i\} \cup \boldsymbol{M}_i$ and x is $t^\sigma_{\mathfrak{M}_i}(x)$-maximal in $\mathfrak{M}_i$, then $x \in \boldsymbol{S}_i$;
3. $t^\sigma_{\mathfrak{M}_i}\big(\{x_i\} \cup \boldsymbol{M}_i \cup \boldsymbol{S}_i\big) \subseteq t^\sigma_{\mathfrak{M}_i}(\boldsymbol{S}_i)$;
4. there is a bijection $f\colon \boldsymbol{S}_1 \to \boldsymbol{S}_2$ with $t^\sigma_{\mathfrak{M}_1}(y) = t^\sigma_{\mathfrak{M}_2}\big(f(y)\big)$, for every $y \in \boldsymbol{S}_1$.

(a) Let $x \in \boldsymbol{S}_i \cap \boldsymbol{b}_i^j$. By Lemma 4.13 ($e$), there is $y \in C^+_{\boldsymbol{b}_i^j}$ with $t^\sigma_{\mathfrak{M}_i}(y) = t^\sigma_{\mathfrak{M}_i}(x)$. Then $C(x) = C(y)$ follows from 1., and so $x \in C^+_{\boldsymbol{b}_i^j}$.

(b) Take $x \in \big(\{x_i\} \cup \boldsymbol{M}_i\big) \cap C^+_{\boldsymbol{b}_i^j}$. By 3., there is $y \in \boldsymbol{S}_i$ with $t^\sigma_{\mathfrak{M}_i}(y) = t^\sigma_{\mathfrak{M}_i}(x)$. By 1., y is $t^\sigma_{\mathfrak{M}_i}(y)$-maximal in $\mathfrak{M}_i$. Thus, by **(block)** and Lemma 4.13 (e), $y \in C^+_{\boldsymbol{b}_i^j}$. It follows that x is $t^\sigma_{\mathfrak{M}_i}(x)$-maximal in $\mathfrak{M}_i$, and so $x \in \boldsymbol{S}_i$ by 2.

(c) We show that $t^\sigma_{\mathfrak{M}_i}\big(\big(\{x_i\} \cup \boldsymbol{M}_i \cup \boldsymbol{S}_i\big) \cap \boldsymbol{b}_i^j\big) \subseteq t^\sigma_{\mathfrak{M}_i}\big(\big(\{x_i\} \cup \boldsymbol{M}_i \cup \boldsymbol{S}_i\big) \cap C^+_{\boldsymbol{b}_i^j}\big)$. Then ($c$) follows from ($b$). To this end, take $x \in \big(\{x_i\} \cup \boldsymbol{M}_i \cup \boldsymbol{S}_i\big) \cap \boldsymbol{b}_i^j$. By 3., there is $y \in \boldsymbol{S}_i$ with $t^\sigma_{\mathfrak{M}_i}(y) = t^\sigma_{\mathfrak{M}_i}(x)$. By 1., y is $t^\sigma_{\mathfrak{M}_i}(y)$-maximal in $\mathfrak{M}_i$. Thus, by **(block)** and Lemma 4.13 (e), $y \in C^+_{\boldsymbol{b}_i^j}$.

(d) Let $f^- = f \restriction_{\boldsymbol{S}_1 \cap C^+_{\boldsymbol{b}_1^j}}$ for the bijection f provided by 4. Then, for every $x \in \boldsymbol{S}_1 \cap C^+_{\boldsymbol{b}_1^j}$, $f^-(x) = f(x) \in \boldsymbol{S}_2$ with $t^\sigma_{\mathfrak{M}_2}(f(x)) = t^\sigma_{\mathfrak{M}_1}(x)$. By Lemma 4.15 ($a$), $t^\sigma_{\mathfrak{M}_2}(f(x)) \in t^\sigma_{\mathfrak{M}_2}(\boldsymbol{b}_2^j)$, so $f(x) \in \boldsymbol{b}_2^j$ follows by **(block)**. Thus, $f(x) \in C^+_{\boldsymbol{b}_2^j}$ by (a).

(e) follows from (c) and (d). ⊣

We are now in a position to partition each of the $\mathfrak{M}_i$ into the same polynomial number N of closed intervals $\mathcal{I}_i = \{I_i^\ell \in \mathcal{P}_i \mid \ell < N\}$ such that $\mathfrak{M}_1 \restriction_{I_1^\ell}$ and $\mathfrak{M}_2 \restriction_{I_2^\ell}$ are globally σ-bisimilar, for every $\ell < N$, even if there are infinitely many σ-blocks in each $\mathfrak{M}_i$ and not all of them are definable in $\mathfrak{M}_i$.

Definition 4.18. We define the partitions $\mathcal{I}_i$ of $\mathfrak{M}_i$, $i = 1, 2$, in three steps. In each step, we add interval-pairs (I_1, I_2) to $\mathcal{I}_1 \times \mathcal{I}_2$ in such a way that

(a) I_i is a closed interval whose final cluster is a non-limit cluster, for $i = 1, 2$;
(b) there are $j, j' \in F$ such that $I_i = \bigcup_{j \preceq k \preceq j'} \boldsymbol{b}_i^k$, for $i = 1, 2$.

It follows then from (a) and Lemma 4.12 that all intervals in $\mathcal{I}_i$ are definable in $\mathfrak{M}_i$. Also, it follows from (b) and Lemma 4.15 (a) that

(12) $\big\{(y_1, y_2) \in I_1^\ell \times I_2^\ell \mid t^\sigma_{\mathfrak{M}_1}(y_1) = t^\sigma_{\mathfrak{M}_2}(y_2)\big\}$ is a global σ-bisimulation between $\mathfrak{M}_1 \restriction_{I_1^\ell}$ and $\mathfrak{M}_2 \restriction_{I_2^\ell}$, for every $\ell < N$.

The three steps are as follows:

($\mathbf{s}_1$) First, suppose $\boldsymbol{b}_1^j$, $j \in F$, is a relevant σ-block that is definable in $\mathfrak{M}_1$. By Lemmas 4.17 (e) and 4.15 (e), $\boldsymbol{b}_2^j$ is also a relevant σ-block definable in $\mathfrak{M}_2$. We put into $\mathcal{I}_i$ all those relevant σ-blocks $\boldsymbol{b}_i^j$ that are definable in $\mathfrak{M}_i$, for $i = 1, 2$. Then (b) clearly holds, and (a) holds by Lemma 4.13 (a), (d).

($\mathbf{s}_2$) Next, suppose $\boldsymbol{b}_1^j$, $j \in F$, is a relevant σ-block that is not definable in $\mathfrak{M}_1$. By Lemmas 4.17 (e) and 4.15 (e), $\boldsymbol{b}_2^j$ is also a relevant σ-block that is not definable in $\mathfrak{M}_2$. By Lemma 4.13 (d), each $C^+_{\boldsymbol{b}_i^j}$ is a limit cluster in $\mathfrak{F}_i$, and so j is a $\succ$-limit. We pick some $\ell \succ j$ such that the σ-blocks $\boldsymbol{b}_i^k$, for $j \prec k \preceq \ell$, are all irrelevant, for $i = 1, 2$. Such an ℓ must exist as j is a $\succ$-limit and the number of relevant points is finite, but this ℓ is not unique. Let $F^- = \{k \in F \mid j \preceq k \preceq \ell\}$ and $\succ^- = \succ \restriction_{F^-}$. By Lemmas 2.4 ($a$) and 2.7, there is an isomorphism f from some countable ordinal γ to $(F^-, \succ^-)$. As j is a $\succ$-limit, $\gamma \geq \omega$. Take $f(n)$, $n < \omega$. There are two cases:

1. There exists m, $0 < m < \omega$, such that $\boldsymbol{b}_1^{f(n)}$ is a degenerate σ-block for every n with $m \leq n < \omega$. Then, by Lemma 4.15 (b), $\boldsymbol{b}_2^{f(n)}$ is a degenerate σ-block, for every n with $m \leq n < \omega$. We set $j' = f(m)$.
2. For every $n < \omega$, there is m_n, $n \leq m_n < \omega$, such that $\boldsymbol{b}_1^{f(m_n)}$ is a non-degenerate σ-block. Then, by Lemma 4.15 (b), $\boldsymbol{b}_2^{f(m_n)}$ is a non-degenerate σ-block as well. Note that if $n \geq 1$, then $f(m_n)$ is not a $\succ$-limit. Thus, $C^+_{\boldsymbol{b}_i^{f(m_n)}}$ is not a limit cluster, and so it is definable in $\mathfrak{M}_i$ by Lemma 2.6. We set $j' = f(m_1)$.

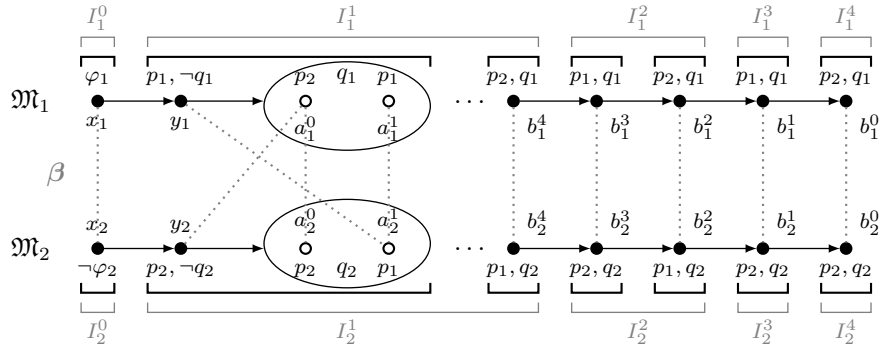
In both cases, we put the intervals $\bigcup_{j \preceq k \preceq j'} \boldsymbol{b}_i^k$ into $\mathcal{I}_i$, $i = 1, 2$, and say that they *extend* the relevant non-definable σ-blocks $\boldsymbol{b}_i^j$. Then (a), (b) hold.

$(\mathbf{s}_3)$ Finally, suppose that, for $i = 1, 2$, the intervals $I_i = \bigcup_{n_1 \preceq k \preceq n_2} \boldsymbol{b}_i^k$ and $J_i = \bigcup_{j_1 \preceq k \preceq j_2} \boldsymbol{b}_i^k$ are such that there is k with $n_2 \prec k \prec j_1$, $I_i, J_i \in \mathcal{I}_i$, and there is no interval in $\mathcal{I}_i$ intersecting the 'gap' between I_i and J_i (that is, any $\boldsymbol{b}_i^k$ with $n_2 \prec k \prec j_1$). By $I_i \in \mathcal{I}_i$ and (i), n_2 is not a $\succ$-limit. Let n_2^+ be the immediate $\prec$-successor of n_2 and j_1^- the immediate $\prec$-predecessor of j_1. Then we put the (irrelevant) interval $\bigcup_{n_2^+ \preceq k \preceq j_1^-} \boldsymbol{b}_i^k$ into $\mathcal{I}_i$, for $i = 1, 2$. Then (b) clearly holds, and (a) holds as j_1^- is not a $\succ$-limit. By doing this for all the gaps, we end up with the required partition $\mathcal{I}_i$ of $\mathfrak{M}_i$.

The number of intervals added in steps $(\mathbf{s}_1)$ and $(\mathbf{s}_2)$ together cannot exceed the number of relevant σ-blocks, and so it is bounded by $\boldsymbol{k}(\varphi_1, \varphi_2)$. As the $\prec_{\mathfrak{F}_i}$-smallest and $\prec_{\mathfrak{F}_i}$-largest σ-blocks are relevant, the number of intervals added in step $(\mathbf{s}_3)$ is bounded by $\boldsymbol{k}(\varphi_1, \varphi_2) - 1$, so altogether the (same) number N of intervals in each $\mathcal{I}_i$ does not exceed $2\boldsymbol{k}(\varphi_1, \varphi_2)$.

The following example illustrates Definition 4.18.

EXAMPLE 4.19. For models $\mathfrak{M}_i$, $i = 1, 2$, from Example 3.6 (a) and σ-blocks from Example 4.14 for $\sigma = \{p_1, p_2\}$, we can pick the intervals I_i^j, $j \leq 4$, shown below, where I_i^2 are irrelevant and all other intervals are relevant (cf. Example 4.16). The choice of the infinite intervals I_i^1 extending the non-definable σ-blocks till b_i^4 is arbitrary. We could make them shorter or, on the contrary, extend until b_i^2, in which case there would be no gap between these intervals (extending relevant non-definable σ-blocks) and the next relevant interval. ⊣



4.3. Simplifying interval-based models. Consider again our σ-bisimilar δ-models $\mathfrak{M}_i$, $i = 1, 2$, that are based on finitely $\mathfrak{M}_i$-generated descriptive frames $\mathfrak{F}_i = (W_i, R_i, \mathcal{P}_i)$ for L with roots $x_i \in W_i$ and witness the lack of an interpolant for φ_1 and φ_2, where $\delta = sig(\varphi_1) \cup sig(\varphi_2)$ and $\sigma = sig(\varphi_1) \cap sig(\varphi_2)$. In Definition 4.18, we determined $N < 2\boldsymbol{k}(\varphi_1, \varphi_2)$, for the polynomial number $\boldsymbol{k}(\varphi_1, \varphi_2)$ from (8), and constructed the partitions $\mathcal{I}_i = \{I_i^\ell \in \mathcal{P}_i \mid \ell < N\}$ of $\mathfrak{M}_i$ with $I_i^0 \prec_{\mathfrak{F}_i} \cdots \prec_{\mathfrak{F}_i} I_i^{N-1}$ satisfying (12). We now use these partitions to prove Theorem 4.5. First, in Lemma 4.21, we transform each pair $(\mathfrak{M}_1 \restriction_{I_1^\ell}, \mathfrak{M}_2 \restriction_{I_2^\ell})$, $\ell < N$, into a pair $(\mathfrak{N}_1^\ell, \mathfrak{N}_2^\ell)$ of models meeting the list of requirements in Definition 4.20. Then, in Lemma 4.22, we show that these requirements ensure that $\mathfrak{N}_i = \mathfrak{N}_i^0 \lhd \cdots \lhd \mathfrak{N}_i^{N-1}$, $i = 1, 2$, satisfy all conditions in Theorem 4.5.

For all $i = 1, 2$ and $\ell < N$, the frame $\mathfrak{H}_i^\ell = (H_i^\ell, R_i^\ell, \mathcal{P}_i^\ell)$ underlying $\mathfrak{N}_i^\ell$ is such that $H_i^\ell \subseteq I_i^\ell$ is definable in $\mathfrak{M}_i$ and $R_i^\ell = R_i \restriction_{H_i^\ell}$, but $\mathfrak{H}_i^\ell$ is not necessarily a subframe of $\mathfrak{F}_i \restriction_{I_i^\ell}$. However, we require each $\mathfrak{H}_i^\ell$ to meet some conditions making sure that the canonical formulas refuted in $\mathfrak{H}_i = \mathfrak{H}_i^0 \lhd \cdots \lhd \mathfrak{H}_i^{N-1}$ are also refuted in $\mathfrak{F}_i$ (and so $L \subseteq \mathsf{Log}(\mathfrak{F}_i) \subseteq \mathsf{Log}(\mathfrak{H}_i)$). Another feature of the construction is that the atomic type of some points in $\mathfrak{N}_i$ could be different from their atomic type in $\mathfrak{M}_i$. We prove $\mathfrak{N}_1, x_1 \models \varphi_1$ and $\mathfrak{N}_2, x_2 \models \neg\varphi_2$ by ensuring that no new (compared to $\mathfrak{M}_i$) atomic types are introduced in $\mathfrak{N}_i$, and the distribution of old atomic types in $\mathfrak{N}_i$ properly matches their distribution in $\mathfrak{M}_i$. We achieve this by introducing functions h_i^ℓ that assign to each point x in $\mathfrak{N}_i^\ell$ a unique 'parent' point in $\mathfrak{M}_i \restriction_{I_i^\ell}$ whose $\mathfrak{M}_i$-behaviour x is intended to mimic in $\mathfrak{N}_i$.

Definition 4.20. Suppose $I \in \mathcal{P}_i$, $i = 1, 2$, is an interval in $\mathfrak{F}_i$. We say that a model $\mathfrak{N}$ based on a frame $\mathfrak{H} = (H, S, \mathcal{P}')$ is (I, i)-*nice* if the following hold:

(13) $H \subseteq I$ and $S = R_i \restriction_H$;

(14) $\big(\{x_i\} \cup \boldsymbol{M}_i \cup \boldsymbol{S}_i\big) \cap I \subseteq H$;

(15) the final cluster in $\mathfrak{H}$ is a subset of the final cluster in $\mathfrak{F}_i \restriction_I$;

(16) if the root cluster C in $\mathfrak{H}$ is degenerate,
then C is the root cluster in $\mathfrak{F}_i \restriction_I$;

(17) for every $x \in H$, if $\{x\}$ is a degenerate non-root cluster in $\mathfrak{H}$
and $C \subseteq I$ is the immediate predecessor of $\{x\}$ in $\mathfrak{F}_i$, then
$C \cap H$ is the immediate predecessor of $\{x\}$ in $\mathfrak{H}$;

(18) for every $x \in H$, if $\{x\} \in \mathcal{P}'$, then $\{x\} \in \mathcal{P}_i$;

there is a function $\mathsf{h}\colon H \to H$ such that

(19) $\mathsf{h}(x) = x$ for all $x \in \big(\{x_i\} \cup \boldsymbol{M}_i \cup \boldsymbol{S}_i\big) \cap H$;

(20) $at_{\mathfrak{N}}(x) = at_{\mathfrak{N}}\big(\mathsf{h}(x)\big) = at_{\mathfrak{M}_i}\big(\mathsf{h}(x)\big)$ for all $x \in H$;

(21) if xR_iy, then $\mathsf{h}(x)R_i\mathsf{h}(y)$ for all $x, y \in H$;

(22) if $\mathsf{h}(x)R_iy$, then xR_iy for all $x \in H$, $y \in \boldsymbol{M}_i \cap H$.

Lemma 4.21. *For all $i = 1, 2$ and $\ell < N$, there exist models $\mathfrak{N}_i^\ell$ based on frames $\mathfrak{H}_i^\ell = (H_i^\ell, S_i^\ell, \mathcal{P}_i^\ell)$, and numbers $\boldsymbol{n}_i^\ell > 0$ with $\sum_{\ell<N} \boldsymbol{n}_i^\ell \leq 3\boldsymbol{k}(\varphi_1, \varphi_2) - 1$ such that the following hold*:

(*a*) $\mathfrak{N}_i^\ell$ *is* (I_i^ℓ, i)*-nice*;
(*b*) $\mathfrak{N}_i^\ell$ *is the ordered sum of* $\boldsymbol{n}_i^\ell$*-many simple* δ*-models based on atomic frames*;
(*c*) *the pair* $(\mathfrak{N}_1^\ell, \mathfrak{N}_2^\ell)$ *is* σ*-matching.*

PROOF. We consider three ***Cases*** I–III, depending on the step the pair (I_1^ℓ, I_2^ℓ) is added to $\mathcal{I}_1 \times \mathcal{I}_2$ in Definition 4.18.

Case I: (I_1^ℓ, I_2^ℓ) is added in step $(\mathbf{s}_3)$, so I_i^ℓ are irrelevant intervals. We let $\boldsymbol{n}_1^\ell = \boldsymbol{n}_2^\ell = 1$ and define $\mathfrak{N}_1^\ell$ and $\mathfrak{N}_2^\ell$ as follows. Let $Z_i^\ell = \{z_i^j \mid j < m_i\}$, for $i = 1, 2$, be the tail of $\mathfrak{F}_i \restriction_{I_i^\ell}$, for some $m_i \leq \omega$, with $z_i^j R_i^s z_i^{j-1}$, $0 < j < m_i$. By (12), $\big\{(y_1, y_2) \in I_1^\ell \times I_2^\ell \mid t^\sigma_{\mathfrak{M}_1}(y_1) = t^\sigma_{\mathfrak{M}_2}(y_2)\big\}$ is a global σ-bisimulation between $\mathfrak{M}_1 \restriction_{I_1^\ell}$ and $\mathfrak{M}_2 \restriction_{I_2^\ell}$. It is straightforward to see that because of this we must have $|Z_1^\ell| = |Z_2^\ell| = m$, for some $m \leq \omega$, and $Z_1^\ell = I_1^\ell$ iff $Z_2^\ell = I_2^\ell$. Also, if $Z_i^\ell \neq I_i^\ell$, then there exist w_i^ℓ in the head of Z_i^ℓ with $t^\sigma_{\mathfrak{M}_1}(w_1^\ell) = t^\sigma_{\mathfrak{M}_2}(w_2^\ell)$. For $i = 1, 2$, let

$$H_i^\ell = \begin{cases} Z_i^\ell, & \text{if } Z_i^\ell = I_i^\ell, \\ \{w_i^\ell\} \cup Z_i^\ell, & \text{otherwise,} \end{cases}$$

$S_i^\ell = R_i \restriction_{H_i^\ell}$, and let $\mathcal{P}_i^\ell$ consist of all finite subsets of Z_i^ℓ and their complements in H_i^ℓ. Then $f\colon H_1^\ell \to H_2^\ell$ defined by $f(z_1^j) = z_2^j$, $j < m$, and $f(w_1^\ell) = w_2^\ell$ is an isomorphism between the resulting frames $\mathfrak{H}_1^\ell$ and $\mathfrak{H}_2^\ell$, which are isomorphic to

(*i*) $m^<$, when $Z_i^\ell = I_i^\ell$;
(*ii*) $① \lhd m^<$, when $Z_i^\ell \neq I_i^\ell$ and $m < \omega$;
(*iii*) $\mathfrak{C}(①, \bullet)$, when Z_i^ℓ is infinite (as $w_i^\ell R_i w_i^\ell$ by (5)).

This gives (13)–(18) for $\mathfrak{H} = \mathfrak{H}_i^\ell$ and $I = I_i^\ell$ (we have (18) because of (3) and Lemma 2.5). For $p \in \delta$, let $\mathfrak{w}_i^\ell(p) = \mathfrak{v}_i(p) \cap H_i^\ell$ in cases (*i*) and (*ii*), and

$$\mathfrak{w}_i^\ell(p) = \begin{cases} H_i^\ell, & \text{if } w_i^\ell \in \mathfrak{v}_i(p), \\ \emptyset, & \text{otherwise} \end{cases}$$

in case (*iii*). In all cases, $\mathfrak{w}_i^\ell(p) \in \mathcal{P}_i^\ell$ and (*b*) holds for $\mathfrak{N}_i^\ell = (\mathfrak{H}_i^\ell, \mathfrak{w}_i^\ell)$. Also, the pair $(\mathfrak{N}_1^\ell, \mathfrak{N}_2^\ell)$ is of type (*a*) in Definition 4.3, and so (*c*) of the lemma holds. Finally, for $x \in H_i^\ell$, we let $\mathsf{h}_i^\ell(x) = x$ in cases (*i*) and (*ii*), and $\mathsf{h}_i^\ell(x) = w_i^\ell$ in case (*iii*). It is straightforward to check that (20) and (21) hold for $\mathsf{h} = \mathsf{h}_i^\ell$. Note that (19) and (22) hold vacuously, as $H_i^\ell \subseteq I_i^\ell$ and $\big(\{x_i\} \cup \boldsymbol{M}_i \cup \boldsymbol{S}_i\big) \cap I_i^\ell = \emptyset$, $i = 1, 2$. Thus, we have (*a*) of the lemma.

Case II: (I_1^ℓ, I_2^ℓ) is added in step $(\mathbf{s}_1)$. For $i = 1, 2$, let $\boldsymbol{b}_i$ be the relevant σ-blocks such that $t^\sigma_{\mathfrak{M}_1}(\boldsymbol{b}_1) = t^\sigma_{\mathfrak{M}_2}(\boldsymbol{b}_2)$ and $I_i^\ell = \boldsymbol{b}_i$ is definable in $\mathfrak{M}_i$. For $\ell < N$, let $\boldsymbol{r}_i^\ell$ denote the number of relevant clusters in I_i^ℓ, and let $C_i^{\ell,j}$, $j < \boldsymbol{r}_i^\ell$, be the sequence (ordered by $<_{R_i}$) of all relevant clusters in $\boldsymbol{b}_i$ (that intersect with $\{x_i\} \cup \boldsymbol{M}_i \cup \boldsymbol{S}_i$). Then $C_i^{\ell, \boldsymbol{r}_i^\ell - 1}$ is the final cluster $C^+_{\boldsymbol{b}_i}$ of $\boldsymbol{b}_i$.

Case II.1: Observe that, by Lemmas 4.13 (*c*) and 4.15 (*b*), $C_1^{\ell, \boldsymbol{r}_1^\ell - 1} = C^+_{\boldsymbol{b}_1}$ is degenerate iff $C_2^{\ell, \boldsymbol{r}_2^\ell - 1} = C^+_{\boldsymbol{b}_2}$ is degenerate iff both $\boldsymbol{b}_1 = C^+_{\boldsymbol{b}_1}$ and $\boldsymbol{b}_2 = C^+_{\boldsymbol{b}_2}$ are degenerate σ-blocks, and so $\boldsymbol{r}_i^\ell = 1$. So, in this case, we just set $\boldsymbol{n}_i^\ell = 1$, $\mathfrak{H}_i^\ell = \mathfrak{F}_i \restriction_{\boldsymbol{b}_i}$, $\mathfrak{N}_i^\ell = \mathfrak{M}_i \restriction_{\boldsymbol{b}_i}$ and $\mathsf{h}_i(z_i) = z_i$ for the only point z_i in $\boldsymbol{b}_i$, $i = 1, 2$. It is straightforward to check that (*a*)–(*c*) of the lemma hold. In particular, (*c*) holds because the pair $(\mathfrak{N}_1^\ell, \mathfrak{N}_2^\ell)$ is of type (*a*) in Definition 4.3.

Case II.2: So, let $C_i^{\ell,\boldsymbol{r}_i^\ell-1} = C^+_{\boldsymbol{b}_i}$ be non-degenerate, for $i = 1, 2$. We may assume that, for any $j < \boldsymbol{r}_i^\ell - 1$, $C_i^{\ell,j}$ is a non-limit cluster. (For $j > 0$, this follows from Lemma 2.6, as $C_i^{\ell,j} \cap \boldsymbol{M}_i \neq \emptyset$ by Lemma 4.17 (a). However, if C_i^0 is the root cluster in $\mathfrak{F}_i$, it can happen that $(\{x_i\} \cup \boldsymbol{M}_i) \cap C_i^0 = \{x_i\}$, $x_i \notin \boldsymbol{M}_i$ and C_i^0 is a limit cluster. We may exclude this case by Lemma 3.3.) Also, as $\boldsymbol{b}_i$ is definable in $\mathfrak{M}_i$, $C_i^{\ell,\boldsymbol{r}_i^\ell-1} = C^+_{\boldsymbol{b}_i}$ is a non-limit cluster by Lemma 4.13 (d). Below, we define sets $A_i^\ell \subseteq C^+_{\boldsymbol{b}_i}$, intervals $J_i^{\ell,j} \subseteq I_i^\ell$, and models $\mathfrak{N}_i^{\ell,j} = (\mathfrak{H}_i^{\ell,j}, \mathfrak{w}_i^{\ell,j})$ with $\mathfrak{H}_i^{\ell,j} = (H_i^{\ell,j}, R_i \restriction_{H_i^{\ell,j}}, \mathcal{P}_i^{\ell,j})$, for $i = 1, 2$ and $j < \boldsymbol{r}_i^\ell$, such that the following hold:

(23) $\mathfrak{N}_i^{\ell,j}$ is $\big(J_i^{\ell,j}, i\big)$-nice, for $j < \boldsymbol{r}_i^\ell$;

(24) $\mathfrak{N}_i^{\ell,j}$ is the ordered sum of at most two simple δ-models based on atomic frames, for $j < \boldsymbol{r}_i^\ell$;

(25) $\{J_i^{\ell,j} \mid j < \boldsymbol{r}_i^\ell\}$ is a partition of I_i^ℓ with $J_i^{\ell,0} \prec_{\mathfrak{F}_i} \cdots \prec_{\mathfrak{F}_i} J_i^{\ell,\boldsymbol{r}_i^\ell-1}$;

(26) there is a σ-type preserving bijection between A_1^ℓ and A_2^ℓ;

(27) $t^\sigma_{\mathfrak{M}_1}\Big(\bigcup_{j<\boldsymbol{r}_1^\ell-1} H_1^{\ell,j}\Big) \subseteq t^\sigma_{\mathfrak{M}_2}\big(A_2^\ell\big)$ and $t^\sigma_{\mathfrak{M}_2}\Big(\bigcup_{j<\boldsymbol{r}_2^\ell-1} H_2^{\ell,j}\Big) \subseteq t^\sigma_{\mathfrak{M}_1}\big(A_1^\ell\big)$.

Then we show that (23)–(27) imply (a)–(c) for $\mathfrak{N}_i^\ell = \mathfrak{N}_i^{\ell,0} \lhd \cdots \lhd \mathfrak{N}_i^{\ell,\boldsymbol{r}_i^\ell-1}$ and some $\boldsymbol{n}_i^\ell \leq 2\boldsymbol{r}_i^\ell$. In particular, (c) because $(\mathfrak{N}_1^\ell, \mathfrak{N}_2^\ell)$ is of type (b) in Definition 4.3.

To this end, we cover first the cases when $j < \boldsymbol{r}_i^\ell - 1$ and then, separately, the case $j = \boldsymbol{r}_i^\ell - 1$. So suppose first that $j < \boldsymbol{r}_i^\ell - 1$, and let $J_i^{\ell,j} = [D_i^{\ell,j}, C_i^{\ell,j}]$, where $D_i^{\ell,0}$ is the root cluster in $\mathfrak{F}_i \restriction_{I_i^\ell}$ and $D_i^{\ell,j}$ is the immediate successor of the non-limit cluster $C_i^{\ell,j-1}$, $0 < j < \boldsymbol{r}_i^\ell - 1$. Observe that $\big(\{x_i\} \cup \boldsymbol{M}_i \cup \boldsymbol{S}_i\big) \cap J_i^{\ell,j} \subseteq C_i^{\ell,j}$. We consider four subcases (i)–(iv), depending on the tail $Z_i^{\ell,j}$ of $\mathfrak{F}_i \restriction_{J_i^{\ell,j}}$.

- (i) $Z_i^{\ell,j} = \emptyset$, so $C_i^{\ell,j}$ is non-degenerate. Let $H_i^{\ell,j} = \big(\{x_i\} \cup \boldsymbol{M}_i \cup \boldsymbol{S}_i\big) \cap C_i^{\ell,j}$ and $\mathcal{P}_i^{\ell,j} = 2^{H_i^{\ell,j}}$. Then $\mathfrak{H}_i^j$ is isomorphic to $\textcircled{k}$, for $k = |H_i^{\ell,j}|$. We set $\mathsf{h}_i^{\ell,j}(x) = x$, for $x \in H_i^{\ell,j}$, and $\mathfrak{w}_i^{\ell,j}(p) = \mathfrak{v}_i(p) \cap H_i^{\ell,j}$, for $p \in \delta$.
- (ii) If $0 < |Z_i^{\ell,j}| = m < \omega$ and $Z_i^{\ell,j} = J_i^{\ell,j}$, then by taking $H_i^{\ell,j} = J_i^{\ell,j}$ and $\mathcal{P}_i^{\ell,j} = 2^{H_i^{\ell,j}}$ we obtain $\mathfrak{H}_i^{\ell,j}$ isomorphic to $m^<$. We set $\mathsf{h}_i^{\ell,j}(x) = x$, for $x \in H_i^{\ell,j}$, and $\mathfrak{w}_i^{\ell,j}(p) = \mathfrak{v}_i(p) \cap H_i^{\ell,j}$, for $p \in \delta$.
- (iii) If $0 < |Z_i^{\ell,j}| = m < \omega$ and $Z_i^{\ell,j} \neq J_i^{\ell,j}$, then setting $H_i^{\ell,j} = \{w_i^{\ell,j}\} \cup Z_i^{\ell,j}$, for any $w_i^{\ell,j}$ in the head of $Z_i^{\ell,j}$, and $\mathcal{P}_i^{\ell,j} = 2^{H_i^{\ell,j}}$ gives $\mathfrak{H}_i^{\ell,j}$ isomorphic to $\textcircled{1} \lhd m^<$. Let $\mathsf{h}_i^{\ell,j}(x) = x$, for $x \in H_i^{\ell,j}$, and $\mathfrak{w}_i^{\ell,j}(p) = \mathfrak{v}_i(p) \cap H_i^{\ell,j}$, for $p \in \delta$.
- (iv) If $Z_i^{\ell,j}$ is infinite, then let $H_i^{\ell,j} = \{w_i^{\ell,j}\} \cup Z_i^{\ell,j}$, for any $w_i^{\ell,j}$ in the head of $Z_i^{\ell,j}$, and $\mathcal{P}_i^{\ell,j}$ consist of all finite subsets of $H_i^{\ell,j}$ and their complements in $H_i^{\ell,j}$. By (5), the resulting $\mathfrak{H}_i^{\ell,j}$ is isomorphic to $\mathfrak{C}(\textcircled{1}, \bullet) \lhd 1^<$. In this case, $C_i^{\ell,j} = \{y_i^{\ell,j}\}$ is a degenerate cluster for some $y_i^{\ell,j} \in \{x_i\} \cup \boldsymbol{M}_i \cup \boldsymbol{S}_i$. We set $\mathsf{h}_i^{\ell,j}(y_i^{\ell,j}) = y_i^{\ell,j}$ and $\mathsf{h}_i^{\ell,j}(x) = w_i^{\ell,j}$ for all $x \in H_i^{\ell,j} \setminus \{y_i^{\ell,j}\}$. For $p \in \delta$, let

$$\mathfrak{w}_i^{\ell,j}(p) = \begin{cases} \big(H_i^{\ell,j} \setminus \{y_i^{\ell,j}\}\big) \cup \big(\mathfrak{v}_i(p) \cap \{y_i^{\ell,j}\}\big), & \text{if } w_i^{\ell,j} \in \mathfrak{v}_i(p), \\ \mathfrak{v}_i(p) \cap \{y_i^{\ell,j}\}, & \text{otherwise.} \end{cases}$$

Then it is not hard to check that, in all (i)–(iv), we have $\mathfrak{w}_i^{\ell,j}(p) \in \mathcal{P}_i^{\ell,j}$, (24) for $\mathfrak{N}_i^{\ell,j} = (\mathfrak{H}_i^{\ell,j}, \mathfrak{w}_i^{\ell,j})$, and (13)–(22) hold for $\mathfrak{H} = \mathfrak{H}_i^{\ell,j}$, $\mathfrak{N} = \mathfrak{N}_i^{\ell,j}$, $\mathsf{h} = \mathsf{h}_i^{\ell,j}$, and $I = J_i^{\ell,j}$. In particular, in (i)–(iii), we have (18) by Lemma 2.5. In (iv), we also need (3) to obtain (18), and the fact that $\boldsymbol{M}_i \cap H_i^{\ell,j} = \{y_i^{\ell,j}\}$ to obtain (22). Therefore, we have (23) for $j < \boldsymbol{r}_i^\ell - 1$.

Now, consider $j = \boldsymbol{r}_i^\ell - 1$. First, we let $J_i^{\ell,\boldsymbol{r}_i^\ell-1} = [D_i^{\ell,\boldsymbol{r}_i^\ell-1}, C_i^{\ell,\boldsymbol{r}_i^\ell-1}]$, where $D_i^{\ell,\boldsymbol{r}_i^\ell-1} = C_i^{\ell,\boldsymbol{r}_i^\ell-1}$ if $\boldsymbol{r}_i^\ell = 1$ and $D_i^{\ell,\boldsymbol{r}_i^\ell-1}$ is the immediate successor of the non-limit cluster $C_i^{\ell,\boldsymbol{r}_i^\ell-2}$ otherwise. Then we have (25). We let $Y_i^\ell = \bigcup_{j<\boldsymbol{r}_1^\ell-1} Y_i^{\ell,j}$, where $Y_i^{\ell,j} = H_i^{\ell,j}$ in cases (i)–(iii) above, and $Y_i^{\ell,j} = \{w_i^{\ell,j}, z_i^{\ell,j}\}$ in case (iv). So Y_i^ℓ is finite. Set $\Theta = \{t^\sigma_{\mathfrak{M}_1}(x) \mid x \in Y_1^\ell\} \cup \{t^\sigma_{\mathfrak{M}_2}(x) \mid x \in Y_2^\ell\}$. Let A_i^ℓ be the smallest set such that $\big(\{x_i\} \cup \boldsymbol{M}_i \cup \boldsymbol{S}_i\big) \cap C_i^{\ell,\boldsymbol{r}_i^\ell-1} \subseteq A_i^\ell \subseteq C_i^{\ell,\boldsymbol{r}_i^\ell-1}$ and A_i^ℓ contains a point z_t with $t^\sigma_{\mathfrak{M}_i}(z_t) = t$, for each $t \in \Theta$. As $Y_i^\ell \subseteq I_i^\ell = \boldsymbol{b}_i$, $C_i^{\ell,\boldsymbol{r}_i^\ell-1} = C^+_{\boldsymbol{b}_i}$, and $t^\sigma_{\mathfrak{M}_1}(\boldsymbol{b}_1) = t^\sigma_{\mathfrak{M}_2}(\boldsymbol{b}_2)$, such A_i^ℓ exist by Lemma 4.13 (e). Observe that not only $t^\sigma_{\mathfrak{M}_1}(A_1^\ell) = t^\sigma_{\mathfrak{M}_2}(A_2^\ell)$ but, by Lemma 4.17 (b), (d), we actually have (26). Then $k := \big|A_1^\ell\big| = \big|A_2^\ell\big| \leq 2^{|\delta|}$, by Lemma 2.4 (b), and also

$$k \leq |Y_1^\ell| + |Y_2^\ell| + \boldsymbol{k}(\varphi_1, \varphi_2). \tag{28}$$

By taking $H_i^{\ell,\boldsymbol{r}_i^\ell-1} = A_i^\ell$ and $\mathcal{P}_i^{\ell,\boldsymbol{r}_i^\ell-1} = 2^{A_i^\ell}$, $i = 1, 2$, we obtain $\mathfrak{H}_1^{\ell,\boldsymbol{r}_1^\ell-1}$ and $\mathfrak{H}_2^{\ell,\boldsymbol{r}_2^\ell-1}$ both isomorphic to ⓚ. (The sets A_i^ℓ are used differently in Case III.) Then we have (27). For $p \in \delta$, set $\mathfrak{w}_i^{\ell,\boldsymbol{r}_i^\ell-1}(p) = \mathfrak{v}_i(p) \cap H_i^{\ell,\boldsymbol{r}_i^\ell-1}$ and $\mathsf{h}_i^{\ell,\boldsymbol{r}_i^\ell-1}(x) = x$ for all $x \in H_i^{\ell,\boldsymbol{r}_i^\ell-1}$. Then we clearly have $\mathfrak{w}_i^{\ell,\boldsymbol{r}_i^\ell-1}(p) \in \mathcal{P}_i^{\ell,\boldsymbol{r}_i^\ell-1}$, (24) for $\mathfrak{N}_i^{\ell,\boldsymbol{r}_i^\ell-1} = (\mathfrak{H}_i^{\ell,\boldsymbol{r}_i^\ell-1}, \mathfrak{w}_i^{\ell,\boldsymbol{r}_i^\ell-1})$, and (13)–(22) hold for $\mathfrak{H} = \mathfrak{H}_i^{\ell,\boldsymbol{r}_i^\ell-1}$, $\mathfrak{N} = \mathfrak{N}_i^{\ell,\boldsymbol{r}_i^\ell-1}$, $\mathsf{h} = \mathsf{h}_i^{\ell,\boldsymbol{r}_i^\ell-1}$ and $I = J_i^{\ell,\boldsymbol{r}_i^\ell-1}$ ((18) is by Lemma 2.5). This gives (23) for $j = \boldsymbol{r}_i^\ell - 1$.

Finally, we claim that (a)–(c) hold for $\mathfrak{N}_i^\ell = \mathfrak{N}_i^{\ell,0} \lhd \cdots \lhd \mathfrak{N}_i^{\ell,\boldsymbol{r}_i^\ell-1}$ and some $\boldsymbol{n}_i^\ell$ with $0 < \boldsymbol{n}_i^\ell \leq 2\boldsymbol{r}_i^\ell$. Indeed, (b) is by the definition of $\lhd$ and (24). For (c): The final cluster in $\mathfrak{N}_i^\ell$ = final cluster in $\mathfrak{N}_i^{\ell,\boldsymbol{r}_i^\ell-1}$ = the non-degenerate cluster A_i^ℓ. So the requirements in Definition 4.3 (b) follow from (26) and (27). For (a): By (23), each $\mathfrak{N}_i^{\ell,j}$ is $\big(J_i^{\ell,j}, i\big)$-nice, for $j < \boldsymbol{r}_i^\ell$, that is, conditions (13)–(22) are satisfied for $\mathfrak{N} = \mathfrak{N}_i^{\ell,j}$, $\mathfrak{H} = \mathfrak{H}_i^{\ell,j}$, $I = J_i^{\ell,j}$, and $\mathsf{h} = \mathsf{h}_i^{\ell,j}$ (as defined above). We claim that (13)–(22) are satisfied for $\mathfrak{N} = \mathfrak{N}_i^\ell$, $\mathfrak{H}$ = the frame $\mathfrak{H}_i^\ell$ underlying $\mathfrak{N}_i^\ell$, $I = I_i^\ell$ and $\mathsf{h}_i^\ell = \bigcup_{j<\boldsymbol{r}_i^\ell} \mathsf{h}_i^{\ell,j}$. Indeed, (13), (14), (18)–(20), clearly follow from (25), the definition of $\lhd$, and the corresponding properties for $\mathfrak{N}_i^{\ell,j}$, $\mathfrak{H}_i^{\ell,j}$, $J_i^{\ell,j}$, and $\mathsf{h}_i^{\ell,j}$, $j < \boldsymbol{r}_i^\ell$; (15) follows from (15) for $\mathfrak{H}_i^{\ell,\boldsymbol{r}_i^\ell-1}$ and $J^{\ell,\boldsymbol{r}_i^\ell-1}$; and (16) follows from (16) for $\mathfrak{H}_i^{\ell,0}$ and $J^{\ell,0}$. For (17): Suppose $x \in H_i^\ell$, $\{x\}$ is a degenerate non-root cluster in $\mathfrak{H}_i^\ell$ and $C \subseteq I_i^\ell$ is the immediate predecessor of $\{x\}$ in $\mathfrak{F}_i$. Let $j < \boldsymbol{r}_i^\ell$ be such that $x \in H_i^{\ell,j}$. If $\{x\}$ is the root cluster in $\mathfrak{H}_i^{\ell,j}$, then $j > 0$ and $\{x\}$ is the root cluster in $\mathfrak{F}_i \restriction_{J_i^{\ell,j}}$ by (16) for $\mathfrak{H}_i^{\ell,j}$ and $J_i^{\ell,j}$. Thus, the final cluster $C^- \subseteq H_i^{\ell,j-1} \subseteq H_i^\ell$ of $\mathfrak{H}_i^{\ell,j-1}$ is a subset of C by (15) for $\mathfrak{H}_i^{\ell,j-1}$ and $J_i^{\ell,j-1}$. If $\{x\}$ is a non-root cluster in $\mathfrak{H}_i^{\ell,j}$, then $C \subseteq J_i^{\ell,j}$ and (17) for $\mathfrak{H}_i^\ell$ and I_i^ℓ follows from $H_i^{\ell,j} \subseteq H_i^\ell$ and (17) for $\mathfrak{H}_i^{\ell,j}$ and $J_i^{\ell,j}$. For (21): Suppose $x, y \in H_i^\ell$, xR_iy

and let $j \leq j' < \boldsymbol{r}_i^\ell$ be such that $x \in H_i^{\ell,j}$ and $y \in H_i^{\ell,j'}$. Then $\mathsf{h}_i^\ell(x) R_i \mathsf{h}_i^\ell(y)$ follows by (21) for $\mathsf{h}_i^{\ell,j}$ when $j = j'$, and by the definition of $\lhd$ when $j < j'$. For (22): Suppose $x, y \in H_i^\ell$, $y \in \boldsymbol{M}_i$, $\mathsf{h}_i^\ell(x) R_i y$, and let $j \leq j' < \boldsymbol{r}_i^\ell$ be such that $x \in H_i^{\ell,j}$ and $y \in H_i^{\ell,j'}$. Then $x R_i y$ follows by (22) for $\mathsf{h}_i^{\ell,j}$ when $j = j'$, and by the definition of $\lhd$ when $j < j'$.

Case III: (I_1^ℓ, I_2^ℓ) is added in step $(\mathbf{s}_2)$. For $i = 1, 2$, let $\boldsymbol{b}_i$ be the relevant σ-blocks such that $t^\sigma_{\mathfrak{M}_1}(\boldsymbol{b}_1) = t^\sigma_{\mathfrak{M}_2}(\boldsymbol{b}_2)$ and I_i^ℓ is extending $\boldsymbol{b}_i$ that is not definable in $\mathfrak{M}_i$. We use the notation from Case II. As explained in Case II, we may again assume that, for every $j < \boldsymbol{r}_i^\ell - 1$, $C_i^{\ell,j}$ is a non-limit cluster. However, as now $\boldsymbol{b}_i$ is not definable in $\mathfrak{M}_i$, $C_i^{\ell,\boldsymbol{r}_i^\ell - 1} = C^+_{\boldsymbol{b}_i}$ is a limit cluster by Lemma 4.13 (d). We again define sets $A_i^\ell \subseteq C^+_{\boldsymbol{b}_i}$, intervals $J_i^{\ell,j} \subseteq I_i^\ell$, and models $\mathfrak{N}_i^{\ell,j} = (\mathfrak{H}_i^{\ell,j}, \mathfrak{w}_i^{\ell,j})$ with $\mathfrak{H}_i^{\ell,j} = (H_i^{\ell,j}, R_i \restriction_{H_i^{\ell,j}}, \mathcal{P}_i^{\ell,j})$ such that (23)–(27) hold. Then we show that (a)–(c) of the lemma hold for $\mathfrak{N}_i^\ell = \mathfrak{N}_i^{\ell,0} \lhd \cdots \lhd \mathfrak{N}_i^{\ell,\boldsymbol{r}_i^\ell - 1}$. This time, $(\mathfrak{N}_1^\ell, \mathfrak{N}_2^\ell)$ is σ-matching because it is of type (c) in Definition 4.3.

To this end, for any $i = 1, 2$ and $j < \boldsymbol{r}_i^\ell - 1$, we define everything like in II.2. For $j = \boldsymbol{r}_i^\ell - 1$, we set $J_i^{\ell,\boldsymbol{r}_i^\ell - 1} = [D_i^\ell, E_i^\ell]$, where D_i^ℓ is the root cluster in $\mathfrak{F}_i \restriction_{I_i^\ell}$ if $\boldsymbol{r}_i^\ell = 1$ and the immediate successor of the non-limit cluster $C_i^{\ell,\boldsymbol{r}_i^\ell - 2}$ if $\boldsymbol{r}_i^\ell > 1$, and E_i^ℓ is the final cluster in I_i^ℓ. We clearly have (25) and can define the sets Y_i^ℓ and A_i^ℓ in the same way as in II.2. However, for property (18) to hold for $\mathfrak{H} = \mathfrak{H}_i^{\ell,\boldsymbol{r}_i^\ell - 1}$, we need to define $\mathfrak{H}_i^{\ell,\boldsymbol{r}_i^\ell - 1}$ differently. We consider the two cases in step $(\mathbf{s}_2)$ of Definition 4.18:

1. The tail of $\mathfrak{M}_i \restriction_{I_i^\ell}$ is $\{b_i^n \in I_i^\ell \setminus \boldsymbol{b}_i \mid n < \omega\}$ with $b_i^n R_i b_i^{n-1}$, $0 < n < \omega$. (Using the notation of Definition 4.18: $\{b_i^n\} = \boldsymbol{b}_i^{f(m+n)}$, $n < \omega$.)
2. There is a sequence of non-degenerate clusters $D_i^n \subseteq I_i^\ell \setminus \boldsymbol{b}_i$ definable in $\mathfrak{M}_i$, $n < \omega$, with D_i^0 being the final cluster in $\mathfrak{M}_i \restriction_{I_i^\ell}$ and $D_i^n <_{R_i} D_i^{n-1}$, $0 < n < \omega$. (Using the notation of Definition 4.18: $D_i^n = C^+_{\boldsymbol{b}_i^{f(m_{n+1})}}$.) For $n < \omega$, we pick some $b_i^n \in D_i^n$.

In both cases, we set $H_i^{\ell,\boldsymbol{r}_i^\ell - 1} = A_i^\ell \cup \{b_i^m \mid m < \omega\}$. Take the $k < \omega$ with $|A_1^\ell| = |A_2^\ell| = k$, and suppose $A_i^\ell = \{a_i^0, \dots, a_i^{k-1}\}$, $i = 1, 2$. We let $\mathcal{P}_i^{\ell,\boldsymbol{r}_i^\ell - 1}$ be generated in $(H_i^{\ell,\boldsymbol{r}_i^\ell - 1}, R_i \restriction_{H_i^{\ell,\boldsymbol{r}_i^\ell - 1}})$ by the sets $\{b_i^n\}$, $n < \omega$, and X_i^s, $s < k$, where $X_i^s = \{a_i^s\} \cup \{b_i^n \mid n < \omega,\ n \equiv s \pmod k\}$ (see Example 2.2). The resulting $\mathfrak{H}_i^{\ell,\boldsymbol{r}_i^\ell - 1}$ are both isomorphic to $\mathfrak{C}(\textcircled{k}, \bullet)$ in case 1., and to $\mathfrak{C}(\textcircled{k}, \circ)$ in case 2. For $p \in \delta$, we set $\mathfrak{w}_i^{\ell,\boldsymbol{r}_i^\ell - 1}(p) = \bigcup_{a_i^s \in \mathfrak{v}_i(p)} X_i^s$. For every $x \in H_i^{\ell,\boldsymbol{r}_i^\ell - 1}$, we set

$$\mathsf{h}_i^{\ell,\boldsymbol{r}_i^\ell - 1}(x) = \begin{cases} x, & \text{if } x = a_i^s, \text{ for } s < k, \\ a_i^s & \text{if } x = b_i^n,\ n < \omega \text{ and } n \equiv s \pmod k. \end{cases}$$

Then clearly $\mathfrak{w}_i^{\ell,\boldsymbol{r}_i^\ell - 1}(p) \in \mathcal{P}_i^{\ell,\boldsymbol{r}_i^\ell - 1}$ and (24) holds for $\mathfrak{N}_i^{\ell,\boldsymbol{r}_i^\ell - 1} = (\mathfrak{H}_i^{\ell,\boldsymbol{r}_i^\ell - 1}, \mathfrak{w}_i^{\ell,\boldsymbol{r}_i^\ell - 1})$. It is not hard to check that (23) holds for $\mathfrak{N}_i^{\ell,\boldsymbol{r}_i^\ell - 1}$ as well. In particular, (18) for $\mathfrak{H} = \mathfrak{H}_i^{\ell,\boldsymbol{r}_i^\ell - 1}$ follows from (3) and Lemma 2.5. Also, as $A_i^\ell \subseteq C^+_{\boldsymbol{b}_i}$ and $C^+_{\boldsymbol{b}_i}$ is

a limit cluster, $\boldsymbol{M}_i \cap H_i^{\ell,\boldsymbol{r}_i^\ell-1} = \boldsymbol{M}_i \cap A_i^\ell = \emptyset$ follows by Lemma 2.6, and so we also have (22) for $H = H_i^{\ell,\boldsymbol{r}_i^\ell-1}$ and $\mathsf{h} = \mathsf{h}_i^{\ell,\boldsymbol{r}_i^\ell-1}$.

Next, the arguments showing that (a) and (b) of the lemma hold for the models $\mathfrak{N}_i^\ell = \mathfrak{N}_i^{\ell,0} \lhd \cdots \lhd \mathfrak{N}_i^{\ell,\boldsymbol{r}_i^\ell-1}$ and some $\boldsymbol{n}_i^\ell$ with $0 < \boldsymbol{n}_i^\ell \leq 2\boldsymbol{r}_i^\ell$ are the same as in Case II.2. To establish (c), we show that the pair $(\mathfrak{N}_1^\ell, \mathfrak{N}_2^\ell)$ is of type (c) in Definition 4.3. Indeed, observe that the last $\lhd$-components of $\mathfrak{N}_i^\ell$ are $\mathfrak{N}_i^{\ell,\boldsymbol{r}_i^\ell-1}$ whose underlying frames $\mathfrak{H}_i^{\ell,\boldsymbol{r}_i^\ell-1}$ are both isomorphic to the same atomic frame of the form $\mathfrak{C}(\textcircled{k}, \bullet)$ or $\mathfrak{C}(\textcircled{k}, \circ)$, with $0 < k \leq 2^{|\delta|}$. Also, the $\textcircled{k}$-cluster in $\mathfrak{H}_i^{\ell,\boldsymbol{r}_i^\ell-1}$ is A_i^ℓ, and so the requirements in Definition 4.3 (c) follow from (26) and (27).

Finally, observe that $\boldsymbol{n}_i^\ell = 1$ if (I_1^ℓ, I_2^ℓ) is added in step $\mathbf{(s_3)}$ of Definition 4.18 (see Case I), and $\boldsymbol{n}_i^\ell \leq 2\boldsymbol{r}_i^\ell$ if (I_1^ℓ, I_2^ℓ) is added in steps $\mathbf{(s_1)}$ or $\mathbf{(s_2)}$ (see II and III). So $\sum_{\ell<N} \boldsymbol{n}_i^\ell \leq (\boldsymbol{k}(\varphi_1, \varphi_2)-1)+\sum_{\ell<N} 2\boldsymbol{r}_i^\ell \leq 3\boldsymbol{k}(\varphi_1, \varphi_2)-1$, as required. $\dashv$

We now complete the proof of Theorem 4.5. In Definition 4.18, we partitioned the models $\mathfrak{M}_1, x_1$ and $\mathfrak{M}_2, x_2$ witnessing the lack of interpolants for φ_1, φ_2 into the same polynomial number N of intervals. For each $\ell < N$, Lemma 4.21 gave us a pair of models $(\mathfrak{N}_1^\ell, \mathfrak{N}_2^\ell)$. Let $\mathfrak{N}_i = \mathfrak{N}_i^0 \lhd \cdots \lhd \mathfrak{N}_i^{N-1}$, for $i = 1, 2$.

LEMMA 4.22. *Conditions (a)–(d) in Theorem 4.5 hold for $\mathfrak{N}_1, x_1$ and $\mathfrak{N}_2, x_2$.*

PROOF. We use the notation of the proof of Lemma 4.21. By Lemma 4.21 (a), each $\mathfrak{N}_i^\ell$ is (I_i^ℓ, i)-nice, that is, conditions (13)–(22) are satisfied for $\mathfrak{N} = \mathfrak{N}_i^\ell$, $\mathfrak{H} = \mathfrak{H}_i^\ell$, $I = I_i^\ell$, and $\mathsf{h} = \mathsf{h}_i^\ell$.

(a) We show by induction that $\mathfrak{M}_i, \mathsf{h}_i^\ell(x) \models \tau$ iff $\mathfrak{N}_i, x \models \tau$, for any $i = 1, 2$, $\ell < N$, $\tau \in \mathit{sub}(\varphi_i)$, and $x \in H_i^\ell$. Then $\mathfrak{N}_1, x_1 \models \varphi_1$ and $\mathfrak{N}_2, x_2 \models \neg\varphi_2$ follow from $\mathfrak{M}_1, x_1 \models \varphi_1$ and $\mathfrak{M}_2, x_2 \models \neg\varphi_2$, as we have $x_i \in H_i^0$ and $\mathsf{h}_i^0(x_i) = x_i$ by (14) and (19). For $\tau = p \in \delta$, the statement follows from (20). The Boolean cases are straightforward, so suppose $\tau = \Diamond\psi$.

$(\Rightarrow)$ If $\mathfrak{M}_i, \mathsf{h}_i^\ell(x) \models \Diamond\psi$, then there are $k \geq \ell$ and $y_\psi \in \boldsymbol{M}_i \cap I_i^k$ with $\mathsf{h}_i^\ell(x) R_i y_\psi$ and $\mathfrak{M}_i, y_\psi \models \psi$. We have $y_\psi \in H_i^k$ by (14), and so $\mathfrak{N}_i, y_\psi \models \psi$ by (19) and IH. We claim that xR_iy_ψ, and so $\mathfrak{N}_i, x \models \Diamond\psi$. Indeed, for $k > \ell$, this follows from the definition of $\lhd$, and for $\ell = k$, by (22).

$(\Leftarrow)$ If $\mathfrak{N}_i, x \models \Diamond\psi$, then there are $k \geq \ell$ and $y \in H_i^k$ with xR_iy and $\mathfrak{N}_i, y \models \psi$. We have $\mathfrak{M}_i, \mathsf{h}_i^k(y) \models \psi$ by IH, and $\mathsf{h}_i^\ell(x) R_i \mathsf{h}_i^k(y)$ by the definition of $\lhd$ when $k > \ell$, and by (21) when $k = \ell$. Hence $\mathfrak{M}_i, \mathsf{h}_i(x) \models \Diamond\psi$.

(c) follows from (14), (19), (20) and $\mathfrak{M}_1, x_1 \sim_\sigma \mathfrak{M}_2, x_2$.

(d) It is shown in Definition 4.18 that $0 < N < 2\boldsymbol{k}(\varphi_1, \varphi_2)$. The rest of (d) follows from Lemma 4.21 (b),(c).

(b) We use the refutation criteria for the canonical formulas to show that the frame $\mathfrak{H}_i$ underlying $\mathfrak{N}_i$ is a frame for L, $i = 1, 2$. To this end, we prove that,

(29) for any canonical formula $\alpha(\mathfrak{G}, \mathfrak{D}, \bot)$, if f is an injection from $\mathfrak{G}$ to $\mathfrak{H}_i$ satisfying $\mathbf{(cf_1)}$–$\mathbf{(cf_4)}$, then the same f is an injection from $\mathfrak{G}$ to $\mathfrak{F}_i$ satisfying $\mathbf{(cf_1)}$–$\mathbf{(cf_4)}$.

Indeed, $\mathbf{(cf_1)}$ holds by (13) and the definition of $\lhd$; $\mathbf{(cf_2)}$ holds, as the final cluster in $\mathfrak{H}_i$ = final cluster in $\mathfrak{H}_i^{N-1} \subseteq$ final cluster in $\mathfrak{F}_i \restriction_{I_i^{N-1}}$ = final cluster in $\mathfrak{M}_i$, by (15). Condition $\mathbf{(cf_4)}$ holds by (18) and the definition of $\lhd$. For $\mathbf{(cf_3)}$,

suppose $x \in \mathfrak{D}$, $C(y)$ is the immediate predecessor of $C(x) = \{x\}$ in $\mathfrak{G}$ and $C(f(y))$ is the immediate predecessor of $C(f(x)) = \{f(x)\}$ in $\mathfrak{H}_i$. Let $\ell < N$ be such that $x \in H_i^\ell$. If $\{x\}$ is the root cluster in $\mathfrak{H}_i^\ell$, then $\ell > 0$ and $\{x\}$ is the root cluster in $\mathfrak{F}_i \restriction_{I_i^\ell}$ by (16). Thus, $C(f(y))$ in $\mathfrak{H}_i$ = final cluster in $\mathfrak{H}_i^{\ell-1} \subseteq$ final cluster in $\mathfrak{F}_i \restriction_{I_i^{\ell-1}} = C(f(y))$ in $\mathfrak{F}_i$, by (15). If $\{x\}$ is a non-root cluster in $\mathfrak{H}_i^\ell$, then $C(f(y))$ in $\mathfrak{H}_i = C(f(y))$ in $\mathfrak{H}_i^\ell \subseteq C(f(y))$ in $\mathfrak{F}_i \restriction_{I_i^\ell} = C(f(y))$ in $\mathfrak{F}_i$, by (17). Now, (29) implies that $L \subseteq \mathsf{Log}(\mathfrak{F}_i) \subseteq \mathsf{Log}(\mathfrak{H}_i)$, as required. $\dashv$

4.4. Proofs of Theorems 4.6 and 4.7. Suppose the finitely axiomatisable logic L is given by its canonical axioms as $L = \mathsf{K4.3} \oplus \{\alpha(\mathfrak{G}_j, \mathfrak{D}_j, \bot) \mid j \in J_L\}$, for some finite index set J_L and $\mathfrak{G}_j = (V_j, S_j)$, $j \in J_L$. Let $\boldsymbol{c}_L = \max_{j \in J_L} |V_j|$. Given formulas φ_1, φ_2 without an interpolant in L, let $0 < N < 2\boldsymbol{k}(\varphi_1, \varphi_2)$ and $\mathfrak{N}_i = \mathfrak{N}_i^0 \lhd \dots \lhd \mathfrak{N}_i^{N-1}$ with root x_i $i = 1, 2$, be the models satisfying the conditions of Theorem 4.5 and obtained via Lemma 4.21. In particular, the underlying frame $\mathfrak{H}_i$ of each $\mathfrak{N}_i$ is a frame for L. We show in Lemma 4.23 below that the proof of Lemma 4.21 can be refined to yield polynomial-size models $\mathfrak{N}_i^{\star\ell}$, $\ell < N$. However, $\mathfrak{N}_i^{\star\ell}$ is no longer (I_i^ℓ, i)-nice, as conditions (16) and (17) in Definition 4.20 do not necessarily hold for $\mathfrak{H} = \mathfrak{H}_i^{\star\ell}$ underlying $\mathfrak{N}_i^{\star\ell}$ and $I = I_i^\ell$. Thus, we do not have (29) in the proof of Lemma 4.22 for the frames $\mathfrak{H}_i^\star$ underlying $\mathfrak{N}_i^\star = \mathfrak{N}_i^{\star 0} \lhd \dots \lhd \mathfrak{N}_i^{\star N-1}$. We prove that $\mathfrak{H}_i^\star$, $i = 1, 2$, are frames for L (as required by Theorem 4.6 (b)) by using Lemma 4.24 below instead.

Take the number N, $0 < N < 2\boldsymbol{k}(\varphi_1, \varphi_2)$, provided by Definition 4.18, the numbers $\boldsymbol{n}_i^\ell > 0$ with $\sum_{\ell < N} \boldsymbol{n}_i^\ell \leq 3\boldsymbol{k}(\varphi_1, \varphi_2) - 1$, and sets H_i^ℓ, $i = 1, 2$, $\ell < N$, from Lemma 4.21.

LEMMA 4.23. *If $L \supseteq \mathsf{K4.3}$ is finitely axiomatisable, then, for $i = 1, 2$, $\ell < N$, there exist sets $H_i^{\star\ell} \subseteq H_i^\ell$ and models $\mathfrak{N}_i^{\star\ell}$ based on frames $\mathfrak{H}_i^{\star\ell} = (H_i^{\star\ell}, S_i^{\star\ell}, \mathcal{P}_i^{\star\ell})$ such that the following hold*:

- (a) *$\mathfrak{N}_i^{\star\ell}$ is 'almost' (I_i^ℓ, i)-nice in the sense that* (13)–(15), (18)–(22) *hold for $\mathfrak{N} = \mathfrak{N}_i^{\star\ell}$ and $I = I_i^\ell$*;
- (b) *$\mathfrak{N}_i^{\star\ell}$ is the ordered sum of $\boldsymbol{n}_i^\ell$-many simple δ-models based on L-bounded atomic frames*;
- (c) *the pair $(\mathfrak{N}_1^{\star\ell}, \mathfrak{N}_2^{\star\ell})$ is σ-matching.*

PROOF. We go through Cases I–III in the proof of Lemma 4.21 and make the necessary modifications.

Case I: (I_1^ℓ, I_2^ℓ) is added in step $(\mathbf{s}_3)$ of Definition 4.18. An inspection of this part of the proof of Lemma 4.21 reveals that $m^<$ or $① \lhd m^<$ is used in cases (i) and (ii), and in both cases *all* the m elements of the finite non-empty tails Z_i^ℓ of $\mathfrak{F}_i \restriction_{I_i^\ell}$ are put into the chosen subset H_i^ℓ of I_i^ℓ. Now, we choose a subset $H_i^{\star\ell} \subseteq H_i^\ell$ with $|H_i^{\star\ell}| \leq \boldsymbol{c}_L + 2$ as follows. Suppose $Z_i^\ell = \{z_i^a \mid a < m\}$ with $z_i^a R_i^s z_i^{a-1}$, $0 < a < m$, and let $m' = \min(m, \boldsymbol{c}_L + 1)$. We set $H_i^{\star\ell} = \{z_i^a \mid a < m'\}$ in case (i), and $H_i^{\star\ell} = \{w_i^\ell\} \cup \{z_i^a \mid a < m'\}$, for the chosen w_i^ℓ from the head of Z_i^ℓ in case (ii). In case (iii), we let $H_i^{\star\ell} = H_i^\ell$. Then, in all cases (i)–(iii), we let $\mathfrak{H}_i^{\star\ell} = \mathfrak{H}_i^\ell \restriction_{H_i^{\star\ell}}$ and $\mathfrak{N}_i^{\star\ell} = \mathfrak{N}_i^\ell \restriction_{H_i^{\star\ell}}$. Observe that we have $\mathsf{h}_i^\ell(x) \in H_i^{\star\ell}$, for every $x \in H_i^{\star\ell}$, and so $\mathsf{h}_i^{\star\ell} = \mathsf{h}_i^\ell \restriction_{H_i^{\star\ell}}$ is a $H_i^{\star\ell} \to H_i^{\star\ell}$ function. It is straightforward to check that (13)–(15), (18)–(22) hold for $\mathfrak{N} = \mathfrak{N}_i^{\star\ell}$, $\mathfrak{H} = \mathfrak{H}_i^{\star\ell}$, $\mathsf{h} = \mathsf{h}_i^{\star\ell}$, and $I = I_i^\ell$.

Note that all non-degenerate clusters in $\mathfrak{H}_i^{\star\ell}$ are of the form ① this case, and so $\mathfrak{N}_i^{\star\ell}$ is a simple δ-model based on an L-bounded atomic frame.

Case II and III: (I_1^ℓ, I_2^ℓ) is added in steps $(\mathbf{s}_1)$ or $(\mathbf{s}_2)$. An inspection of these parts of the proof of Lemma 4.21 reveals that $m^<$ or $① \lhd m^<$ is used only when $\boldsymbol{b}_i$ is non-degenerate, in cases (ii) and (iii) of the definition of $\mathfrak{H}_i^{\ell,j}$ for some $j < \boldsymbol{r}_i^\ell - 1$. (Recall that $\boldsymbol{r}_i^\ell$ denotes the number of relevant clusters in I_i^ℓ.) In both cases (ii) and (iii), *all* the m elements of the finite non-empty tail $Z_i^{\ell,j}$ of $\mathfrak{F}_i \restriction_{J_i^{\ell,j}}$ are put into the chosen subset $H_i^{\ell,j}$ of $J_i^{\ell,j}$, for some subinterval $J_i^{\ell,j}$ of I_i^ℓ. We repeat the trick from Case I above. Suppose $Z_i^{\ell,j} = \{z^a \mid a < m\}$ with $z^a R_i^s z^{a-1}$, $0 < a < m$, and let $m' = \min(m, \boldsymbol{c}_L + 1)$. We set $H_i^{\star\ell,j} = \{z^a \mid a < m'\}$ in case (ii), and $H_i^{\star\ell,j} = \{w_i^{\ell,j}\} \cup \{z^a \mid a < m'\}$, for the chosen $w_i^{\ell,j}$ from the head of $Z_i^{\ell,j}$ in case (iii). In cases (i) and (iv) of II and III, we let $H_i^{\star\ell,j} = H_i^{\ell,j}$. Then, in all cases (i)–(iv), we let $\mathfrak{H}_i^{\star\ell,j} = \mathfrak{H}_i^{\ell,j} \restriction_{H_i^{\star\ell,j}}$ and $\mathfrak{N}_i^{\star\ell,j} = \mathfrak{N}_i^{\ell,j} \restriction_{H_i^{\star\ell,j}}$, for $j < \boldsymbol{r}_i^\ell - 1$. One can see that $\mathsf{h}_i^{\ell,j}(x) \in H_i^{\star\ell,j}$, for every $x \in H_i^{\star\ell,j}$, and so $\mathsf{h}_i^{\star\ell,j} = \mathsf{h}_i^{\ell,j} \restriction_{H_i^{\star\ell,j}}$ is an $H_i^{\star\ell,j} \to H_i^{\star\ell,j}$ function. It is straightforward to check that, for all $j < \boldsymbol{r}_i^\ell - 1$, (13)–(15), (18)–(22) hold for $\mathfrak{N} = \mathfrak{N}_i^{\star\ell,j}$, $\mathfrak{H} = \mathfrak{H}_i^{\star\ell,j}$, $\mathsf{h} = \mathsf{h}_i^{\star\ell,j}$, and $I = J_i^{\ell,j}$ (but (16) and (17) do not necessarily hold). Note that the size of non-degenerate clusters in these $\mathfrak{H}_i^{\star\ell,j}$ is bounded by $\boldsymbol{k}(\varphi_1, \varphi_2)$, and so $\mathfrak{N}_i^{\star\ell,j}$ is the ordered sum of at most two simple δ-models based on L-bounded atomic frames.

We also need to adjust the definitions of $\mathfrak{H}_i^{\ell,\boldsymbol{r}_i^\ell-1}$, $\mathfrak{N}_i^{\ell,\boldsymbol{r}_i^\ell-1}$ and $\mathsf{h}_i^{\ell,\boldsymbol{r}_i^\ell-1}$. We define the sets $Y_i^{\star\ell} \subseteq Y_i^\ell$ and $A_i^{\star\ell} \subseteq A_i^\ell$ from $H_i^{\star\ell,j}$, $j < \boldsymbol{r}_i^\ell - 1$, in the same way as Y_i^ℓ and A_i^ℓ were defined from $H_i^{\ell,j}$, $j < \boldsymbol{r}_i^\ell - 1$, resulting in (26) and (27). Let $k^\star = \big|A_1^{\star\ell}\big| = \big|A_2^{\star\ell}\big|$. By (28),

$$k^\star \leq |Y_1^{\star\ell|} + |Y_2^{\star\ell}| + \boldsymbol{k}(\varphi_1, \varphi_2) \leq \\ 2(\boldsymbol{k}(\varphi_1, \varphi_2) - 1) \cdot \max\big(\boldsymbol{c}_L + 2, \boldsymbol{k}(\varphi_1, \varphi_2)\big) + \boldsymbol{k}(\varphi_1, \varphi_2) = \boldsymbol{p}_L(\varphi_1, \varphi_2).$$

In Case II, we set $\mathfrak{H}_i^{\star\ell,\boldsymbol{r}_i^\ell-1} = \mathfrak{H}_i^{\ell,\boldsymbol{r}_i-1} \restriction_{H_i^{\star\ell,\boldsymbol{r}_i^\ell-1}}$, $\mathfrak{N}_i^{\star\ell,\boldsymbol{r}_i^\ell-1} = \mathfrak{N}_i^{\ell,\boldsymbol{r}_i-1} \restriction_{H_i^{\star\ell,\boldsymbol{r}_i^\ell-1}}$, and $\mathsf{h}_i^{\star\ell,\boldsymbol{r}_i^\ell-1} = \mathsf{h}_i^{\ell,\boldsymbol{r}_i-1} \restriction_{H_i^{\star\ell,\boldsymbol{r}_i^\ell-1}}$. In Case III, the definitions of $\mathfrak{H}_i^{\ell,\boldsymbol{r}_i^\ell-1}$, $\mathfrak{N}_i^{\ell,\boldsymbol{r}_i^\ell-1}$, $\mathsf{h}_i^{\ell,\boldsymbol{r}_i^\ell-1}$ need to be mimicked for $k^\star$ in place of k to obtain $\mathfrak{H}_i^{\star\ell,\boldsymbol{r}_i^\ell-1}$, $\mathfrak{N}_i^{\star\ell,\boldsymbol{r}_i^\ell-1}$, $\mathsf{h}_i^{\star\ell,\boldsymbol{r}_i^\ell-1}$. It is straightforward to check now that (13)–(15), (18)–(22) hold for $\mathfrak{N} = \mathfrak{N}_i^{\star\ell,\boldsymbol{r}_i^\ell-1}$, $\mathfrak{H} = \mathfrak{H}_i^{\star\ell,\boldsymbol{r}_i^\ell-1}$, $\mathsf{h} = \mathsf{h}_i^{\star\ell,\boldsymbol{r}_i^\ell-1}$, and $I = J_i^{\ell,\boldsymbol{r}_i^\ell-1}$. Note that the size $k^\star$ of the root cluster in $\mathfrak{H}_i^{\star\ell,\boldsymbol{r}_i^\ell-1}$ is bounded by $\boldsymbol{p}_L(\varphi_1, \varphi_2)$ and every other non-degenerate cluster in it is of the form ①, so $\mathfrak{N}_i^{\star\ell,\boldsymbol{r}_i^\ell-1}$ is a simple δ-model based on an L-bounded atomic frame.

Therefore, $\mathfrak{N}_i^{\star\ell} = \mathfrak{N}_i^{\star\ell,0} \lhd \cdots \lhd \mathfrak{N}_i^{\star\ell,\boldsymbol{r}_i^\ell-1}$, for $i = 1, 2$, $\ell < N$, is the ordered sum of $\boldsymbol{n}_i^\ell$-many simple δ-models based on L-bounded atomic frames, for the same $\boldsymbol{n}_i^\ell$ as in Lemma 4.21, and so we have (b) of the lemma. Finally, by the same arguments as in the proof of Lemma 4.21, we obtain (a) and (c). ⊣

LEMMA 4.24. *For $i = 1, 2$, $\ell < N$, take the frames $\mathfrak{H}_i^\ell$ and $\mathfrak{H}_i^{\star\ell}$ provided by Lemmas 4.21 and 4.23. Let $\mathfrak{H}_i = \mathfrak{H}_i^0 \lhd \cdots \lhd \mathfrak{H}_i^{N-1}$ and $\mathfrak{H}_i^\star = \mathfrak{H}_i^{\star 0} \lhd \cdots \lhd \mathfrak{H}_i^{\star N-1}$.*

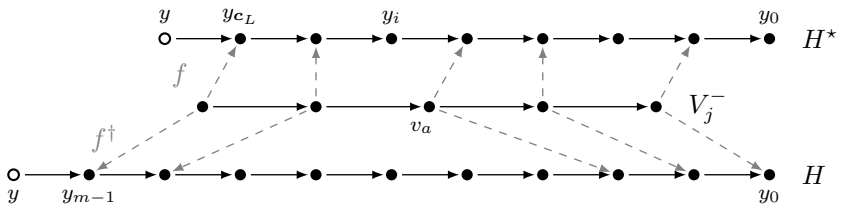
Then, for any $j \in J_L$, if there is an injection f from $\mathfrak{G}_j$ to $\mathfrak{H}_i^\star$ satisfying $(\mathbf{cf}_1)$–$(\mathbf{cf}_4)$ *for $\alpha(\mathfrak{G}_j, \mathfrak{D}_j, \bot)$, then there is an injection $f^\dagger$ from $\mathfrak{G}_j$ to $\mathfrak{H}_i$ also satisfying* $(\mathbf{cf}_1)$–$(\mathbf{cf}_4)$ *for $\alpha(\mathfrak{G}_j, \mathfrak{D}_j, \bot)$. Thus, $\mathfrak{H}_i \models L$ implies $\mathfrak{H}_i^\star \models L$.*

PROOF. Fix some $i \in \{1, 2\}$ and $j \in J_L$. Suppose that f is an injection from $\mathfrak{G}_j = (V_j, S_j)$ to $\mathfrak{H}_i^\star$ satisfying $(\mathbf{cf}_1)$–$(\mathbf{cf}_4)$ for $\alpha(\mathfrak{G}_j, \mathfrak{D}_j, \bot)$. For every atomic $\lhd$-component $\mathfrak{F}^\star = (H^\star, R_i \restriction_{H^\star})$ in $\mathfrak{H}_i^\star$ such that

1. $\mathfrak{F}^\star$ is obtained from the atomic $\lhd$-component $\mathfrak{F} = (H, R_i \restriction_H)$ in $\mathfrak{H}_i$ of the form $m^<$ or $① \lhd m^<$, and
2. there is $v \in V_j$ such that $f(v_j)$ is an irreflexive point in $\mathfrak{F}^\star$,

we proceed as follows. Suppose $H = \{y_0, \dots, y_{m-1}\}$ or $H = \{y, y_0, \dots, y_{m-1}\}$ with $yR_iyR_iy_{m-1}R_i \dots R_iy_0$, and so $H^\star = \{y_0, \dots, y_{\boldsymbol{c}_L}\}$ or $H^\star = \{y, y_0, \dots, y_{\boldsymbol{c}_L}\}$. Let $V_j^- = \big\{v \in V_j \mid f(v) \in H^\star \setminus \{y\}\big\}$. Then $|V_j^-| \leq |V_j| \leq \boldsymbol{c}_L$. Thus, by the pigeonhole principle, there is $i \leq \boldsymbol{c}_L$ with $y_i \notin f(V_i) \cap (H^\star \setminus \{y\})$. Suppose $V_j^- = \{v_0, \dots, v_{s-1}\}$, for some $s \leq \boldsymbol{c}_L$ with $v_{s-1}S_j \dots S_jv_0$. Let a be the largest $k < s$ with $y_iR_if(v_k)$. As f satisfies $(\mathbf{cf}_3)$, $v_a \notin \mathfrak{D}_j$. Now, for $k < s$, we set

$$f^\dagger(v_k) = \begin{cases} y_k, & \text{if } k \leq a, \\ y_{m-(s-k)}, & \text{if } a+1 \leq k < s. \end{cases}$$



We do this for every $\mathfrak{F}^\star$ having 1. and 2. above, and set $f^\dagger(x) = f(x)$, for any other $x \in V_j$. It is straightforward to check that the resulting $f^\dagger$ is an injection from $\mathfrak{G}_j$ to $\mathfrak{H}_i$ satisfying $(\mathbf{cf}_1)$–$(\mathbf{cf}_4)$ for $\alpha(\mathfrak{G}_j, \mathfrak{D}_j, \bot)$. ⊣

This completes the proof of Theorem 4.6. We obtain Theorem 4.7 using Lemma 4.4 as we have, for $i = 1, 2$:

$$\|\mathfrak{N}_i^\star\| = \|\mathfrak{N}_i^{\star 0}\| + \dots + \|\mathfrak{N}_i^{\star N-1}\| \leq \sum_{\ell < N} \boldsymbol{n}_i^\ell \cdot \max\big(\boldsymbol{c}_L + 2, \boldsymbol{p}_L(\varphi_1, \varphi_2)\big) \leq (3\boldsymbol{k}(\varphi_1, \varphi_2) - 1) \cdot \max\big(\boldsymbol{c}_L + 2, \boldsymbol{p}_L(\varphi_1, \varphi_2)\big).$$

4.5. Cofinal subframe logics. By Theorem 3.5 (a), all d-persistent cofinal subframe logics $L \supseteq \mathsf{K4.3}$ have the polysize bisimilar model property, with the polynomial $\boldsymbol{k}(\varphi_1, \varphi_2)$ (defined in (8)) not dependent on L. We show now that, for arbitrary, not necessarily d-persistent cofinal subframe L, it is enough to replace polysize in Theorem 3.5 (a) by quasi-polysize:

THEOREM 4.25. *All cofinal subframe logics $L \supseteq \mathsf{K4.3}$ have the quasi-polysize bisimilar model property, with the size of witnessing models bounded by $\boldsymbol{k}(\varphi_1, \varphi_2)$.*

This follows from the following special case of the 'structural' Theorem 4.5:

THEOREM 4.26. *For any cofinal subframe logic $L \supseteq \mathsf{K4.3}$ and formulas φ_1, φ_2 without an interpolant in L, there are rooted δ-models $\mathfrak{N}_1, x_1$ and $\mathfrak{N}_2, x_2$ satisfying (a)–(c) from Theorem 4.5 as well as conditions (d) and (e) below*:

(d) *there is $M \leq \boldsymbol{k}(\varphi_1, \varphi_2)$ such that $\mathfrak{N}_i = \mathfrak{N}_i^0 \lhd \cdots \lhd \mathfrak{N}_i^{M-1}$ and, for all $j < M$,*
   1. *$\mathfrak{N}_i^j$ is the ordered sum of simple δ-models based on atomic frames*;
   2. *the pair $(\mathfrak{N}_1^j, \mathfrak{N}_2^j)$ is σ-matching*;

(e) *$\{x_i\} \cup \boldsymbol{M}_i \cup \boldsymbol{S}_i$ coincides with the set of points in $\mathfrak{N}_i$, $i = 1, 2$ that are not in the $\{b_i^n \mid n < \omega\}$-part of some $\lhd$-component based on a $\mathfrak{C}(\textcircled{k}, *)$.*

It follows from (e) that $\|\mathfrak{N}_i\| = |\{x_i\} \cup \boldsymbol{M}_i \cup \boldsymbol{S}_i| \leq \boldsymbol{k}(\varphi_1, \varphi_2)$.

PROOF. As in the proof of Theorem 4.5, we take any σ-bisimilar witness models $\mathfrak{M}_i, x_i$, $i = 1, 2$, based on frames $\mathfrak{F}_i = (W_i, R_i, \mathcal{P}_i)$ for L. Let M be the number of relevant σ-blocks in $\mathfrak{M}_1$ (or $\mathfrak{M}_2$, by Lemma 4.17 (e)). For $i = 1, 2$, consider the partitions $\mathcal{I}_i = \{I_i^\ell \in \mathcal{P}_i \mid \ell < N\}$ of $\mathfrak{M}_i$ given by Definition 4.18, and let $0 = \ell_0 < \cdots < \ell_{M-1} = N - 1$ be the list of indices such that the pair $(I_1^{\ell_j}, I_2^{\ell_j})$ is added to $\mathcal{I}_1 \times \mathcal{I}_2$ in steps $(\mathbf{s}_1)$ or $(\mathbf{s}_2)$, and $I_i^{\ell_0} \prec_{\mathfrak{F}_i} \cdots \prec_{\mathfrak{F}_i} I_i^{\ell_{M-1}}$. We define $\mathfrak{N}_i^{\ell_z}$, $z < M$, by choosing *fewer* points from $I_i^{\ell_z}$ than in Cases II.2 and III in the proof of Lemma 4.21, and we also define functions $\mathsf{h}_i^{\ell_z}$. Let $\ell = \ell_z$, for $z < M$, let $C_i^{\ell,j}$, $j < \boldsymbol{r}_i^\ell$, be the sequence (ordered by $<_{R_i}$) of all relevant clusters in I_i^ℓ, and $D_i^{\ell,j} = C_i^{\ell,j} \cap \big(\{x_i\} \cup \boldsymbol{M}_i \cup \boldsymbol{S}_i\big)$. Three cases are possible now, the first of which coincides with Case II.1, while the other two select fewer points for $\mathfrak{N}_i^{\ell_z}$ than Cases II.2 and III:

(*i*) If (I_1^ℓ, I_2^ℓ) is added in step $(\mathbf{s}_1)$ and I_i^ℓ consists of a degenerate cluster, then, like in Case II.1, we let $\mathfrak{N}_i^\ell = \mathfrak{M}_i \restriction_{I_i^\ell}$ and h_i^ℓ be the identity function on $\mathfrak{N}_i^\ell$.

(*ii*) If $C_i^{\ell,\boldsymbol{r}_i^\ell - 1}$ is non-degenerate and (I_1^ℓ, I_2^ℓ) is added in step $(\mathbf{s}_1)$ as in Case II.2, then $C_i^{\ell,\boldsymbol{r}_i^\ell - 1}$ is definable in $\mathfrak{M}_i$. We let $\mathfrak{N}_i^\ell = \mathfrak{M}_i \restriction_{D_i^{\ell,0}} \lhd \cdots \lhd \mathfrak{M}_i \restriction_{D_i^{\ell,\boldsymbol{r}_i^\ell - 1}}$ and h_i^ℓ be the identity function on $\mathfrak{N}_i^\ell$.

(*iii*) If (I_1^ℓ, I_2^ℓ) is added in $(\mathbf{s}_2)$ like in Case III, then $C_i^{\ell,\boldsymbol{r}_i^\ell - 1}$ is a not definable in $\mathfrak{M}_i$. As shown in Case III, there is an infinite sequence of irrelevant points $\{b_i^n \in I_i^\ell \mid n < \omega\}$ such that $b_i^n R_i b_i^{n-1}$, $C_i^{\ell,\boldsymbol{r}_i^\ell - 1} <_{R_i} C(b_n^i)$ and $C(b_n^i) \in \mathcal{P}_i$, $n < \omega$, and the b_i^n are either 1) all irreflexive or 2) all reflexive. By Lemma 4.17, there is $k \leq \boldsymbol{k}(\varphi_1, \varphi_2)$ with $|D_1^{\ell,\boldsymbol{r}_1^\ell - 1}| = |D_2^{\ell,\boldsymbol{r}_2^\ell - 1}| = k$. Suppose $D_i^{\ell,\boldsymbol{r}_i^\ell - 1} = \{a_i^0, \dots, a_i^{k-1}\}$. We let $H_i^{\ell,\boldsymbol{r}_i^\ell - 1} = D_i^\ell \cup \{b_i^n \mid n < \omega\}$ and $\mathcal{P}_i^{\ell,\boldsymbol{r}_i^\ell - 1}$ be generated in $(H_i^{\ell,\boldsymbol{r}_i^\ell - 1}, R_i \restriction_{H_i^{\ell,\boldsymbol{r}_i^\ell - 1}})$ by the sets $\{b_i^n\}$, $n < \omega$, and $X_i^s = \{a_i^s\} \cup \{b_i^n \mid n < \omega,\ n \equiv s \pmod k\}$, $s < k$ (see Example 2.2). The resulting $\mathfrak{H}_i^{\ell,\boldsymbol{r}_i^\ell - 1}$ are both isomorphic to $\mathfrak{C}(\textcircled{k}, \bullet)$ in case 1), and to $\mathfrak{C}(\textcircled{k}, \circ)$ in case 2). We then set $\mathfrak{w}_i^{\ell,\boldsymbol{r}_i^\ell - 1}(p) = \bigcup_{a_i^s \in \mathfrak{v}_i(p)} X_i^s$ and $\mathfrak{N}_i^{\ell,\boldsymbol{r}_i^\ell - 1} = \big((H_i^{\ell,\boldsymbol{r}_i^\ell - 1}, R_i \restriction_{H_i^{\ell,\boldsymbol{r}_i^\ell - 1}}, \mathcal{P}^{\ell,\boldsymbol{r}_i^\ell - 1}), \mathfrak{w}_i^{\ell,\boldsymbol{r}_i^\ell - 1}\big)$. Finally, we set $\mathfrak{N}_i^\ell = \mathfrak{M}_i \restriction_{D_i^{\ell,0}} \lhd \cdots \lhd \mathfrak{M}_i \restriction_{D_i^{\ell,\boldsymbol{r}_i^\ell - 2}} \lhd \mathfrak{N}_i^{\ell,\boldsymbol{r}_i^\ell - 1}$ and define h_i^ℓ as the identity on relevant points in $\mathfrak{N}_i^\ell$ and $\mathsf{h}_i^\ell(b_i^n) = a_i^s$, for $n < \omega$ with $n \equiv s \pmod k$.

Clearly, (c), $(d).1$ and (e) hold for $\mathfrak{N}_i = \mathfrak{N}_i^{\ell_0} \lhd \cdots \lhd \mathfrak{N}_i^{\ell_{M-1}}$. Condition (a) is shown like in Lemma 4.22 using that (19)–(22) hold for $\mathsf{h} = \mathsf{h}_i^{\ell_z}$ and $H = H_i^{\ell_z}$, $z < M$. Condition (b) is proved via (29): $\mathbf{(cf_1)}$ clearly holds; $\mathbf{(cf_2)}$ holds as the final cluster in $\mathfrak{F}_i$ is always relevant; and $\mathbf{(cf_4)}$ holds, as $\{x\}$ being definable in $\mathfrak{N}_i^{\ell_z}$ implies $\{x\} \in \mathcal{P}_i$, for all $z < M$ and x in $\mathfrak{N}_i^{\ell_z}$. As L is a cofinal subframe logic, $\mathfrak{D} = \emptyset$, so $\mathbf{(cf_3)}$ holds vacuously. Finally, to show $(d).2$, observe that $(\mathfrak{N}_1^{\ell_z}, \mathfrak{N}_2^{\ell_z})$ is σ-matching as it always meets one of the conditions in Definition 4.3: in case (i), it meets (a); in case (ii), it meets (b); and in case (iii), it meets (c). ⊣

EXAMPLE 4.27. By Example 2.10 (a), given any formulas φ_1 and φ_2 without an interpolant in $\mathsf{GL.3}$, one can always find witnessing models $\mathfrak{N}_i$, $i = 1, 2$, of size $\leq \boldsymbol{k}(\varphi_1, \varphi_2)$ that are ordered sums of simple models based on $m^<$ or $\mathfrak{C}(ⓚ, \bullet)$ (see e.g. the models depicted in Fig. 1 in Example 3.6 (a)). ⊣

We emphasise that the construction in the proof of Theorem 4.26 does not work for non-cofinal subframe logics, in which case $\mathfrak{D} \neq \emptyset$; see also the special treatment of the density axiom in the proof of Theorem 5.9 below.

§5. The IEP for standard Priorean temporal logics. *Priorean temporal logics* [36] deal with the operators 'sometime in the future' denoted $\Diamond_\mathsf{F}$, 'sometime in the past' denoted $\Diamond_\mathsf{P}$, and their duals 'always in the future' $\Box_\mathsf{F}$ and 'always in the past' $\Box_\mathsf{P}$. *Temporal formulas*—propositional bimodal formulas with these operators—are interpreted over general *temporal frames* of the form $\mathfrak{F} = (W, R, R^-, \mathcal{P})$ representing various *flows of time* in such a way that (W, R) is transitive and connected (2), R is the 'future-time' accessibility relation for $\Diamond_\mathsf{F}$, $\Box_\mathsf{F}$, its inverse R^- is the 'past-time' accessibility relation for $\Diamond_\mathsf{P}$, $\Box_\mathsf{P}$, and the internal sets $\mathcal{P} \subseteq 2^W$ are closed under the Booleans and the operators

$$\Diamond_\mathsf{F}^\mathfrak{F} X = \{x \in W \mid \exists y \in X \, xRy\}, \quad \Diamond_\mathsf{P}^\mathfrak{F} X = \{x \in W \mid \exists y \in X \, xR^-y\}.$$

To simplify notation, we omit R^- and write $\mathfrak{F} = (W, R, \mathcal{P})$. Also, as before, if $\mathcal{P} = 2^W$, we call $\mathfrak{F}$ a *Kripke frame* and write $\mathfrak{F} = (W, R)$. The *universal modality* 'always' can be introduced as an abbreviation $\Box\varphi = \varphi \wedge \Box_\mathsf{F}\varphi \wedge \Box_\mathsf{P}\varphi$. Descriptive temporal frames are defined in the same way as in §2. Note that tightness condition **(tig)** for R^- actually follows from **(tig)** for R.

In fact, many results from §2, 3 straightforwardly generalise to the temporal setting. Let $\mathfrak{M}$ be a *temporal model*—that is, a model based on some temporal frame $\mathfrak{F} = (W, R, \mathcal{P})$—and let Γ be a set of temporal formulas. A point $x \in W$ is Γ-*minimal in* $\mathfrak{M}$ if $\mathfrak{M}, x \models \Gamma$ and whenever $x'Rx$ and $\mathfrak{M}, x' \models \Gamma$, then xRx'. Denote by $\min_\mathfrak{M} \Gamma$ the set of all Γ-minimal points in $\mathfrak{M}$. (The definition of $\max_\mathfrak{M} \Gamma$ remains the same.) In the temporal case, Lemma 2.3 generalises to

LEMMA 5.1. *Suppose* Γ *is a set of temporal formulas and* $\mathfrak{M}$ *a model based on a descriptive temporal frame* $\mathfrak{F} = (W, R, \mathcal{P})$. *Then the following hold*:

(temporal saturation) *If* $\mathfrak{M}, x \models \Diamond_\mathsf{F} \bigwedge \Gamma'$ *for every finite* $\Gamma' \subseteq \Gamma$, *then there is* y *with* xRy *and* $\mathfrak{M}, y \models \Gamma$. *If* $\mathfrak{M}, x \models \Diamond_\mathsf{P} \bigwedge \Gamma'$ *for every finite* $\Gamma' \subseteq \Gamma$, *then there is* y *with* xR^-y *and* $\mathfrak{M}, y \models \Gamma$.

(maximal and minimal points) *If there is* x *with* $\mathfrak{M}, x \models \Gamma$, *then* $\max_\mathfrak{M} \Gamma \neq \emptyset$ *and* $\min_\mathfrak{M} \Gamma \neq \emptyset$.

A relation $\boldsymbol{\beta} \subseteq W_1 \times W_2$ is a *temporal σ-bisimulation* between temporal models $\mathfrak{M}_1$ and $\mathfrak{M}_2$ based on respective frames $\mathfrak{F}_i = (W_i, R_i, \mathcal{P}_i)$, $i = 1, 2$, if it satisfies **(atom)**, **(move)** and its past-time couterpart: whenever $x_1 \boldsymbol{\beta} x_2$, then

(move$^-$) $x_1 R_1^- y_1$ implies $y_1 \boldsymbol{\beta} y_2$, for some $y_2 \in W_2$ with $x_2 R_2^- y_2$; conversely, $x_2 R_2^- y_2$ implies $y_1 \boldsymbol{\beta} y_2$, for some $y_1 \in W_1$ with $x_1 R_1^- y_1$.

The relation $\mathfrak{M}_1, x_1 \equiv_\sigma \mathfrak{M}_2, x_2$, saying that temporal models $\mathfrak{M}_1$ and $\mathfrak{M}_2$ satisfy the same temporal σ-formulas at x_1 and x_2, respectively, is characterised in terms of temporal σ-bisimulations: it is readily seen that, with this modification, Lemma 3.1 and Theorem 3.2 continue to hold for all Priorean temporal logics. (As temporal frames are transitive and connected, any of their points can be regarded as a root with respect to the relation $R \cup R^-$.)

In this article, we consider the Priorean temporal logics of five most popular classes of temporal Kripke frames [7]:

$$
\begin{aligned}
&\mathsf{Lin} = \{\varphi \mid \mathfrak{F} \models \varphi,\ \mathfrak{F} = (W, R) \text{ is any temporal Kripke frame}\}\\
&\quad = \mathsf{K4}_2 \oplus p \to \Box_\mathsf{F}\Diamond_\mathsf{P} p \oplus p \to \Box_\mathsf{P}\Diamond_\mathsf{F} p \oplus \Diamond_\mathsf{F}\Diamond_\mathsf{P} p \vee \Diamond_\mathsf{P}\Diamond_\mathsf{F} p \to p \vee \Diamond_\mathsf{F} p \vee \Diamond_\mathsf{P} p;\\
&\mathsf{Lin}_\mathbb{Q} = \{\varphi \mid (\mathbb{Q}, <) \models \varphi\}\\
&\quad = \mathsf{Lin} \oplus \Diamond_\mathsf{F}\top \oplus \Diamond_\mathsf{P}\top \oplus \Diamond_\mathsf{F} p \to \Diamond_\mathsf{F}\Diamond_\mathsf{F} p;\\
&\mathsf{Lin}_\mathbb{R} = \{\varphi \mid (\mathbb{R}, <) \models \varphi\}\\
&\quad = \mathsf{Lin}_\mathbb{Q} \oplus \Box(\Box_\mathsf{P} p \to \Diamond_\mathsf{F}\Box_\mathsf{P} p) \to (\Box_\mathsf{P} p \to \Box_\mathsf{F} p);\\
&\mathsf{Lin}_{<\omega} = \{\varphi \mid \mathfrak{F} \models \varphi,\ \mathfrak{F} = (W, <) \text{ any } \textit{finite} \text{ strict linear order}\}\\
&\quad = \mathsf{Lin} \oplus \big\{\Box_\mathsf{X}(\Box_\mathsf{X} p \to p) \to \Box_\mathsf{X} p \mid \mathsf{X} \in \{\mathsf{F}, \mathsf{P}\}\big\};\\
&\mathsf{Lin}_\mathbb{Z} = \{\varphi \mid (\mathbb{Z}, <) \models \varphi\}\\
&\quad = \mathsf{Lin} \oplus \Diamond_\mathsf{F}\top \oplus \Diamond_\mathsf{P}\top \oplus \big\{\Box_\mathsf{X}(\Box_\mathsf{X} p \to p) \to (\Diamond_\mathsf{X}\Box_\mathsf{X} p \to \Box_\mathsf{X} p) \mid \mathsf{X} \in \{\mathsf{F}, \mathsf{P}\}\big\},
\end{aligned}
$$

where $\mathsf{K4}_2$ is the bimodal version of $\mathsf{K4}$ (with $\Diamond_\mathsf{F}$ and $\Diamond_\mathsf{P}$). None of these five logics (and any other temporal logic with frames of unbounded depth) has the CIP [14, 9], and our aim in this section is to prove that the IEP for each of them is decidable in coNP. The following example illustrates the new semantic phenomena of temporal logics compared to modal logics containing $\mathsf{K4.3}$ that we need to address in order to achieve this aim.

Example 5.2. (*a*) Consider the formulas φ_1 and φ_2 from Example 3.6 (*a*) in the context of $\mathsf{Lin}_{<\omega}$ in place of $\mathsf{GL.3}$, reading $\Diamond$ as $\Diamond_\mathsf{F}$ and $\Box$ as $\Box_\mathsf{F}$:

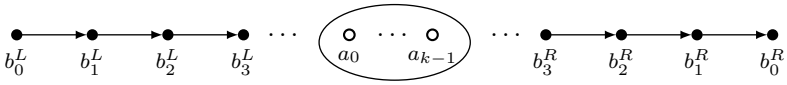
$$
\begin{aligned}
\varphi_1 &= \Diamond_\mathsf{F}(p_1 \wedge \Diamond_\mathsf{F}^+ \neg q_1) \wedge \Box_\mathsf{F}(p_2 \to \Box_\mathsf{F}^+ q_1) \wedge \Box_\mathsf{F}(p_1 \to \neg p_2),\\
\varphi_2 &= \neg[\Diamond_\mathsf{F}(p_2 \wedge \Diamond_\mathsf{F}^+ \neg q_2) \wedge \Box_\mathsf{F}(p_1 \to \Box_\mathsf{F}^+ q_2)].
\end{aligned}
$$

We clearly have $(\varphi_1 \to \varphi_2) \in \mathsf{Lin}_{<\omega}$. Using Theorem 3.2, we show that φ_1 and φ_2 have no interpolant in $\mathsf{Lin}_{<\omega}$. The argument from Example 3.6 (*a*) shows that any models $\mathfrak{M}_i$ meeting the criterion of Theorem 3.2 cannot be based on a Kripke frame for $\mathsf{Lin}_{<\omega}$. However, the descriptive frame $\bullet \lhd \bullet \lhd \mathfrak{C}(\textcircled{2}, \bullet)$ we employed for $\mathsf{GL.3}$ in Example 3.6 (*a*) does not help now, because it refutes $\Box_\mathsf{P}(\Box_\mathsf{P} p \to p) \to \Box_\mathsf{P} p$ at any point save the first two under the valuation below:

To fix this issue, we modify $\mathfrak{C}(②, \bullet)$ by making it symmetric in both directions. Consider the frame $\mathfrak{F}_k = (W'_k, R_{\bullet k \bullet}, \mathcal{P}'_k)$, $k > 0$, in which the points in

$$W'_k = \{a_0, \dots, a_{k-1}\} \cup \{b^L_n, b^R_n \mid n < \omega\}$$

are ordered as shown in the picture below



or, more formally, $xR_{\bullet k \bullet}y$ iff ($x = b^L_n$, $y = b^L_m$ for $n < m$), or ($x = b^L_n$, $y = a_i$), or ($x = b^L_n$, $y = b^R_m$), or ($x = a_i$, $y = a_j$) or ($x = a_i$, $y = b^R_n$), or ($x = b^R_n$, $y = b^R_m$, for $n > m$). The internal sets in $\mathcal{P}_k$ are generated by

$$X_i = \{a_i\} \cup \{b^L_n, b^R_n \mid n < \omega,\ n \equiv i \pmod k\}, \quad \text{for } i < k. \tag{30}$$

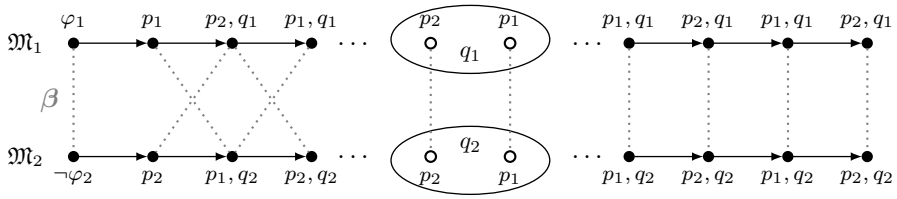
Observe that $\{b^L_n\}, \{b^R_n\} \in \mathcal{P}'_k$, for all $n < \omega$. It is not hard to see that $\mathfrak{F}_k$ is a descriptive frame; we denote it by $\mathfrak{C}(\bullet, ⓚ, \bullet)$. As an exercise, the reader can check that, for any natural numbers $k, l, \dots, m, n > 0$,

$$\mathfrak{C}(\bullet, ⓚ, \bullet) \lhd \cdots \lhd \mathfrak{C}(\bullet, ⓝ, \bullet) \models \mathsf{Lin}_{<\omega}, \tag{31}$$

$$\mathfrak{C}(ⓚ, \bullet) \lhd \mathfrak{C}(\bullet, ①, \bullet) \lhd \cdots \lhd \mathfrak{C}(\bullet, ⓜ, \bullet) \lhd \mathfrak{C}(\bullet, ⓝ) \models \mathsf{Lin}_{\mathbb{Z}}, \tag{32}$$

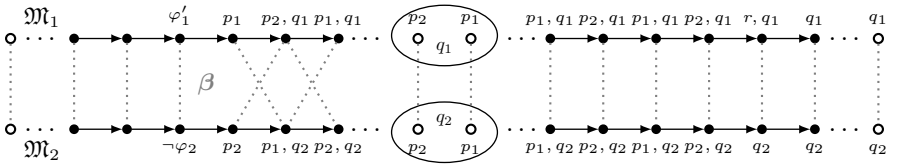
where $\mathfrak{C}(\bullet, ⓝ)$ is the mirror image of $\mathfrak{C}(ⓝ, \bullet)$; see also Lemma 5.6.

The picture below shows models $\mathfrak{M}_1$ and $\mathfrak{M}_2$ based on $\mathfrak{C}(\bullet, ②, \bullet)$ and satisfying the conditions of Theorem 3.2 for φ_1 and φ_2:



By (31), $\mathfrak{C}(\bullet, ②, \bullet) \models \mathsf{Lin}_{<\omega}$, so φ_1 and φ_2 do not have an interpolant in $\mathsf{Lin}_{<\omega}$.

(*b*) Consider next the temporal version of the implication $\varphi'_1 \to \varphi_2$ from Example 3.6 (*b*), which is clearly valid in $\mathsf{Lin}_{\mathbb{Z}}$. To demonstrate that φ'_1 and φ_2 have no interpolant in $\mathsf{Lin}_{\mathbb{Z}}$, we can use $\mathfrak{C}(①, \bullet) \lhd \mathfrak{C}(\bullet, ②, \bullet) \lhd \mathfrak{C}(\bullet, ①)$, which is a frame for $\mathsf{Lin}_{\mathbb{Z}}$ by (32). The models $\mathfrak{M}_1$ and $\mathfrak{M}_2$ depicted below



satisfy the conditions of Theorem 3.2 for φ'_1 and φ_2. ⊣

As illustrated by Example 5.2, the temporal frames $\mathfrak{F} = (W, R, \mathcal{P})$ we need for checking the criterion of Theorem 3.2 may contain both infinite descending and ascending chains of clusters (and so the $\mathfrak{F}_c^{-1}$ are not necessarily isomorphic to ordinals). Accordingly, we now have *R-final* and *R^--final clusters* as well as two types of limit clusters: an *R-limit cluster* is a non-R^--final cluster without

an immediate R^--successor; an *R^--limit cluster* is a non-R-final cluster without an immediate R-successor. Some clusters can be both R- and R^--limit clusters.

We say that a set $S \neq \emptyset$ of clusters in $\mathfrak{F}$ is *R-unbounded* (*R^--unbounded*) if there is no $C \in S$ such that $C' \leq_R C$ (respectively, $C \leq_R C'$), for all $C' \in S$. A cluster C is the *R-limit of* an R-unbounded set S if $C' <_R C$ for all $C' \in S$ and there is no cluster C'' with $C' <_R C'' <_R C$ for all $C' \in S$; the *R^--limit of* an R^--unbounded set S is defined symmetrically by replacing R with R^-. It is straightforward to see that each R-limit cluster C is the R-limit of the R-unbounded set $\{C' \mid C' <_R C\}$, and each R^--limit cluster D is the R^--limit of the R^--unbounded set $\{D' \mid D <_R D'\}$. For any cluster C, we let $(C, +\infty) = \{x \mid C <_R C(x)\}$ and $(-\infty, C) = \{x \mid C(x) <_R C\}$.

LEMMA 5.3. *Suppose $\mathfrak{F} = (W, R, \mathcal{P})$ is a temporal n-generated descriptive frame, for some $n < \omega$. Then*

(*a*) *every cluster in $\mathfrak{F}$ has at most 2^n points*;
(*b*) *every R-unbounded (R^--unbounded) set of clusters in $\mathfrak{F}$ has an R-limit (R^--limit) in $\mathfrak{F}$, and so $\mathfrak{F}$ contains both R- and R^--final clusters.*

PROOF. (*a*) is proved similarly to Lemma 2.4 (*b*).

(*b*) Suppose $\mathfrak{F}$ is $\mathfrak{M}$-generated, for some model $\mathfrak{M}$. Let S be an R-unbounded set of clusters in $\mathfrak{F}$ with $y_C \in C$, $C \in S$, and let

$$\Gamma = \bigcup_{C \in S} \Diamond_{\mathsf{P}} t_{\mathfrak{M}}(y_C) \cup \bigcup_{C \in S} \{\psi \mid \Box_{\mathsf{F}} \psi \in t_{\mathfrak{M}}(y_C)\}.$$

Clearly, Γ is finitely satisfiable in $\mathfrak{M}$, and so by **(com)** and Lemma 5.1, there is a Γ-minimal point x in $\mathfrak{M}$. By **(tig)**, $y_C R x$ for all $C \in S$. Now suppose that y is such that $y_C R y$, for all $C \in S$, and yRx. Then $\Gamma \subseteq t_{\mathfrak{M}}(y)$, and so xRy by the Γ-minimality of x. Thus, $C(x)$ is the R-limit of S. The existence of R^--limits of R^--unbounded S is symmetric. ⊣

A cluster C is called *minimal* (*maximal*) in a temporal model $\mathfrak{M}$ if there is a formula μ such that $C \cap \min_{\mathfrak{M}}\{\mu\} \neq \emptyset$ ($C \cap \max_{\mathfrak{M}}\{\mu\} \neq \emptyset$). If there is such a σ-formula μ, for some signature σ, we call C *σ-minimal* (*σ-maximal*) *in* $\mathfrak{M}$.

LEMMA 5.4. *Suppose $\mathfrak{M}$ is a model based on a finitely $\mathfrak{M}$-generated temporal descriptive frame $\mathfrak{F}$. Then*

(*a*) *every degenerate cluster in $\mathfrak{F}$ is both maximal and minimal in $\mathfrak{M}$*;
(*b*) *a cluster is maximal* (*minimal*) *in $\mathfrak{M}$ iff either it is R-final* (*respectively, R^--final*) *or has an immediate R-successor* (*respectively, R^--successor*);
(*c*) *a cluster is definable in $\mathfrak{M}$ iff it is both maximal and minimal in $\mathfrak{M}$.*

It follows that the R- and R^--limit clusters are not definable and not degenerate; *all other clusters are definable in $\mathfrak{M}$. We also have that*

(*d*) *for any clusters $C <_R C'$ in $\mathfrak{F}$, the interval $[C, C']$ contains a maximal cluster and also a minimal one*;
(*e*) *if C is not an R-limit cluster and C' is not an R^--limit cluster, then the closed interval $[C, C']$ is definable in $\mathfrak{M}$.*

PROOF. Items (*a*)–(*c*) are proved in the same way as Lemma 2.6. Item (*d*) follows from **(tig)**, which gives formulas φ and ψ with $\mathfrak{M}, x \not\models \Box_{\mathsf{F}}\varphi$, $\mathfrak{M}, y \models \Box_{\mathsf{F}}\varphi$

and $\mathfrak{M}, x \models \Box_{\mathsf{P}}\psi$, $\mathfrak{M}, y \not\models \Box_{\mathsf{P}}\psi$, and so $[C(x), C(y)]$ contains a $\Box_{\mathsf{F}}\varphi$-minimal cluster and a $\Box_{\mathsf{P}}\psi$-maximal one. Item (e): by (b), C is λ-minimal and C' is μ-maximal for some λ, μ. Then $[C, C']$ is defined in $\mathfrak{M}$ by $\Diamond^+_{\mathsf{P}}\lambda \wedge \Diamond^+_{\mathsf{F}}\mu$. $\dashv$

The following temporal analogue is harder to prove than Lemma 2.7:

LEMMA 5.5. *If $\mathfrak{F} = (W, R, \mathcal{P})$ is a finitely generated temporal descriptive frame, then W is countable.*

PROOF. By Lemma 5.3 (a), it suffices to show that $\mathfrak{F}_c = (W_c, <_R)$ is countable. Suppose $\mathfrak{F}$ is $\mathfrak{M}$-generated, for some δ-model $\mathfrak{M} = (\mathfrak{F}, \mathfrak{v})$ and finite signature δ. First, observe that, by Lemma 5.4 (b), each non-R-limit cluster C is μ_C-minimal in $\mathfrak{M}$ for some μ_C. Thus, the internal set $X_C = \mathfrak{v}(\Diamond^+_{\mathsf{P}}\mu_C)$ distinguishes C from every D with $D <_R C$, and so $X_C \neq X_D$ whenever $C \neq D$. As $\mathcal{P}$ is countable, the number of non-R-limit clusters in $\mathfrak{F}_c$ is countable. Similarly, there are countably-many non-R^--limit clusters in $\mathfrak{F}_c$. So it is enough to show that the number of clusters in $\mathfrak{F}_c$ that are both R- and R^--limits is countable. We refer to such clusters as simply *limit clusters*. Call an interval $[C^-, C^+]$ a *neighbourhood* of a limit cluster C if $C^- <_R C <_R C^+$. By Lemma 5.4, every limit cluster C has a *nice* neighbourhood $N_C = [C^-, C^+]$ with non-limit clusters C^- and C^+. As the number of different nice N_C is countable, it follows that

(33) every uncountable interval $[D, D']$ contains a limit cluster C all of whose neighbourhoods are uncountable

(otherwise all limit clusters in $[D, D']$ would belong to the countable union of the countable intervals N_C, and so $[D, D']$ were countable).

By an *atomic type* we mean any $at^{\delta}_{\mathfrak{M}}(x)$ with $x \in W$. For any cluster C, we set $at(C) = \{at^{\delta}_{\mathfrak{M}}(x) \mid x \in C\}$. Let C be an R-limit cluster. We say that an atomic type a *occurs infinitely R-close to* C if, for every $C' <_R C$, there is C'' such that $C' <_R C'' <_R C$ and $a \in at(C'')$. Similarly, a *occurs infinitely R^--close to* an R^--limit cluster C if whenever $C <_R C'$, then there is C'' such that $C <_R C'' <_R C'$ and $a \in at(C'')$. We claim that

(34) if a occurs infinitely R-close to an R-limit cluster C, then $a \in at(C)$.

Indeed, let S be an R-unbounded set of clusters with R-limit C and $y_D \in D$, $D \in S$, and let

$$\Gamma_a = a \cup \bigcup_{D \in S} \Diamond_{\mathsf{P}} t_{\mathfrak{M}}(y_D) \cup \bigcup_{D \in S} \{\psi \mid \Box_{\mathsf{F}}\psi \in t_{\mathfrak{M}}(y_D)\}.$$

If a occurs infinitely R-close to C, it can be shown similarly to the proof of Lemma 5.3 (b) that there is a Γ_a-minimal point $x \in C$, so $a = at^{\delta}_{\mathfrak{M}}(x) \in at(C)$.

The converse of (34) also holds:

(35) if $a \in at(C)$, for an R-limit C, then a occurs infinitely R-close to C.

Indeed, suppose there is $C' <_R C$ with $a \notin at(C'')$, for any C'' in the interval $C' <_R C'' <_R C$. By Lemma 5.4 (d), there is a cluster C'' in $[C', C]$ that is μ-minimal in $\mathfrak{M}$ for some formula μ. But then C is $\Diamond_{\mathsf{P}}\mu \wedge \bigwedge a$-minimal, contrary to Lemma 5.4 (b). Symmetric variants of (34) and (35) hold for R^--limit clusters.

Call non-degenerate clusters $C' <_R C''$ *twins* if $at(C') = at(C'')$ and, for every C in $[C', C'']$, we have $at(C) \subseteq at(C') = at(C'')$. We claim that

(36) there are no twins.

Indeed, suppose C', C'' are twins. By induction on the construction of a δ-formula α, we see that if $x, y \in [C', C'']$ with xRy and $at^{\delta}_{\mathfrak{M}}(x) = at^{\delta}_{\mathfrak{M}}(y)$, then $\mathfrak{M}, x \models \alpha$ iff $\mathfrak{M}, y \models \alpha$. We only consider one of the nontrivial cases. Let $\mathfrak{M}, x \models \Diamond_{\mathsf{F}}\alpha$. Then there is z with xRz and $\mathfrak{M}, z \models \alpha$. If yRz, then clearly $\mathfrak{M}, y \models \Diamond_{\mathsf{F}}\alpha$. Otherwise, $z \in [C', C'']$, so $at^{\delta}_{\mathfrak{M}}(z) = at^{\delta}_{\mathfrak{M}}(z')$, for some $z' \in C''$. Thus, by IH, $\mathfrak{M}, z' \models \alpha$, which implies $\mathfrak{M}, y \models \Diamond_{\mathsf{F}}\alpha$ as C'' is non-degenerate. It follows that there are $x \in C'$ and $y \in C''$ with $t_{\mathfrak{M}}(x) = t_{\mathfrak{M}}(y)$, contrary to **(dif)**.

We can now prove that $\mathfrak{F}_c$ is countable. Suppose $\mathfrak{F}_c$ is uncountable. By (33) and Lemma 5.3 (b), $\mathfrak{F}_c$ contains a limit cluster C whose neighbourhoods are all uncountable. Let C be such a cluster with a minimal $at(C)$. As δ is finite, C has a neighbourhood N such that, for any $D \in N$ with $D <_R C$, every $a \in at(D)$ occurs infinitely R-close to C, and, for any $D \in N$ with $C <_R D$, every $a \in at(D)$ occurs infinitely R^--close to C. We call such N a *close proximity of* C. As N is uncountable, either $[C^-, C)$ or $(C, C^+]$ is uncountable. We only consider the former case, as the latter is similar. We claim that

(37) for every cluster C' in $[C^-, C)$, the interval $[C^-, C']$ is countable.

Indeed, take such C'. As $[C^-, C']$ is contained in the close proximity N, for every limit cluster D in $[C^-, C']$, we have $at(D) \subsetneq at(C)$, by (34) and (36). So by the $at(C)$-minimality of C among limit clusters with only uncountable neighbourhoods, every limit cluster D in $[C^-, C']$ has a countable neighbourhood. Thus, $[C^-, C']$ is countable by (33).

By (35), there is a countably infinite ascending chain $C_1 <_R C_2 <_R \dots$ of clusters in $[C^-, C)$ such that, for every $a \in at(C)$ and every $n < \omega$, there is m with $n < m < \omega$ and $a \in at(C_m)$. Let C' be the R-limit of the R-unbounded set $\{C_n \mid n < \omega\}$ (which exists by Lemma 5.3 (b)). Then $C' \leq_R C$. Also, every $a \in at(C)$ occurs infinitely R-close to C', and so $at(C) \subseteq at(C')$ by (34). We cannot have $C' <_R C$ since otherwise (as C' belongs to the close proximity N of C) every $a \in at(C')$ occurred infinitely R-close to C, resulting in $at(C) = at(C')$ by (34), and so C' and C were twins, contrary to (36). It follows that $C' = C$, and so $[C^-, C) = \bigcup_{n<\omega}[C^-, C_n]$. As each $[C^-, C_n]$ is countable by (37), $[C^-, C)$ is also countable, which is a contradiction. ⊣

Using Lemmas 5.3 and 5.4, we can also obtain elegant characterisations of descriptive frames for $\mathsf{Lin}_{\mathbb{Q}}$, $\mathsf{Lin}_{\mathbb{R}}$, $\mathsf{Lin}_{<\omega}$ and $\mathsf{Lin}_{\mathbb{Z}}$ (cf. [8, 19, 37, 40]):

LEMMA 5.6. *Let $\mathfrak{F} = (W, R, \mathcal{P})$ be any finitely generated temporal descriptive frame. Then*

$\mathsf{Lin}_{\mathbb{Q}}$**:** $\mathfrak{F} \models \mathsf{Lin}_{\mathbb{Q}}$ *iff* $\mathfrak{F}$ *is* serial *in both directions—i.e., the R- and R^--final clusters in $\mathfrak{F}$ are both non-degenerate, and* $\mathfrak{F}$ *is* dense*—i.e., there is a non-degenerate cluster between any two distinct degenerate ones*;

$\mathsf{Lin}_{\mathbb{R}}$**:** $\mathfrak{F} \models \mathsf{Lin}_{\mathbb{R}}$ *iff* $\mathfrak{F}$ *is serial, dense, and Dedekind-complete in the sense that there is a degenerate cluster between any two distinct non-degenerate ones*;

$\mathsf{Lin}_{<\omega}$: $\mathfrak{F} \models \mathsf{Lin}_{<\omega}$ *iff* $\mathfrak{F}$ *does not contain a non-degenerate cluster* C *such that* $(-\infty, C) \in \mathcal{P}$ *or* $(C, +\infty) \in \mathcal{P}$ (*in particular, the* R- *and* R^--*final clusters in* $\mathfrak{F}$ *are degenerate*);

$\mathsf{Lin}_{\mathbb{Z}}$: $\mathfrak{F} \models \mathsf{Lin}_{\mathbb{Z}}$ *iff* $\mathfrak{F}$ *is serial and does not contain a non-degenerate cluster* C *with* $\emptyset \neq (-\infty, C) \in \mathcal{P}$ *or* $\emptyset \neq (C, +\infty) \in \mathcal{P}$ (*a single non-degenerate cluster is a frame for* $\mathsf{Lin}_{\mathbb{Z}}$ *but not for* $\mathsf{Lin}_{<\omega}$).

PROOF. We only show the ($\Rightarrow$)-directions, leaving the converses to the reader. Suppose $\mathfrak{F}$ is $\mathfrak{M}$-generated, for some model $\mathfrak{M} = (\mathfrak{F}, \mathfrak{v})$.

$\mathsf{Lin}_{\mathbb{Q}}$: As $\mathfrak{F} \models \Diamond_{\mathsf{F}}\top$ ($\mathfrak{F} \models \Diamond_{\mathsf{P}}\top$), Lemma 5.1 gives a $\{\Diamond_{\mathsf{F}}\top\}$-maximal ($\{\Diamond_{\mathsf{P}}\top\}$-minimal) point x in $\mathfrak{M}$ with R-final (R^--final) and non-degenerate $C(x)$. Thus, $\mathfrak{F}$ is serial. Suppose $\{x\}$, $\{y\}$ are degenerate clusters with xRy. Lemma 5.4 gives formulas ψ_x and ψ_y defining $\{x\}$ and $\{y\}$ in $\mathfrak{M}$. As $\mathfrak{M}, x \models \Diamond_{\mathsf{F}}\psi_y$ and $\mathfrak{F} \models \Diamond_{\mathsf{F}}\psi_y \to \Diamond_{\mathsf{F}}\Diamond_{\mathsf{F}}\psi_y$, the formula $\Diamond_{\mathsf{F}}\psi_y \wedge \Diamond_{\mathsf{P}}\psi_x$ is satisfiable in $\mathfrak{M}$. Let z be $\{\Diamond_{\mathsf{F}}\psi_y \wedge \Diamond_{\mathsf{P}}\psi_x\}$-maximal in $\mathfrak{M}$. Then $xRzRy$. As $\mathfrak{M}, z \models \Diamond_{\mathsf{F}}(\Diamond_{\mathsf{F}}\psi_y \wedge \Diamond_{\mathsf{P}}\psi_x)$ by $\mathfrak{F} \models \Diamond_{\mathsf{F}}\psi_y \to \Diamond_{\mathsf{F}}\Diamond_{\mathsf{F}}\psi_y$, the cluster $C(z)$ is non-degenerate.

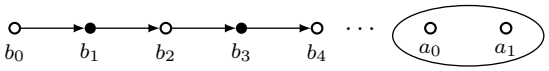
$\mathsf{Lin}_{\mathbb{R}}$: Non-degenerate $C(x) <_R C(y)$ cannot be $<_R$-consecutive because otherwise, by Lemma 5.4, $C(x)$ were ψ-maximal in $\mathfrak{M}$ for some formula ψ, and so $\mathfrak{M}, x \not\models \Box(\Box_{\mathsf{P}}\Diamond_{\mathsf{F}}\psi \to \Diamond_{\mathsf{F}}\Box_{\mathsf{P}}\Diamond_{\mathsf{F}}\psi) \to (\Box_{\mathsf{P}}\Diamond_{\mathsf{F}}\psi \to \Box_{\mathsf{F}}\Diamond_{\mathsf{F}}\psi)$, contrary to $\mathfrak{F} \models \mathsf{Lin}_{\mathbb{R}}$. Thus, there is z with $C(x) <_R C(z) <_R C(y)$. If z is irreflexive, we are done. Otherwise, by **(tig)**, there is some formula χ with $\Box_{\mathsf{F}}\chi \in t_{\mathfrak{M}}(y)$ and $\chi \notin t_{\mathfrak{M}}(z)$, and so $\mathfrak{M}, z \models \Diamond_{\mathsf{F}}\neg\chi$. Let z' be a $\Diamond_{\mathsf{F}}\neg\chi$-maximal point in $\mathfrak{M}$. Clearly, $C(x) <_R C(z') <_R C(y)$. If z' is irreflexive, we are done. Otherwise, we take the immediate R-successor z'' of z', which exists by Lemma 5.4. As $\mathfrak{M}, z'' \models \Box_{\mathsf{F}}\chi \wedge \neg\chi$, point z'' is irreflexive and $C(z'') <_R C(y)$.

$\mathsf{Lin}_{<\omega}$: If there existed a non-degenerate cluster $C(x)$ and a formula ψ with $(-\infty, C(x)) = \mathfrak{v}(\psi)$, then $\mathfrak{M}, x \not\models \Box_{\mathsf{P}}(\Box_{\mathsf{P}}\psi \to \psi) \to \Box_{\mathsf{P}}\psi$, contrary to $\mathfrak{F} \models \mathsf{Lin}_{<\omega}$.

$\mathsf{Lin}_{\mathbb{Z}}$: If there existed a non-degenerate cluster $C(x)$ and some formula ψ with $\emptyset \neq (C(x), +\infty) = \mathfrak{v}(\psi)$, then $\mathfrak{M}, x \not\models \Box_{\mathsf{F}}(\Box_{\mathsf{F}}\psi \to \psi) \to (\Diamond_{\mathsf{F}}\Box_{\mathsf{F}}\psi \to \Box_{\mathsf{F}}\psi)$, contrary to $\mathfrak{F} \models \mathsf{Lin}_{\mathbb{Z}}$. ⊣

Note that Lin and $\mathsf{Lin}_{\mathbb{Q}}$ are d-persistent while the other three logics are not [37].

EXAMPLE 5.7. The descriptive frame $\mathfrak{F} = (W_2, R_{\circ\bullet}, \mathcal{P}_2)$ with $(W_2, R_{\circ\bullet})$ depicted below and $\mathcal{P}_2$ defined in Example 2.2 is serial, dense and Dedekind-complete, so $\mathfrak{F} \models \mathsf{Lin}_{\mathbb{R}}$.



It is readily seen, however, that $(W_2, R_{\circ\bullet}) \not\models \mathsf{Lin}_{\mathbb{R}}$, so $\mathsf{Lin}_{\mathbb{R}}$ is not d-persistent. ⊣

The notion of σ-block from §4.2 also needs a modification for temporal models. Namely, a set $\boldsymbol{b} \subseteq W$ is a σ-*block* in a temporal model $\mathfrak{M}$ based on $\mathfrak{F} = (W, R, \mathcal{P})$ if $\boldsymbol{b} = \boldsymbol{b}^{\sigma}_{\mathfrak{M}}(x)$, for some $x \in W$, where

$$\boldsymbol{b}^{\sigma}_{\mathfrak{M}}(x) = \big\{y \in W \mid \Diamond_{\mathsf{X}} t^{\sigma}_{\mathfrak{M}}(y) \subseteq t^{\sigma}_{\mathfrak{M}}(x) \;\&\; \Diamond_{\mathsf{X}} t^{\sigma}_{\mathfrak{M}}(x) \subseteq t^{\sigma}_{\mathfrak{M}}(y), \text{ for } \mathsf{X} \in \{\mathsf{F}, \mathsf{P}\}\big\},$$

if both $\Diamond_{\mathsf{F}} t^{\sigma}_{\mathfrak{M}}(x) \subseteq t^{\sigma}_{\mathfrak{M}}(x)$ and $\Diamond_{\mathsf{P}} t^{\sigma}_{\mathfrak{M}}(x) \subseteq t^{\sigma}_{\mathfrak{M}}(x)$ hold; otherwise $\boldsymbol{b}^{\sigma}_{\mathfrak{M}}(x) = \{x\}$. Then we have the following temporal analogue of Lemma 4.13:

LEMMA 5.8. *Suppose $\mathfrak{M}$ is a model based on a finitely $\mathfrak{M}$-generated temporal descriptive frame $\mathfrak{F} = (W, R, \mathcal{P})$. Then, for any σ-block $\boldsymbol{b}$ in $\mathfrak{M}$, there exist clusters $C^-_{\boldsymbol{b}}$ and $C^+_{\boldsymbol{b}}$ in $\mathfrak{F}$ such that the following hold*:

(*a*) $\boldsymbol{b} = \big[C^-_{\boldsymbol{b}}, C^+_{\boldsymbol{b}}\big]$;
(*b*) *if cluster $C^-_{\boldsymbol{b}}$ (cluster $C^+_{\boldsymbol{b}}$) is minimal (respectively, maximal) in $\mathfrak{M}$, then it is σ-minimal (respectively, σ-maximal) in $\mathfrak{M}$*;
(*c*) *if $\boldsymbol{b}$ is non-degenerate, then both $C^-_{\boldsymbol{b}}$ and $C^+_{\boldsymbol{b}}$ are non-degenerate*;
(*d*) *$\boldsymbol{b}$ is definable in $\mathfrak{M}$ iff $C^-_{\boldsymbol{b}}$ is not an R-limit cluster and $C^+_{\boldsymbol{b}}$ is not an R^--limit cluster*;
(*e*) $t^\sigma_{\mathfrak{M}}(\boldsymbol{b}) = t^\sigma_{\mathfrak{M}}\big(C^-_{\boldsymbol{b}}\big) = t^\sigma_{\mathfrak{M}}\big(C^+_{\boldsymbol{b}}\big)$.

PROOF. This can be proved similarly to Lemma 4.13, using Lemmas 5.3, 5.4, and 5.1, in place of Lemmas 2.4, 2.6, and 2.3, respectively. ⊣

Given σ-bisimilar models $\mathfrak{M}_i$, $i = 1, 2$, based on finitely $\mathfrak{M}_i$-generated temporal frames, we can adapt Lemma 4.15 to the temporal setting to show that σ-blocks in $\mathfrak{M}_1$ and $\mathfrak{M}_2$ always come in σ-bisimilar pairs $\boldsymbol{b}$, $\boldsymbol{\beta}(\boldsymbol{b})$. Being equipped with these modifications, we show first how to extend the selection procedure from the proof of Theorem 3.5 to Lin, $\mathsf{Lin}_{\mathbb{Q}}$ and $\mathsf{Lin}_{\mathbb{R}}$.

THEOREM 5.9. *Each $L \in \{\mathsf{Lin}, \mathsf{Lin}_{\mathbb{Q}}, \mathsf{Lin}_{\mathbb{R}}\}$ has the polysize bisimilar model property, and the IEP for L is* coNP*-complete.*

PROOF. Suppose φ_1 and φ_2 have no interpolant in L, $\sigma = \mathit{sig}(\varphi_1) \cap \mathit{sig}(\varphi_2)$, and $\delta = \mathit{sig}(\varphi_1) \cup \mathit{sig}(\varphi_2)$. By Theorem 3.2, there are δ-models $\mathfrak{M}_i$, for $i = 1, 2$, based on $\mathfrak{M}_i$-generated temporal descriptive frames $\mathfrak{F}_i = (W_i, R_i, \mathcal{P}_i)$ for L with $\mathfrak{M}_1, x_1 \sim_\sigma \mathfrak{M}_2, x_2$, $\mathfrak{M}_1, x_1 \models \varphi_1$ and $\mathfrak{M}_2, x_2 \models \neg\varphi_2$. Let $\boldsymbol{\beta}$ be the largest σ-bisimulation between $\mathfrak{M}_1$ and $\mathfrak{M}_2$, that is, $y_1 \boldsymbol{\beta} y_2$ iff $t^\sigma_{\mathfrak{M}_1}(y_1) = t^\sigma_{\mathfrak{M}_2}(y_2)$, for all $y_i \in W_i$. We show that there exist such $\mathfrak{M}_i$ of polynomial size in $\max(|\varphi_1|, |\varphi_2|)$.

For any $i = 1, 2$ and $\tau \in \mathit{sub}(\varphi_i)$ satisfied in $\mathfrak{M}_i$, we take one $\{\tau\}$-maximal and one $\{\tau\}$-minimal point in $\mathfrak{M}_i$. Let $\boldsymbol{M}_i$ be the set of all selected points and let

$$T = \big\{t^\sigma_{\mathfrak{M}_1}(x) \mid x \in \{x_1\} \cup \boldsymbol{M}_1\big\} \cup \big\{t^\sigma_{\mathfrak{M}_2}(x) \mid x \in \{x_2\} \cup \boldsymbol{M}_2\big\}.$$

For each $t \in T$, we take a smallest set $\boldsymbol{S}_i \subseteq W_i$ containing one t-maximal and one t-minimal point in $\mathfrak{M}_i$.

Let $W'_i = \{x_i\} \cup \boldsymbol{M}_i \cup \boldsymbol{S}_i$, $R'_i = R_i \restriction_{W'_i}$, $\mathfrak{F}'_i = (W'_i, R'_i)$, let $\mathfrak{M}'_i$ be the restriction of $\mathfrak{M}_i$ to $\mathfrak{F}'_i$, and let $x'_1 \boldsymbol{\beta}' x'_2$ iff $t^\sigma_{\mathfrak{M}_1}(x'_1) = t^\sigma_{\mathfrak{M}_2}(x'_2)$, for all $x'_1 \in W'_1$, $x'_2 \in W'_2$. Following the proof of Lemma 3.4, we see that $\mathfrak{M}'_1, x_1 \models \varphi_1$, $\mathfrak{M}'_2, x_2 \models \neg\varphi_2$, and $\boldsymbol{\beta}'$ is a σ-bisimulation between $\mathfrak{M}'_1$ and $\mathfrak{M}'_2$ with $x_1 \boldsymbol{\beta}' x_2$. Clearly, $\mathfrak{F}'_i \models \mathsf{Lin}$ and the $\mathfrak{M}_i$ are of polynomial size in $\max(|\varphi_1|, |\varphi_2|)$.

For $L = \mathsf{Lin}_{\mathbb{Q}}$, we do not necessarily have $\mathfrak{F}'_i \models L$. To fix this, we add some extra points from W_i to W'_i. As $\mathfrak{F}_i \models \mathsf{Lin}_{\mathbb{Q}}$, the R- and R^--final clusters in $\mathfrak{F}_i$ are non-degenerate and, as observed in the selection procedure from §3, W'_i contains some points from these final clusters. Thus, $\mathfrak{F}'_i \not\models \mathsf{Lin}_{\mathbb{Q}}$ iff $\mathfrak{F}'_i$ contains an irreflexive point x with an immediate irreflexive R'_i-successor y. We call such pair x, y an *irr-defect in $\mathfrak{F}'_i$*. We are going to 'cure' one irr-defect after the other without introducing new irr-defects in either frame.

Given an irr-defect u_1, v_1 in $\mathfrak{F}'_1$, we find an R_1-reflexive z_1 with $u_1 R_1 z_1 R_1 v_1$, which exists by $\mathfrak{F}_1 \models \mathsf{Lin}_{\mathbb{Q}}$ and Lemma 5.6. Let $t = t^\sigma_{\mathfrak{M}_1}(z_1)$ and $\boldsymbol{b} = \boldsymbol{b}^\sigma_{\mathfrak{M}_1}(z_1)$. As

$\Diamond_F t \subseteq t$ and $\Diamond_P t \subseteq t$, $\boldsymbol{b}$ is a non-degenerate σ-block in $\mathfrak{M}_1$. By Lemma 5.8, there are t-minimal and t-maximal points z_1^- and z_1^+ in the non-degenerate clusters $C^-_{\boldsymbol{b}}$ and $C^+_{\boldsymbol{b}}$. As $\boldsymbol{\beta}(\boldsymbol{b})$ is a non-degenerate σ-block in $\mathfrak{M}_2$ by Lemma 4.15, there are t-minimal and t-maximal points z_2^- and z_2^+ in the non-degenerate clusters $C^-_{\boldsymbol{\beta}(\boldsymbol{b})}$ and $C^+_{\boldsymbol{\beta}(\boldsymbol{b})}$. By adding z_1, z_1^-, z_1^+ to W_1' and z_2^-, z_2^+ to W_2' we cure the irr-defect u_1, v_1 without creating a new irr-defect in either frame. Let W_i'', $i = 1, 2$, be the sets we obtain after curing all irr-defects in both frames in this way, $R_i'' = R_i \restriction_{W_i''}$, $\mathfrak{F}_i'' = (W_i'', R_i'')$, let $\mathfrak{M}_i''$ the restriction of $\mathfrak{M}_i$ to $\mathfrak{F}_i''$, and $x_1' \boldsymbol{\beta}'' x_2'$ iff $t^\sigma_{\mathfrak{M}_1}(x_1') = t^\sigma_{\mathfrak{M}_2}(x_2')$, for all $x_1' \in W_1''$, $x_2' \in W_2''$. Then $\mathfrak{F}_i'' \models \mathsf{Lin}_{\mathbb{Q}}$, by Lemma 5.6, and

(minmax) for all $x \in W_1'' \cup W_2''$ and $i = 1, 2$, the set W_1'' contains $t^\sigma_{\mathfrak{M}_i}(x)$-minimal and $t^\sigma_{\mathfrak{M}_i}(x)$-maximal points in $\mathfrak{M}_1$, and W_2'' contains $t^\sigma_{\mathfrak{M}_i}(x)$-minimal and $t^\sigma_{\mathfrak{M}_i}(x)$-maximal points in $\mathfrak{M}_2$.

So it is readily seen (similarly to the proof of Lemma 3.4) that $\mathfrak{M}_1'', x_1 \models \varphi_1$, $\mathfrak{M}_2'', x_2 \models \neg\varphi_2$, and $\boldsymbol{\beta}''$ is a σ-bisimulation between $\mathfrak{M}_1''$ and $\mathfrak{M}_2''$ with $x_1 \boldsymbol{\beta}'' x_2$.

Finally, let $L = \mathsf{Lin}_{\mathbb{R}}$. Since $\mathsf{Lin}_{\mathbb{Q}} \subseteq \mathsf{Lin}_{\mathbb{R}}$, we first cure the irr-defects in the $\mathfrak{F}_i'$, $i = 1, 2$, as described above. Let $\mathfrak{F}_i''$, $i = 1, 2$, be the resulting serial and dense frames. Thus, $\mathfrak{F}_i'' \not\models L$ iff $\mathfrak{F}_i''$ contains two $<_{R_i''}$-consecutive non-degenerate clusters $C(x) \neq C(y)$. We call such x, y a *ref-defect in* $\mathfrak{F}_i''$. We show that the ref-defects can also be cured in a step-by-step manner without introducing new defects of either type, while maintaining **(minmax)**.

If u_1, v_1 is a ref-defect in $\mathfrak{F}_1''$, Lemma 5.6 provides an irreflexive $z_1 \in W_1$ with $u_1 R_1 z_1 R_1 v_1$. Let $t = t^\sigma_{\mathfrak{M}_1}(z_1)$. The insertion of extra points into W_1'' depends on whether u_1 and v_1 are in the same σ-block in $\mathfrak{M}_1$ or not.

Case 1: $u_1, v_1 \in \boldsymbol{b}$, for some σ-block $\boldsymbol{b}$ in $\mathfrak{M}_1$. By Lemma 5.8, $\boldsymbol{b}$ is non-degenerate, and there are t-minimal and t-maximal points z_1^- and z_1^+ in the non-degenerate clusters $C^-_{\boldsymbol{b}}$ and $C^+_{\boldsymbol{b}}$. By Lemma 4.15, $\boldsymbol{\beta}(\boldsymbol{b})$ is a non-degenerate σ-block in $\mathfrak{M}_2$, so there are t-minimal and t-maximal points z_2^- and z_2^+ in the non-degenerate clusters $C^-_{\boldsymbol{\beta}(\boldsymbol{b})}$ and $C^+_{\boldsymbol{\beta}(\boldsymbol{b})}$. By adding z_1, z_1^-, z_1^+ to W_1'' and z_2^-, z_2^+ to W_2'' we cure the ref-defect u_1, v_1 in $\mathfrak{F}_1''$ and maintain **(minmax)**. Also, as **(minmax)** held in $\mathfrak{F}_i''$, by Lemma 5.8 we already had some points from $C^-_{\boldsymbol{b}}$ and $C^+_{\boldsymbol{b}}$ in W_1'' and some points from $C^-_{\boldsymbol{\beta}(\boldsymbol{b})}$ and $C^+_{\boldsymbol{\beta}(\boldsymbol{b})}$ in W_2''. So we did not create new defects in either frame, and the property **(minmax)** is maintained.

Case 2: $u_1 \in \boldsymbol{b}^{u_1}$, $v_1 \in \boldsymbol{b}^{v_1}$, for σ-blocks $\boldsymbol{b}^{u_1} \neq \boldsymbol{b}^{v_1}$ in $\mathfrak{M}_1$. By the definition of W_1'' and $C(u_1)$, $C(v_1)$ being $<_{R_1''}$-consecutive, $C(u_1) = C^+_{\boldsymbol{b}^{u_1}}$ and $C(v_1) = C^-_{\boldsymbol{b}^{v_1}}$, so $z_1 \notin \boldsymbol{b}^{u_1}$. We claim that there is an irreflexive $z \in W_1$ such that $u_1 R_1 z R_1 v_1$ and z is either $t^\sigma_{\mathfrak{M}_1}(z)$-maximal or $t^\sigma_{\mathfrak{M}_1}(z)$-minimal. Indeed, as $u_1 R_1 z_1$, we have $\Diamond_{\mathsf{F}} t^\sigma_{\mathfrak{M}_1}(z_1) \subseteq t^\sigma_{\mathfrak{M}_1}(u_1)$ and $\Diamond_{\mathsf{P}} t^\sigma_{\mathfrak{M}_1}(u_1) \subseteq t^\sigma_{\mathfrak{M}_1}(z_1)$. As $z_1 \notin \boldsymbol{b}^{u_1}$, there can be two cases: either (i) $\Diamond_{\mathsf{F}} t^\sigma_{\mathfrak{M}_1}(u_1) \not\subseteq t^\sigma_{\mathfrak{M}_1}(z_1)$ or (ii) $\Diamond_{\mathsf{P}} t^\sigma_{\mathfrak{M}_1}(u_1) \not\subseteq t^\sigma_{\mathfrak{M}_1}(z_1)$. In case (i), there is a σ-formula χ with $\mathfrak{M}_1, u_1 \models \Diamond_{\mathsf{F}}\chi$ but $\mathfrak{M}_1, z_1 \not\models \Diamond_{\mathsf{F}}\chi$. Take a $\{\Diamond_{\mathsf{F}}\chi\}$-maximal point z'. Clearly, $u_1 R_1 z' R_1 v_1$. If z' is irreflexive, we set $z = z'$ as it is $t^\sigma_{\mathfrak{M}_1}(z')$-maximal. Otherwise, Lemma 5.4 gives an immediate degenerate $<_{R_1}$-successor $C(z)$ of $C(z')$ such that z is $t^\sigma_{\mathfrak{M}_1}(z)$-maximal. In case (ii), there is a σ-formula χ with $\mathfrak{M}_1, u_1 \not\models \Diamond_{\mathsf{P}}\chi$ but $\mathfrak{M}_1, z_1 \models \chi$, and so $\mathfrak{M}_1, v_1 \models \Diamond_{\mathsf{P}}\chi$. Take a $\{\Diamond_{\mathsf{P}}\chi\}$-minimal point z'. Clearly, $u_1 R_1 z' R_1 v_1$. If z' is irreflexive, we set $z = z'$

as it is $t^{\sigma}_{\mathfrak{M}_1}(z')$-minimal. Otherwise, Lemma 5.4 gives an immediate degenerate $<_{R_1}$-predecessor $C(z)$ of $C(z')$ such that z is $t^{\sigma}_{\mathfrak{M}_1}(z)$-minimal.

Let $\boldsymbol{b} = \boldsymbol{b}^{\sigma}_{\mathfrak{M}_1}(z)$. Then $\boldsymbol{b}$ is a degenerate σ-block in $\mathfrak{M}_1$ by Lemma 5.8. By Lemma 4.15, $\boldsymbol{\beta}(\boldsymbol{b})$ is a degenerate σ-block in $\mathfrak{M}_2$ with $\boldsymbol{\beta}(\boldsymbol{b}^{u_1}) \prec_{\mathfrak{F}_2} \boldsymbol{\beta}(\boldsymbol{b}) \prec_{\mathfrak{F}_2} \boldsymbol{\beta}(\boldsymbol{b}^{v_1})$. Also, by **(minmax)** in $\mathfrak{F}''_i$, $C^{+}_{\boldsymbol{\beta}(\boldsymbol{b}^{u_1})}$ and $C^{-}_{\boldsymbol{\beta}(\boldsymbol{b}^{v_1})}$ are $<_{R''_2}$-consecutive non-degenerate clusters. Therefore, by adding z to W''_1 and z_2 with $C(z_2) = \boldsymbol{\beta}(\boldsymbol{b})$ to W''_2, we cured the ref-defect u_1, v_1 in $\mathfrak{F}''_1$ and we did not create new defects of either kind in either frame while maintaining **(minmax)**. So again it is readily seen (similarly to the proof of Lemma 3.4) that, after fixing all defects, we end up with a pair of models as required that are based on frames for $\mathsf{Lin}_{\mathbb{R}}$ by Lemma 5.6.

This establishes the polysize bisimilar model property of $L \in \{\mathsf{Lin}, \mathsf{Lin}_{\mathbb{Q}}, \mathsf{Lin}_{\mathbb{R}}\}$. We show that the IEP for L is in coNP using the description of finite frames for L in Lemma 5.6. $\dashv$

The finitary selection construction in the proof above does not work for logics $L \in \{\mathsf{Lin}_{<\omega}, \mathsf{Lin}_{\mathbb{Z}}\}$. In fact, these logics do not have the polysize bisimilar model property. However, below we show that they still have a kind of quasi-finite bisimilar model property similar to Definition 4.1 in the following sense. We can always witness the lack of an interpolant for φ_1, φ_2 in L by a pair of temporal models that are based on frames for L, and assembled from $\mathcal{O}\big(\max(|\varphi_1|, |\varphi_2|)\big)$-many 'simple' models (like those in Example 5.2) that are based *atomic* descriptive frames of the forms $m^{<}$, $\textcircled{k}$, $\mathfrak{C}(\textcircled{k}, \bullet)$, $\mathfrak{C}(\bullet, \textcircled{k}, \bullet)$, and $\mathfrak{C}(\bullet, \textcircled{k})$, for $m, k = \mathcal{O}\big(\max(|\varphi_1|, |\varphi_2|)\big)$, $k > 0$.

Given φ_i, $\mathfrak{M}_i$, x_i, for $i = 1, 2$, as above, let $\boldsymbol{M}_i$, $\boldsymbol{S}_i$, and $W'_i = \{x_i\} \cup \boldsymbol{M}_i \cup \boldsymbol{S}_i$ be as defined in the proof of Theorem 5.9. As before, we call the points from W'_i *relevant in* $\mathfrak{M}_i$. A cluster or a σ-block in $\mathfrak{M}_i$ is *relevant* if it contains a relevant point in $\mathfrak{M}_i$. Given any pair $\boldsymbol{b}$, $\boldsymbol{\beta}(\boldsymbol{b})$ of σ-bisimilar σ-blocks in $\mathfrak{M}_1$ and $\mathfrak{M}_2$, we can now have the temporal analogue of Lemma 4.17, dealing not only with $\boldsymbol{S}_1 \cap C^{+}_{\boldsymbol{b}}$ and $\boldsymbol{S}_2 \cap C^{+}_{\boldsymbol{\beta}(\boldsymbol{b})}$ but also with $\boldsymbol{S}_1 \cap C^{-}_{\boldsymbol{b}}$ and $\boldsymbol{S}_2 \cap C^{-}_{\boldsymbol{\beta}(\boldsymbol{b})}$. In particular,

(38) there are σ-type preserving bijections $f^{-}\colon \boldsymbol{S}_1 \cap C^{-}_{\boldsymbol{b}} \to \boldsymbol{S}_2 \cap C^{-}_{\boldsymbol{\beta}(\boldsymbol{b})}$ and $f^{+}\colon \boldsymbol{S}_1 \cap C^{+}_{\boldsymbol{b}} \to \boldsymbol{S}_2 \cap C^{+}_{\boldsymbol{\beta}(\boldsymbol{b})}$;

(39) $|\boldsymbol{S}_1 \cap C^{-}_{\boldsymbol{b}}| = |\boldsymbol{S}_1 \cap C^{+}_{\boldsymbol{b}}|$ and $|\boldsymbol{S}_2 \cap C^{-}_{\boldsymbol{\beta}(\boldsymbol{b})}| = |\boldsymbol{S}_2 \cap C^{+}_{\boldsymbol{\beta}(\boldsymbol{b})}|$;

(40) $\boldsymbol{b}$ is relevant in $\mathfrak{M}_1$ iff $\boldsymbol{\beta}(\boldsymbol{b})$ is relevant in $\mathfrak{M}_2$.

THEOREM 5.10. *The IEPs for* $\mathsf{Lin}_{<\omega}$ *and* $\mathsf{Lin}_{\mathbb{Z}}$ *are both* coNP*-complete.*

PROOF. Let $\boldsymbol{b}^0_1, \dots, \boldsymbol{b}^N_1$ be all the relevant σ-blocks in $\mathfrak{M}_1$ ordered by $\prec_{\mathfrak{F}_1}$, for some $N = \mathcal{O}\big(\big(\max(|\varphi_1|, |\varphi_2|)\big)\big)$. By (40) and Lemma 4.15, the $\prec_{\mathfrak{F}_2}$-ordered list of all relevant σ-blocks in $\mathfrak{M}_2$ is $\boldsymbol{b}^0_2, \dots, \boldsymbol{b}^N_2$, where $\boldsymbol{b}^j_2 = \boldsymbol{\beta}(\boldsymbol{b}^j_1)$, for $j \leq N$. By (38) and (39), for every $j \leq N$ there is $k^j > 0$ with $k^j = |\boldsymbol{S}_1 \cap C^{-}_{\boldsymbol{b}^j_1}| = |\boldsymbol{S}_1 \cap C^{+}_{\boldsymbol{b}^j_1}| = |\boldsymbol{S}_2 \cap C^{-}_{\boldsymbol{b}^j_2}| = |\boldsymbol{S}_2 \cap C^{+}_{\boldsymbol{b}^j_2}|$. Also, by Lemma 4.15, $\boldsymbol{b}^j_1$ is degenerate iff $\boldsymbol{b}^j_2$ is degenerate, for $j \leq N$.

Case $L = \mathsf{Lin}_{<\omega}$: By Lemmas 5.6 and 5.8, $\boldsymbol{b}^0_i$ and $\boldsymbol{b}^N_i$, $i = 1, 2$, are degenerate. By Lemmas 5.4, 5.6, 5.8, if $\boldsymbol{b}^j_i$ is non-degenerate, then $C^{-}_{\boldsymbol{b}^j_i}$ and $C^{+}_{\boldsymbol{b}^j_i}$ are R^{-}- and

R-limit clusters, and $C \cap \boldsymbol{M}_i = \emptyset$, for every non-degenerate cluster C in $\boldsymbol{b}_i^j$. (It can happen that x_i is in a non-degenerate cluster in $\boldsymbol{b}_i^j$ different from $C^-_{\boldsymbol{b}_i^j}$, $C^+_{\boldsymbol{b}_i^j}$.)

For all $i = 1, 2$ and $j \leq N$, we let $m_i^j = \big|\big((\{x_i\} \cup \boldsymbol{M}_i) \cap \boldsymbol{b}_i^j\big) \setminus (C^-_{\boldsymbol{b}_i^j} \cup C^+_{\boldsymbol{b}_i^j})\big|$ and define an atomic frame $\mathfrak{H}_i^j = (H_i^j, R_i^j, \mathcal{P}_i^j)$ by taking

$$\mathfrak{H}_i^j = \begin{cases} \bullet, & \text{if } \boldsymbol{b}_i^j \text{ is degenerate;} \\ \mathfrak{C}\big(\bullet, \textcircled{k^j}, \bullet\big), & \text{if } C^-_{\boldsymbol{b}_i^j} = C^+_{\boldsymbol{b}_i^j} \text{ is non-degenerate;} \\ \mathfrak{C}\big(\bullet, \textcircled{k^j}, \bullet\big) \lhd (m_i^j)^< \lhd \mathfrak{C}\big(\bullet, \textcircled{k^j}, \bullet\big), & \text{otherwise.} \end{cases}$$

Note that m_1^j and m_2^j might be different, and $(\{x_i\} \cup \boldsymbol{M}_i) \cap \boldsymbol{b}_i^j = \emptyset$ (and so $m_i^j = 0$) can happen even when $C^-_{\boldsymbol{b}_i^j} \neq C^+_{\boldsymbol{b}_i^j}$. Let $\mathfrak{H}_i = (H_i, R_i', \mathcal{P}_i') = \mathfrak{H}_i^0 \lhd \cdots \lhd \mathfrak{H}_i^N$. It is readily seen that $\mathfrak{H}_i$ is a frame for $\mathsf{Lin}_{<\omega}$, for $i = 1, 2$. Next, we define a 'parent' function $\mathsf{h}_i \colon H_i \to W_i'$ such that, for all $x \in H_i$,

(41) for all $j \leq N$, if $x \in H_i^j$ then $\mathsf{h}_i(x) \in W_i' \cap \boldsymbol{b}_i^j$,

(42) for all $y \in H_i$, if $x R_i' y$ then $\mathsf{h}_i(x) R_i \mathsf{h}_i(y)$,

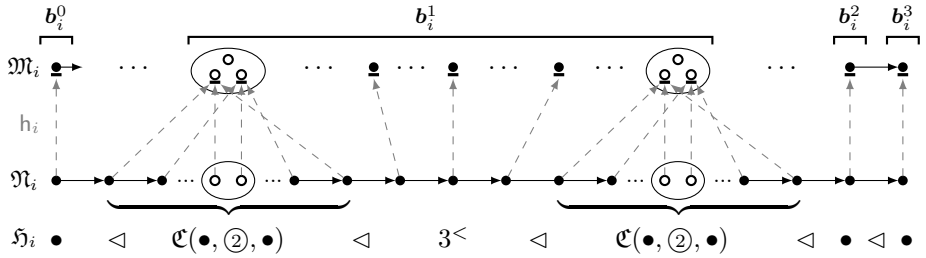
(43) for all $y \in \boldsymbol{M}_i$, if $\mathsf{h}_i(x) R_i y$ then $x R_i' z$ and $\mathsf{h}_i(z) = y$ for some z.

Finally, for $j \leq N$, we define a model $\mathfrak{N}_i^j$ based on $\mathfrak{H}_i^j$ by taking, for all $x \in H_i^j$,

$$at_{\mathfrak{N}_i^j}(x) = at_{\mathfrak{M}_i}\big(\mathsf{h}_i(x)\big), \tag{44}$$

and let $\mathfrak{N}_i = \mathfrak{N}_i^0 \lhd \cdots \lhd \mathfrak{N}_i^N$.

Instead of giving the general definitions of h_i and $\mathfrak{N}_i$, we illustrate the construction in the picture below, where $\mathfrak{M}_i$ has three degenerate σ-blocks $\boldsymbol{b}_i^0, \boldsymbol{b}_i^2$ and $\boldsymbol{b}_i^3$ and one non-definable non-degenerate σ-block $\boldsymbol{b}_i^1$; the relevant points in $\mathfrak{M}_i$ are underlined; $k^0 = k^2 = k^3 = 1$, $k^1 = 2$ and $m_i^1 = 3$.



It is readily seen that this way (41)–(43) hold and $\mathfrak{N}_i^j$ is based on $\mathfrak{H}_i^j$, for $j \leq N$. Thus, $\mathfrak{H}_i^0 \lhd \cdots \lhd \mathfrak{H}_i^N$, $i = 1, 2$, is a frame for $\mathsf{Lin}_{<\omega}$ by Lemma 5.6. Using (41)–(43), a proof similar to that of Lemma 4.22 (a) shows that each point x in $\mathfrak{N}_i$ makes true exactly the same formulas in $sub(\varphi_i)$ as its parent $\mathsf{h}_i(x)$ in $\mathfrak{M}_i$. It follows that $\mathfrak{N}_1, x_1' \models \varphi_1$ and $\mathfrak{N}_2, x_2' \models \neg\varphi_2$, where $x_i = \mathsf{h}_i(x_i')$.

Further, the construction and (38) guarantee that each pair $(\mathfrak{N}_1^j, \mathfrak{N}_2^j)$, for $j \leq N$, satisfies an obvious condition similar to Definition 4.1 (a) or (c). Then a proof similar to that of Lemma 4.4 shows that $\mathfrak{N}_1^j$ and $\mathfrak{N}_2^j$ are globally σ-bisimilar for every $j \leq N$, and so $\mathfrak{N}_1, x_1' \sim_\sigma \mathfrak{N}_2, x_2'$.

Case $L = \mathsf{Lin}_{\mathbb{Z}}$: While the definitions of $\mathfrak{H}_i^j$, for $0 < j < N$, are the same as above, for $j = 0, N$ we need new ones. Now, by Lemmas 5.6 and 5.8, $\boldsymbol{b}_i^0$ and $\boldsymbol{b}_i^N$

are non-degenerate, for $i = 1, 2$. Also, by Lemmas 5.4, 5.6, and 5.8, the R^--final cluster $C^-_{\boldsymbol{b}^0_i}$ in $\mathfrak{F}_i$ is an R^--limit cluster, and the R-final cluster $C^+_{\boldsymbol{b}^N_i}$ in $\mathfrak{F}_i$ is an R-limit cluster, for $i = 1, 2$. There are several cases. If $N = 0$ (that is, $\boldsymbol{b}^0_i = W_i$) and $C^-_{\boldsymbol{b}^0_i} = C^+_{\boldsymbol{b}^0_i}$, then we let $\mathfrak{H}^0_i = \mathfrak{C}\big(\textcircled{k^0}\big)$. If $N = 0$ and $C^-_{\boldsymbol{b}^0_i} \neq C^+_{\boldsymbol{b}^0_i}$, then we let $\mathfrak{H}^0_i = \mathfrak{C}\big(\textcircled{k^0}, \bullet\big) \lhd (m^0_i)^< \lhd \mathfrak{C}\big(\bullet, \textcircled{k^0}\big)$. If $N > 0$ then

$$\mathfrak{H}^0_i = \begin{cases} \mathfrak{C}\big(\textcircled{k^0}, \bullet\big), & \text{if } C^-_{\boldsymbol{b}^0_i} = C^+_{\boldsymbol{b}^0_i}; \\ \mathfrak{C}\big(\textcircled{k^0}, \bullet\big) \lhd (m^0_i)^< \lhd \mathfrak{C}\big(\bullet, \textcircled{k^0}, \bullet\big), & \text{otherwise,} \end{cases}$$

and

$$\mathfrak{H}^N_i = \begin{cases} \mathfrak{C}\big(\bullet, \textcircled{k^N}\big), & \text{if } C^-_{\boldsymbol{b}^N_i} = C^+_{\boldsymbol{b}^N_i}; \\ \mathfrak{C}\big(\bullet, \textcircled{k^N}, \bullet\big) \lhd (m^N_i)^< \lhd \mathfrak{C}\big(\bullet, \textcircled{k^N}\big), & \text{otherwise.} \end{cases}$$

This way, by Lemma 5.6, $\mathfrak{H}_i = \mathfrak{H}^0_i \lhd \cdots \lhd \mathfrak{H}^N_i$ is a frame for $\mathsf{Lin}_{\mathbb{Z}}$, for $i = 1, 2$. Apart from these modifications, everything is similar to the $\mathsf{Lin}_{<\omega}$ case.

A coNP-algorithm deciding interpolant existence in $\mathsf{Lin}_{<\omega}$ or $\mathsf{Lin}_{\mathbb{Z}}$ is an obvious adaptation of the algorithm detailed in the proof of Theorem 4.9. ⊣

We conjecture that the IEP for every consistent finitely axiomatisable Priorean temporal logic is coNP-complete.

§6. Outlook and open problems. We have turned the lack of the CIP into a research question by asking whether deciding interpolant existence becomes harder than validity for modal logics without the CIP. As argued in [33, 35] for the closely related problem of separability of disjoint regular languages using a smaller language class (such as first-order definable languages), this question can be understood as a generalisation of satisfiability that provides new insights into the expressivity of the logic in question. We have shown that, in contrast to modal logics with nominals, the product modal logic $\mathsf{S5}\times\mathsf{S5}$, and the guarded and two-variable fragments of first-order logic, the complexity of deciding interpolant existence in finitely axiomatisable modal logics of linear frames is in coNP and, therefore, of the same complexity as validity. This appears to be the first general result about Craig interpolants for logics lacking the CIP. It gives rise to many further questions of which we mention only a few:

Q1: Is there a decidable modal logic above GL, $\mathsf{K4}$, or K with the undecidable IEP? Currently, the only known example of a decidable logic with the undecidable IEP is the two-variable fragment of first-order logic with two equivalence relations [41].

Q2: Do all d-persistent (cofinal) subframe logics above $\mathsf{K4}$ have the finite bisimilar model property? Can one show a quasi-finite bisimilar model property for all (cofinal) subframe logics above $\mathsf{K4}$ and use it to prove that interpolant existence is decidable for all finitely axiomatisable ones?

Q3: What is the situation with the IEP for propositional superintuitionistic (aka intermediate) logics and (super)intuitionistic modal logics without the CIP? Note that the Gödel translation reduces the IEP for propositional superintuitionistic logics to the IEP for (certain fragments of) modal logics above $\mathsf{S4}$; see the proof of [9, Theorem 14.9].

Q4: Our proof is not constructive in the sense that is does not provide a non-trivial algorithm for computing interpolants if they exist (beyond exhaustive search) nor any upper bounds on their size. It would be of great interest to develop such algorithms. First steps towards computing interpolants in description logics without CIP are presented in [23].

Descriptive frames have been crucial for our proofs. It would therefore be interesting and in line with the modal logic tradition to characterise logics for which descriptive frames can be replaced by Kripke (or even finite) frames in Theorem 3.2. While d-persistence is clearly a sufficient condition, $\mathsf{Lin}_{\mathbb{R}}$ shows that it is not a necessary one; see Example 5.7. It is known, however, that $\mathsf{Lin}_{\mathbb{R}}$ is strongly complete [37], which suggests the conjecture that, in Theorem 3.2, descriptive frames for L can be replaced by Kripke frames iff L is strongly Kripke complete (in the sense that every L-consistent set of formulas is satisfiable in a Kripke frame for L). Note that a logic is strongly Kripke complete iff the corresponding variety of modal algebras is complex [18, 37].

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