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A NON-UNIFORM VIEW OF CRAIG INTERPOLATION IN MODAL LOGICS WITH LINEAR FRAMES

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Abstract. Normal modal logics extending the logic K4.3 of linear transitive frames are known to lack the Craig interpolation property, except some logics of bounded depth such as S5. We turn this 'negative' fact into a research question and pursue a non-uniform approach to Craig interpolation by investigating the following interpolant existence problem: decide whether there exists a Craig interpolant between two given formulas in any fixed logic above K4.3. Using a bisimulation-based characterisation of interpolant existence for descriptive frames, we show that this problem is decidable and coNP-complete for all finitely axiomatisable normal modal logics containing K4.3. It is thus not harder than entailment in these logics, which is in sharp contrast to other recent non-uniform interpolation results. We also extend our approach to Priorean temporal logics (with both past and future modalities) over the standard time flows—the integers, rationals, reals, and finite strict linear orders—none of which is blessed with the Craig interpolation property.

§1. Introduction. Unlike classical and intuitionistic first-order and propositional logics, numerous modal logics, L, do not enjoy the Craig interpolation property (CIP): they contain valid implications $\varphi \to \psi$ without an interpolant in L—a formula ι in the shared signature of φ and ψ such that both $\varphi \to \iota$ and $\iota \to \psi$ are also valid in L. Typical examples of such L are first-order modal logics with constant domains between K and S5 [12] and propositional modal logics with linear transitive Kripke frames of unbounded depth [13, 35]. There have been various attempts to classify propositional modal logics with the CIP, successful for extensions of S4 and unsuccessful for extensions of K4 or GL, where the CIP turned out to be undecidable; see [13, 8] and further references therein.

While establishing the CIP of a logic L typically gives rise to further research problems—develop proof systems that admit efficient/elegant interpolant computation [25, 2], investigate the complexity of computing interpolants from proofs [23, Sections 17, 18], consider restrictions on the shape of interpolants such as in, say, Lyndon's interpolation [27], or employ the CIP to investigate related properties such as Beth definability [9, 10]—a counterexample to the CIP has usually terminated further research of Craig interpolants and their applications for the unfortunate logic in question.

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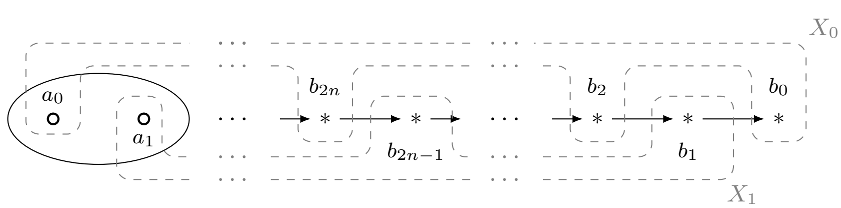
In this article, we take a different, non-uniform view of Craig interpolation and aim to understand interpolants also for logics L without the CIP. We consider the following *interpolant existence problem* (IEP) for L: given formulas φ and ψ, decide whether $\varphi \to \psi$ has an interpolant in L. For L without the CIP, the existence of an interpolant for φ and ψ does not follow from the validity of $\varphi \to \psi$ in L, and so the IEP does not reduce to validity checking. A first question then is whether the former problem is harder than the latter one. Recent results show that this is indeed the case for the one-variable fragment of modal logic S5, modal logics with nominals, and the two-variable and guarded fragments of first-order logic [22, 1, 24].

Here, we show that the opposite is true of propositional modal logics containing K4.3, the logic of linear transitive frames: while none of these logics with frames of unbounded depth has the CIP [13, 35], interpolant existence is nevertheless decidable in coNP for finitely axiomatisable logics, and so is as hard as validity [26]. This is the first general result on Craig interpolant existence covering a large family of modal logics and, potentially, a step towards a classification of modal logics according to the complexity of the IEP.

We proceed as follows. To begin with, we give a 'folklore' characterisation of interpolant existence via bisimulations between models based on descriptive frames: $\varphi \to \psi$ does not have an interpolant in L iff φ and $\neg\psi$ can be satisfied in $sig(\varphi) \cap sig(\psi)$-bisimilar models based on descriptive frames for L. If L had the CIP, we could merge (amalgamate) these two models into a single one satisfying $\varphi \land \neg\psi$, which is impossible in our case. Instead, we aim to understand the fine-grained structure of the required bisimilar models and use it to decide their existence. We show that, for some logics (such as first-order definable cofinal subframe logics), any pair of bisimilar models can be transformed into bisimilar models of polynomial size; in other words, such logics enjoy the polysize bisimilar model property. However, for other logics like GL.3, not even models based on infinite Kripke frames are enough despite GL.3 having the finite model property.

We prove, nevertheless, that every pair of bisimilar models satisfying φ and $\neg\psi$ and based on descriptive frames for a finitely axiomatisable L can be converted to a pair of such models with an understandable structure. In a nutshell, their underlying frames look like a polynomial-size chain of polynomial-size clusters and tadpole-like descriptive frames that comprise a nondegenerate cluster $\{a_0, \dots, a_{k-1}\}$, for some polynomial-size $k > 0$, followed by an infinite descending chain of points b_n, $n < \omega$, which are all irreflexive or all reflexive, with the internal sets (restricting possible valuations) generated as a modal algebra by the singletons $\{b_n\}$ and the k-many pairwise disjoint infinite sets $X_i = \{a_i\} \cup \{b_n \mid n \equiv i \pmod{k}\}$. The picture below illustrates the underlying Kripke frame and the generators of the tadpole descriptive frame with $k = 1$.



Because of this, we say that all finitely axiomatisable $L \supseteq \mathsf{K4.3}$ have the quasi-polysize bisimilar model property. We show that the existence of such quasi-polysize bisimilar models can be checked in NP in the size of φ and ψ, for any finitely axiomatisable L.

Finally, we extend the developed techniques to analyse the IEP for a few Priorean temporal logics with past and future modal operators: the logic Lin of all linear frames, the logic $\mathsf{Lin}_{<\omega}$ of all finite strict linear orders, and the logics $\mathsf{Lin}_{\mathbb{Q}}$ of the rationals, $\mathsf{Lin}_{\mathbb{R}}$ of the reals, and $\mathsf{Lin}_{\mathbb{Z}}$ of the integers. We prove that Lin, $\mathsf{Lin}_{\mathbb{Q}}$, and $\mathsf{Lin}_{\mathbb{R}}$ have the polysize bisimilar model property, while $\mathsf{Lin}_{<\omega}$ and $\mathsf{Lin}_{\mathbb{Z}}$ have the quasi-polysize one, with the IEP being coNP-complete.

The remainder of the article is organised as follows. The introduction is concluded with a brief discussion of related work. §2 contains the necessary modal logic preliminaries. §3 gives the bisimulation-based criterion of interpolant existence and applies it to first-order definable cofinal subframe logics above $\mathsf{K4.3}$. It also provides illustrative examples explaining why the same method does not work in general and what kind of descriptive frames might be needed. §4 establishes the quasi-finite bisimilar model property of all logics above $\mathsf{K4.3}$ and the quasi-polysize bisimilar model property of all finitely axiomatisable ones; for the latter, it gives a coNP-algorithm for deciding the IEP. §5 extends the developed techniques to the Priorean temporal logics mentioned above.

1.1. Related work. The IEP for some logics of linear frames turns out to be closely related to separability of regular languages by first-order definable languages. Formally, the separability problem is to decide whether two input regular languages L_1 and L_2 can be separated by some language L in a given class $\mathcal{L}$ in the sense that $L_1 \subseteq L$ and $L \cap L_2 = \emptyset$. If $\mathcal{L}$ is the class of first-order definable languages over finite words, the separability problem is easily seen to be equivalent to the IEP for the linear temporal logic LTL extending modal logic with the operators 'next' and 'until' over finite strict linear orders. For regular languages of infinite words, the separability problem is equivalent to the IEP for LTL over the natural numbers. It was shown in [19, 20, 30] that both of these separability problems are decidable in 2ExpTime in the size of NFAs defining L_1 and L_2. It follows that the corresponding IEPs are decidable in 3ExpTime in the size of LTL-formulas. (Separability by other language classes $\mathcal{L}$ are discussed in [29, 31].) These separability results have been obtained using algebraic machinery from semigroup theory, which seems to be orthogonal to our model-theoretic approach to the IEP developed to deal with all modal logics of linear orders. However, for finite strict linear orders and the natural numbers, the algebraic approach also provides an upper bound for the size of interpolants.

It is also worth mentioning that, for these two frame classes, the smallest modal logic with the CIP is LTL extended with fixpoint operators or, equivalently, monadic second-order logic (under very mild conditions on the definition of what a logic is) [14]. Thus, to 'repair' the CIP by extending the expressive power of the logic, we require the addition of second-order features.

§2. Preliminaries. This section provides the basic definitions that will be used later on in the article. For more details the reader is referred to [8, 17, 4, 5].

2.1. Descriptive frames for $\mathsf{K4.3}$**.** The formulas, φ, of propositional unimodal logics are built from propositional variables $p_i \in \mathcal{V}$, for some countably-infinite set $\mathcal{V} = \{p_i \mid i < \omega\}$, and constants $\top$, $\bot$ using the Boolean connectives $\neg$, $\wedge$, and the unary possibility operator $\Diamond$. The other Booleans and the necessity operator $\Box$ dual to $\Diamond$ are defined as standard abbreviations. We also use $\Diamond^+\varphi = \varphi \vee \Diamond\varphi$, $\Box^+\varphi = \varphi \wedge \Box\varphi$, and $\Diamond\Gamma = \{\Diamond\varphi \mid \varphi \in \Gamma\}$, for a set Γ of formulas. By a *signature* we mean any set $\sigma \subseteq \mathcal{V}$, denoting by $sig(\varphi)$ the (finite) set of variables in a formula φ. If $sig(\varphi) \subseteq \sigma$, we call φ a σ*-formula*. We denote by $sub(\varphi)$ the set of subformulas of φ together with their negations, and let $|\varphi| = |sub(\varphi)|$.

A (*normal*) *modal logic*, L, is any set of formulas that contains all Boolean tautologies, the modal axiom $\Box(p_0 \to p_1) \to (\Box p_0 \to \Box p_1)$, and is closed under the rules of *modus ponens*, uniform substitution of formulas in place of variables, and necessitation $\varphi/\Box\varphi$. The smallest modal logic is known as K. Given a formula φ and a modal logic L, the smallest modal logic to contain L and φ is denoted by $L \oplus \varphi$. For example,

$$\begin{aligned}
\mathsf{K4} &= \mathsf{K} \oplus \Box p_0 \to \Box\Box p_0,\\
\mathsf{K4.3} &= \mathsf{K4} \oplus \Box(\Box^+ p_0 \to p_1) \vee \Box(\Box^+ p_1 \to p_0),\\
\mathsf{GL.3} &= \mathsf{K4.3} \oplus \Box(\Box p_0 \to p_0) \to \Box p_0,\\
\mathsf{Log}\{(\mathbb{N}, <)\} &= \mathsf{K4.3} \oplus \Diamond\top \oplus \Box(\Box p \to p) \to (\Diamond\Box p \to \Box p).
\end{aligned}$$

All logics considered in this article are extensions of $\mathsf{K4.3}$.

We interpret formulas in (*general*) *frames* $\mathfrak{F} = (W, R, \mathcal{P})$, where R is a binary (accessibility) relation on a nonempty set W (of worlds or, more neutrally, points) and $\mathcal{P} \subseteq 2^W$ contains $\emptyset$, W and is closed under $\cap$, $\neg$, and the operator

$$\Diamond^{\mathfrak{F}} X = \{x \in W \mid \exists y \in X\; xRy\}.$$

The structure $\mathfrak{F}^+ = (\mathcal{P}, \cap, \neg, \emptyset, W, \Diamond^{\mathfrak{F}})$ is a Boolean algebra $(\mathcal{P}, \cap, \neg, \emptyset, W)$ with a normal and additive operator $\Diamond^{\mathfrak{F}}$ (BAO, for short). If $\mathfrak{F}^+$ is generated by a set $\mathcal{X} \subseteq \mathcal{P}$ as a BAO, we say that the frame $\mathfrak{F}$ (or the set $\mathcal{P}$) is *generated by* $\mathcal{X}$. If $|\mathcal{X}| = n$, for some $n < \omega$, we call $\mathfrak{F}$ n*-generated* or *finitely generated*. The elements of $\mathcal{P}$ are called *internal sets* in $\mathfrak{F}$. If $\mathcal{P} = 2^W$, $\mathfrak{F}$ is known as a *Kripke frame*; in this case, we drop $\mathcal{P}$ and write $\mathfrak{F} = (W, R)$. A frame $\mathfrak{F} = (W, R, \mathcal{P})$ is *descriptive* if the following conditions hold, for any $x, y \in W$ and any $\mathcal{X} \subseteq \mathcal{P}$:

(dif) $x = y$ iff $\forall X \in \mathcal{P}\,(x \in X \leftrightarrow y \in X)$,
(ref) xRy iff $\forall X \in \mathcal{P}\,(y \in X \to x \in \Diamond^{\mathfrak{F}} X)$,
(com) if $\mathcal{X} \subseteq \mathcal{P}$ has the *finite intersection property* (*fip*, for short)—that is, $\bigcap \mathcal{X}' \neq \emptyset$ for every finite $\mathcal{X}' \subseteq \mathcal{X}$—then $\bigcap \mathcal{X} \neq \emptyset$.

(Frames with **(dif)** are called *differentiated*, with **(ref)** *refined*, and with **(com)** *compact*.) Every BAO is isomorphic to $\mathfrak{F}^+$, for some descriptive frame $\mathfrak{F}$.

Given a signature σ, a σ*-model* based on a frame $\mathfrak{F} = (W, R, \mathcal{P})$ is a pair $\mathfrak{M} = (\mathfrak{F}, \mathfrak{v})$ with a *valuation* $\mathfrak{v}\colon \sigma \to \mathcal{P}$. The *atomic* σ*-type* of $x \in W$ in $\mathfrak{M}$ is

$$at^{\sigma}_{\mathfrak{M}}(x) = \{p_i \mid p_i \in \sigma,\ x \in \mathfrak{v}(p_i)\} \cup \{\neg p_i \mid p_i \in \sigma,\ x \notin \mathfrak{v}(p_i)\}.$$

We omit $\sigma = \mathcal{V}$, saying simply *model* and writing $at_{\mathfrak{M}}(x)$. The *value* of a formula φ in $\mathfrak{M}$ is the set $\mathfrak{v}(\varphi) \in \mathcal{P}$ computed inductively in the obvious way starting from $\mathfrak{v}(p_i)$, $\mathfrak{v}(\top) = W$ and $\mathfrak{v}(\bot) = \emptyset$. A set $X \subseteq W$ is *definable* in $\mathfrak{M}$ if

$X = \mathfrak{v}(\varphi)$, for some formula φ, in which case $X \in \mathcal{P}$. If every internal set $X \in \mathcal{P}$ is definable in $\mathfrak{M}$, we say that $\mathfrak{F}$ is *$\mathfrak{M}$-generated*. Every $\mathfrak{F}$ with countable $\mathcal{P}$ is clearly $\mathfrak{M}$-generated, for some model $\mathfrak{M}$.

A formula φ is *true* at x in $\mathfrak{M}$ if $x \in \mathfrak{v}(\varphi)$, also written $\mathfrak{M}, x \models \varphi$. The *$\sigma$-type of x in $\mathfrak{M}$* is the set $t^{\sigma}_{\mathfrak{M}}(x)$ of all σ-formulas that are true at x in $\mathfrak{M}$. For a set X of points in $\mathfrak{M}$, we let $t^{\sigma}_{\mathfrak{M}}(X) = \{t^{\sigma}_{\mathfrak{M}}(x) \mid x \in X\}$. As before, we drop $\sigma = \mathcal{V}$.

A set Γ of formulas is *finitely satisfiable* in $\mathfrak{M}$ if, for every finite subset $\Gamma' \subseteq \Gamma$, there is $x' \in W$ such that $\Gamma' \subseteq t_{\mathfrak{M}}(x')$; Γ is *satisfiable in* $\mathfrak{M}$ if $\Gamma \subseteq t_{\mathfrak{M}}(x)$, for some $x \in W$. Using these definitions and notations, we can equivalently reformulate conditions **(dif)**, **(ref)**, and **(com)** for $\mathfrak{M}$-generated frames as follows: for any $x, y \in W$ and any set Γ of formulas,

(dif) $x = y$ iff $t_{\mathfrak{M}}(x) = t_{\mathfrak{M}}(y)$,
(ref) xRy iff $\Diamond t_{\mathfrak{M}}(y) \subseteq t_{\mathfrak{M}}(x)$ iff $\{\varphi \mid \Box\varphi \in t_{\mathfrak{M}}(x)\} \subseteq t_{\mathfrak{M}}(y)$,
(com) if Γ is finitely satisfiable in $\mathfrak{M}$, then Γ is satisfiable in $\mathfrak{M}$.

A frame $\mathfrak{F}$ *satisfies* Γ if there is a model $\mathfrak{M}$ based on $\mathfrak{F}$ satisfying Γ. Further, φ is *valid* in $\mathfrak{F}$, written $\mathfrak{F} \models \varphi$, if $\mathfrak{M}, x \models \varphi$ for any model $\mathfrak{M}$ based on $\mathfrak{F}$ and any $x \in W$. We call $\mathfrak{F}$ a *frame for* a logic L and write $\mathfrak{F} \models L$ if $\mathfrak{F} \models \varphi$ for all $\varphi \in L$. Conversely, any class $\mathcal{S}$ of general frames *determines* the modal logic $\mathsf{Log}\,\mathcal{S} = \{\varphi \mid \forall \mathfrak{F} \in \mathcal{S}\, \mathfrak{F} \models \varphi\}$. We write $\mathsf{Log}(\mathfrak{F})$ for $\mathsf{Log}(\{\mathfrak{F}\})$. If x is a point in $\mathfrak{F}$, then $\mathsf{Log}(\mathfrak{F}, x) = \{\varphi \mid \mathfrak{M}, x \models \varphi$ for all models $\mathfrak{M}$ based on $\mathfrak{F}\}$.

A set Γ of formulas is *L-consistent* if $(\bigwedge \Gamma' \to \bot) \notin L$, for any finite $\Gamma' \subseteq \Gamma$. We require the following well-known fact [8, 4]:

LEMMA 2.1. *For any modal logic L and any finite signature σ, if Σ is an L-consistent set of σ-formulas, then Σ is satisfiable in a σ-model $\mathfrak{M}$ based on a finitely $\mathfrak{M}$-generated descriptive frame for L.*

Denote by $\mathsf{Dfr}\,L$ and $\mathsf{Kfr}\,L$ the classes of all descriptive and Kripke frames for L, respectively. By Lemma 2.1, $L = \mathsf{Log}\,\mathsf{Dfr}\,L$, for every modal logic L. A logic L is *Kripke complete* if $L = \mathsf{Log}\,\mathsf{Kfr}\,L$. L is *d-persistent* (aka *canonical*) if $(W, R, \mathcal{P}) \models L$ implies $(W, R) \models L$, for any descriptive frame $(W, R, \mathcal{P})$. L has the *finite model property* (*fmp*) if it is determined by its finite (Kripke) frames.

The smallest logic $\mathsf{K4.3}$ we are interested in is d-persistent; its descriptive and Kripke frames $\mathfrak{F} = (W, R, \mathcal{P})$ are *transitive* and *weakly connected*, that is,

$$\forall x, y, z \in W\,(xRy \land yRz \to xRz),$$
$$\forall x, y, z \in W(xRy \land xRz \to y = z \lor yRz \lor zRy).$$

$\mathsf{GL.3}$, on the contrary, is not d-persistent yet has the fmp. In fact, all extensions of $\mathsf{K4.3}$ are Kripke complete [11]. From now on, all frames are assumed to be transitive and weakly connected. Such a frame $\mathfrak{F}$ is *rooted* if there is $r \in W$, a *root* of $\mathfrak{F}$, with $W = \{x \in W \mid rR^+x\}$, where R^+ is the *reflexive closure* of R, that is, $R^+ = R \cup \{(x, x) \mid x \in W\}$. Every rooted $\mathfrak{F}$ for $\mathsf{K4.3}$ is *connected*:

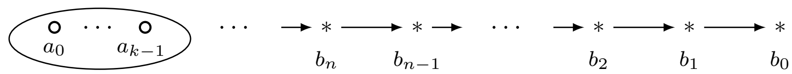
$$\forall x, y \in W\,\big(xRy \lor x = y \lor yRx\big). \tag{1}$$

A *cluster* in $\mathfrak{F}$ is any set of the form $C(x) = \{x\} \cup \{y \in W \mid xRy \land yRx\}$ with $x \in W$. If x is *irreflexive*, i.e., xRx does not hold, $C(x)$ is called a *degenerate cluster* and depicted as $\bullet$; a *reflexive* x (for which xRx) is depicted as $\circ$. A non-degenerated cluster with $k \geq 1$ (reflexive) points is depicted as $\textcircled{k}$. The next example illustrates the definitions and will be used many times in what follows.

EXAMPLE 2.2. Consider the frame $\mathfrak{F} = (W_k, R_{k\bullet}, \mathcal{P}_k)$, where $k > 0$,

$$W_k = A_k \cup \{b_n \mid n < \omega\}, \qquad A_k = \{a_0, \dots, a_{k-1}\},$$
$$xR_{k\bullet}y \text{ iff either } x = a_i \text{ or } x = b_n,\ y = b_m \text{ and } m < n,$$

and $\mathcal{P}_k$ is generated by the sets $X_i = \{a_i\} \cup \{b_n \mid n < \omega,\ n \equiv i \pmod{k}\}$, for $i < k$, and $\{b_n\}$, for $n < \omega$. The underlying Kripke frame $(W_k, R_{k\bullet})$ is shown in the picture below, where all $*$ are $\bullet$.



It is not hard to see that

$$\text{for any } X \in \mathcal{P}_k,\ X \text{ is infinite iff } A_k \cap X \neq \emptyset, \tag{2}$$

and so $A_k \notin \mathcal{P}_k$. (For instance, $\mathcal{P}_1$ consists of finite subsets of $\{b_n \mid n < \omega\}$ and their complements in W_1.) For every nonempty $X \in \mathcal{P}_k$, $\Diamond^{\mathfrak{F}} X$ is cofinite in W_k. Using these, it is readily checked that $\mathfrak{F}$ is a descriptive frame for $\mathsf{GL.3}$; we denote it by $\mathfrak{C}(\textcircled{k}, \bullet)$. Clearly, $\mathfrak{C}(\textcircled{k}, \bullet)$ is $\mathfrak{M}$-generated for $\mathfrak{M}$ with $\mathfrak{v}(p_i) = X_i$ if $i < k$, and $\mathfrak{v}(p_i) = \emptyset$ otherwise. The descriptive frame $(W_k, R_{k\circ}, \mathcal{P}_k)$ with $R_{k\circ} = R_{k\bullet} \cup \{(b_n, b_n) \mid n < \omega\}$ is denoted by $\mathfrak{C}(\textcircled{k}, \circ)$; $(W_k, R_{k\circ})$ looks like the picture above, where all $*$ are $\circ$. Note that $\mathfrak{C}(\textcircled{k}, \circ) \not\models \mathsf{GL.3}$. $\dashv$

Given frames $\mathfrak{F} = (W, R, \mathcal{P})$ and $\mathfrak{F}' = (W', R', \mathcal{P}')$, a surjection $f\colon W \to W'$ is a *p-morphism* from $\mathfrak{F}$ onto $\mathfrak{F}'$ if, for all $x, y \in W$ and $X' \in \mathcal{P}'$,

- xRy implies $f(x)R'f(y)$,
- $f(x)R'y$ implies that there is $z \in W$ with xRz and $f(z) = y$,
- $f^{-1}(X') \in \mathcal{P}$.

If $\mathfrak{F}$ and $\mathfrak{F}'$ are rooted and f is such that, for every root r' in $\mathfrak{F}'$, there is a root r in $\mathfrak{F}$ with $f(r) = r'$, we call f *root-mapping*. If there is a root-mapping p-morphism from $\mathfrak{F}$ onto $\mathfrak{F}'$, we write $\mathfrak{F} \rightarrowtail \mathfrak{F}'$. The following is well-known [8, 4]:

LEMMA 2.3. *If $\mathfrak{F} \rightarrowtail \mathfrak{F}'$, then $\mathsf{Log}(\mathfrak{F}) \subseteq \mathsf{Log}(\mathfrak{F}')$, and for every root r' in $\mathfrak{F}'$, there is a root r in $\mathfrak{F}$ with $\mathsf{Log}(\mathfrak{F}, r) \subseteq \mathsf{Log}(\mathfrak{F}', r')$.*

A frame $\mathfrak{F}' = (W', R', \mathcal{P}')$ is a *subframe* of a frame $\mathfrak{F} = (W, R, \mathcal{P})$ if $W' \subseteq W$, $R' = R\restriction_{W'} = R \cap (W' \times W')$, and $\mathcal{P}' \subseteq \mathcal{P}$. For every internal set $V \in \mathcal{P}$, the frame $\mathfrak{F}\restriction_V = \big(V, R\restriction_V, \mathcal{P}\restriction_V\big)$ with $\mathcal{P}\restriction_V = \{V \cap X \mid X \in \mathcal{P}\}$ is a subframe of $\mathfrak{F}$. For a model $\mathfrak{M} = (\mathfrak{F}, \mathfrak{v})$, we let $\mathfrak{M}\restriction_V = (\mathfrak{F}\restriction_V, \mathfrak{v}\restriction_V)$, where $\mathfrak{v}\restriction_V(p) = V \cap \mathfrak{v}(p)$. A *point-generated subframe* of $\mathfrak{F} = (W, R, \mathcal{P})$ takes the form $\mathfrak{F}_x = (W_x, R_x, \mathcal{P}_x)$, for some $x \in W$, with $W_x = \{x\} \cup \{y \in W \mid xRy\}$, $R_x = R\restriction_{W_x}$, and $\mathcal{P}_x = \mathcal{P}\restriction_{W_x}$. Any point-generated subframe of a descriptive transitive $\mathfrak{F}$ is also descriptive.

The next lemma, originating in [11], will play a key role in our subsequent constructions. Let $\mathfrak{M}$ be a model based on a rooted frame $\mathfrak{F} = (W, R, \mathcal{P})$ for $\mathsf{K4.3}$, and let Γ be a set of formulas. A point $x \in W$ is called Γ-*maximal in* $\mathfrak{M}$ if $\mathfrak{M}, x \models \Gamma$, and whenever xRy and $\mathfrak{M}, y \models \Gamma$, then yRx. We denote by $\max_{\mathfrak{M}} \Gamma$ the set of all Γ-maximal points in $\mathfrak{M}$.

LEMMA 2.4. *Suppose* Γ *is a set of formulas and* $\mathfrak{M}$ *a model based on a rooted descriptive frame* $\mathfrak{F} = (W, R, \mathcal{P})$ *for* $\mathsf{K4.3}$. *Then the following hold*:

(modal saturation) *if* $\mathfrak{M}, x \models \Diamond \bigwedge \Gamma'$ *for every finite* $\Gamma' \subseteq \Gamma$, *then there is* y *with* xRy *and* $\mathfrak{M}, y \models \Gamma$;

(maximal points) *if there is* x *with* $\mathfrak{M}, x \models \Gamma$, *then* $\max_{\mathfrak{M}} \Gamma \neq \emptyset$.

2.2. Building linear models from pieces. Let $\mathfrak{F} = (W, R, \mathcal{P})$ be a rooted frame for $\mathsf{K4.3}$. An *interval in* $\mathfrak{F}$ is any subset $I \subseteq W$ such that $xRyRz$ and $x, z \in I$ imply $y \in I$, for all $x, y, z \in W$. If $I \cap C \neq \emptyset$, for a cluster C, then clearly $C \subseteq I$. Let $R^s = \{(x, y) \in R \mid (y, x) \notin R\}$ be the *strict* R-accessibility in $\mathfrak{F}$. Sometimes it will be convenient to view (W, R) as a strict linear order[1] $\mathfrak{F}_c = (W_c, <_R)$ of clusters, where $W_c = \{C(x) \mid x \in W\}$ and $C(x) <_R C(y)$ iff xR^sy. A cluster C is *final* in $\mathfrak{F}$ if there is no cluster C' with $C <_R C'$. A cluster C is a *root cluster* if there is no cluster C' with $C' <_R C$, in which case $C <_R C'$ for every $C' \neq C$ in $\mathfrak{F}$. A cluster C' is an *immediate successor* of a cluster C in $\mathfrak{F}$ if $C <_R C'$ and there is no C'' with $C <_R C'' <_R C'$, in which case C is an *immediate predecessor* of C'. We require the following four types of intervals:

$$(C, C') = \bigcup \{D \mid C <_R D <_R C'\}, \qquad [C, C') = (C, C') \cup C,$$
$$(C, C'] = (C, C') \cup C', \qquad [C, C'] = (C, C'] \cup C.$$

Intervals of the form $[C, C']$ are called *closed*. Given two closed intervals I, I' in $\mathfrak{F}$, we write $I \prec_{\mathfrak{F}} I'$ if I and I' are disjoint and xRx' for all $x \in I$, $x \in I'$. Observe that if I is a closed internal interval in $\mathfrak{F}$, then $\mathfrak{F}\!\restriction_I$ is also a rooted frame for $\mathsf{K4.3}$. Also, if $\mathfrak{F}$ is descriptive, then $\mathfrak{F}\!\restriction_I$ is descriptive as well. And if $\mathfrak{F}$ is finitely $\mathfrak{M}$-generated for some model $\mathfrak{M}$, then $\mathfrak{F}\!\restriction_I$ is finitely $\mathfrak{M}\!\restriction_I$-generated.

DEFINITION 2.5. The *ordered sum* $\mathfrak{F}_0 \lhd \cdots \lhd \mathfrak{F}_{n-1} = (W, R, \mathcal{P})$ of rooted frames $\mathfrak{F}_i = (W_i, R_i, \mathcal{P}_i)$, $i < n$, for $\mathsf{K4.3}$ with pairwise disjoint W_i is defined by

$$W = \bigcup_{i<n} W_i, \quad R = \bigcup_{i<n} R_i \cup \bigcup_{i<j<n} (W_i \times W_j), \quad \mathcal{P} = \{X_0 \cup \cdots \cup X_{n-1} \mid X_i \in \mathcal{P}_i\}.$$

(It is not hard to see that if the $\mathfrak{F}_i$ are descriptive, then $\mathfrak{F}_0 \lhd \cdots \lhd \mathfrak{F}_{n-1}$ is also descriptive.) If $\mathfrak{M}_i = (\mathfrak{F}_i, \mathfrak{v}_i)$, then $\mathfrak{M}_0 \lhd \cdots \lhd \mathfrak{M}_{n-1}$ is the model based on $\mathfrak{F}_0 \lhd \cdots \lhd \mathfrak{F}_{n-1}$ with the valuation $\mathfrak{v}(p) = \bigcup_{i<n} \mathfrak{v}_i(p)$, for any $p \in \mathcal{V}$.

LEMMA 2.6. *If* $\mathsf{Log}(\mathfrak{F}_i) \subseteq \mathsf{Log}(\mathfrak{G}_i)$, $i < n$, *and for every root* y_i *in* $\mathfrak{G}_i$, *there is a root* z_i *in* $\mathfrak{F}_i$ *with* $\mathsf{Log}(\mathfrak{F}_i, z_i) \subseteq \mathsf{Log}(\mathfrak{G}_i, y_i)$, *then*

$$\mathsf{Log}(\mathfrak{F}_0 \lhd \cdots \lhd \mathfrak{F}_{n-1}) \subseteq \mathsf{Log}(\mathfrak{G}_0 \lhd \cdots \lhd \mathfrak{G}_{n-1})$$

and, for every root y_0 *in* $\mathfrak{G}_0$, *there is a root* z_0 *in* $\mathfrak{F}_0$ *with*

$$\mathsf{Log}(\mathfrak{F}_0 \lhd \cdots \lhd \mathfrak{F}_{n-1}, z_0) \subseteq \mathsf{Log}(\mathfrak{G}_0 \lhd \cdots \lhd \mathfrak{G}_{n-1}, y_0).$$

PROOF. Let $\mathfrak{F} = \mathfrak{F}_0 \lhd \cdots \lhd \mathfrak{F}_{n-1}$ and $\mathfrak{G} = \mathfrak{G}_0 \lhd \cdots \lhd \mathfrak{G}_{n-1}$. We show that if $\mathfrak{M}, y \models \varphi$, for some formula φ, model $\mathfrak{M}$ based on $\mathfrak{G}$ and point y in $\mathfrak{G}$, then there exist a model $\mathfrak{N}$ based on $\mathfrak{F}$ and a point z in $\mathfrak{F}$ such that $\mathfrak{N}, z \models \varphi$. By introducing abbreviations for nonatomic $\psi \in sub(\varphi)$, we may assume that $\varphi = p \wedge \Box^+ \chi$, where

[1] An irreflexive and transitive relation $<$ is a *strict linear order* if, for all $x \neq y$, we have either $x < y$ or $y < x$.

$p \in \mathcal{V}$ and χ is a conjunction of formulas of the form $p_1 \leftrightarrow \Diamond p_2$, $p_1 \leftrightarrow p_2 \wedge p_3$, and $p_1 \leftrightarrow \neg p_2$. Suppose $\mathfrak{F}_j = (W_j, R_j, \mathcal{P}_j)$, $\mathfrak{G}_j = (V_j, S_j, \mathcal{P}'_j)$, $\mathfrak{M}_j = \mathfrak{M}{\restriction}_{V_j}$, $j < n$, and $y \in V_i$. We let $y_i = y$, and for every j with $i < j < n$, we pick a root y_j in $\mathfrak{G}_j$. For $j \geq i$, we set

$$t_j = at^{sig(\varphi)}_{\mathfrak{M}}(y_j), \qquad T_j = \big\{ at^{sig(\varphi)}_{\mathfrak{M}}(x) \mid x \in V_j,\ y_j R_j^+ x \big\},$$

and define φ_j to be the conjunction of

- $\Box \bigvee_{t \in T_j} \bigwedge t$,
- $\bigwedge_{t \in T_j} \Diamond \bigwedge t$,
- $\bigwedge \Box^+ \psi$, where $\mathfrak{M}_j, y_j \models \Box^+ \psi$ and ψ is of the form $p_1 \to \Diamond p_2$ or $\Diamond p_1 \to p_2$.

Then we clearly have $\mathfrak{M}_j, y_j \models \varphi_j \wedge \bigwedge t_j$, for $j \geq i$.

By our assumptions, there exist $z_i \in W_i$, roots z_j in $\mathfrak{F}_j$, for $j > i$, and models $\mathfrak{N}_j$ based on $\mathfrak{F}_j$, for $j \geq i$, such that $\mathfrak{N}_j, z_j \models \varphi_j \wedge \bigwedge t_j$, for all $j \geq i$. (Moreover, if y_i is a root in $\mathfrak{G}_i$ then z_i can be chosen to be a root in $\mathfrak{F}_i$.) Now, for $j < i$, take arbitrary models $\mathfrak{N}_j$ based on $\mathfrak{F}_j$, and let $\mathfrak{N} = \mathfrak{N}_0 \lhd \cdots \lhd \mathfrak{N}_{n-1}$. We claim that $\mathfrak{N}, z_i \models \varphi$. To see this, consider a conjunct of χ of the form $p_1 \to \Diamond p_2$. As $\mathfrak{M}, y_i \models \varphi$, we have $\mathfrak{M}, y_i \models \Box^+(p_1 \to \Diamond p_2)$. We want to show that $\mathfrak{N}, z_i \models \Box^+(p_1 \to \Diamond p_2)$. So suppose $\mathfrak{N}, x \models p_1$, for some $x \in W_j$ with $j \geq i$ and $z_j R_j^+ x$. If $\mathfrak{N}_j, z_j \models \Box^+(p_1 \to \Diamond p_2)$, then there is $w \in W_j$ such that $x R_j w$ and $\mathfrak{N}_j, w \models p_2$. So we have $\mathfrak{N}, x \models \Diamond p_2$, as required. If $\mathfrak{N}_j, z_j \not\models \Box^+(p_1 \to \Diamond p_2)$, then $\Box^+(p_1 \to \Diamond p_2)$ is not a conjunct of φ_j, and so $\mathfrak{M}_j, y_j \not\models \Box^+(p_1 \to \Diamond p_2)$. But, as $\mathfrak{M}, y_i \models \Box^+(p_1 \to \Diamond p_2)$, there exist $k > j$ and $v \in V_k$ with $\mathfrak{M}, v \models p_2$. As $p_2 \in at^{sig(\varphi)}_{\mathfrak{M}}(v)$ and $at^{sig(\varphi)}_{\mathfrak{M}}(v) \in T_k$, $\Diamond p_2$ is implied by φ_k. As $\mathfrak{N}_k, z_k \models \varphi_k$, there is $w \in W_k$ with $\mathfrak{N}_k, w \models p_2$, and so $\mathfrak{N}, x \models \Diamond p_2$, as required.

The remaining types of conjuncts of χ are considered in a similar way. ⊣

We also clearly have the following:

Lemma 2.7. *Let I be an interval in a frame $\mathfrak{F}$ that is partitioned as $\{I_j \mid j < n\}$, $n < \omega$, with the I_j being internal intervals in $\mathfrak{F}$ and $I_j \prec_{\mathfrak{F}} I_k$ iff $j < k$. Then*

(a) $\mathfrak{F}{\restriction}_I = \mathfrak{F}{\restriction}_{I_0} \lhd \cdots \lhd \mathfrak{F}{\restriction}_{I_{n-1}}$;

(b) *if $\mathfrak{M}$ is a model based on $\mathfrak{F}$, then* $\mathfrak{M}{\restriction}_I = \mathfrak{M}{\restriction}_{I_0} \lhd \cdots \lhd \mathfrak{M}{\restriction}_{I_{n-1}}$.

2.3. Canonical formulas. As shown in [38, 34, 8] (see also [3]), every finitely axiomatisable logic $L \supseteq \mathsf{K4.3}$ can be efficiently represented in the form

$$L = \mathsf{K4.3} \oplus \{\alpha(\mathfrak{G}_i, \mathfrak{D}_i, \bot) \mid i < m_L\}, \quad \text{for some } m_L < \omega, \tag{3}$$

where each $\alpha(\mathfrak{G}_i, \mathfrak{D}_i, \bot)$ is a (*canonical*) formula based on a finite rooted Kripke frame $\mathfrak{G}_i = (V_i, S_i)$ for $\mathsf{K4.3}$ and a (possibly empty) set $\mathfrak{D}_i \subseteq V_i$ of *irreflexive* points in $\mathfrak{G}_i$. The formulas $\alpha(\mathfrak{G}_i, \mathfrak{D}_i, \bot)$ are constructed so that, for any finitely generated descriptive frame $\mathfrak{F} = (W, R, \mathcal{P})$ for $\mathsf{K4.3}$, we have $\mathfrak{F} \not\models \alpha(\mathfrak{G}_i, \mathfrak{D}_i, \bot)$ iff there is an injective function $f \colon V_i \to W$ with the following properties, for all $x, y \in V_i$:

- $x S_i y$ iff $f(x) R f(y)$;
- if $C(x)$ is the final cluster in $\mathfrak{G}_i$, then $C(f(x))$ is the final cluster in $\mathfrak{F}$;
- if $x \in \mathfrak{D}_i$ and $C(y)$ is the immediate predecessor of $\{x\}$ in $\mathfrak{G}_i$, then $C(f(y))$ is the immediate predecessor of $\{f(x)\}$ in $\mathfrak{F}$;
- $\{f(x)\} \in \mathcal{P}$.

For example,

$$\mathsf{GL.3} = \mathsf{K4.3} \oplus \alpha(\circ, \emptyset, \bot) \oplus \alpha(\circ \lhd \bullet, \emptyset, \bot),$$
$$\mathsf{Log}\{(\mathbb{N}, <)\} = \mathsf{K4.3} \oplus \alpha(\bullet, \emptyset, \bot) \oplus \alpha(\circ \lhd \circ, \emptyset, \bot).$$

Canonical formulas of the form $\alpha(\mathfrak{G}, \emptyset, \bot)$ axiomatise exactly *cofinal subframe logics* whose frames are closed under taking cofinal subframes. We remind the reader [8] that a subframe $\mathfrak{F}' = (W', R', \mathcal{P}')$ of a frame $\mathfrak{F} = (W, R, \mathcal{P})$ is called *cofinal* if W' is cofinal in $\mathfrak{F}$ in the sense that, for any $x \in W'$ and $y \in W$, whenever xRy then either $y \in W'$ or there is $z \in W'$ with yRz. Cofinal subframe logics enjoy the fmp, and so are decidable if finitely axiomatisable [37]. A prominent example of a non-cofinal subframe logic is $\mathsf{K4.3} \oplus \Diamond p \to \Diamond\Diamond p$; see Section 5.

§3. Craig interpolant existence: warming up. A formula ι is called a *Craig interpolant* of formulas φ_1 and φ_2 in a logic L if $sig(\iota) \subseteq sig(\varphi_1) \cap sig(\varphi_2)$ and both $\varphi_1 \to \iota$ and $\iota \to \varphi_2$ are in L. We say that L has the *Craig interpolation property* (*CIP*) if an interpolant for φ_1 and φ_2 exists whenever $(\varphi_1 \to \varphi_2) \in L$.

Many standard modal logics have the CIP, including K, $\mathsf{K4}$, $\mathsf{S4}$. In fact, there are a continuum of logics containing $\mathsf{K4}$ with the CIP. However, none of the continuum-many extensions of $\mathsf{K4.3}$ with frames of unbounded depth has the CIP, and very few—not more than 37—out of the continuum-many logics containing $\mathsf{S4}$ enjoy the CIP (deciding whether a finitely axiomatisable logic above $\mathsf{S4}$ has the CIP is in coNExpTime and PSpace-hard). The reader can find proofs of these results and further references in [13, 8].

We now introduce the model-theoretic notions and tools that are needed in our non-uniform approach to deciding interpolant existence in modal logics.

Given two models $\mathfrak{M}_i$, $i = 1, 2$, based on $\mathfrak{F}_i = (W_i, R_i, \mathcal{P}_i)$ with $x_i \in W_i$, we write $\mathfrak{M}_1, x_1 \equiv_\sigma \mathfrak{M}_2, x_2$, for a signature σ, if $t^\sigma_{\mathfrak{M}_1}(x_1) = t^\sigma_{\mathfrak{M}_2}(x_2)$. The equivalence relation $\equiv_\sigma \subseteq W_1 \times W_2$ can be characterised in terms of bisimulations. Namely, a relation $\boldsymbol{\beta} \subseteq W_1 \times W_2$ is called a *σ-bisimulation* between $\mathfrak{M}_1$ and $\mathfrak{M}_2$ if the following conditions hold whenever $x_1 \boldsymbol{\beta} x_2$:

(atom) $at^\sigma_{\mathfrak{M}_1}(x_1) = at^\sigma_{\mathfrak{M}_2}(x_2)$;

(move) if $x_1 R_1 y_1$, then there is y_2 such that $x_2 R_2 y_2$ and $y_1 \boldsymbol{\beta} y_2$; and, conversely, if $x_2 R_2 y_2$, then there is y_1 with $x_1 R_1 y_1$ and $y_1 \boldsymbol{\beta} y_2$.

If there is such $\boldsymbol{\beta}$ with $z_1 \boldsymbol{\beta} z_2$, we write $\mathfrak{M}_1, z_1 \sim_\sigma \mathfrak{M}_2, z_2$. We call $\boldsymbol{\beta}$ *global* if, for every $x_1 \in W_1$, there is $x_2 \in W_2$ with $x_1 \boldsymbol{\beta} x_2$, and, for every $x_2 \in W_2$, there is $x_1 \in W_1$ with $x_1 \boldsymbol{\beta} x_2$. In this case, we say that $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are *globally σ-bisimilar* and write $\mathfrak{M}_1 \sim_\sigma \mathfrak{M}_2$.

We employ the following characterisation of $\equiv_\sigma$ (see [18] for a further discussion of the relationship between bisimulations and modal equivalence):

Lemma 3.1. *For any signature σ, any models $\mathfrak{M}_i$, $i = 1, 2$, based on descriptive frames $\mathfrak{F}_i = (W_i, R_i, \mathcal{P}_i)$, and any $x_i \in W_i$,*

$$\mathfrak{M}_1, x_1 \equiv_\sigma \mathfrak{M}_2, x_2 \quad \textit{iff} \quad \mathfrak{M}_1, x_1 \sim_\sigma \mathfrak{M}_2, x_2.$$

The implication ($\Leftarrow$) holds for arbitrary models.

Proof. ($\Rightarrow$) We show that $\{(y_1, y_2) \in W_1 \times W_2 \mid t^\sigma_{\mathfrak{M}_1}(y_1) = t^\sigma_{\mathfrak{M}_2}(y_2)\}$ is a σ-bisimulation between $\mathfrak{M}_1$ and $\mathfrak{M}_2$. Condition **(atom)** is obvious. For **(move)**, suppose $y_1 R_1 z_1$ and $t^\sigma_{\mathfrak{M}_1}(y_1) = t^\sigma_{\mathfrak{M}_2}(y_2)$. Let $\Gamma = t^\sigma_{\mathfrak{M}_1}(z_1)$. Then, for every finite $\Gamma' \subseteq \Gamma$, we have $\mathfrak{M}_1, y_1 \models \Diamond \bigwedge \Gamma'$, and so $\mathfrak{M}_2, y_2 \models \Diamond \bigwedge \Gamma'$ as well. Since $\mathfrak{F}_2$ is descriptive, Lemma 2.4 gives us z_2 with $y_2 R_2 z_2$ and $\mathfrak{M}_2, z_2 \models \Gamma$. It follows that $t^\sigma_{\mathfrak{M}_1}(z_1) = t^\sigma_{\mathfrak{M}_2}(z_2)$, as required. The implication ($\Leftarrow$) is straightforward. $\dashv$

Note that if $\mathcal{B}$ is a set of σ-bisimulations between $\mathfrak{M}_1$ and $\mathfrak{M}_2$, then $\bigcup_{\boldsymbol{\beta} \in \mathcal{B}} \boldsymbol{\beta}$ is also a σ-bisimulation between $\mathfrak{M}_1$ and $\mathfrak{M}_2$. It follows that there is always a *largest* σ-bisimulation between $\mathfrak{M}_1$ and $\mathfrak{M}_2$ (which is $\equiv_\sigma$ if both $\mathfrak{M}_i$ are based on descriptive frames).

Variations of the following criterion of interpolant (non-)existence are implicit in various (dis-)proofs of the CIP in modal logics [28, 18].

THEOREM 3.2. *Formulas φ_1 and φ_2 do not have an interpolant in a modal logic L iff there are models $\mathfrak{M}_i$, $i = 1, 2$, based on finitely $\mathfrak{M}_i$-generated descriptive frames $\mathfrak{F}_i = (W_i, R_i, \mathcal{P}_i)$ for L with roots $x_i \in W_i$ such that*

- $\mathfrak{M}_1, x_1 \models \varphi_1$,
- $\mathfrak{M}_2, x_2 \models \neg\varphi_2$,
- $\mathfrak{M}_1, x_1 \sim_\sigma \mathfrak{M}_2, x_2$, *where* $\sigma = sig(\varphi_1) \cap sig(\varphi_2)$.

PROOF. ($\Leftarrow$) is straightforward (and holds for arbitrary frames for L). For ($\Rightarrow$), consider the signature $\delta = sig(\varphi_1) \cup sig(\varphi_2)$ and the set

$$\Sigma = \{\chi \mid \chi \text{ is a } \sigma\text{-formula and } (\varphi_1 \to \chi) \in L\} \cup \{\neg\varphi_2\}$$

of δ-formulas. As φ_1 and φ_2 have no interpolant in L, Σ is L-consistent, and so, by Lemma 2.1, there exist a δ-model $\mathfrak{M}_2$ based on a finitely $\mathfrak{M}_2$-generated descriptive frame and a point x_2 with $\mathfrak{M}_2, x_2 \models \Sigma$. Let $\Sigma' = t^\sigma_{\mathfrak{M}_2}(x_2) \cup \{\varphi_1\}$. As Σ' is an L-consistent set of δ-formulas, Lemma 2.1 gives a δ-model $\mathfrak{M}_1$ based on a finitely $\mathfrak{M}_1$-generated descriptive frame and an x_1 in $\mathfrak{M}_1$ such that $\mathfrak{M}_1, x_1 \models \Sigma'$. We clearly have $t^\sigma_{\mathfrak{M}_1}(x_1) = t^\sigma_{\mathfrak{M}_2}(x_2)$, and so $\mathfrak{M}_1, x_1 \sim_\sigma \mathfrak{M}_2, x_2$ by Lemma 3.1. $\dashv$

We begin our study of the interpolant existence problem (IEP) by showing how the criterion of Theorem 3.2 can be used to decide whether given formulas have an interpolant in a given d-persistent cofinal subframe logic $L \supseteq \mathsf{K4.3}$; see Section 2.3. Suppose φ_1 and φ_2 do not have an interpolant in L. Let $\sigma = sig(\varphi_1) \cap sig(\varphi_2)$. By Theorem 3.2, there are models $\mathfrak{M}_i$, $i = 1, 2$, based on descriptive frames $\mathfrak{F}_i = (W_i, R_i, \mathcal{P}_i)$ for L with roots $x_i \in W_i$ such that $\mathfrak{M}_1, x_1 \sim_\sigma \mathfrak{M}_2, x_2$, $\mathfrak{M}_1, x_1 \models \varphi_1$ and $\mathfrak{M}_2, x_2 \models \neg\varphi_2$. We may assume that $\boldsymbol{\beta}$ is the largest σ-bisimulation $\equiv_\sigma$ between $\mathfrak{M}_1$ and $\mathfrak{M}_2$ (for which $x_1 \boldsymbol{\beta} x_2$ of course). We show how to extract from the $\mathfrak{M}_i$ polynomial-size models $\mathfrak{M}'_i$ that still witness that φ_1 and φ_2 lack an interpolant in L. We proceed in two steps.

Step 1: For each $i = 1, 2$ and each $\tau \in sub(\varphi_i)$ satisfied in $\mathfrak{M}_i$, we take a $\{\tau\}$-maximal point $y_\tau \in W_i$ (which exists by Lemma 2.4), and denote the set of all these y_τ by $M_i \subseteq W_i$. Note that M_i is cofinal in $\mathfrak{F}_i$ because each point in $W_i \setminus M_i$ has a $\{\varphi_i\}$- or $\{\neg\varphi_i\}$-maximal R_i-successor. Set

$$T = \big\{t^\sigma_{\mathfrak{M}_1}(x) \mid x \in \{x_1\} \cup M_1\big\} \cup \big\{t^\sigma_{\mathfrak{M}_2}(x) \mid x \in \{x_2\} \cup M_2\big\}. \tag{4}$$

Step 2: As $\mathfrak{M}_1, x_1 \sim_\sigma \mathfrak{M}_2, x_2$ and $\boldsymbol{\beta}$ is the largest σ-bisimulation, each $t \in T$ is satisfied in both $\mathfrak{M}_i$. For $i = 1, 2$, we take a smallest set $S_i \subseteq W_i$ containing a t-maximal point z_t in $\mathfrak{M}_i$ (which exists by Lemma 2.4), for each $t \in T$.

Now, let $W_i' = \{x_i\} \cup M_i \cup S_i$, $R_i' = R_i \restriction_{W_i'}$, $\mathfrak{F}_i' = (W_i', R_i')$, and let $\mathfrak{M}_i'$ be the restriction of $\mathfrak{M}_i$ to $\mathfrak{F}_i'$. As L is d-persistent, $(W_i, R_i) \models L$. By the construction, $\mathfrak{F}_i'$ is a cofinal subframe of (W_i, R_i), and so $\mathfrak{F}_i' \models L$ since L is a cofinal subframe logic. Clearly, $|W_i'| \leq 3+3\max(|\varphi_1|, |\varphi_2|)$. Finally, we define $\boldsymbol{\beta}'$ as the restriction of $\boldsymbol{\beta}$ to $W_1' \times W_2'$, that is, $x_1'\boldsymbol{\beta}'x_2'$ iff $t^\sigma_{\mathfrak{M}_1}(x_1') = t^\sigma_{\mathfrak{M}_2}(x_2')$, for all $x_1' \in W_1'$, $x_2' \in W_2'$.

Lemma 3.3. *(a) $\mathfrak{M}_1', x_1 \models \varphi_1$, $\mathfrak{M}_2', x_2 \models \neg\varphi_2$, and (b) $\boldsymbol{\beta}'$ is a σ-bisimulation between $\mathfrak{M}_1'$ and $\mathfrak{M}_2'$ with $x_1\boldsymbol{\beta}'x_2$.*

Proof. (a) follows from the fact that, for any $\tau \in \mathit{sub}(\varphi_i)$ and $x \in W_i'$, $\mathfrak{M}_i, x \models \tau$ iff $\mathfrak{M}_i', x \models \tau$, which can be established by a straightforward induction on the construction of φ_1 and φ_2. We only show $(\Rightarrow)$ for $\tau = \Diamond\psi$. If $\mathfrak{M}_i, x \models \Diamond\psi$, then there is $y \in W_i$ with xR_iy and $\mathfrak{M}_i, y \models \psi$. Take $y_\psi \in M_i \subseteq W_i'$. By the $\{\psi\}$-maximality of y_ψ, either $y = y_\psi$ or yR_iy_ψ, and so $xR_i'y_\psi$ and $\mathfrak{M}_i', x \models \Diamond\psi$.

(b) Condition **(atom)** follows from the definition. To establish **(move)**, assume $x\boldsymbol{\beta}'x'$ and $xR_1'y$. Let $t = t^\sigma_{\mathfrak{M}_1}(y)$. Then $t \in T$, and so there is a t-maximal $z_t \in S_2 \subseteq W_2'$ in $\mathfrak{M}_2$. In particular, $t^\sigma_{\mathfrak{M}_2}(z_t) = t$, and so $y\boldsymbol{\beta}'z_t$. As $x\boldsymbol{\beta}x'$ and $\boldsymbol{\beta}$ is the largest σ-bisimulation, there is $z \in W_2$ with $x'R_2z$ and $t^\sigma_{\mathfrak{M}_2}(z) = t$. It follows from the t-maximality of z_t that $z = z_t$ or zR_2z_t, and so $x'R_2'z_t$, as required. ⊣

Thus, the fact that φ_1 and φ_2 have no interpolant in L can always be witnessed (in the sense of Theorem 3.2) by models $\mathfrak{M}_i$ of size polynomial in $\max(|\varphi_1|, |\varphi_2|)$, and so we can say that L has the *polysize bisimilar model property*. This gives the first claim of the following theorem:

Theorem 3.4. *(a) All d-persistent cofinal subframe logics $L \supseteq \mathsf{K4.3}$ have the polysize bisimilar model property. (b) If such an L is consistent and finitely axiomatisable, then the IEP for L is coNP-complete.*

Proof. It is readily seen that $(a) \Rightarrow (b)$; cf. Theorem 4.6 in Section 4. Indeed, suppose L is given by (3) (with $\mathfrak{D}_i = \emptyset$, for all $i < m_L$). To decide whether formulas φ_1 and φ_2 do not have an interpolant in L, we guess polynomial-size models $\mathfrak{M}_i$ based on $\mathfrak{F}_i$ and restricted to the variables in φ_1 and φ_2. Checking the first two items in Theorem 3.2 needs polynomial time; the third one is also polynomially checkable using dynamic programming. Finally, we check $\mathfrak{F}_i \models L$ in polynomial time using the refutability criterion for canonical formulas. ⊣

We now give examples explaining why the construction above does not work for logics that are not d-persistent. Prominent specimens of such logics are $\mathsf{GL.3}$ and $\mathsf{Log}\{(\mathbb{N}, <)\}$. Kripke frames $\mathfrak{F} = (W, R)$ for $\mathsf{GL.3}$ do not contain infinite ascending R-chains $x_0Rx_1Rx_2R\dots$ of not necessarily distinct points; in other words, rooted Kripke frames for $\mathsf{GL.3}$ are finite chains of $\bullet$. $\mathsf{Log}\{(\mathbb{N}, <)\}$ is determined by the class of finite *lassos* (aka *balloons*)—finite chains of $\bullet$ followed by some non-degenerate cluster ⓚ, for $k \geq 1$. (See [16, 8] for more details.)

Example 3.5. Consider the following formulas φ_1 and φ_2:

$$\varphi_1 = \Diamond(p_1 \wedge \Diamond^+\neg q_1) \wedge \Box(p_2 \to \Box^+ q_1) \wedge \Box(p_1 \to \neg p_2),$$

$$\varphi_2 = \neg[\Diamond(p_2 \wedge \Diamond^+\neg q_2) \wedge \Box(p_1 \to \Box^+ q_2)]. \tag{5}$$

It is not hard to see that $(\varphi_1 \to \varphi_2) \in \mathsf{K4.3} \subseteq \mathsf{GL.3}$. Indeed, suppose otherwise. Then there exist a model $\mathfrak{M}$ based on a frame $\mathfrak{F} = (W, R)$ for $\mathsf{K4.3}$ and $z \in W$

such that $\mathfrak{M}, z \models \varphi_1 \wedge \neg\varphi_2$. We then have $zRxR^+x'$ and $zRyR^+y'$, for some $x, x', y, y' \in W$, with

$$\mathfrak{M}, x \models p_1 \wedge \neg p_2, \quad \mathfrak{M}, x' \models \neg q_1 \quad \text{and} \quad \mathfrak{M}, y \models p_2 \wedge \neg p_1, \quad \mathfrak{M}, y' \models \neg q_2.$$

Since $\mathfrak{F}$ is weakly connected, either xRy or yRx. However, neither of these is possible in view of the boxed subformulas of φ_1 and φ_2 according to which xRy implies $\mathfrak{M}, y' \models q_2$, and yRx implies $\mathfrak{M}, x' \models q_1$.

We now use Theorem 3.2 to show that φ_1 and φ_2 do not have an interpolant in $\mathsf{GL.3}$. Let $\sigma = sig(\varphi_1) \cap sig(\varphi_2) = \{p_1, p_2\}$. Observe that any models $\mathfrak{M}_i$ meeting the conditions of Theorem 3.2 cannot be based on a Kripke frame. Indeed, suppose $\boldsymbol{\beta}$ is the corresponding bisimulation. Then $\mathfrak{M}_1, x_1 \models \varphi_1$ implies that there is $x_1^1 \in W_1$ with $x_1 R_1 x_1^1$ and $\mathfrak{M}_1, x_1^1 \models p_1$; we must also have $\mathfrak{M}_1, y_1 \models \neg q_1$, for some y_1 with $x_1^1 R_1^+ y_1$. Similarly, $\mathfrak{M}_2, x_2 \models \neg\varphi_2$ implies that there is $x_2^1 \in W_2$ with $x_2 R_1 x_2^1$ and $\mathfrak{M}_2, x_2^1 \models p_2$, and we also have $\mathfrak{M}_2, y_2 \models \neg q_2$, for some y_2 with $x_2^1 R_2^+ y_2$. Since $x_1 \boldsymbol{\beta} x_2$ and $x_1 R_1 x_1^1$, there is x_2^2 with $x_2 R_2 x_2^2$ and $x_1^1 \boldsymbol{\beta} x_2^2$. But then $\mathfrak{M}_2, x_2^2 \models p_1$, and so $x_2 R_2 x_2^1 R_2 x_2^2$ because $\mathfrak{F}_2$ is a frame for $\mathsf{K4.3}$ and in view of the second conjunct of φ_2. Symmetrically, we find $x_1 R_1 x_1^1 R_1 x_1^2$ with $x_1^2 \boldsymbol{\beta} x_2^1$. Using **(move)**, we construct infinite ascending chains of (non-necessarily distinct) points as shown in Fig. 1. It follows that neither $\mathfrak{F}_1$ nor $\mathfrak{F}_2$ is a frame for $\mathsf{GL.3}$.

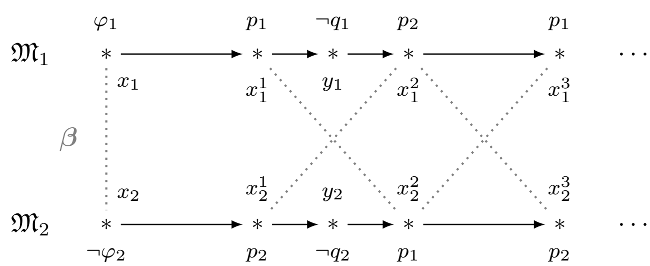


FIGURE 1. Infinite ascending chains in σ-bisimilar models $\mathfrak{M}_i$.

We now construct a descriptive frame for $\mathsf{GL.3}$ that can be used to show that φ_1 and φ_2 do not have an interpolant in $\mathsf{GL.3}$. Take the descriptive frame $\mathfrak{C}(②, \bullet)$ defined in Example 2.2 and consider the frame $\mathfrak{F} = \bullet \lhd \bullet \lhd \mathfrak{C}(②, \bullet)$ (see Definition 2.5). It is readily seen that $\mathfrak{F}$ is a frame for $\mathsf{GL.3}$. Indeed, suppose $\boldsymbol{la} = \Box(\Box p \to p) \to \Box p$ and there is a model $\mathfrak{M} = (\mathfrak{F}, \mathfrak{v})$ for which $\mathfrak{v}(\neg\boldsymbol{la}) \neq \emptyset$. As $\mathfrak{v}(\neg\boldsymbol{la})$ is an internal set in $\mathfrak{F}$, every $x \in \mathfrak{v}(\neg\boldsymbol{la})$ has a successor in $\mathfrak{v}(\neg\boldsymbol{la})$. On the other hand, by Lemma 2.4, $\mathfrak{v}(\neg\boldsymbol{la})$ has a maximal point, which can only be a_0 or a_1, contrary to the definition of $\mathfrak{F}$.

Consider the models $\mathfrak{M}_i$ with root x_i, $i = 1, 2$, shown in Fig. 2, both of which are based on a frame isomorphic to $\bullet \lhd \bullet \lhd \mathfrak{C}(②, \bullet)$. It is readily checked that $\mathfrak{M}_1, x_1 \models \varphi_1$, $\mathfrak{M}_2, x_2 \models \neg\varphi_2$, and the depicted relation $\boldsymbol{\beta}$ is a σ-bisimulation between $\mathfrak{M}_1$ and $\mathfrak{M}_2$ with $x_1 \boldsymbol{\beta} x_2$. ⊣

EXAMPLE 3.6. Consider now the logic $\mathsf{Log}\{(\mathbb{N}, <)\}$ and show that the formulas

$$\varphi_1' = \Diamond(p_1 \wedge \Diamond^+ \neg q_1) \wedge \Box(p_2 \to \Box^+ q_1) \wedge \Box(p_1 \to \neg p_2) \wedge \Diamond r \wedge \neg\Diamond(r \wedge \Diamond p_1)$$

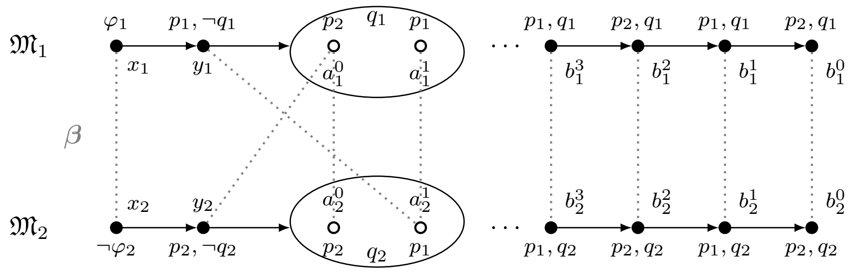
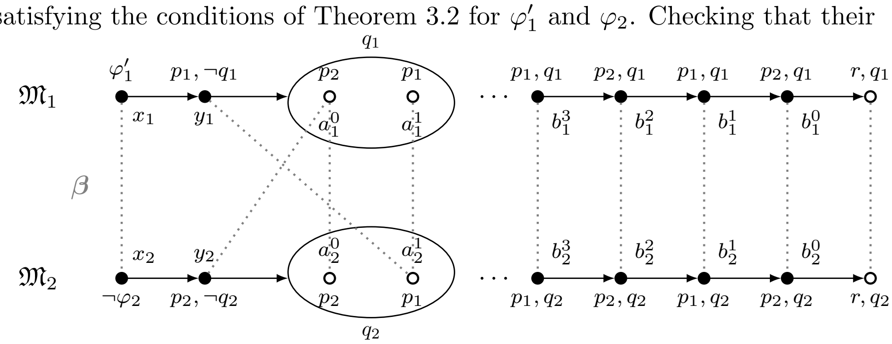


FIGURE 2. σ-bisimilar models based on a descriptive frame for $\mathsf{GL.3}$.

and φ_2 given by (5) do not have an interpolant in it. It is easy to see that $\varphi_1' \to \varphi_2$ is valid in all finite lassos, and so $(\varphi_1' \to \varphi_2) \in \mathsf{Log}\{(\mathbb{N}, <)\}$.

As in Example 3.5, any models $\mathfrak{M}_i$, $i = 1, 2$, satisfying the conditions of Theorem 3.2 for φ_1' and φ_2 cannot be based on Kripke frames, however the reason for this is slightly different. Suppose $\boldsymbol{\beta}$ is a bisimulation witnessing these conditions. Then the models $\mathfrak{M}_i$ must contain infinite ascending chains such as those in Fig. 1. Also, the model $\mathfrak{M}_1$ with $\mathfrak{M}_1, x_1 \models \varphi_1'$ must contain a point z such that $x_1 R_1 z$ and $\mathfrak{M}_1, z \models r \wedge \Box\neg p_1$, which means that z is located after all of the x_1^j, $j < \omega$. But then the Kripke frame $\mathfrak{F}_1$ underlying $\mathfrak{M}_1$ is not a frame for $\mathsf{Log}\{(\mathbb{N}, <)\}$, as it refutes its axiom $\Box(\Box p \to p) \to (\Diamond\Box p \to \Box p)$ if we make p true everywhere after the initial ascending chain in $\mathfrak{F}_1$ and false elsewhere.

The picture below shows models $\mathfrak{M}_1$ and $\mathfrak{M}_2$ based on $\bullet \lhd \bullet \lhd \mathfrak{C}(②, \bullet) \lhd \circ$ and satisfying the conditions of Theorem 3.2 for φ_1' and φ_2. Checking that their



underlying frame is a frame for $\mathsf{Log}\{(\mathbb{N}, <)\}$ is left to the reader. $\dashv$

Thus, establishing model-theoretically that given formulas do not have an interpolant even in logics with the fmp may require infinite descriptive frames. Fortunately, the structure of the required frames is perfectly understandable.

§4. Interpolant existence in logics above $\mathsf{K4.3}$. In this section, we generalise Theorem 3.4 to all finitely axiomatisable logics containing $\mathsf{K4.3}$.

4.1. The quasi-polysize bisimilar model property. Suppose $L \supseteq \mathsf{K4.3}$ and φ_1, φ_2 are formulas without an interpolant in L. For $0 < m < \omega$, we let $m^< = \underbrace{\bullet \lhd \cdots \lhd \bullet}_{m}$. An *atomic frame* for φ_1, φ_2 takes one of the forms

$$m^<, \quad ① \lhd m^<, \quad ⓚ, \quad \mathfrak{C}(ⓚ, \bullet), \quad \mathfrak{C}(ⓚ, \circ), \tag{6}$$

where $0 < m < \omega$ and $0 < k = \mathcal{O}\big(\max(|\varphi_1|, |\varphi_2|)\big)$ (in fact, as we shall see below, $k \le 3 + 3\max(|\varphi_1|, |\varphi_2|)$). If $m = 0$, $m^<$ is regarded to be empty. The *size* $|\mathfrak{F}|$ of an atomic frame $\mathfrak{F}$ is defined by taking $|m^<| = m$, $|\textcircled{1} \lhd m^<| = 1 + m$, and $|\textcircled{k}| = |\mathfrak{C}(\textcircled{k}, \bullet)| = |\mathfrak{C}(\textcircled{k}, \circ)| = k$. For a model $\mathfrak{M}$ based on $\mathfrak{F}$ we set $|\mathfrak{M}| = |\mathfrak{F}|$.

A model $\mathfrak{M} = (\mathfrak{F}, \mathfrak{v})$ based on atomic $\mathfrak{F}$ and its valuation $\mathfrak{v}$ are *simple* if either $\mathfrak{F}$ is finite or, for every $p \in \mathcal{V}$, there is $A_p \subseteq \{0, \dots, k-1\}$ with $\mathfrak{v}(p) = \bigcup_{i \in A_p} X_i$, where the X_i are the infinite generators of the internal sets in $\mathfrak{C}(\textcircled{k}, *)$; see Example 2.2. (Thus, if $k = 1$ and $\mathfrak{v}$ is simple, then $\mathfrak{v}(p)$ is either empty or the whole frame.) Note that, even though the atomic frame $\mathfrak{C}(\textcircled{k}, *)$ is infinite, for a finite signature σ, a simple σ-model based on it is fully determined by the *finitary* information provided by the finite sets A_p, $p \in \sigma$.

A *basic frame* for φ_1, φ_2 is any frame $\mathfrak{F}_0 \lhd \dots \lhd \mathfrak{F}_{n-1}$, where each $\mathfrak{F}_j$, $j < n$, is an atomic frame for φ_1, φ_2 and $0 < n = \mathcal{O}\big(\max(|\varphi_1|, |\varphi_2|)\big)$. If each $\mathfrak{M}_j$ based on atomic $\mathfrak{F}_j$, $j < n$, is a simple model, we say that $\mathfrak{M}_0 \lhd \dots \lhd \mathfrak{M}_{n-1}$ is *simple*.

The first main result in this section is that one can always assemble models witnessing the absence of an interpolant for φ_1 and φ_2 in any $L \supseteq \mathsf{K4.3}$ from polynomially-many (at most $2k+1$) simple models based on atomic frames. We can therefore say that all $L \supseteq \mathsf{K4.3}$ have 'quasi-finite bisimilar model property', with the word 'quasi' indicating that the models are infinite but finitely presentable. In fact, we prove a more structured property, which is defined below.

Definition 4.1. A logic $L \supseteq \mathsf{K4.3}$ has the *quasi-finite bisimilar model property* if, for any formulas φ_1, φ_2 without an interpolant in L, there are rooted models $\mathfrak{N}_1, x_1$ and $\mathfrak{N}_2, x_2$ satisfying (a)–(d) below, for $\sigma = sig(\varphi_1) \cap sig(\varphi_2)$:

- (a) $\mathfrak{N}_1, x_1 \models \varphi_1$ and $\mathfrak{N}_2, x_2 \models \neg\varphi_2$;
- (b) each $\mathfrak{N}_i$, $i = 1, 2$, is based on a frame for L;
- (c) $at^\sigma_{\mathfrak{N}_1}(x_1) = at^\sigma_{\mathfrak{N}_2}(x_2)$;
- (d) there is N, $0 < N \le 2k+1$, such that $\mathfrak{N}_i = \mathfrak{N}_i^0 \lhd \dots \lhd \mathfrak{N}_i^{N-1}$, $i = 1, 2$, and, for each $\ell < N$, the pair $(\mathfrak{N}_1^\ell, \mathfrak{N}_2^\ell)$ satisfies one of the following conditions (d_1)–(d_3):
  - (d_1)
    1. $\mathfrak{N}_1^\ell$ and $\mathfrak{N}_2^\ell$ are simple models based on the same atomic frame $\mathfrak{H}^\ell$;
    2. $at^\sigma_{\mathfrak{N}_1^\ell}(y) = at^\sigma_{\mathfrak{N}_2^\ell}(y)$, for every point y in $\mathfrak{H}^\ell$;
  - (d_2) $\mathfrak{N}_i^\ell = \mathfrak{N}_i^{0,\ell} \lhd \dots \lhd \mathfrak{N}_i^{n_i^\ell - 1,\ell}$, for $i = 1, 2$ and $0 < n_i^\ell \le k$, where
    1. each $\mathfrak{N}_i^{j,\ell}$, for $j < n_i^\ell$, is a simple model based on an atomic frame;
    2. for every point y_1 in $\mathfrak{N}_1^\ell$, there is a point y_2 in the final cluster of $\mathfrak{N}_2^{n_2^\ell - 1,\ell}$ with $at^\sigma_{\mathfrak{N}_1^\ell}(y_1) = at^\sigma_{\mathfrak{N}_2^\ell}(y_2)$, and for every y_2 in $\mathfrak{N}_2^\ell$, there is y_1 in the final cluster of $\mathfrak{N}_1^{n_1^\ell - 1,\ell}$ with $at^\sigma_{\mathfrak{N}_2^\ell}(y_2) = at^\sigma_{\mathfrak{N}_1^\ell}(y_1)$;
  - (d_3) $\mathfrak{N}_i^\ell = \mathfrak{N}_i^{0,\ell} \lhd \dots \lhd \mathfrak{N}_i^{n_i^\ell - 1,\ell}$, for $i = 1, 2$ and $0 < n_i^\ell \le k+1$, where
    1. each $\mathfrak{N}_i^{j,\ell}$, for $j < n_i^\ell$, is a simple model based on an atomic frame;
    2. $\mathfrak{N}_1^{n_1^\ell - 1,\ell}$ and $\mathfrak{N}_2^{n_2^\ell - 1,\ell}$ are based on the same atomic frame $\mathfrak{G}^\ell$ of the form $\mathfrak{C}(\textcircled{k}, \bullet)$ or $\mathfrak{C}(\textcircled{k}, \circ)$;
    3. for every point y in the $\textcircled{k}$-cluster of $\mathfrak{G}^\ell$, $at^\sigma_{\mathfrak{N}_1^\ell}(y) = at^\sigma_{\mathfrak{N}_2^\ell}(y)$;

4. for every point y_1 in $\mathfrak{N}_1^\ell$, there is a point y_2 in the ⓚ-cluster of $\mathfrak{N}_2^{n_2^\ell-1,\ell}$ with $at^\sigma_{\mathfrak{N}_1^\ell}(y_1) = at^\sigma_{\mathfrak{N}_2^\ell}(y_2)$, and for every y_2 in $\mathfrak{N}_2^\ell$, there is y_1 in the ⓚ-cluster of $\mathfrak{N}_1^{n_1^\ell-1,\ell}$ with $at^\sigma_{\mathfrak{N}_2^\ell}(y_2) = at^\sigma_{\mathfrak{N}_1^\ell}(y_1)$.

The role of conditions (c) and (d) in Definition 4.1 is explained by the following:

Lemma 4.2. *If* $\mathfrak{N}_1, x_1$ *and* $\mathfrak{N}_2, x_2$ *satisfy* (c) *and* (d)*, then* $\mathfrak{N}_1, x_1 \sim_\sigma \mathfrak{N}_2, x_2$.

Proof. Let $\mathfrak{N}_i = \mathfrak{N}_i^0 \lhd \cdots \lhd \mathfrak{N}_i^{N-1}$, for $i = 1, 2$. We claim that condition (d) implies that, for every $\ell < N$, there is a global σ-bisimulation between $\mathfrak{N}_1^\ell$ and $\mathfrak{N}_2^\ell$. Indeed, in case (d_1), the identity function on $\mathfrak{H}^\ell$ is such a bisimulation. For (d_2) and (d_3), suppose $\mathfrak{N}_i^\ell = \mathfrak{N}_i^{0,\ell} \lhd \cdots \lhd \mathfrak{N}_i^{n_i^\ell-1,\ell}$, $i = 1, 2$. In case (d_2), take

$$\boldsymbol{\beta}_1^\ell = \big\{(y_1, y_2) \mid y_1 \text{ in } \mathfrak{N}_1^\ell,\ y_2 \text{ in the last cluster of } \mathfrak{N}_2^{n_2^\ell-1,\ell},\ at^\sigma_{\mathfrak{N}_1^\ell}(y_1) = at^\sigma_{\mathfrak{N}_2^\ell}(y_2)\big\},$$

$$\boldsymbol{\beta}_2^\ell = \big\{(y_1, y_2) \mid y_2 \text{ in } \mathfrak{N}_2^\ell,\ y_1 \text{ in the last cluster of } \mathfrak{N}_1^{n_1^\ell-1,\ell},\ at^\sigma_{\mathfrak{N}_1^\ell}(y_1) = at^\sigma_{\mathfrak{N}_2^\ell}(y_2)\big\}.$$

Then $\boldsymbol{\beta}_1^\ell \cup \boldsymbol{\beta}_2^\ell$ is a global σ-bisimulation between $\mathfrak{N}_1^\ell$ and $\mathfrak{N}_2^\ell$. In case (d_3), let

$$\boldsymbol{\beta}_1^\ell = \big\{(y_1, y_2) \mid y_1 \text{ in } \mathfrak{N}_1^\ell,\ y_2 \text{ in the ⓚ-cluster of } \mathfrak{N}_2^{n_2^\ell-1,\ell},\ at^\sigma_{\mathfrak{N}_1^\ell}(y_1) = at^\sigma_{\mathfrak{N}_2^\ell}(y_2)\big\},$$

$$\boldsymbol{\beta}_2^\ell = \big\{(y_1, y_2) \mid y_2 \text{ in } \mathfrak{N}_2^\ell,\ y_1 \text{ in the ⓚ-cluster of } \mathfrak{N}_1^{n_1^\ell-1,\ell},\ at^\sigma_{\mathfrak{N}_1^\ell}(y_1) = at^\sigma_{\mathfrak{N}_2^\ell}(y_2)\big\}.$$

We claim that

$$at^\sigma_{\mathfrak{N}_1^\ell}(b_n) = at^\sigma_{\mathfrak{N}_2^\ell}(b_n), \quad \text{for all } n < \omega, \tag{7}$$

where $\{a_s \mid s < k\} \cup \{b_n \mid n < \omega\}$ are all the points of $\mathfrak{C}(ⓚ, *)$ underlying both $\mathfrak{N}_1^{n_1^\ell-1,\ell}$ and $\mathfrak{N}_2^{n_2^\ell-1,\ell}$ (see Example 2.2). Indeed, as $\mathfrak{N}_i^{n_i^\ell-1,\ell}$ is a simple model by (d_3).1, for every $n < \omega$, there is $s < k$ with $at_{\mathfrak{N}_i^\ell}(b_n) = at_{\mathfrak{N}_i^\ell}(a_s)$. So (7) follows from (d_3).3. Now, by (7) and (d_3).4, $\boldsymbol{\beta}_1^\ell \cup \boldsymbol{\beta}_2^\ell \cup \big\{(b_n, b_n) \mid n < \omega\big\}$ is a global σ-bisimulation between $\mathfrak{N}_1^\ell$ and $\mathfrak{N}_2^\ell$.

Finally, if $\boldsymbol{\beta}^0$ is a global σ-bisimulation between $\mathfrak{N}_1^0$ and $\mathfrak{N}_2^0$, then $\boldsymbol{\beta}^0 \cup \{(x_1, x_2)\}$ is also a global σ-bisimulation between $\mathfrak{N}_1^0$ and $\mathfrak{N}_2^0$ in view of (c) in Definition 4.1. The union of the constructed bisimulations is clearly a global bisimulation between $\mathfrak{N}_1$ and $\mathfrak{N}_2$, and so $\mathfrak{N}_1, x_1 \sim_\sigma \mathfrak{N}_2, x_2$ as required. ⊣

The first main result will be proved in Sections 4.3–4.4:

Theorem 4.3. *All* $L \supseteq \mathsf{K4.3}$ *have the quasi-finite bisimilar model property.*

Note, however, that the models provided by the quasi-finite bisimilar model property might have $\lhd$-components like the first two types of atomic frames in (6), where their respective sizes m or $1 + m$ are finite but unbounded. Our second main result shows that all *finitely axiomatisable* logics $L \supseteq \mathsf{K4.3}$ have the stronger *quasi-polysize bisimilar model property* in the sense that the models $\mathfrak{N}_i$, $i = 1, 2$, in Definition 4.1 can be chosen so that $m \le c_L$, for some constant $c_L < \omega$ depending only on L, whenever an atomic frame of the form $m^<$ occurs as a $\lhd$-component in the underlying basic frames of the $\mathfrak{N}_i$. We call such atomic and basic frames *L-bounded*. Thus, the size of the models $\mathfrak{N}_i$ does not exceed $p(\varphi_1, \varphi_2) = (2k + 1)\max(c_L + 1, k)$. In Section 4.5, we prove:

Theorem 4.4. *All finitely axiomatisable $L \supseteq \mathsf{K4.3}$ have the quasi-polysize bisimilar model property, with the size of witnessing models bounded by $p(\varphi_1, \varphi_2)$.*

Remark 4.5. As a consequence we obtain that each finitely axiomatisable logic $L \supseteq \mathsf{K4.3}$ has the *quasi-polysize model property*: $\varphi \in L$ iff φ is true in all simple models $\mathfrak{M}$ based on a basic frame for L of size $\mathcal{O}(|\varphi|^2)$; cf. [38, 26].

In the remainder of Section 4.1 we show how Theorem 4.4 implies the following:

Theorem 4.6. *The IEP for any finitely axiomatisable logic $L \supseteq \mathsf{K4.3}$ is* coNP-*complete.*

Proof. We describe an NP-algorithm deciding the complement of the IEP for any fixed finitely axiomatisable L. Given φ_1 and φ_2, let $\delta = sig(\varphi_1) \cup sig(\varphi_2)$. We guess polynomial-size N. Then, for each $\ell < N$, we guess $z_\ell \in \{1, 2, 3\}$, and if $z_\ell = 1$, we let $n_1^\ell = n_2^\ell = 1$; otherwise, we guess polynomial-size n_i^ℓ for $i = 1, 2$; we also guess polynomial-size simple δ-models $\mathfrak{N}_i^{j,\ell}$, for $\ell < N$, $i = 1, 2$, and $j < n_i^\ell$, based on L-bounded atomic frames, and respective roots x_i in $\mathfrak{N}_i^{0,0}$. Checking conditions (c) and (d) in Definition 4.1 can clearly be done in time polynomial in $\max(|\varphi_1|, |\varphi_2|)$. We check condition (a) using

Lemma 4.7. *Checking whether $\mathfrak{M}_0 \lhd \cdots \lhd \mathfrak{M}_{n-1}, x \models \varphi$, for simple $sig(\varphi)$-models $\mathfrak{M}_j$, $j < n$, based on atomic frames with root x in $\mathfrak{M}_0$, can be done in time polynomial in $|\varphi|$ and $|\mathfrak{M}_0| + \cdots + |\mathfrak{M}_{n-1}|$.*

Proof. Let $\mathfrak{M} = \mathfrak{M}_0 \lhd \cdots \lhd \mathfrak{M}_{n-1}$. Suppose $\mathfrak{M}_j$ is based on the frame $\mathfrak{C}(ⓚ, *)$ defined in Example 2.2 with points a_s, $s < k$, and b_ℓ, $\ell < \omega$. Using the definition of a simple model, it is readily shown by structural induction that, for any formula $\psi \in sub(\varphi)$, ψ is satisfiable in $\mathfrak{M}_j$ iff there is $\ell < k + md(\psi)$ with $\mathfrak{M}_j, b_\ell \models \psi$, where $md(\psi)$, the *modal depth* of ψ, is the maximal number of nested modal operators in ψ. The required algorithm is now obvious. ⊣

Condition (b) for the fixed finitely axiomatisable $L \supseteq \mathsf{K4.3}$ is checked using

Lemma 4.8. *Checking whether $\mathfrak{F}_0 \lhd \cdots \lhd \mathfrak{F}_{n-1} \models L$, for atomic frames $\mathfrak{F}_j$, $j < n$, can be done in time polynomial in $|\mathfrak{F}_0| + \cdots + |\mathfrak{F}_{n-1}|$.*

Proof. According to Section 2.3, we can axiomatise L by canonical formulas $\alpha(\mathfrak{G}_i, \mathfrak{D}_i, \bot)$, $i < m_L$, as in (3). It remains to observe that, using the refutability criterion for $\alpha(\mathfrak{G}_i, \mathfrak{D}_i, \bot)$, we can decide whether $\mathfrak{F}_0 \lhd \cdots \lhd \mathfrak{F}_{n-1} \models \alpha(\mathfrak{G}_i, \mathfrak{D}_i, \bot)$, for atomic $\mathfrak{F}_j$, in time polynomial in $|\mathfrak{F}_0| + \cdots + |\mathfrak{F}_{n-1}|$. ⊣

If all checks come back positive, then there is no interpolant of φ_1 and φ_2 in L because of Lemma 4.2 and the criterion of Theorem 3.2. ⊣

The remainder of Section 4 contains the proofs of Theorems 4.3 and 4.4. In a nutshell, our plan is as follows. First, in Section 4.2, we establish a few general facts about the structure of finitely generated descriptive frames for $\mathsf{K4.3}$ that are needed for our construction. Given φ_1 and φ_2 without an interpolant in $L \supseteq \mathsf{K4.3}$, the criterion of Theorem 3.2 supplies some pair $\mathfrak{M}_i$, $i = 1, 2$, of σ-bisimilar models that are based on respective finitely $\mathfrak{M}_i$-generated descriptive frames $\mathfrak{F}_i$ for L and witnessing the lack of an interpolant in L. In Section 4.3, we partition each of these $\mathfrak{M}_i$ into the same polynomial number N of closed intervals $\mathcal{I}_i = \{I_i^\ell \mid \ell < N\}$ such that, for every $\ell < N$,

(I1) I_i^ℓ is definable in $\mathfrak{M}_i$, and so $\mathfrak{F}_i = (\mathfrak{F}_i\!\upharpoonright_{I_i^0}) \lhd \cdots \lhd (\mathfrak{F}_i\!\upharpoonright_{I_i^{N-1}})$, for $i = 1, 2$;

(I2) $\big\{(y_1, y_2) \in I_1^\ell \times I_2^\ell \mid t^\sigma_{\mathfrak{M}_1}(y_1) = t^\sigma_{\mathfrak{M}_2}(y_2)\big\}$ is a global σ-bisimulation between $\mathfrak{M}_1\!\upharpoonright_{I_1^\ell}$ and $\mathfrak{M}_2\!\upharpoonright_{I_2^\ell}$.

The partitions are built around the sets M_i and S_i of maximal points selected in the proof of Theorem 3.4 (a) and the σ-types of points in the $\mathfrak{M}_i$. Then, in Section 4.4, we complete the proof of Theorem 4.3 by transforming the models $\mathfrak{M}_i\!\upharpoonright_{I_i^\ell}$ into simple models $\mathfrak{N}_i^\ell$ based on basic frames such that the conditions in Definition 4.1 are satisfied. We prove Theorem 4.4 in Section 4.5.

4.2. The structure of linear descriptive frames. Until the end of Section 4, every frame $\mathfrak{F} = (W, R, \mathcal{P})$ is assumed to be a rooted frame for $\mathsf{K4.3}$. Recall from Section 2.2 that (W, R) can be regarded as the strict linear order $\mathfrak{F}_c = (W_c, <_R)$ of clusters, where $W_c = \{C(x) \mid x \in W\}$ and $C(x) <_R C(y)$ iff xR^sy. A sequence C_n, $n < \omega$, of clusters in $\mathfrak{F}_c$ is an *infinite ascending chain* if $C_n <_R C_{n+1}$ for all $n < \omega$. $\mathfrak{F}_c$ is *converse well-founded* if it has no infinite ascending chain of clusters.

The next lemma follows from, e.g., the more general [8, Theorems 10.34, 10.35]:

LEMMA 4.9. *If $\mathfrak{F}$ is a rooted n-generated descriptive frame for* $\mathsf{K4.3}$*, for some $n < \omega$, then*

(a) *$\mathfrak{F}_c$ is converse well-founded, and so the strict linear order $\mathfrak{F}_c^{-1} = (W_c, >_R)$ is isomorphic to some ordinal*;

(b) *every cluster in $\mathfrak{F}$ has at most 2^n points.*

PROOF. Let $\mathfrak{F} = (W, R, \mathcal{P})$, let $\leq_R$ be the reflexive closure of $<_R$, and let $\mathcal{G}$ be a finite set generating $\mathcal{P}$ with $|\mathcal{G}| = n$. For $x, y \in W$, we write $x \sim_\mathcal{G} y$ in case $x \in G$ iff $y \in G$, for all $G \in \mathcal{G}$, and denote by $[x]_\mathcal{G}$ the $\sim_\mathcal{G}$-class of x. Clearly, $\big|\big\{[x]_\mathcal{G} \mid x \in W\big\}\big| = 2^{|\mathcal{G}|} = 2^n$.

(a) Suppose on the contrary that $C(x_n)$, $n < \omega$, is an infinite ascending chain in $\mathfrak{F}_c$. Call $x \in W$ a *middle-point* if $C(x_0) \leq_R C(x) \leq_R C(x_n)$, for some $n < \omega$. Let $V_x = \{[y]_\mathcal{G} \mid y \text{ a middle-point with } xRy\}$. Since $V_x \supseteq V_y$ whenever xRy and each V_x is finite, there is $m < \omega$ such that $V_y = V_{x_m}$, for every middle-point y with $C(x_m) \leq_R C(y)$. By induction on the construction of $X \in \mathcal{P}$ from the generators in $\mathcal{G}$, it is readily seen that

(8) if y, z are middle-points, $C(x_m) \leq_R C(y)$, $C(x_m) \leq_R C(z)$, and $y \sim_\mathcal{G} z$, then $y \in X$ iff $z \in X$, for all $X \in \mathcal{P}$.

(Indeed, the only non-trivial case is when $X = \Diamond^\mathfrak{F} Y$, yRz and $y \in \Diamond^\mathfrak{F} Y$. Then there is $x \in Y$ with yRx. If zRx, we are done. Otherwise, x is a middle-point. As $V_y = V_z$, there is a middle-point x' with zRx' and $x \sim_\mathcal{G} x'$. By IH, $x' \in Y$.) As there are finitely many $\sim_\mathcal{G}$-classes, there exist $k \neq \ell \geq m$ such that $x_k \sim_\mathcal{G} x_\ell$, and so $x_k \in X$ iff $x_\ell \in X$ for all $X \in \mathcal{P}$, by (8). But this contradicts **(dif)**.

(b) It is straightforward to show that if $C(x) = C(y)$ and $x \sim_\mathcal{G} y$, then $x \in X$ iff $y \in X$, for all $X \in \mathcal{P}$. So by **(dif)**, every cluster in $\mathfrak{F}$ has at most $2^{|\mathcal{G}|}$ points. ⊣

Note that the existence of maximal points (Lemma 2.4) in models based on rooted finitely generated descriptive frames for $\mathsf{K4.3}$ also follows from Lemma 4.9. Another consequence is that such an $\mathfrak{F}$ contains a final cluster, and any non-root cluster in $\mathfrak{F}$ has an immediate predecessor. We refer to clusters that are images by the above isomorphism of a non-zero limit ordinal as *limit clusters*. Clearly, a non-final cluster is a limit cluster iff it does not have an immediate successor.

Now, suppose $\mathfrak{M}$ is a model based on a rooted finitely $\mathfrak{M}$-generated descriptive frame $\mathfrak{F} = (W, R, \mathcal{P})$ for $L \supseteq \mathsf{K4.3}$. Given a formula μ, we call a cluster C *μ-maximal in* $\mathfrak{M}$ if there is $x \in C$ such that $\mathfrak{M}, x \models \mu$ but $\mathfrak{M}, y \models \neg\mu$ whenever $xR^s y$ (that is, $C \cap \max_{\mathfrak{M}}\{\mu\} \neq \emptyset$). We call C *maximal in* $\mathfrak{M}$ if it is μ-maximal in $\mathfrak{M}$ for some μ. If there is such a σ-formula μ, for some signature σ, we call C *σ-maximal in* $\mathfrak{M}$. Every definable in $\mathfrak{M}$ cluster is clearly maximal in $\mathfrak{M}$. The next lemma says that the converse is also true:

LEMMA 4.10. *Suppose $\mathfrak{M}$ is a model based on a rooted finitely $\mathfrak{M}$-generated descriptive frame $\mathfrak{F} = (W, R, \mathcal{P})$ for* $\mathsf{K4.3}$*. Then*

(a) *every degenerate cluster in $\mathfrak{F}$ is maximal in $\mathfrak{M}$*;

(b) *a cluster is maximal in $\mathfrak{M}$ iff either it is final or has an immediate successor*;

(c) *a cluster is definable in $\mathfrak{M}$ iff it is maximal in $\mathfrak{M}$.*

So limit clusters are not definable and not degenerate, and every other cluster is definable in $\mathfrak{M}$. Also,

(d) *every interval $[C, C']$ in $\mathfrak{F}$ with a non-limit cluster C' is definable in $\mathfrak{M}$.*

PROOF. (a) If $C(x)$ is degenerate, then $\Diamond t_{\mathfrak{M}}(x) \not\subseteq t_{\mathfrak{M}}(x)$, by **(ref)**. So there is a formula μ with $\mathfrak{M}, x \models \mu$ but $\mathfrak{M}, x \not\models \Diamond\mu$.

($b, \Rightarrow$) Let $C(x)$ be maximal in $\mathfrak{M}$ with $\mathfrak{M}, x \models \mu$ and $\mathfrak{M}, y \not\models \mu$ whenever $xR^s y$. Suppose $C(x)$ is a limit cluster. Let $S = \{C \in W_c \mid C(x) <_R C\}$ with $y_C \in C$, for $C \in S$. Consider

$$\Gamma = \bigcup_{C \in S} \Diamond t_{\mathfrak{M}}(y_C) \cup \{\psi \mid \Box\psi \in t_{\mathfrak{M}}(x)\} \cup \{\Box\neg\mu\}.$$

Clearly, Γ is finitely satisfiable in $\mathfrak{M}$, and so by **(com)**, $\Gamma \subseteq t_{\mathfrak{M}}(y)$, for some y. Thus, by **(ref)**, $xRyRy_C$ for all $C \in S$, and so $yR^s y_C$ for all $C \in S$ and yRx. But we also have $\mathfrak{M}, y \models \Box\neg\mu$, contrary to $\mathfrak{M}, x \models \mu$.

($b, \Leftarrow$) The (unique) final cluster is maximal in $\mathfrak{M}$ for $\top$. Suppose $C(y)$ is an immediate successor of $C(x)$. If $C(y)$ is degenerate, then $C(y)$ is maximal in $\mathfrak{M}$ by (a), and so there is μ with $\mathfrak{M}, y \models \mu \wedge \neg\Diamond\mu$. It follows that $C(x)$ is $\Diamond(\mu \wedge \neg\Diamond\mu)$-maximal in $\mathfrak{M}$. If $C(y)$ is non-degenerate and $C(x)$ is not maximal in $\mathfrak{M}$, then $\Diamond t_{\mathfrak{M}}(x) \subseteq t_{\mathfrak{M}}(y)$, and so yRx by **(ref)**, contrary to $xR^s y$.

($c, \Leftarrow$) Let $C(x)$ be μ-maximal in $\mathfrak{M}$. If $C(x)$ is degenerate, it is defined by $\mu \wedge \neg\Diamond\mu$. Otherwise, take the immediate predecessor $C(y)$ of $C(x)$. By (b), $C(y)$ is τ-maximal in $\mathfrak{M}$ for some τ, so $\Box^+\neg\tau \wedge \Diamond\mu$ defines $C(x)$. ($c, \Rightarrow$) is obvious.

(d) By (a)–(c), the non-limit C' is defined in $\mathfrak{M}$ by some γ. Let $\delta = \bot$ if C is the root cluster, and let δ define the immediate predecessor of C in $\mathfrak{M}$ otherwise (which exists by Lemma 4.9 (a) and is definable by (a)–(c)). Then $[C, C']$ is defined in $\mathfrak{M}$ by $\neg\Diamond^+\delta \wedge \Diamond^+\gamma$. ⊣

LEMMA 4.11. *If $\mathfrak{F} = (W, R, \mathcal{P})$ is a rooted finitely generated descriptive frame for* $\mathsf{K4.3}$*, then W is countable.*

PROOF. By Lemma 4.9, it suffices to show that the ordinal γ isomorphic to $\mathfrak{F}_c^{-1} = (W_c, >_R)$ is countable. So suppose $W_c = \{C_\alpha \mid \alpha < \gamma\}$ and $C_\alpha <_R C_\beta$ iff

$\beta < \alpha < \gamma$. Let $Z = \{\alpha + 1 \mid \alpha < \gamma,\ \alpha + 1 \neq \gamma\}$ be the set of successor ordinals $< \gamma$. Then $|Z| = |\gamma|$ and, for any $\beta \in Z$, we have $C_\beta \in \mathcal{P}$ by Lemma 4.10 (a)–(c). As $\mathfrak{F}$ is finitely generated, $\mathcal{P}$ is countable, and so are Z and W. ⊣

If C_n, $n < \omega$, are such that $C_{n+1} <_R C_n$ for all $n < \omega$, they form an *infinite descending chain* of clusters in $\mathfrak{F}_c$. A cluster C is the *limit of* an infinite descending chain of clusters C_n, $n < \omega$, in $\mathfrak{F}_c$ if $C <_R C_n$ for all $n < \omega$, and there is no cluster C' with $C <_R C'$ and $C' <_R C_n$ for all $n < \omega$. By Lemmas 4.9, 4.11 and the fact that the cofinality of any countable limit ordinal is the cardinality $\aleph_0$ of ω [21], every limit cluster is the limit of some infinite descending chain of clusters. We say that a limit cluster is

- *of type* $\circ$ if it is the limit of an infinite descending chain of non-degenerate clusters; and
- *of type* $\bullet$ if it is the limit of an infinite descending chain C_n, $n < \omega$, of degenerate clusters, where C_{n+1} is the immediate predecessor of C_n, $n < \omega$.

Lemma 4.12. *If $\mathfrak{F}$ is a rooted finitely generated descriptive frame for* $\mathsf{K4.3}$, *then every limit cluster C in $\mathfrak{F}$ is either of type $\circ$ or of type $\bullet$* (*but not both*).

Proof. Suppose C is the limit of an infinite descending chain C_n, $n < \omega$, of clusters. If this chain contains infinitely many non-degenerate clusters, then C is of type $\circ$. So suppose every chain with limit C contains only finitely many non-degenerate clusters. Let C' be the $<_R$-smallest non-degenerate cluster with $C <_R C'$ if there is such, and let C' be the final cluster of $\mathfrak{F}$ otherwise. Let $H = \{C\} \cup \{E \mid C <_R E <_R C'\}$ and $> \ = >_R\restriction_H$. As every cluster E with $C <_R E <_R C'$ is degenerate, it is not a limit cluster by Lemma 4.10. Therefore, the countable ordinal isomorphic to $(H, >)$ must be ω. Thus, C is the limit of an infinite descending chain D_n, $n < \omega$, of degenerate clusters for which D_{n+1} is the immediate predecessor of D_n for all $n < \omega$, and so C is of type $\circ$. ⊣

We can now proceed with the plan outlined at the end of Section 4.1.

4.3. Partitioning the models into globally σ-bisimilar intervals. Suppose $\mathfrak{F} = (W, R, \mathcal{P})$ is a rooted frame for $\mathsf{K4.3}$. The *tail of* $\mathfrak{F}$ is the smallest (possibly empty) interval Z in $\mathfrak{F}$ such that

- if the final cluster of $\mathfrak{F}$ is degenerate, then it is included in Z;
- if $z \in Z$ and a degenerate cluster $\{z'\}$ is the immediate predecessor of $\{z\}$, then $z' \in Z$.

Note that, as $\mathfrak{F}$ is rooted, if Z is infinite, then $Z \neq W$. Observe also that every finite subset of Z and its complement in W are in $\mathcal{P}$ (Example 2.2). If Z is the tail of $\mathfrak{F}$ and $Z \neq W$, then the *face of* Z is the unique cluster C such that $C <_R \{z\}$, for all $z \in Z$, and there is no C' with $C <_R C' <_R \{z\}$, for all $z \in Z$. In particular, if $Z = \emptyset$, then its face is the final (non-degenerate) cluster in $\mathfrak{F}$, and if $Z \neq W$ and $Z \neq \emptyset$ is finite, then its face is the immediate predecessor of the $<_R$-smallest cluster in Z. If $\mathfrak{F}$ is a rooted finitely $\mathfrak{M}$-generated descriptive frame, for some model $\mathfrak{M}$, then the face is always non-degenerate by Lemma 4.10.

It is not hard to see the following:

Lemma 4.13. *Suppose $\boldsymbol{\beta}$ is a global σ-bisimulation between models $\mathfrak{M}$ and $\mathfrak{M}'$ that are based on frames $\mathfrak{F} = (W, R, \mathcal{P})$ and $\mathfrak{F}' = (W', R', \mathcal{P}')$, respectively. If $Z = \{z_i \mid i < n\}$ is the tail of $\mathfrak{F}$, for some $n \leq \omega$, and $z_i R^s z_{i-1}$, $0 < i < n$, then*

(a) *the tail Z' of $\mathfrak{F}'$ is of the form $Z' = \{z'_i \mid i < n\}$ with $z'_i {R'}^s z'_{i-1}$ for $0 < i < n$*;
(b) *for all $i < n$, $w \in W$, $w' \in W'$, $z_i \boldsymbol{\beta} w'$ iff $w' = z'_i$, and $w \boldsymbol{\beta} z'_i$ iff $w = z_i$*;
(c) *if $Z \neq W$, then $Z' \neq W'$ and there exist u in the face of Z and u' in the face of Z' such that $u \boldsymbol{\beta} u'$.*

Given $x \in W$ and a signature σ, the *σ-block $\boldsymbol{b}^\sigma_{\mathfrak{M}}(x)$ of x in* $\mathfrak{M}$ is defined as

$$\boldsymbol{b}^\sigma_{\mathfrak{M}}(x) = \begin{cases} \{y \in W \mid \Diamond t^\sigma_{\mathfrak{M}}(y) \subseteq t^\sigma_{\mathfrak{M}}(x),\ \Diamond t^\sigma_{\mathfrak{M}}(x) \subseteq t^\sigma_{\mathfrak{M}}(y)\}, & \text{if } \Diamond t^\sigma_{\mathfrak{M}}(x) \subseteq t^\sigma_{\mathfrak{M}}(x); \\ \{x\}, & \text{otherwise;} \end{cases}$$

in the latter case—when $\{x\}$ is irreflexive—the σ-block $\boldsymbol{b}^\sigma_{\mathfrak{M}}(x)$ is called *degenerate*. (It can happen that $\Diamond t^\sigma_{\mathfrak{M}}(x) \subseteq t^\sigma_{\mathfrak{M}}(x)$ and not xRx.) We call a set $\boldsymbol{b} \subseteq W$ a *σ-block in* $\mathfrak{M}$ if $\boldsymbol{b} = \boldsymbol{b}^\sigma_{\mathfrak{M}}(x)$, for some x. It is readily seen that the relation $x \approx y$ iff $\boldsymbol{b}^\sigma_{\mathfrak{M}}(x) = \boldsymbol{b}^\sigma_{\mathfrak{M}}(y)$ is an equivalence relation on W, and every σ-block $\boldsymbol{b}$ is an interval in $\mathfrak{F}$. See Example 4.15 below for an illustration. Observe that

(block) for all σ-blocks $\boldsymbol{b}$ in $\mathfrak{M}$ and $y \in W$, if $y \notin \boldsymbol{b}$ then $t^\sigma_{\mathfrak{M}}(y) \notin t^\sigma_{\mathfrak{M}}(\boldsymbol{b})$.

For degenerate σ-blocks this follows from the definability of degenerate clusters (Lemma 4.10), and for other σ-blocks it is straightforward from the definitions.

Lemma 4.14. *Suppose $\mathfrak{M}$ is a model based on a rooted finitely $\mathfrak{M}$-generated descriptive frame $\mathfrak{F} = (W, R, \mathcal{P})$ for* K4.3. *For any σ-block $\boldsymbol{b}$ in $\mathfrak{M}$ there exist clusters $C^-_{\boldsymbol{b}}$, $C^+_{\boldsymbol{b}}$ in $\mathfrak{F}$ such that the following hold*:
(a) $\boldsymbol{b} = \big[C^-_{\boldsymbol{b}}, C^+_{\boldsymbol{b}}\big]$;
(b) *if $C^+_{\boldsymbol{b}}$ is maximal in $\mathfrak{M}$ then it is σ-maximal in* $\mathfrak{M}$;
(c) *if $C^+_{\boldsymbol{b}}$ is degenerate, then $\boldsymbol{b} = C^+_{\boldsymbol{b}}$*;
(d) *$\boldsymbol{b}$ is definable in $\mathfrak{M}$ iff $C^+_{\boldsymbol{b}}$ is not a limit cluster*;
(e) $t^\sigma_{\mathfrak{M}}(\boldsymbol{b}) = t^\sigma_{\mathfrak{M}}\big(C^+_{\boldsymbol{b}}\big)$.

Proof. (a) By Lemma 4.9, we may assume that $\boldsymbol{b}$ is either of the form $[C, C']$ or of the form $(C_1, C_2]$, for some limit cluster C_1, as the other kinds of intervals are clearly expressible in these forms. Suppose on the contrary that $\boldsymbol{b} = (C_1, C_2]$ for some limit cluster C_1. If $y \in C_1$, then $t^\sigma_{\mathfrak{M}}(y) \notin t^\sigma_{\mathfrak{M}}(\boldsymbol{b})$ by **(block)**. So there is a σ-formula μ such that $\sigma \in t^\sigma_{\mathfrak{M}}(y)$ and $\Diamond\sigma \notin t^\sigma_{\mathfrak{M}}(x)$ for any $x \in \boldsymbol{b}$, and so for any x with $yR^s x$. As C_1 is non-degenerate by Lemma 4.10, it follows that C_1 is $\Diamond\mu$-maximal in $\mathfrak{M}$, contrary to Lemma 4.10 (b).

(b) If $C^+_{\boldsymbol{b}}$ is maximal in $\mathfrak{M}$, then either it is final or has an immediate successor, by Lemma 4.10 (b). If $C^+_{\boldsymbol{b}}$ is final, then it is $\top$-maximal in $\mathfrak{M}$. So suppose that $C(y)$ is an immediate successor of $C^+_{\boldsymbol{b}} = C(x)$. If $C^+_{\boldsymbol{b}}$ is not degenerate, then $\Diamond t^\sigma_{\mathfrak{M}}(x) \not\subseteq t^\sigma_{\mathfrak{M}}(y)$ follows from $y \notin \boldsymbol{b}$. So there is a σ-formula μ such that $\mathfrak{M}, x \models \mu$ and $\mathfrak{M}, y \not\models \Diamond\mu$. If $\mathfrak{M}, y \models \mu$, then $C^+_{\boldsymbol{b}}$ is $\Diamond\mu$-maximal in $\mathfrak{M}$. And if $\mathfrak{M}, y \not\models \mu$, then $C^+_{\boldsymbol{b}}$ is μ-maximal in $\mathfrak{M}$. If $C^+_{\boldsymbol{b}}$ is degenerate, we cannot have $\Diamond t^\sigma_{\mathfrak{M}}(x) \subseteq t^\sigma_{\mathfrak{M}}(x)$, for otherwise $t^\sigma_{\mathfrak{M}}(x) \subseteq t^\sigma_{\mathfrak{M}}(y)$, contrary to **(block)**. Thus, $\Diamond t^\sigma_{\mathfrak{M}}(x) \not\subseteq t^\sigma_{\mathfrak{M}}(x)$, and so there is σ-formula μ such that $\mathfrak{M}, x \models \mu$ and $\mathfrak{M}, x \not\models \Diamond\mu$. Therefore, $C^+_{\boldsymbol{b}}$ is μ-maximal in $\mathfrak{M}$.

(c) Suppose on the contrary that $C^+_{\boldsymbol{b}} = \{x\} \neq \boldsymbol{b}$. Then $|\boldsymbol{b}| > 1$, and so $\Diamond t^\sigma_{\mathfrak{M}}(x) \subseteq t^\sigma_{\mathfrak{M}}(x)$ follows from $\boldsymbol{b} = \boldsymbol{b}^\sigma_{\mathfrak{M}}(x)$. So, for every σ-formula μ, if $\mathfrak{M}, x \models \mu$ then $\mathfrak{M}, x \models \Diamond\mu$. On the other hand, $C^+_{\boldsymbol{b}}$ is maximal in $\mathfrak{M}$ by Lemma 4.10 (a), and so σ-maximal in $\mathfrak{M}$ by (b), which is a contradiction.

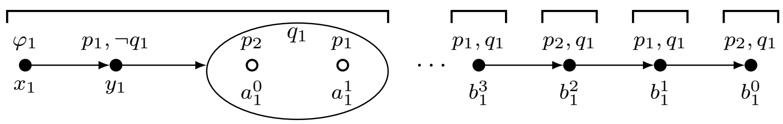
($d, \Leftarrow$) This is by (a) and Lemma 4.10 (d).

($d, \Rightarrow$) Suppose that $\boldsymbol{b}$ is defined in $\mathfrak{M}$ by some ψ. Then $C^+_{\boldsymbol{b}}$ is ψ-maximal in $\mathfrak{M}$, and so cannot be a limit cluster by Lemma 4.10 (b).

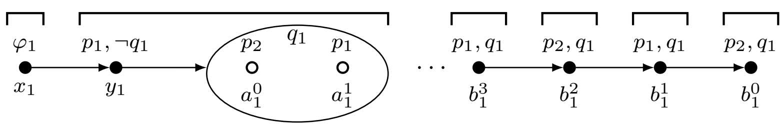
(e) If $C^+_{\boldsymbol{b}}$ is degenerate, then this is obvious by (c). So suppose $C^+_{\boldsymbol{b}} = C(y)$ is non-degenerate and $x \in \boldsymbol{b}$. Then $\Diamond t^\sigma_{\mathfrak{M}}(x) \subseteq t^\sigma_{\mathfrak{M}}(y)$, and so $\Diamond \bigwedge \Gamma \in t^\sigma_{\mathfrak{M}}(y)$ for every finite $\Gamma \subseteq t^\sigma_{\mathfrak{M}}(x)$. By Lemma 2.4 ($a$), there is z such that yRz and $t^\sigma_{\mathfrak{M}}(z) = t^\sigma_{\mathfrak{M}}(x)$. By **(block)**, we have $z \in \boldsymbol{b}$, and so $z \in C^+_{\boldsymbol{b}}$. ⊣

Example 4.15. The model $\mathfrak{M}_1$ in Fig. 2 from Example 3.5 is partitioned into the following σ-blocks (indicated by the brackets), for three different σ:

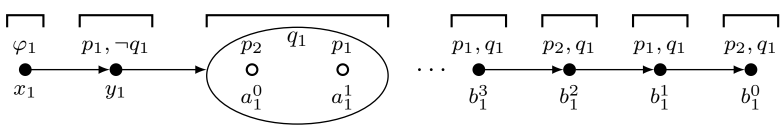
$\sigma = \emptyset$



$\sigma = \{p_1, p_2\}$



$\sigma = \{p_1, p_2, q_1, q_2\}$



To show this for $\sigma = \emptyset$, observe that, for every $n > 0$, we have $\Diamond^n \top \in t^\sigma_{\mathfrak{M}_1}(b^n_1)$, $\Diamond^{n+1} \top \notin t^\sigma_{\mathfrak{M}_1}(b^n_1)$, $\neg \Diamond \top \in t^\sigma_{\mathfrak{M}_1}(b^0_1)$, and $\Diamond^n \top \in t^\sigma_{\mathfrak{M}_1}(a^0_1)$. Note that the cluster $C(a^0_1)$ is not maximal in $\mathfrak{M}_1$: any formula α that is true at a^0_1 or a^1_1 is also true at b^n_1 for some $n < \omega$ (which is seen by induction on the structure of α.) The model $\mathfrak{M}_1$ in Example 3.6 has only one $\emptyset$-block comprising all of its points. ⊣

By Lemma 4.14 (a), σ-blocks in each of our models $\mathfrak{M}_i$, $i = 1, 2$, are closed intervals that form a partition of W_i (with not all of them being necessarily definable in $\mathfrak{M}_i$). We now show that we have the same number of σ-blocks in both models. Indeed, suppose that W_1 is partitioned as $\{\boldsymbol{b}^j \mid j \in F\}$ into σ-blocks in $\mathfrak{M}_1$, for some countable set F. For each $j \in F$, we let

$$\boldsymbol{\beta}(\boldsymbol{b}^j) = \{y \in W_2 \mid t^\sigma_{\mathfrak{M}_2}(y) \in t^\sigma_{\mathfrak{M}_1}(\boldsymbol{b}^j)\}. \tag{9}$$

Lemma 4.16. *For all $j \in F$, the following hold*:

(a) *for every $y_1 \in \boldsymbol{b}^j$, there is $y_2 \in \boldsymbol{\beta}(\boldsymbol{b}^j)$ with $t^\sigma_{\mathfrak{M}_1}(y_1) = t^\sigma_{\mathfrak{M}_2}(y_2)$, and, for every $y_2 \in \boldsymbol{\beta}(\boldsymbol{b}^j)$, there is $y_1 \in \boldsymbol{b}^j$ with $t^\sigma_{\mathfrak{M}_1}(y_1) = t^\sigma_{\mathfrak{M}_2}(y_2)$*;

(b) *$\boldsymbol{\beta}(\boldsymbol{b}^j)$ is a σ-block in $\mathfrak{M}_2$, and $\boldsymbol{b}^j$ is degenerate iff $\boldsymbol{\beta}(\boldsymbol{b}^j)$ is degenerate*;

(c) *$\{\boldsymbol{\beta}(\boldsymbol{b}^j) \mid j \in F\}$ is a partition of W_2*;

(d) *$\boldsymbol{b}^j \prec_{\mathfrak{F}_1} \boldsymbol{b}^k$ iff $\boldsymbol{\beta}(\boldsymbol{b}^j) \prec_{\mathfrak{F}_2} \boldsymbol{\beta}(\boldsymbol{b}^k)$, for $j, k \in F$*;

(e) *$\boldsymbol{b}^j$ is definable in $\mathfrak{M}_1$ iff $\boldsymbol{\beta}(\boldsymbol{b}^j)$ is definable in $\mathfrak{M}_2$*;

(*f*) *for every* $* \in \{\circ, \bullet\}$, $C^+_{\boldsymbol{b}^j}$ *is a limit cluster of type* $*$ *iff* $C^+_{\boldsymbol{\beta}(\boldsymbol{b}^j)}$ *is a limit cluster of type* $*$.

Proof. (*a*) This follows from $\mathfrak{M}_1, x_1 \sim_\sigma \mathfrak{M}_2, x_2$ and Lemma 3.1.

(*b*) Let $j \in F$. As $\mathfrak{M}_1, x_1 \sim_\sigma \mathfrak{M}_2, x_2$, $\boldsymbol{\beta}(\boldsymbol{b}^j) \neq \emptyset$. Take some $y \in \boldsymbol{\beta}(\boldsymbol{b}^j)$. We show that $\boldsymbol{\beta}(\boldsymbol{b}^j) = \boldsymbol{b}^\sigma_{\mathfrak{M}_2}(y)$. Indeed, this is straightforward from the definitions if $\Diamond t^\sigma_{\mathfrak{M}_2}(y) \subseteq t^\sigma_{\mathfrak{M}_2}(y)$. If $\Diamond t^\sigma_{\mathfrak{M}_2}(y) \not\subseteq t^\sigma_{\mathfrak{M}_2}(y)$, then $\boldsymbol{b}^\sigma_{\mathfrak{M}_2}(y) = \{y\}$. Take some $x \in \boldsymbol{b}^j$ with $t^\sigma_{\mathfrak{M}_1}(x) = t^\sigma_{\mathfrak{M}_2}(y)$. Then $\Diamond t^\sigma_{\mathfrak{M}_1}(x) \not\subseteq t^\sigma_{\mathfrak{M}_1}(x)$, and so $\boldsymbol{b}^j = \{x\}$. Thus, $\boldsymbol{\beta}(\boldsymbol{b}^j) = \{z \in W_2 \mid t^\sigma_{\mathfrak{M}_2}(z) = t^\sigma_{\mathfrak{M}_2}(y)\}$, and so there is a σ-formula μ such that $\mu \in t^\sigma_{\mathfrak{M}_2}(z) = t^\sigma_{\mathfrak{M}_2}(y)$ and $\Diamond\mu \notin t^\sigma_{\mathfrak{M}_2}(z) = t^\sigma_{\mathfrak{M}_2}(y)$. Suppose there is $z \in \boldsymbol{\beta}(\boldsymbol{b}^j)$, $z \neq y$. Then either zR_2y or yR_2z, which is a contradiction.

(*c*) As $\boldsymbol{\beta}(\boldsymbol{b}^j)$ and $\boldsymbol{\beta}(\boldsymbol{b}^k)$ are disjoint for $j \neq k$ by (*a*) and **(block)**, the relation '$y \approx y'$ iff there is $j \in F$ with $y, y' \in \boldsymbol{\beta}(\boldsymbol{b}^j)$' is an equivalence relation on W_2.

(*d*) This follows from $\mathfrak{M}_1, x_1 \sim_\sigma \mathfrak{M}_2, x_2$, (*a*) and **(block)**.

(*e*) This follows from (*b*)–(*d*) and Lemma 4.14 (*a*) and (*d*).

(*f*) As every degenerate cluster is a degenerate σ-block by Lemma 4.14 (*c*), it follows from (*b*)–(*e*) that $C^+_{\boldsymbol{b}^j}$ is a limit cluster of type $\bullet$ iff $C^+_{\boldsymbol{\beta}(\boldsymbol{b}^j)}$ is a limit cluster of type $\bullet$. Now (*f*) follows from Lemma 4.12. ⊣

So from now on we assume that we have a countable strict linear order $(F, \prec)$ such that each W_i, $i = 1, 2$, is partitioned as $\{\boldsymbol{b}^j_i \mid j \in F\}$ into σ-blocks in $\mathfrak{M}_i$ with $j \prec k$ iff $\boldsymbol{b}^j_1 \prec_{\mathfrak{F}_1} \boldsymbol{b}^k_1$ iff $\boldsymbol{b}^j_2 \prec_{\mathfrak{F}_2} \boldsymbol{b}^k_2$, for $j, k \in F$. Also, by Lemma 4.16 (*a*), we have $t^\sigma_{\mathfrak{M}_1}(\boldsymbol{b}^j_1) = t^\sigma_{\mathfrak{M}_2}(\boldsymbol{b}^j_2)$, for every $j \in F$.

Our aim is to achieve **(I1)** and **(I2)** above, even if not all σ-blocks $\boldsymbol{b}^j_i$ are definable in $\mathfrak{M}_i$. To begin with, **Steps 1** and **2** from Section 3 give us the sets M_i containing the $\{\psi\}$-maximal points in $\mathfrak{M}_i$ that satisfy each formula ψ in $sub(\varphi_i)$ that is satisfiable in $\mathfrak{M}_i$; the set T of the σ-types of points in $\{x_1, x_2\} \cup M_1 \cup M_2$ (cf. (4)); and also the sets $S_i \subseteq W_i$ of t-maximal points in $\mathfrak{M}_i$ satisfying the σ-types t from T. Points in $\{x_i\} \cup M_i \cup S_i$ are called *relevant in* $\mathfrak{M}_i$. A cluster or an interval is *relevant in* $\mathfrak{M}_i$ if it contains a relevant point. The number of relevant clusters (and of relevant σ-blocks) in $\mathfrak{M}_i$ is bounded by $\mathcal{O}\big(\max(|\varphi_1|, |\varphi_2|)\big)$.

Lemma 4.17. *Let* $j \in F$. *For every* $y_1 \in \big(\{x_1\} \cup M_1 \cup S_1\big) \cap \boldsymbol{b}^j_1$, *there are* $y'_1 \in S_1 \cap C^+_{\boldsymbol{b}^j_1}$ *and* $y_2 \in S_2 \cap C^+_{\boldsymbol{b}^j_2}$ *with* $t^\sigma_{\mathfrak{M}_1}(y_1) = t^\sigma_{\mathfrak{M}_1}(y'_1) = t^\sigma_{\mathfrak{M}_2}(y_2)$, *and, for every* $y_2 \in \big(\{x_2\} \cup M_2 \cup S_2\big) \cap \boldsymbol{b}^j_2$, *there are* $y'_2 \in S_2 \cap C^+_{\boldsymbol{b}^j_2}$ *and* $y_1 \in S_1 \cap C^+_{\boldsymbol{b}^j_1}$ *with* $t^\sigma_{\mathfrak{M}_2}(y_2) = t^\sigma_{\mathfrak{M}_2}(y'_2) = t^\sigma_{\mathfrak{M}_1}(y_1)$.

Proof. Let $y_1 \in \big(\{x_1\} \cup M_1 \cup S_1\big) \cap \boldsymbol{b}^j_1$ and $t = t^\sigma_{\mathfrak{M}_1}(x)$. Then $t \in T$, and so there are $y'_1 \in S_1$, $y_2 \in S_2$ such that y'_1 is t-maximal in $\mathfrak{M}_1$ and y_2 t-maximal in $\mathfrak{M}_2$. Then $y'_1 \in \boldsymbol{b}^j_1$ by **(block)**, so $y'_1 \in C^+_{\boldsymbol{b}^j_1}$ by Lemma 4.14 (*e*). By Lemma 4.16 (*a*), there is $y \in \boldsymbol{b}^j_2$ with $t = t^\sigma_{\mathfrak{M}_2}(y)$, and so $y_2 \in \boldsymbol{b}^j_2$ by Lemma 4.16 (*b*) and **(block)**. Thus, $y_2 \in C^+_{\boldsymbol{b}^j_2}$ by Lemma 4.14 (*e*). The other direction is similar. ⊣

It follows that, for all $j \in F$,

(10) $\boldsymbol{b}_1^j$ is relevant in $\mathfrak{M}_1$ iff $\boldsymbol{b}_2^j$ is relevant in $\mathfrak{M}_2$;

(11) if $\boldsymbol{b}_1^j$ and $\boldsymbol{b}_2^j$ are relevant, then $|S_1 \cap C^+_{\boldsymbol{b}_1^j}| = |S_2 \cap C^+_{\boldsymbol{b}_2^j}| \neq 0$.

EXAMPLE 4.18. For the models $\mathfrak{M}_i$ shown in Fig. 2 from Example 3.5 and $\sigma = \{p_1, p_2\}$, we have $M_i = \{x_i, y_i, b_i^1, b_i^0\}$, $S_1 = \{x_1, y_1, a_1^1, b_1^1, b_1^0\}$, and $S_2 = \{x_2, y_2, a_2^0, b_2^1, b_2^0\}$, so only the first two and the last two σ-blocks in the $\mathfrak{M}_i$ are relevant (cf. Example 4.15 for the σ-blocks). ⊣

DEFINITION 4.19. We now define the partitions $\mathcal{I}_i$, $i = 1, 2$, satisfying conditions **(I1)** and **(I2)** (at the end of Section 4.1) in three steps as follows:

(i) First, we put into $\mathcal{I}_i$ all those finitely-many relevant σ-blocks that are definable in $\mathfrak{M}_i$ (by (10), $\boldsymbol{b}_1^j$ is such iff $\boldsymbol{b}_2^j$ is such).

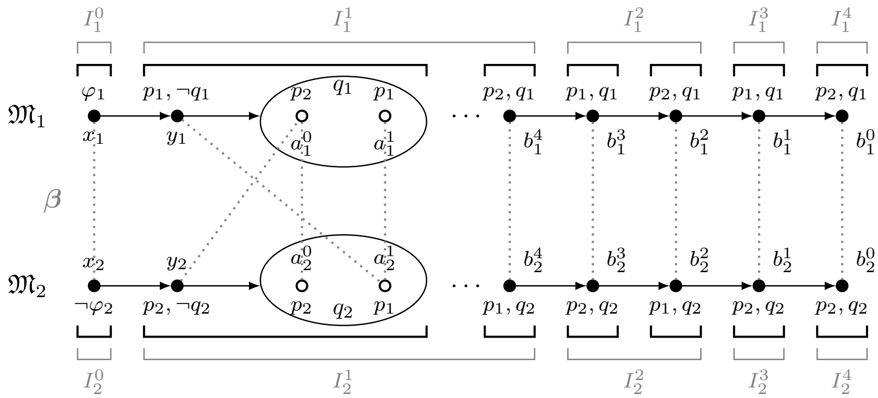
(ii) Next, suppose $\boldsymbol{b}_1^j$, $j \in F$, is a relevant σ-block that is not definable in $\mathfrak{M}_1$. By (10) and Lemma 4.16 (e), $\boldsymbol{b}_2^j$ is also a relevant σ-block that is not definable in $\mathfrak{M}_2$. By Lemma 4.14 (d), each $C^+_{\boldsymbol{b}_i^j}$ is a limit cluster in $\mathfrak{F}_i$, with both of them being the same type $\circ$ or $\bullet$ by Lemma 4.16 (f). We pick some $k \succ j$ such that that the intervals $\big(C^+_{\boldsymbol{b}_i^j}, C^-_{\boldsymbol{b}_i^k}\big)$ are irrelevant. Such a k must exist because the number of relevant points is finite, but is not necessarily unique. For $i = 1, 2$, let E_i be the immediate predecessor of $C^-_{\boldsymbol{b}_i^k}$. We put the interval $\big[C^-_{\boldsymbol{b}_i^j}, E_i\big]$ into $\mathcal{I}_i$, $i = 1, 2$, and say that such intervals *extend* the relevant non-definable σ-blocks $\boldsymbol{b}_i^j$.

(iii) Finally, suppose intervals $I_i^1 = [C_i, C_i']$ and $I_i^2 = [D_i, D_i']$ with $I_i^1 \prec_{\mathfrak{F}_i} I_i^2$ are already in $\mathcal{I}_i$ and there is no interval in $\mathcal{I}_i$ intersecting the gap between I_i^1 and I_i^2—the interval $[C_i'', D_i'']$ with the immediate successor C_i'' of the non-limit cluster C_i' and the immediate predecessor D_i'' of D_i. Then we put $[C_i'', D_i'']$ into $\mathcal{I}_i$. Clearly, such an interval is a union of irrelevant σ-blocks, and D_i'' is non-limit cluster.

Observe that this way every interval in $\mathcal{I}_i$ is of the form $[C, C']$, where C' is a non-limit cluster in $\mathfrak{F}_i$. So **(I1)** holds by Lemma 4.10 (d), and **(I2)** follows from Lemma 4.16 and the fact that each interval in $\mathcal{I}_i$ is a union of σ-blocks.

The following example illustrates Definition 4.19.

EXAMPLE 4.20. For models $\mathfrak{M}_i$, $i = 1, 2$, from Example 3.5 and σ-blocks from Example 4.15 for $\sigma = \{p_1, p_2\}$, we can pick the intervals I_i^j, $j = 0, \dots, 4$, shown in the picture below. The intervals I_i^2 are irrelevant, while all other intervals are relevant (cf. Example 4.18). The choice of the infinite intervals I_i^1 extending the non-definable σ-blocks till points b_i^4 is arbitrary. We could make them shorter or, on the contrary, extend until points b_i^2, in which case there would be no gap between these intervals (extending relevant non-definable σ-blocks) and the next relevant interval. ⊣



4.4. Simplifying interval-based models. The partitions $\mathcal{I}_i = \{I_i^\ell \in \mathcal{P}_i \mid \ell < N\}$, $i = 1, 2$, we constructed in Definition 4.19 are such that the intervals in each pair (I_1^ℓ, I_2^ℓ), $\ell < N$, are either (i) both original relevant σ-blocks, or (ii) both extensions of relevant non-definable σ-blocks, or (iii) both unions of irrelevant σ-blocks. Below we address case (iii) in Lemma 4.21, and then cases (i) and (ii) in Lemma 4.23. In each case, we transform pairs $(\mathfrak{M}_1 \restriction_{I_1^\ell}, \mathfrak{M}_2 \restriction_{I_2^\ell})$ of globally σ-bisimilar models into pairs $(\mathfrak{N}_1^\ell, \mathfrak{N}_2^\ell)$ of simple models meeting one of (d_1), (d_2), or (d_3) in Definition 4.1 so that the resulting $\mathfrak{N}_i = \mathfrak{N}_i^0 \lhd \cdots \lhd \mathfrak{N}_i^{N-1}$ meet (a)–(c) in Definition 4.1. The basic frames $\mathfrak{H}_i^\ell = (H_i^\ell, R_i^\ell, \mathcal{P}_i^\ell)$ underlying $\mathfrak{N}_i^\ell$ will be such that $H_i^\ell \subseteq I_i^\ell$ and $R_i^\ell = R_i \restriction_{H_i^\ell}$, but $\mathfrak{H}_i^\ell$ is not necessarily a subframe of $\mathfrak{F}_i \restriction_{I_i^\ell}$. We will ensure that no new types are introduced in $\mathfrak{N}_i$ by defining a special 'parent' function h_i from $\mathfrak{N}_i$ to $\mathfrak{M}_i$, for which $at_{\mathfrak{N}_i}(x) = at_{\mathfrak{M}_i}\big(\mathsf{h}_i(x)\big)$.

Lemma 4.21. *Suppose $I_i = I_i^\ell \in \mathcal{I}_i$, for some $\ell < N$, and $I_i \in \mathcal{P}_i$ is a union of irrelevant σ-blocks in $\mathfrak{M}_i$, $i = 1, 2$. Then there exist sets $H_i \subseteq I_i$, $\mathcal{P}_i' \subseteq 2^{H_i}$ and functions $\mathfrak{w}_i \colon \mathcal{V} \to \mathcal{P}_i'$, $\mathsf{h}_i \colon H_i \to H_i$ such that the following hold, for $i = 1, 2$:*

- (a) *there is an isomorphism f between the frames $\mathfrak{H}_1 = (H_1, R_1 \restriction_{H_1}, \mathcal{P}_1')$ and $\mathfrak{H}_2 = (H_2, R_2 \restriction_{H_2}, \mathcal{P}_2')$, which are isomorphic to the same atomic frame*;
- (b) $\mathfrak{F}_i \restriction_{I_i} \rightarrowtail \mathfrak{H}_\mathfrak{i}$;
- (c) $\mathfrak{N}_1 = (\mathfrak{H}_1, \mathfrak{w}_1)$, $\mathfrak{N}_2 = (\mathfrak{H}_2, \mathfrak{w}_2)$ *are simple models with* $at^\sigma_{\mathfrak{N}_1}(y) = at^\sigma_{\mathfrak{N}_2}\big(f(y)\big)$, *for all $y \in H_1$, and so $(\mathfrak{N}_1, \mathfrak{N}_2)$ satisfies (d_1) in Definition 4.1*;
- (d) $\big(\{x_i\} \cup M_i \cup S_i\big) \cap I_i \subseteq H_i$;
- (e) *for all $x, y \in H_i$,*
  1. *if $x \in \{x_i\} \cup M_i \cup S_i$, then $\mathsf{h}_i(x) = x$;*
  2. $at_{\mathfrak{N}_i}(x) = at_{\mathfrak{N}_i}\big(\mathsf{h}_i(x)\big) = at_{\mathfrak{M}_i}\big(\mathsf{h}_i(x)\big)$;
  3. *if xR_iy, then $\mathsf{h}_i(x)R_i\mathsf{h}_i(y)$;*
  4. *for all $y \in M_i \cap H_i$, if $\mathsf{h}_i(x)R_iy$, then xR_iy.*

Proof. Let $Z_i = \{z_i^j \mid j < m_i\}$ be the tail of $\mathfrak{F}_i \restriction_{I_i}$, for $i = 1, 2$ and $m_i \leq \omega$, with $z_i^j R^s z_i^{j-1}$, for $0 < j < m_i$. By **(I2)**, $\big\{(y_1, y_2) \in I_1 \times I_2 \mid t^\sigma_{\mathfrak{M}_1}(y_1) = t^\sigma_{\mathfrak{M}_2}(y_2)\big\}$ is a global σ-bisimulation between $\mathfrak{M}_1 \restriction_{I_1}$ and $\mathfrak{M}_2 \restriction_{I_2}$. So by Lemma 4.13, we have $|Z_1| = |Z_2| = m$, for some $m \leq \omega$, and $Z_1 = I_1$ iff $Z_2 = I_2$. Also, for

$i = 1, 2$, there exist u_i in the face of Z_i such that $t^\sigma_{\mathfrak{M}_1}(u_1) = t^\sigma_{\mathfrak{M}_2}(u_2)$. Let

$$H_i = \begin{cases} Z_i, & \text{if } Z_i = I_i, \\ \{u_i\} \cup Z_i, & \text{otherwise,} \end{cases}$$

and let $\mathcal{P}'_i$ consist of all finite subsets of Z_i and their complements in H_i. Then the function $f\colon H_1 \to H_2$ defined by $f(z^j_1) = z^j_2$, $j < m$, and $f(u_1) = u_2$ is an isomorphism between the resulting frames $\mathfrak{H}_1$ and $\mathfrak{H}_2$, which are isomorphic to

(*i*) $m^<$, when $Z_i = I_i$;

(*ii*) $① \lhd m^<$, when $Z_i \neq I_i$ and $m < \omega$;

(*iii*) $\mathfrak{C}(①, \bullet)$, when Z_i is infinite.

This gives (*a*); (*b*) is straightforward (in (*iii*), this is because all finite subsets of Z_i and their complements in I_i are in $\mathcal{P}_i$ by Lemma 4.10, so the function mapping $I_i \setminus Z_i$ to u_i and being identical on Z_i is a root-mapping p-morphism from $\mathfrak{F}_i\restriction_{I_i}$ onto $\mathfrak{H}_i$). For $p \in \mathcal{V}$, let $\mathfrak{w}_i(p) = \mathfrak{v}_i(p) \cap H_i$ in case (*i*), and

$$\mathfrak{w}_i(p) = \begin{cases} H_i, & \text{if } u_i \in \mathfrak{v}_i(p), \\ \emptyset, & \text{otherwise} \end{cases}$$

in cases (*ii*) and (*iii*). Then $\mathfrak{w}_i(p) \in \mathcal{P}'_i$ and (*c*) holds. For $x \in H_i$, we let $\mathsf{h}_i(x) = x$ in case (*i*), and $\mathsf{h}_i(x) = u_i$ in cases (*ii*) and (*iii*). Then we also have (*e*).2 and (*e*).3. Note that (*d*), (*e*).1 and (*e*).4 hold vacuously as each I_i is a union of irrelevant σ-blocks in $\mathfrak{M}_i$, and so $\big(\{x_i\} \cup M_i \cup S_i\big) \cap I_i = \emptyset$. ⊣

We next deal with the remaining two cases in the construction of Definition 4.19. The next lemma makes it possible to replace certain infinite 'bits' in our frames by tadpoles $\mathfrak{C}(ⓚ, *)$, $* \in \{\bullet, \circ\}$, with points $W_k = A_k \cup \{b_n \mid n < \omega\}$, for $A_k = \{a_0, \dots, a_{k-1}\}$, and the internal sets $\mathcal{P}_k$ (as defined in Example 2.2).

Lemma 4.22. *Suppose $\mathfrak{G} = (W, R, \mathcal{P})$ is a finitely generated descriptive frame for* K4.3 *with a root cluster $A_\ell = \{a_0, \dots, a_{\ell-1}\}$, for some $\ell < \omega$, such that $A_k \subseteq A_\ell$, A_ℓ is the limit of an infinite descending chain of clusters C_n, $n < \omega$, which are either* (*i*) *all degenerate or* (*ii*) *all non-degenerate, and $b_n \in C_n$, for $n < \omega$. Then $\mathsf{Log}(\mathfrak{G}) \subseteq \mathsf{Log}\big(\mathfrak{C}(ⓚ, *)\big)$ and $\mathsf{Log}(\mathfrak{G}, a_s) \subseteq \mathsf{Log}\big(\mathfrak{C}(ⓚ, *), a_s\big)$, for $s < k$, where $* = \bullet$ in case* (*i*) *and $* = \circ$ in case* (*ii*).

Proof. For $n < \omega$, we let $J_n = [C_n, D_n]$, where D_0 is the final cluster of $\mathfrak{G}$ and D_n is the immediate predecessor of D_{n-1}. Then, for any finite $X \subseteq \{b_n \mid n < \omega\}$, we let $J_X = \bigcup_{b_n \in X} J_n$. By Lemma 4.10 (*d*), $J_n \in \mathcal{P}$ for every $n < \omega$, and so

(12) $\qquad J_X \in \mathcal{P}$, for every finite subset of X of $\{b_n \mid n < \omega\}$.

Suppose there is a valuation $\mathfrak{v}\colon \mathcal{V} \to \mathcal{P}_k$ with $x \in \mathfrak{v}(\varphi)$, for some $x \in W_k$ and formula φ. Our aim is to define $\mathfrak{w}\colon \mathcal{V} \to \mathcal{P}$ such that $x \in \mathfrak{w}(\varphi)$. If $x = b_n$ for some $n < \omega$, then we let $\mathfrak{w}(p) = J_{\{b_m \mid m \leq n\} \cap \mathfrak{v}(p)}$, for $p \in \mathcal{V}$. By (12), $\mathfrak{w}(p) \in \mathcal{P}$. Clearly, $x = b_n \in \mathfrak{w}(\varphi)$, as required. Now suppose $x = a_s$, for some $s < k$. We define $\mathfrak{w}\colon \mathcal{V} \to \mathcal{P}$ such that $a_s \in \mathfrak{w}(\varphi)$. To this end, we let

$$\mathcal{X}_\varphi = \{\mathfrak{v}(\psi) \mid \psi \in sub(\varphi),\ \mathfrak{v}(\psi) \text{ is finite}\} \ \cup \\ \{W_k \setminus \mathfrak{v}(\psi) \mid \psi \in sub(\varphi),\ \mathfrak{v}(\psi) \text{ is cofinite in } W_k\}$$

and $X_\varphi = \bigcup \mathcal{X}_\varphi$. By (2) in Example 2.2, X_φ is a finite subset of $\{b_n \mid n < \omega\}$, and so $J_{X_\varphi} \in \mathcal{P}$ by (12). By **(dif)**, for every $z < \ell$, there is $Z_z \in \mathcal{P}$ such that $A_\ell \cap Z_z = \{a_z\}$, for all $z < \ell$, and $\{Z_z \mid z < \ell\}$ is a partition of W. Now, for every $p \in \mathcal{V}$, we let

- $\mathfrak{w}(p) = J_{\mathfrak{v}(p)}$, if $\mathfrak{v}(p)$ is finite;
- $\mathfrak{w}(p) = W \setminus J_{\mathfrak{v}(p)}$, if $\mathfrak{v}(p)$ is cofinite in W_k; and
- $\mathfrak{w}(p) = \big(\bigcup_{a_z \in \mathfrak{v}(p)} Z_z \cap (W \setminus J_{X_\varphi})\big) \cup J_{\mathfrak{v}(p) \cap X_\varphi}$, if both $\mathfrak{v}(p)$ and $W_k \setminus \mathfrak{v}(p)$ are infinite.

Let $Z = \bigcup_{k \le z < \ell} Z_z$. It can be shown by induction that, for every $\psi \in sub(\varphi)$,

- $\mathfrak{w}(\psi) \setminus Z = J_{\mathfrak{v}(\psi)} \setminus Z$ if $\mathfrak{v}(\psi)$ is finite or cofinite in W_k; and
- $\mathfrak{w}(\psi) \setminus Z = \big(\bigcup_{a_z \in \mathfrak{v}(\psi)} Z_z \cap (W \setminus J_{X_\varphi})\big) \cup J_{\mathfrak{v}(\psi) \cap X_\varphi}$, if both $\mathfrak{v}(\psi)$ and $W_k \setminus \mathfrak{v}(\psi)$ are infinite.

It follows that $a_s \in \mathfrak{w}(\varphi)$, as required. ⊣

The following lemma deals with cases (i) and (ii) in Definition 4.19, which give rise to a pair of simple models satisfying conditions (d_2) and (d_3) in Definition 4.1:

Lemma 4.23. *Suppose $I_i = I_i^\ell \in \mathcal{I}_i$, for some $\ell < N$, and $I_i \in \mathcal{P}_i$, $i = 1, 2$, is an interval that is either a relevant definable σ-block $\boldsymbol{b}_i$ itself or extending the relevant non-definable σ-block $\boldsymbol{b}_i$. Then there exist n_i, $0 < n_i = \mathcal{O}\big(\max(|\varphi_1|, |\varphi_2|)\big)$, pairwise disjoint sets $H_i^j \subseteq I_i$, sets $\mathcal{P}_i^j \subseteq 2^{H_i^j}$, and functions $\mathfrak{w}_i^j \colon \mathcal{V} \to \mathcal{P}_i^j$, $j < n_i$, such that the following hold, for $\mathfrak{H}_i^j = (H_i^j, R_i \restriction_{H_i^j}, \mathcal{P}_i^j)$, $\mathfrak{N}_i^j = (\mathfrak{H}_i^j, \mathfrak{w}_i^j)$, $H_i = \bigcup_{j < n_i} H_i^j$, $\mathfrak{H}_\mathsf{i} = \mathfrak{H}_i^0 \lhd \cdots \lhd \mathfrak{H}_i^{n_i - 1}$; $\mathfrak{N}_\mathsf{i} = \mathfrak{N}_i^0 \lhd \cdots \lhd \mathfrak{N}_i^{n_i - 1}$, and $i = 1, 2$:*

- (a) *$\mathfrak{H}_i^j$ is an atomic frame, for $j < n_i$, and if I_i is extending the relevant non-definable σ-block $\boldsymbol{b}_i$, then $\mathfrak{H}_1^{n_1 - 1}$ and $\mathfrak{H}_2^{n_2 - 1}$ are isomorphic to the same atomic frame of the form $\mathfrak{C}(ⓚ, \bullet)$ or $\mathfrak{C}(ⓚ, \circ)$;*
- (b) *$\mathsf{Log}(\mathfrak{F}_i \restriction_{I_i}) \subseteq \mathsf{Log}(\mathfrak{H}_i)$, and for every root z_i in $\mathfrak{H}_i$ there is a root y_i in $\mathfrak{F}_i \restriction_{I_i}$ with $\mathsf{Log}(\mathfrak{F}_i \restriction_{I_i}, y_i) \subseteq \mathsf{Log}(\mathfrak{H}_i, z_i)$;*
- (c) *$\mathfrak{N}_i^j$ is a simple model, for $j < n_i$, and if I_i is extending the relevant non-definable σ-block $\boldsymbol{b}_i$, then $at^\sigma_{\mathfrak{N}_1}(y) = at^\sigma_{\mathfrak{N}_2}\big(f(y)\big)$, for every y in the ⓚ-cluster of $\mathfrak{H}_1^{n_1 - 1}$ and the isomorphism f between $\mathfrak{H}_1^{n_1 - 1}$ and $\mathfrak{H}_2^{n_2 - 1}$;*
- (d) *$\big(\{x_i\} \cup M_i \cup S_i\big) \cap I_i \subseteq H_i$;*
- (e) *there is a function $\mathsf{h}_i \colon H_i \to H_i$ such that, for all $x, y \in H_i$,*
  1. *if $x \in \{x_i\} \cup M_i \cup S_i$, then $\mathsf{h}_i(x) = x$;*
  2. *$at_{\mathfrak{N}_i}(x) = at_{\mathfrak{N}_i}\big(\mathsf{h}_i(x)\big) = at_{\mathfrak{M}_i}\big(\mathsf{h}_i(x)\big)$;*
  3. *if $x R_i y$, then $\mathsf{h}_i(x) R_i \mathsf{h}_i(y)$;*
  4. *for all $y \in M_i \cap H_i$, if $\mathsf{h}_i(x) R_i y$ then $x R_i y$;*
  5. *$\mathsf{h}_i(x) \in \{x_i\} \cup M_i \cup S_i$.*

Proof. For $i = 1, 2$, let C_i^j, $j < m_i$ be the sequence (ordered by $<_{R_i}$) of all relevant clusters in $\boldsymbol{b}_i$ (that is, those that intersect with $\{x_i\} \cup M_i \cup S_i$). Then $C_i^{m_i - 1}$ is the final cluster $C^+_{\boldsymbol{b}_i}$ of $\boldsymbol{b}_i$. Below, we first partition I_i into closed intervals $\{J_i^j \mid j < n_i\}$ that are definable in $\mathfrak{M}_i \restriction_{I_i}$, where n_i is either m_i or $m_i + 1$, depending on I_i. Then we replace each model $\mathfrak{M}_i \restriction_{J_i^j}$, $j < n_i$, with a simple model $\mathfrak{N}_i^j$ based on an atomic frame $\mathfrak{H}_i^j$.

By Lemma 4.10, C_i^j, is a non-limit cluster, for all j with $0 < j < m_i - 1$. $C_i^{m_i-1}$ is also a non-limit cluster if $I_i = \boldsymbol{b}_i$ for a definable σ-block $\boldsymbol{b}_i$, and $C_i^{m_i-1}$ is a limit cluster if I_i is extending the non-definable σ-block $\boldsymbol{b}_i$. If C_i^0 is the root cluster in $\mathfrak{F}_i$, it can happen that $\big(\{x_i\} \cup M_i\big) \cap C_i^0 = \{x_i\}$, $x_i \notin M_i$, and C_i^0 is a limit cluster. (In this case, $m_i > 0$ by (11).) In any other case, C_i^0 is a non-limit cluster. So we define n_i and $\{J_i^j \mid j < n_i\}$ depending on these cases as follows:

(p1) If $I_i = \boldsymbol{b}_i$, for the definable σ-block $\boldsymbol{b}_i$, then we let $n_i = m_i$ and, for all j, $1 < j < m_i$, $J_i^j = [D_i^j, C_i^j]$, where D_i^j is the immediate successor of the non-limit cluster C_i^{j-1}.
1. If C_i^0 is a non-limit cluster, then we let $J_i^0 = C_i^0$ and $J_i^1 = [D_i^1, C_i^1]$, where D_i^1 is the immediate successor of C_i^0.
2. If C_i^0 is a limit cluster, then we let $J_i^0 = [C_i^0, E_i^1]$, for the immediate predecessor E_i^1 of the non-limit cluster C_i^1, and $J_i^1 = C_i^1$.

(p2) If I_i is extending the non-definable σ-block $\boldsymbol{b}_i$, then, for all $j < m_i - 1$, we define J_i^j as in item (**p1**) above. There are two cases:
1. If $C_i^{m_i-2}$ is the immediate predecessor of $C_i^{m_i-1} = C_{\boldsymbol{b}_i}^+$, then we let $n_i = m_i$ and $J_i^{m_i-1} = [C_i^{m_i-1}, E_i]$, for the final cluster E_i of I_i.
2. Otherwise, we let $n_i = m_i+1$, $J_i^{m_i-1} = [D_i, D_i']$, where D_i is the immediate successor of the non-limit cluster $C_i^{m_i-2}$ and D_i' is the immediate predecessor of the (limit) cluster $C_i^{m_i-1} = C_{\boldsymbol{b}_i}^+$, and $J_i^{m_i} = [C_i^{m_i}, E_i]$ for the final cluster E_i of I_i. Observe that $J_i^{m_i-1}$ is an irrelevant interval.

By Lemma 4.10 (d), J_i^j is definable in $\mathfrak{M}_i \restriction_{I_i}$, for $j < n_i$; all relevant points in $\mathfrak{M}_i$ are included in some J_i^j, and all relevant points in any J_i^j are included in the same cluster in J_i^j. Next, we replace each model $\mathfrak{M}_i \restriction_{J_i^j}$ with a simple model $\mathfrak{N}_i^j$ based on an atomic frame $\mathfrak{H}_i^j$ so that

$$\big(\{x_i\} \cup M_i \cup S_i\big) \cap J_i^j \subseteq H_i^j. \tag{13}$$

There are four cases:

Case 1: $J_i^j = [C, C_i^j]$ for some cluster C, and C_i^j is a non-limit cluster. Suppose Z_i^j is the tail of $\mathfrak{F}_i \restriction_{J_i^j}$. If $Z_i^j \neq \emptyset$, then $C_i^j = \{z_i^j\}$ is a degenerate cluster for some $z_i^j \in \{x_i\} \cup M_i \cup S_i$, by Lemma 4.17, so $\big(\{x_i\} \cup M_i \cup S_i\big) \cap J_i^j = \{z_i^j\}$. We consider several subcases, depending on $|Z_i^j| = m$. In each case, we choose H_i^j and $\mathcal{P}_i^j$ so that (13) holds and

$$\mathfrak{F}_i \restriction_{J_i^j} \rightarrowtail \mathfrak{H}_\mathfrak{i}^\mathfrak{j}. \tag{14}$$

Case 1.1: $Z_i^j = \emptyset$, so C_i^j is non-degenerate. Let $0 < k = \big|\big(\{x_i\} \cup M_i \cup S_i\big) \cap C_i^j\big| = \mathcal{O}\big(\max(|\varphi_1|, |\varphi_2|)\big)$. Let $H_i^j = \big(\{x_i\} \cup M_i \cup S_i\big) \cap C_i^j$ and $\mathcal{P}_i^j = 2^{H_i^j}$. Then $\mathfrak{H}_i^j$ isomorphic to ⓚ, having (a), (13) and (14); by taking $\mathfrak{w}_i^j(p) = \mathfrak{v}_i(p) \cap H_i^j$, $p \in \mathcal{V}$, we have (c). Set $\mathsf{h}_i(x) = x$ for all $x \in H_i^j$.

Case 1.2: If $m < \omega$ and $Z_i^j = J_i^j$, then by taking $H_i^j = J_i^j$ and $\mathcal{P}_i^j = 2^{H_i^j}$ we obtain $\mathfrak{H}_i^j$ isomorphic to $m^<$ having (a), (13) and (14).

Case 1.3: If $m < \omega$ and $Z_i^j \neq J_i^j$, then by taking $H_i^j = Z_i^j \cup \{u\}$ for any u in the face of Z_i^j and $\mathcal{P}_i^j = 2^{H_i^j}$ we obtain $\mathfrak{H}_i^j$ isomorphic to $① \lhd m^<$ having (a), (13) and (14).

Case 1.4: If Z_i^j is infinite, then we let $H_i^j = Z_i^j \cup \{u\}$ for any u in the face of Z_i^j, and let $\mathcal{P}_i^j$ consist of all finite subsets of Z_i^j and their complements in H_i^j. The resulting $\mathfrak{H}_i^j$ is isomorphic to $\mathfrak{C}(①, \bullet)$ having (a) and (13). Observe that all finite subsets of Z_i and their complements in I_i are in $\mathcal{P}_i$ because of Lemma 4.10. So the function that is the identity on Z_i^j and maps $J_i^j \setminus Z_i^j$ to u is a root-mapping p-morphism from $\mathfrak{F}_i\restriction_{J_i^j}$ onto $\mathfrak{H}_i^j$, which gives (14).

In any of *Cases* 1.2–1.4, by taking

$$\mathfrak{w}_i^j(p) = \begin{cases} H_i^j, & \text{if } z_i^j \in \mathfrak{v}_i(p), \\ \emptyset, & \text{otherwise,} \end{cases}$$

for $p \in \mathcal{V}$, we have $\mathfrak{w}_i^j(p) \in \mathcal{P}_i^j$ and (c). We let $\mathsf{h}_i(x) = z_i^j$ for all $x \in H_i^j$.

Case 2: $J_i^{n_i-1} = [C_i^{m_i-1}, E_i]$ for the final cluster E_i of I_i, and $C_i^{m_i-1}$ is a limit cluster, for $i = 1, 2$. (This can happen when $I_i \in \mathcal{P}_i$ is extending the non-definable σ-block $\boldsymbol{b}_i$ with $C_{\boldsymbol{b}_i}^+ = C_i^{m_i-1}$.) By **(I2)** and Lemma 4.16 (f), $C_1^{m_1-1}$ and $C_2^{m_2-1}$ are limit clusters of the same type $* \in \{\bullet, \circ\}$. By (11), $|S_1 \cap C_1^{m_1-1}| = |S_2 \cap C_2^{m_2-1}| = k$ with $0 < k = \mathcal{O}\big(\max(|\varphi_1|, |\varphi_2|)\big)$. By Lemma 4.17, we may assume that $S_i \cap C_i^{m_i-1} = \{a_i^0, \dots, a_i^{k-1}\}$ with $at^\sigma_{\mathfrak{M}_1}(a_1^s) = at^\sigma_{\mathfrak{M}_2}(a_2^s)$ for all $s < k$. There are two subcases:

Case 2.1: The limit clusters $C_1^{m_1-1}$ and $C_2^{m_2-1}$ are both of type $\bullet$. Then, $C_i^{m_i-1}$ is the limit of an infinite descending chain of degenerate clusters $\{b_i^n\}$ in $J_i^{n_i-1}$, $n < \omega$, for which $\{b_i^{n+1}\}$ is the immediate predecessor of $\{b_i^n\}$, $n < \omega$.

Case 2.2: The limit clusters $C_1^{m_1-1}$ and $C_2^{m_2-1}$ are both of type $\circ$. Then $C_i^{m_i-1}$ is the limit of an infinite descending chain of non-degenerate clusters Y_i^n in $J_i^{n_i-1}$, $n < \omega$. For each $i = 1, 2$ and each $n < \omega$, we pick some $b_i^n \in Y_i^n$.

In both of these subcases, we let $H_i^{n_i-1} = \{a_i^0, \dots, a_i^{k-1}\} \cup \{b_i^n \mid n < \omega\}$, and let $\mathcal{P}_i^{n_i-1}$ be generated in $\big(H_i^{n_i-1}, R_i\restriction_{H_i^{n_i-1}}\big)$ by the sets $\{b_i^n\}$, $n < \omega$, and X_i^s, $s < k$, where $X_i^s = \{a_i^s\} \cup \{b_i^n \mid n < \omega,\ n \equiv s \pmod{k}\}$. Then the function $f\colon H_1^{n_1-1} \to H_2^{n_2-1}$ defined by taking $f(a_1^s) = a_2^s$ and $f(b_1^n) = b_2^n$, for $s < k$, $n < \omega$, is an isomorphism between the resulting frames $\mathfrak{H}_1^{n_1-1}$ and $\mathfrak{H}_2^{n_2-1}$, which are both isomorphic to $\mathfrak{C}(ⓚ, \bullet)$ in *Case* 2.1, and to $\mathfrak{C}(ⓚ, \circ)$ in *Case* 2.2. This gives us (a). Since, by Definition 4.19, none of the b_i^n is relevant, we also have (13) for $j = n_i - 1$. By Lemma 4.22, for $s < k$,

$$\text{(15)} \quad \mathsf{Log}(\mathfrak{F}_i\restriction_{J_i^{n_i-1}}) \subseteq \mathsf{Log}(\mathfrak{H}_i^{n_i-1}) \ \text{ and } \ \mathsf{Log}(\mathfrak{F}_i\restriction_{J_i^{n_i-1}}, a_i^s) \subseteq \mathsf{Log}(\mathfrak{H}_i^{n_i-1}, a_i^s).$$

For $p \in \mathcal{V}$, we let $\mathfrak{w}_i^{n_i-1}(p) = \bigcup_{a_i^s \in \mathfrak{v}(p)} X_i^s$, which yields $\mathfrak{w}_i^{n_i-1}(p) \in \mathcal{P}_i^{n_i-1}$ and (c). For $x \in H_i^{n_i-1}$, we let $\mathsf{h}_i(x) = a_i^s$, whenever $x \in X_i^s$.

Case 3: $J_i^0 = [C_i^0, C]$ for some non-limit cluster C, C_i^0 is a limit cluster and $\big(\{x_i\} \cup M_i\big) \cap C_i^0 = \{x_i\}$, $x_i \notin M_i$. By Lemma 4.12, there are two subcases:

Case 3.1: C_i^0 is a limit cluster of type $\bullet$. Then C_i^0 is the limit of an infinite descending chain of degenerate clusters $\{b_i^n\}$ in J_i^0, $n < \omega$, for which $\{b_i^{n+1}\}$ is the immediate predecessor of $\{b_i^n\}$ for all $n < \omega$.

Case 3.2: C_i^0 is a limit cluster of type $\circ$. Then C_i^0 is the limit of an infinite descending chain of non-degenerate clusters Y_i^n in J_i^0, $n < \omega$. For each $n < \omega$, we pick some $b_i^n \in Y_i^n$.

In both of these subcases, we let $H_i^0 = \{x_i\} \cup \{b_i^n \mid n < \omega\}$ and $\mathcal{P}_i^0$ consist of all finite subsets of $\{b_i^n \mid n < \omega\}$ and their complements in H_i^0. The resulting $\mathfrak{H}_i^0 = (H_i^0, R_i\restriction_{H_i^0}, \mathcal{P}_i^0)$ is isomorphic to $\mathfrak{C}(①, \bullet)$ in *Case* 3.1 and to $\mathfrak{C}(①, \circ)$ in *Case* 3.2 having (a) and (13). By Lemma 4.22,

$$\mathsf{Log}(\mathfrak{F}_i\restriction_{J_i^0}) \subseteq \mathsf{Log}(\mathfrak{H}_i^0) \text{ and } \mathsf{Log}(\mathfrak{F}_i\restriction_{J_i^0}, x_i) \subseteq \mathsf{Log}(\mathfrak{H}_i^0, x_i). \tag{16}$$

Also, we let

$$\mathfrak{w}_i^0(p) = \begin{cases} H_i^0, & \text{if } x_i \in \mathfrak{v}_i(p), \\ \emptyset, & \text{otherwise,} \end{cases}$$

for $p \in \mathcal{V}$, obtaining $\mathfrak{w}_i^0(p) \in \mathcal{P}_i^0$ and (c). We let $\mathsf{h}_i(x) = x_i$ for all $x \in H_i^0$.

Case 4: $J_i^{m_i-1}$ is the irrelevant interval defined in **(p2)**.2 above. Suppose Z_i is the tail of $\mathfrak{F}_i\restriction_{J_i^{m_i-1}}$. There are four subcases that are slightly different to the cases in the proof of Lemma 4.21:

Case 4.1: $Z_i = \emptyset$. Then take any point y in the final (non-degenerate) cluster of $J_i^{m_i-1}$. By taking $H_i^{m_i-1} = \{y\}$ and $\mathcal{P}_i^{m_i-1} = 2^{H_i^{m_i-1}}$ we obtain $\mathfrak{H}_i^{m_i-1}$ isomorphic to ① having (a) and (14).

Case 4.2: $|Z_i| = m$ for some $0 < m < \omega$ and $Z_i = J_i^{m_i-1}$. Let $H_i^{m_i-1} = J_i^{m_i-1}$ and $\mathcal{P}_i^{m_i-1} = 2^{H_i^{m_i-1}}$. Then $\mathfrak{H}_i^{m_i-1}$ isomorphic to $m^<$ having (a) and (14).

Case 4.3: $|Z_i| = m$ for some $m < \omega$ and $Z_i \neq J_i^{m_i-1}$. Let $H_i^{m_i-1} = Z_i \cup \{u\}$ for any u in the face of Z_i and $\mathcal{P}_i^{m_i-1} = 2^{H_i^{m_i-1}}$. Then $\mathfrak{H}_i^{m_i-1}$ isomorphic to $① \lhd m^<$ having (a) and (14).

Case 4.4: Z_i is infinite. Let $H_i^{m_i-1} = Z_i \cup \{u\}$ for any u in the face of Z_i and let $\mathcal{P}_i^{m_i-1}$ consist of all finite subsets of Z_i and their complements in $H_i^{m_i-1}$. The resulting $\mathfrak{H}_i^{m_i-1}$ is isomorphic to $\mathfrak{C}(①, \bullet)$ with (a). Note that all finite subsets of Z_i and their complements in $J_i^{m_i-1}$ are in $\mathcal{P}_i$ because of Lemma 4.10. So the function identical on Z_i and mapping $J_i^{m_i-1} \setminus Z_i$ to u is a root-mapping p-morphism from $\mathfrak{F}_i\restriction_{J_i^{m_i-1}}$ onto $\mathfrak{H}_i^{m_i-1}$, giving (14).

Notice that in all four of the above cases we have (13) for $j = m_i - 1$ because the interval $J_i^{m_i-1}$ is irrelevant. Now we define $\mathfrak{w}_i^{m_i-1}$ and h_i differently from the proof of Lemma 4.21. We take the final (limit) cluster $C_i^{m_i-1}$ of $\boldsymbol{b}_i$ and pick a point $z \in S_i \cap C_i^{m_i-1}$ (we can do that by (11)). Then we let, for $p \in \mathcal{V}$,

$$\mathfrak{w}_i^{m_i-1}(p) = \begin{cases} H_i^{m_i-1}, & \text{if } z \in \mathfrak{v}_i(p), \\ \emptyset, & \text{otherwise,} \end{cases}$$

which gives $\mathfrak{w}_i^{m_i-1}(p) \in \mathcal{P}_i^{m_i-1}$ and (c). We let $\mathsf{h}_i(x) = z$ for all $x \in H_i^{m_i-1}$. Note that $z \notin H_i^{m_i-1}$, but $z \in S_i \cap C_i^{m_i-1} \subseteq H_i^{m_i}$ by *Case* 2 above. We need to define h_i this way for (e).4 and (e).5 to hold in this case.

Finally, item (b) of the lemma follows from Lemmas 2.3 and 2.6, as in *Cases* 1 and 4 we have (14), in *Case* 2 we have (15), and in *Case* 3 we have (16). As in each of *Cases* 1–4 we have (13), item (d) follows. And item (e) can be checked by going through the definitions of h_i in each of *Cases* 1–4. ⊣

Lemma 4.24. *Lemmas 4.21 and 4.23 imply conditions* (a)–(d) *of Definition 4.1 for* $\mathfrak{N}_1, x_1$ *and* $\mathfrak{N}_2, x_2$.

Proof. We write 'items (x)' for 'items (x) of Lemmas 4.21 and 4.23'.

(a) We show that, for any $i = 1, 2$, $\ell < N$, $\tau \in sub(\varphi_i)$, and $x \in H_i^\ell$, we have $\mathfrak{M}_i, \mathsf{h}_i(x) \models \tau$ iff $\mathfrak{N}_i, x \models \tau$. Then $\mathfrak{N}_1, x_1 \models \varphi_1$ and $\mathfrak{N}_2, x_2 \models \neg\varphi_2$ follow by items (d) and (e).1. For $\tau = p \in \mathcal{V}$, the statement follows from items (e).2. The Boolean cases are straightforward, so suppose $\tau = \Diamond\psi$.

$(\Rightarrow)$ If $\mathfrak{M}_i, \mathsf{h}_i(x) \models \Diamond\psi$, then there are $k \geq \ell$ and $y_\psi \in M_i \cap I_i^k$ with $\mathsf{h}_i(x) R_i y_\psi$ and $\mathfrak{M}_i, y_\psi \models \psi$. We have $y_\psi \in H_i^\ell$ by items (d), and so $\mathfrak{N}_i, y_\psi \models \psi$ by items (e).1 and IH. We claim that $x R_i y_\psi$ follows, and so $\mathfrak{N}_i, x \models \Diamond\psi$. Indeed, for $k > \ell$, this follows from the definition of $\lhd$, and for $\ell = k$, by items (e).4.

$(\Leftarrow)$ If $\mathfrak{N}_i, x \models \Diamond\psi$, then there are $k \geq \ell$ and $y \in H_i^k$ with $x R_i y$ and $\mathfrak{N}_i, y \models \psi$. We have $\mathfrak{M}_i, \mathsf{h}_i(y) \models \psi$ by IH, and $\mathsf{h}_i(x) R_i \mathsf{h}_i(y)$ by (e).3, and so $\mathfrak{M}_i, \mathsf{h}_i(x) \models \Diamond\psi$.

(b) By items (b) and Lemmas 2.3, 2.6, $\mathfrak{N}_i$, $i = 1, 2$, is based on a frame for L.

(c) It follows from items (d) and (e).1–2 that $at^\sigma_{\mathfrak{N}_1}(x_1) = at^\sigma_{\mathfrak{N}_2}(x_2)$.

(d) There are the three cases:

- If (I_1^ℓ, I_2^ℓ) is a pair of globally σ-bisimilar unions of irrelevant σ-blocks, then we have option (d_1) in Definition 4.1 by Lemma 4.21 (c).
- If (I_1^ℓ, I_2^ℓ) is a pair of globally σ-bisimilar relevant definable σ-blocks, then Lemma 4.23 implies option (d_2) in Definition 4.1. Indeed, (d_2).1 holds because of the first parts of (a) and (c). For (d_2).2, take a point y_1 in H_1. By (e).2, $at_{\mathfrak{N}_1}(y_1) = at_{\mathfrak{N}_1}\big(\mathsf{h}_1(y_1)\big) = at_{\mathfrak{M}_1}\big(\mathsf{h}_1(y_1)\big)$. As $\mathsf{h}_1(y_1) \in \{x_1\} \cup M_1 \cup S_1$ by (e).5, Lemma 4.17 gives $y_2 \in S_2 \cap C^+_{\boldsymbol{b}_2}$ with $at^\sigma_{\mathfrak{M}_1}\big(\mathsf{h}_1(y_1)\big) = at^\sigma_{\mathfrak{M}_2}(y_2)$. By (d), y_2 is in the final cluster of $\mathfrak{N}_2^{n_2-1}$. By (e).1 and (e).2, $at_{\mathfrak{M}_2}(y_2) = at_{\mathfrak{M}_2}\big(\mathsf{h}_2(y_2)\big) = at_{\mathfrak{N}_2}\big(\mathsf{h}_2(y_2)\big) = at_{\mathfrak{N}_2}(y_2)$, and so $at^\sigma_{\mathfrak{N}_1}(y_1) = at^\sigma_{\mathfrak{N}_2}(y_2)$, as required. Starting with a y_2 in H_2 is symmetrical.
- If (I_1^ℓ, I_2^ℓ) is a pair of globally σ-bisimilar extended relevant non-definable σ-blocks, then again Lemma 4.23 implies option (d_3) in Definition 4.1. Indeed, (d_3).1–(d_3).3 hold because of of the second parts of (a) and (c), and (d_3).4 is proved similarly to (d_2).2 above. ⊣

4.5. Proof of Theorem 4.4. We can now show that every finitely axiomatisable logic $L \supseteq \mathsf{K4.3}$ has the quasi-polysize bisimilar model property.

Suppose $L = \mathsf{K4.3} \oplus \gamma_L$, for some formula γ_L. By Theorem 4.3, L has the quasi-finite bisimilar model property. Let $\mathfrak{N}_i$, $i = 1, 2$, be the simple models with underlying basic frames $\mathfrak{G}_i$ provided by Definition 4.1. We show that the size of $\lhd$-component frames in $\mathfrak{G}_i$ of the form $m^<$ can be bounded by $c_L = 2^{|\gamma_L|} + 1$. An inspection of the proofs of Lemmas 4.21 and 4.23 reveals that $m^<$ could be a $\lhd$-component of $\mathfrak{G}_i$ in the following three cases:

(i) We simultaneously replaced (by $m^<$ or $① \lhd m^<$) a pair of globally σ-bisimilar unions I_1 and I_2 of irrelevant σ-blocks (in the proof of Lemma 4.21).

(ii) We replaced (by $m^<$ or $① \lhd m^<$) an interval ending in a relevant cluster, for some subinterval of a relevant σ-block in $\mathfrak{M}_i$ (in *Case* 1 of the proof of Lemma 4.23).

(iii) We replaced (by $m^<$ or $① \lhd m^<$) an irrelevant subinterval of a (non-definable) relevant σ-block in $\mathfrak{M}_i$ (in *Case* 4 of the proof of Lemma 4.23).

In each of these cases, we obtain $m^<$ or $① \lhd m^<$ by putting all the m elements of the tails Z_i of the interval I_i in question into its chosen subset H_i. Alternatively, we can only choose some subset $Y_i = \{y_i^0, \dots, y_i^{c_L-1}\}$ of Z_i of size c_L. Then, in all three cases, we define the new set H_i' by replacing Z_i with Y_i in the definition of H_i. This will work as long as we choose Y_i carefully: In case (i), we need to choose Y_1 and Y_2 in parallel in such a way that $t^\sigma_{\mathfrak{M}_1}(y_1^j) = t^\sigma_{\mathfrak{M}_2}(y_2^j)$, for all $j < c_L$. And in case (ii), we need to include the final cluster of the tail Z_i in Y_i. Finally, in all three cases, the new model over the resulting frame $\mathfrak{H}_i'$ and the function $\mathsf{h}_i' \colon H_i' \to H_i'$ are defined as the restrictions of their previous definitions to H_i'. This way items (a), (c), (d) and (e) of Lemmas 4.21 and 4.23 continue to hold for $\mathfrak{H}_i'$ and the model (or pair of models) based on them. However, we no longer have item (b) of Lemmas 4.21 and 4.23. That the resulting smaller frames $\mathfrak{G}_i' = \mathfrak{G}_i^1 \lhd c_L^< \lhd \mathfrak{G}_i^2$ are frames for L (as required by Definition 4.1 (b)) follows from $\mathfrak{G}_i \models L$ and the observation below:

Lemma 4.25. *If $\mathfrak{G} \lhd (2^{|\gamma|}+1)^< \lhd \mathfrak{G}' \not\models \gamma$ (with possibly empty frames $\mathfrak{G}$ and $\mathfrak{G}'$), then $\mathfrak{G}_1 \lhd m^< \lhd \mathfrak{G}_2 \not\models \gamma$, for any $m > 2^{|\gamma|}+1$.*

Proof. Suppose a model $\mathfrak{M}$ based on $\mathfrak{G}_1 \lhd (2^{|\gamma|}+1)^< \lhd \mathfrak{G}_2$ refutes γ (i.e., satisfies $\neg\gamma$). Let $t_{\mathfrak{M}}(x) = \{\chi \in sub(\gamma) \mid \mathfrak{M}, x \models \chi\}$. By the pigeonhole principle, there are points $x < y$ in $(2^{|\gamma|}+1)^<$ such that $t_{\mathfrak{M}}(x) = t_{\mathfrak{M}}(y)$. We insert $m - (2^{|\gamma|}+1)$ new points directly before y (thus recovering $\mathfrak{G}_1 \lhd m^< \lhd \mathfrak{G}_2$) and extend $\mathfrak{M}$ to a model $\mathfrak{M}'$ by defining the atomic $sig(\gamma)$-type of each new point as $at^{sig(\gamma)}_{\mathfrak{M}}(y)$. It is readily seen by induction that $\mathfrak{M}'$ refutes γ. ⊣

§5. The IEP for standard Priorean temporal logics. *Priorean temporal logics* [32] deal with the operators 'sometime in the future' denoted $\Diamond_{\mathsf{F}}$, 'sometime in the past' denoted $\Diamond_{\mathsf{P}}$, and their duals 'always in the future' $\Box_{\mathsf{F}}$ and 'always in the past' $\Box_{\mathsf{P}}$. *Temporal formulas*—propositional bimodal formulas with these operators—are interpreted over general *temporal frames* of the form $\mathfrak{F} = (W, R, R^-, \mathcal{P})$ representing various *flows of time* in such a way that (W, R) is transitive and connected (1), R is the 'future-time' accessibility relation for $\Diamond_{\mathsf{F}}$, $\Box_{\mathsf{F}}$, its inverse R^- is the 'past-time' accessibility relation for $\Diamond_{\mathsf{P}}$, $\Box_{\mathsf{P}}$, and the internal sets $\mathcal{P} \subseteq 2^W$ are closed under the Booleans and the operators

$$\Diamond^{\mathfrak{F}}_{\mathsf{F}} X = \{x \in W \mid \exists y \in X \; xRy\}, \quad \Diamond^{\mathfrak{F}}_{\mathsf{P}} X = \{x \in W \mid \exists y \in X \; xR^-y\}.$$

To simplify notation, we omit R^- and write $\mathfrak{F} = (W, R, \mathcal{P})$. Also, as before, if $\mathcal{P} = 2^W$, we call $\mathfrak{F}$ a *Kripke frame* and write $\mathfrak{F} = (W, R)$. The *universal modality* 'always' can be introduced as an abbreviation $\Box\varphi = \varphi \wedge \Box_{\mathsf{F}}\varphi \wedge \Box_{\mathsf{P}}\varphi$. Descriptive temporal frames are defined in the same way as in Section 2. Note that condition **(ref)** for R^- actually follows from **(ref)** for R.

In fact, many results from Sections 2 and 3 straightforwardly generalise to the temporal setting. Let $\mathfrak{M}$ be a *temporal model*—that is, a model based on some temporal frame $\mathfrak{F} = (W, R, \mathcal{P})$—and let Γ be a set of temporal formulas. A point $x \in W$ is Γ-*minimal in* $\mathfrak{M}$ if $\mathfrak{M}, x \models \Gamma$ and whenever $x'Rx$ and $\mathfrak{M}, x' \models \Gamma$, then xRx'. Denote by $\min_{\mathfrak{M}} \Gamma$ the set of all Γ-minimal points in $\mathfrak{M}$. (The definition of $\max_{\mathfrak{M}} \Gamma$ remains the same.) In the temporal case, Lemma 2.4 generalises to

LEMMA 5.1. *Suppose Γ is a set of temporal formulas and $\mathfrak{M}$ a model based on a descriptive temporal frame $\mathfrak{F} = (W, R, \mathcal{P})$. Then the following hold:*

(temporal saturation) *If $\mathfrak{M}, x \models \Diamond_{\mathsf{F}} \bigwedge \Gamma'$ for every finite $\Gamma' \subseteq \Gamma$, then there is y with xRy and $\mathfrak{M}, y \models \Gamma$. If $\mathfrak{M}, x \models \Diamond_{\mathsf{P}} \bigwedge \Gamma'$ for every finite $\Gamma' \subseteq \Gamma$, then there is y with xR^-y and $\mathfrak{M}, y \models \Gamma$.*

(maximal and minimal points) *If there is x with $\mathfrak{M}, x \models \Gamma$, then $\max_{\mathfrak{M}} \Gamma \neq \emptyset$ and $\min_{\mathfrak{M}} \Gamma \neq \emptyset$.*

A relation $\boldsymbol{\beta} \subseteq W_1 \times W_2$ is a *temporal σ-bisimulation* between temporal models $\mathfrak{M}_1$ and $\mathfrak{M}_2$ based on respective frames $\mathfrak{F}_i = (W_i, R_i, \mathcal{P}_i)$, $i = 1, 2$, if it satisfies **(atom)**, **(move)** and its past-time couterpart: whenever $x_1 \boldsymbol{\beta} x_2$, then

(move$^-$) $x_1 R_1^- y_1$ implies $y_1 \boldsymbol{\beta} y_2$, for some $y_2 \in W_2$ with $x_2 R_2^- y_2$; conversely, $x_2 R_2^- y_2$ implies $y_1 \boldsymbol{\beta} y_2$, for some $y_1 \in W_1$ with $x_1 R_1^- y_1$.

The relation $\mathfrak{M}_1, x_1 \equiv_\sigma \mathfrak{M}_2, x_2$, saying that temporal models $\mathfrak{M}_1$ and $\mathfrak{M}_2$ satisfy the same temporal σ-formulas at x_1 and x_2, respectively, is characterised in terms of temporal σ-bisimulations: it is readily seen that, with this modification, Lemma 3.1 and Theorem 3.2 continue to hold for all Priorean temporal logics. (As temporal frames are connected, any of their points can be regarded as a root with respect to the relation $R \cup R^-$.)

In this article, we consider the Priorean temporal logics of five most popular classes of temporal Kripke frames [6]:

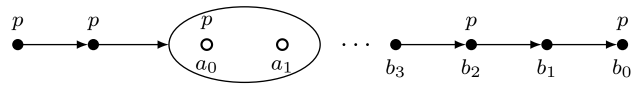
$$\begin{aligned}
\mathsf{Lin} &= \{\varphi \mid \mathfrak{F} \models \varphi,\ \mathfrak{F} = (W, R) \text{ is any temporal Kripke frame}\}\\
&= \mathsf{K4}_2 \oplus p \to \Box_{\mathsf{F}} \Diamond_{\mathsf{P}} p \oplus p \to \Box_{\mathsf{P}} \Diamond_{\mathsf{F}} p \oplus \Diamond_{\mathsf{F}} \Diamond_{\mathsf{P}} p \vee \Diamond_{\mathsf{P}} \Diamond_{\mathsf{F}} p \to p \vee \Diamond_{\mathsf{F}} p \vee \Diamond_{\mathsf{P}} p;\\
\mathsf{Lin}_{\mathbb{Q}} &= \{\varphi \mid (\mathbb{Q}, <) \models \varphi\}\\
&= \mathsf{Lin} \oplus \Diamond_{\mathsf{F}} \top \oplus \Diamond_{\mathsf{P}} \top \oplus \Diamond_{\mathsf{F}} p \to \Diamond_{\mathsf{F}} \Diamond_{\mathsf{F}} p;\\
\mathsf{Lin}_{\mathbb{R}} &= \{\varphi \mid (\mathbb{R}, <) \models \varphi\}\\
&= \mathsf{Lin}_{\mathbb{Q}} \oplus \Box(\Box_{\mathsf{P}} p \to \Diamond_{\mathsf{F}} \Box_{\mathsf{P}} p) \to (\Box_{\mathsf{P}} p \to \Box_{\mathsf{F}} p);\\
\mathsf{Lin}_{<\omega} &= \{\varphi \mid \mathfrak{F} \models \varphi,\ \mathfrak{F} = (W, <) \text{ any } \textit{finite} \text{ strict linear order}\}\\
&= \mathsf{Lin} \oplus \big\{\Box_{\mathsf{X}}(\Box_{\mathsf{X}} p \to p) \to \Box_{\mathsf{X}} p \mid \mathsf{X} \in \{\mathsf{F}, \mathsf{P}\}\big\};\\
\mathsf{Lin}_{\mathbb{Z}} &= \{\varphi \mid (\mathbb{Z}, <) \models \varphi\}\\
&= \mathsf{Lin} \oplus \Diamond_{\mathsf{F}} \top \oplus \Diamond_{\mathsf{P}} \top \oplus \big\{\Box_{\mathsf{X}}(\Box_{\mathsf{X}} p \to p) \to (\Diamond_{\mathsf{X}} \Box_{\mathsf{X}} p \to \Box_{\mathsf{X}} p) \mid \mathsf{X} \in \{\mathsf{F}, \mathsf{P}\}\big\},
\end{aligned}$$

where $\mathsf{K4}_2$ is the bimodal version of $\mathsf{K4}$ (with $\Diamond_{\mathsf{F}}$ and $\Diamond_{\mathsf{P}}$). None of these five logics (and any other temporal logic with frames of unbounded depth) has the CIP [13, 8], and our aim in this section is to prove that the IEP for each of them is decidable in coNP. The following example illustrates the new semantic phenomena of temporal logics compared to modal logics containing $\mathsf{K4.3}$ that we need to address in order to achieve this aim.

Example 5.2. (*i*) Consider the formulas φ_1 and φ_2 from Example 3.5 in the context of $\mathsf{Lin}_{<\omega}$ in place of $\mathsf{GL.3}$, reading $\Diamond$ as $\Diamond_{\mathsf{F}}$ and $\Box$ as $\Box_{\mathsf{F}}$:

$$\varphi_1 = \Diamond_{\mathsf{F}}(p_1 \wedge \Diamond_{\mathsf{F}}^{+}\neg q_1) \wedge \Box_{\mathsf{F}}(p_2 \to \Box_{\mathsf{F}}^{+} q_1) \wedge \Box_{\mathsf{F}}(p_1 \to \neg p_2),$$
$$\varphi_2 = \neg[\Diamond_{\mathsf{F}}(p_2 \wedge \Diamond_{\mathsf{F}}^{+}\neg q_2) \wedge \Box_{\mathsf{F}}(p_1 \to \Box_{\mathsf{F}}^{+} q_2)].$$

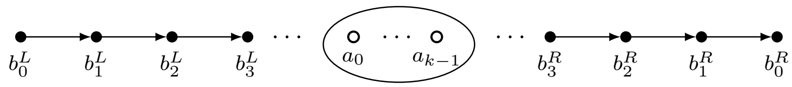
We clearly have $(\varphi_1 \to \varphi_2) \in \mathsf{Lin}_{<\omega}$. Using Theorem 3.2, we show that φ_1 and φ_2 do not have an interpolant in $\mathsf{Lin}_{<\omega}$. The argument from Example 3.5 shows that any models $\mathfrak{M}_i$ meeting the criterion of Theorem 3.2 cannot be based on a Kripke frame for $\mathsf{Lin}_{<\omega}$. However, the descriptive frame $\bullet \lhd \bullet \lhd \mathfrak{C}(②, \bullet)$ we employed for $\mathsf{GL.3}$ in Example 3.5 does not help now, because it refutes $\Box_{\mathsf{P}}(\Box_{\mathsf{P}} p \to p) \to \Box_{\mathsf{P}} p$ at any point save the first two under the valuation shown below:



To fix this issue, we modify $\mathfrak{C}(②, \bullet)$ by making it symmetric in both directions. Consider the frame $\mathfrak{F}_k = (W_k', R_{\bullet k \bullet}, \mathcal{P}_k')$, $k > 0$, in which the points in

$$W_k' = \{a_0, \dots, a_{k-1}\} \cup \{b_n^L, b_n^R \mid n < \omega\}$$

are ordered as shown in the picture below



or, more formally, $xR_{\bullet k \bullet}y$ iff ($x = b_n^L$, $y = b_m^L$ for $n < m$), or ($x = b_n^L$, $y = a_i$), or ($x = b_n^L$, $y = b_m^R$), or ($x = a_i$, $y = a_j$) or ($x = a_i$, $y = b_n^R$), or ($x = b_n^R$, $y = b_m^R$, for $n > m$). The internal sets in $\mathcal{P}_k$ are generated by

(17) $$X_i = \{a_i\} \cup \{b_n^L, b_n^R \mid n < \omega,\ n \equiv i \ (\mathrm{mod}\ k)\}, \quad \text{for } i < k.$$

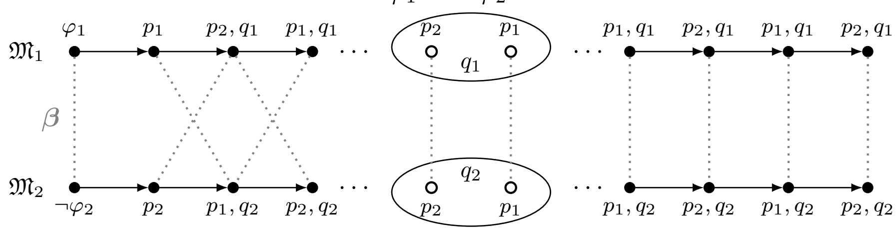
Observe that $\{b_n^L\}, \{b_n^R\} \in \mathcal{P}_k'$, for all $n < \omega$. It is not hard to see that $\mathfrak{F}_k$ is a descriptive frame; we denote it by $\mathfrak{C}(\bullet, ⓚ, \bullet)$. As an exercise, the reader can check that, for any natural numbers $k, l, \dots, m, n > 0$,

(18) $$\mathfrak{C}(\bullet, ⓚ, \bullet) \lhd \dots \lhd \mathfrak{C}(\bullet, ⓝ, \bullet) \models \mathsf{Lin}_{<\omega},$$

(19) $$\mathfrak{C}(ⓚ, \bullet) \lhd \mathfrak{C}(\bullet, ⓛ, \bullet) \lhd \dots \lhd \mathfrak{C}(\bullet, ⓜ, \bullet) \lhd \mathfrak{C}(\bullet, ⓝ) \models \mathsf{Lin}_{\mathbb{Z}},$$

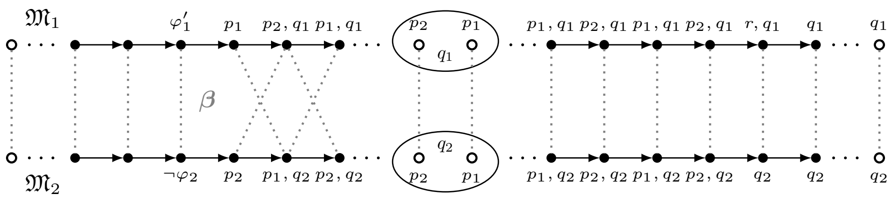
where $\mathfrak{C}(\bullet, ⓝ)$ is the mirror image of $\mathfrak{C}(ⓝ, \bullet)$; see also Lemma 5.6.

The picture below shows models $\mathfrak{M}_1$ and $\mathfrak{M}_2$ based on $\mathfrak{C}(\bullet, ②, \bullet)$ and satisfying the conditions of Theorem 3.2 for φ_1 and φ_2:



By (18), $\mathfrak{C}(\bullet, ②, \bullet) \models \mathsf{Lin}_{<\omega}$, so φ_1 and φ_2 do not have an interpolant in $\mathsf{Lin}_{<\omega}$.

(*ii*) Consider next the temporal version of the implication $\varphi_1' \to \varphi_2$ from Example 3.6, which is clearly valid in $\mathsf{Lin}_{\mathbb{Z}}$. To demonstrate that φ_1' and φ_2 have no interpolant in $\mathsf{Lin}_\mathbb{Z}$, we can use $\mathfrak{C}(①,\bullet) \lhd \mathfrak{C}(\bullet,②,\bullet) \lhd \mathfrak{C}(\bullet,①)$, which is a frame for $\mathsf{Lin}_\mathbb{Z}$ by (19). The models $\mathfrak{M}_1$ and $\mathfrak{M}_2$ depicted below



satisfy the conditions of Theorem 3.2 for φ_1' and φ_2. ⊣

As illustrated by Example 5.2, the temporal frames $\mathfrak{F} = (W, R, \mathcal{P})$ we need for checking the criterion of Theorem 3.2 may contain both infinite descending and ascending chains of clusters (and so the $\mathfrak{F}_c^{-1}$ are not necessarily isomorphic to ordinals). Accordingly, we now have *R-final* and *R^--final clusters* as well as two types of limit clusters: an *R-limit cluster* is a non-R^--final cluster without an immediate R^--successor, and an *R^--limit cluster* is a non-R-final cluster without an immediate R-successor. Some clusters might be both R- and R^--limit clusters.

We say that a set $S \neq \emptyset$ of clusters in $\mathfrak{F}$ is *R-unbounded* (*R^--unbounded*) if there is no $C \in S$ such that $C' \leq_R C$ (respectively, $C \leq_R C'$), for all $C' \in S$. A cluster C is the *R-limit of* an R-unbounded set S if $C' <_R C$ for all $C' \in S$ and there is no cluster C'' with $C' <_R C'' <_R C$ for all $C' \in S$; the *R^--limit of* an R^--unbounded set S is defined symmetrically by replacing R with R^-. It is straightforward to see that each R-limit cluster C is the R-limit of the R-unbounded set $\{C' \mid C' <_R C\}$, and each R^--limit cluster D is the R^--limit of the R^--unbounded set $\{D' \mid D <_R D'\}$. For any cluster C, we let $(C, +\infty) = \{x \mid C <_R C(x)\}$ and $(-\infty, C) = \{x \mid C(x) <_R C\}$.

Lemma 5.3. *Suppose $\mathfrak{F} = (W, R, \mathcal{P})$ is a temporal n-generated descriptive frame, for some $n < \omega$. Then*

(*a*) *every cluster in $\mathfrak{F}$ has at most 2^n points*;

(*b*) *every R-unbounded (R^--unbounded) set of clusters in $\mathfrak{F}$ has an R-limit (R^--limit) in $\mathfrak{F}$, and so $\mathfrak{F}$ contains both R- and R^--final clusters.*

Proof. (*a*) is proved similarly to Lemma 4.9 (*b*).

(*b*) Suppose $\mathfrak{F}$ is $\mathfrak{M}$-generated, for some model $\mathfrak{M}$. Let S be an R-unbounded set of clusters in $\mathfrak{F}$ with $y_C \in C$, $C \in S$, and let

$$\Gamma = \bigcup_{C \in S} \Diamond_{\mathsf{P}} t_{\mathfrak{M}}(y_C) \cup \bigcup_{C \in S} \{\psi \mid \Box_{\mathsf{F}} \psi \in t_{\mathfrak{M}}(y_C)\}.$$

Clearly, Γ is finitely satisfiable in $\mathfrak{M}$, and so by **(com)** and Lemma 5.1, there is a Γ-minimal point x in $\mathfrak{M}$. By **(ref)**, $y_C R x$ for all $C \in S$. Now suppose that y is such that $y_C R y$, for all $C \in S$, and yRx. Then $\Gamma \subseteq t_{\mathfrak{M}}(y)$, and so xRy by the Γ-minimality of x. Thus, $C(x)$ is the R-limit of S. The existence of R^--limits of R^--unbounded S is symmetric. ⊣

A cluster C is called *minimal* (*maximal*) in a temporal model $\mathfrak{M}$ if there is a formula μ such that $C \cap \min_{\mathfrak{M}}\{\mu\} \neq \emptyset$ ($C \cap \max_{\mathfrak{M}}\{\mu\} \neq \emptyset$). If there is such a σ-formula μ, for some signature σ, we call C *σ-minimal* (*σ-maximal*) *in* $\mathfrak{M}$.

Lemma 5.4. *Suppose $\mathfrak{M}$ is a model based on a finitely $\mathfrak{M}$-generated temporal descriptive frame $\mathfrak{F}$. Then*

(a) *every degenerate cluster in $\mathfrak{F}$ is both maximal and minimal in $\mathfrak{M}$;*
(b) *a cluster is maximal (minimal) in $\mathfrak{M}$ iff either it is R-final (respectively, R^--final) or has an immediate R-successor (respectively, R^--successor);*
(c) *a cluster is definable in $\mathfrak{M}$ iff it is both maximal and minimal in $\mathfrak{M}$.*

It follows that the R- and R^--limit clusters are not definable and not degenerate; all other clusters are definable in $\mathfrak{M}$. We also have that

(d) *for any clusters $C <_R C'$ in $\mathfrak{F}$, the interval $[C, C']$ contains a maximal cluster and also a minimal one;*
(e) *if C is not an R-limit cluster and C' is not an R^--limit cluster, then the closed interval $[C, C']$ is definable in $\mathfrak{M}$.*

Proof. Items (a)–(c) are proved in the same way as Lemma 4.10. Item (d) follows from **(ref)**, which gives formulas φ and ψ such that $\mathfrak{M}, x \not\models \Box_{\mathsf{F}}\varphi$, $\mathfrak{M}, y \models \Box_{\mathsf{F}}\varphi$ and $\mathfrak{M}, x \models \Box_{\mathsf{P}}\psi$, $\mathfrak{M}, y \not\models \Box_{\mathsf{P}}\psi$, and so $[C(x), C(y)]$ contains a $\Box_{\mathsf{F}}\varphi$-minimal cluster and a $\Box_{\mathsf{P}}\psi$-maximal one. Item (e): by (b), C is λ-minimal and C' is μ-maximal for some λ, μ. Then $[C, C']$ is defined in $\mathfrak{M}$ by $\Diamond_{\mathsf{P}}^+\lambda \wedge \Diamond_{\mathsf{F}}^+\mu$. ⊣

The following temporal analogue is harder to prove than Lemma 4.11:

Lemma 5.5. *If $\mathfrak{F} = (W, R, \mathcal{P})$ is a finitely generated temporal descriptive frame, then W is countable.*

Proof. By Lemma 5.3 (a), it suffices to show that $\mathfrak{F}_c = (W_c, <_R)$ is countable. Suppose $\mathfrak{F}$ is $\mathfrak{M}$-generated, for some δ-model $\mathfrak{M} = (\mathfrak{F}, \mathfrak{v})$ and finite signature δ. First, observe that each non-R-limit cluster C is μ_C-minimal in $\mathfrak{M}$ for some μ_C, by Lemma 5.4 (b). Thus, the internal set $X_C = \mathfrak{v}(\Diamond_{\mathsf{P}}^+\mu_C)$ distinguishes C from every D with $D <_R C$, and so $X_C \neq X_D$ whenever $C \neq D$. As $\mathcal{P}$ is countable, it follows that the number of non-R-limit clusters in $\mathfrak{F}_c$ is countable. Similarly, there are countably-many non-R^--limit clusters in $\mathfrak{F}_c$. So it is enough to show that the number of clusters in $\mathfrak{F}_c$ that are both R- and R^--limits is countable. We refer to such clusters as simply *limit clusters*. Call an interval $[C^-, C^+]$ a *neighbourhood* of a limit cluster C if $C^- <_R C <_R C^+$. By Lemma 5.4, every limit cluster C has a *nice* neighbourhood $N_C = [C^-, C^+]$ with non-limit clusters C^- and C^+. As the number of different nice N_C is countable, it follows that

(20) every uncountable interval $[D, D']$ contains a limit cluster C all of whose neighbourhoods are uncountable

(otherwise all limit clusters in $[D, D']$ belonged to the countable union of the countable intervals N_C, and so $[D, D']$ were countable).

By an *atomic type* we mean any $at^{\delta}_{\mathfrak{M}}(x)$ with $x \in W$. For any cluster C, we set $at(C) = \{at^{\delta}_{\mathfrak{M}}(x) \mid x \in C\}$. Let C be an R-limit cluster. We say that an atomic type *a occurs infinitely R-close to C* if, for every $C' <_R C$, there is C'' such that $C' <_R C'' <_R C$ and $a \in at(C'')$. Similarly, *a occurs infinitely R^--close to* an R^--limit cluster C if whenever $C <_R C'$, then there is C'' such that $C <_R C'' <_R C'$ and $a \in at(C'')$. We claim that

(21) if a occurs infinitely R-close to an R-limit cluster C, then $a \in at(C)$.

Indeed, let S be an R-unbounded set of clusters with R-limit C and $y_D \in D$, $D \in S$, and let

$$\Gamma_a = a \cup \bigcup_{D \in S} \Diamond_{\mathsf{P}} t_{\mathfrak{M}}(y_D) \cup \bigcup_{D \in S} \{\psi \mid \Box_{\mathsf{F}}\psi \in t_{\mathfrak{M}}(y_D)\}.$$

If a occurs infinitely R-close to C, it can be shown similarly to the proof of Lemma 5.3 (b) that there is a Γ_a-minimal point $x \in C$, so $a = at^{\delta}_{\mathfrak{M}}(x) \in at(C)$.

The converse of (21) also holds:

(22) if $a \in at(C)$, for an R-limit C, then a occurs infinitely R-close to C.

Indeed, suppose there is $C' <_R C$ with $a \notin at(C'')$, for any C'' in the interval $C' <_R C'' <_R C$. By Lemma 5.4 (d), there is a cluster C'' in $[C', C]$ that is μ-minimal in $\mathfrak{M}$ for some formula μ. But then C is $\Diamond_{\mathsf{P}}\mu \wedge \bigwedge a$-minimal, contrary to Lemma 5.4 (b). Symmetric variants of (21) and (22) hold for R^{-}-limit clusters.

We call non-degenerate clusters $C' <_R C''$ *twins* if $at(C') = at(C'')$ and, for every C in $[C', C'']$, $at(C) \subseteq at(C') = at(C'')$. We claim that

(23) there are no twins.

Indeed, suppose C', C'' are twins. By induction on the construction of a δ-formula α we see that if $x, y \in [C', C'']$ with xRy and $at^{\delta}_{\mathfrak{M}}(x) = at^{\delta}_{\mathfrak{M}}(y)$, then $\mathfrak{M}, x \models \alpha$ iff $\mathfrak{M}, y \models \alpha$. We only consider one of the nontrivial cases. Let $\mathfrak{M}, x \models \Diamond_{\mathsf{F}}\alpha$. Then there is z with xRz and $\mathfrak{M}, z \models \alpha$. If yRz, then clearly $\mathfrak{M}, y \models \Diamond_{\mathsf{F}}\alpha$. Otherwise, $z \in [C', C'']$, so $at^{\delta}_{\mathfrak{M}}(z) = at^{\delta}_{\mathfrak{M}}(z')$, for some $z' \in C''$. Thus, by IH, $\mathfrak{M}, z' \models \alpha$, which implies $\mathfrak{M}, y \models \Diamond_{\mathsf{F}}\alpha$ as C'' is non-degenerate. It follows that there are $x \in C'$ and $y \in C''$ with $t_{\mathfrak{M}}(x) = t_{\mathfrak{M}}(y)$, contrary to **(dif)**.

We can now prove that $\mathfrak{F}_c$ is countable. Suppose $\mathfrak{F}_c$ is uncountable. By (20) and Lemma 5.3 (b), $\mathfrak{F}_c$ contains a limit cluster C whose neighbourhoods are all uncountable. Let C be such a cluster with a minimal $at(C)$. As δ is finite, C has a neighbourhood N such that, for any $D \in N$ with $D <_R C$, every $a \in at(D)$ occurs infinitely R-close to C, and, for any $D \in N$ with $C <_R D$, every $a \in at(D)$ occurs infinitely R^{-}-close to C. We call such N a *close proximity of* C. As N is uncountable, either $[C^{-}, C)$ or $(C, C^{+}]$ is uncountable. We only consider the former case, as the latter is similar. We claim that

(24) for every cluster C' in $[C^{-}, C)$, the interval $[C^{-}, C']$ is countable.

Indeed, take such C'. As $[C^{-}, C']$ is contained in the close proximity N, for every limit cluster D in $[C^{-}, C']$, we have $at(D) \subsetneq at(C)$, by (21) and (23). So by the $at(C)$-minimality of C among limit clusters with only uncountable neighbourhoods, every limit cluster D in $[C^{-}, C']$ has a countable neighbourhood. Thus, $[C^{-}, C']$ is countable by (20).

By (22), there is a countably infinite ascending chain $C_1 <_R C_2 <_R \dots$ of clusters in $[C^{-}, C)$ such that, for every $a \in at(C)$ and every $n < \omega$, there is m with $n < m < \omega$ and $a \in at(C_m)$. Let C' be the R-limit of the R-unbounded set $\{C_n \mid n < \omega\}$ (which exists by Lemma 5.3 (b)). Then $C' \leq_R C$. Also, every $a \in at(C)$ occurs infinitely R-close to C', and so $at(C) \subseteq at(C')$ by (21). We cannot have $C' <_R C$ since otherwise (as C' belongs to the close proximity N of C) every $a \in at(C')$ occurred infinitely R-close to C, resulting in $at(C) = at(C')$ by (21), and so C' and C were twins, contrary to (23). It follows that $C' = C$, and so $[C^-, C) = \bigcup_{n<\omega}[C^-, C_n]$. As each $[C^-, C_n]$ is countable by (24), $[C^-, C)$ is also countable, which is a contradiction. ⊣

Using Lemmas 5.3 and 5.4, we can also obtain elegant characterisations of descriptive frames for $\mathsf{Lin}_{\mathbb{Q}}$, $\mathsf{Lin}_{\mathbb{R}}$, $\mathsf{Lin}_{<\omega}$ and $\mathsf{Lin}_{\mathbb{Z}}$ (cf. [7, 16, 33, 36]):

Lemma 5.6. *Let $\mathfrak{F} = (W, R, \mathcal{P})$ be any finitely generated temporal descriptive frame. Then*

- **$\mathsf{Lin}_{\mathbb{Q}}$:** $\mathfrak{F} \models \mathsf{Lin}_{\mathbb{Q}}$ *iff* $\mathfrak{F}$ *is* serial *in both directions—i.e., the R- and R^--final clusters in $\mathfrak{F}$ are both non-degenerate, and $\mathfrak{F}$ is* dense*—i.e., there is a non-degenerate cluster between any two distinct degenerate ones*;
- **$\mathsf{Lin}_{\mathbb{R}}$:** $\mathfrak{F} \models \mathsf{Lin}_{\mathbb{R}}$ *iff $\mathfrak{F}$ is serial, dense, and* Dedekind-complete *in the sense that there is a degenerate cluster between any two distinct non-degenerate ones*;
- **$\mathsf{Lin}_{<\omega}$:** $\mathfrak{F} \models \mathsf{Lin}_{<\omega}$ *iff $\mathfrak{F}$ does not contain a non-degenerate cluster C such that $(-\infty, C) \in \mathcal{P}$ or $(C, +\infty) \in \mathcal{P}$ (in particular, the R- and R^--final clusters in $\mathfrak{F}$ are degenerate)*;
- **$\mathsf{Lin}_{\mathbb{Z}}$:** $\mathfrak{F} \models \mathsf{Lin}_{\mathbb{Z}}$ *iff $\mathfrak{F}$ is serial and does not contain a non-degenerate cluster C with $\emptyset \neq (-\infty, C) \in \mathcal{P}$ or $\emptyset \neq (C, +\infty) \in \mathcal{P}$ (a single non-degenerate cluster is a frame for $\mathsf{Lin}_{\mathbb{Z}}$ but not for $\mathsf{Lin}_{<\omega}$).*

Proof. We only show the ($\Rightarrow$)-directions, leaving the converses to the reader. Suppose $\mathfrak{F}$ is $\mathfrak{M}$-generated, for some model $\mathfrak{M} = (\mathfrak{F}, \mathfrak{v})$.

$\mathsf{Lin}_{\mathbb{Q}}$: As $\mathfrak{F} \models \Diamond_{\mathsf{F}}\top$ ($\mathfrak{F} \models \Diamond_{\mathsf{P}}\top$), Lemma 5.1 gives a $\{\Diamond_{\mathsf{F}}\top\}$-maximal ($\{\Diamond_{\mathsf{P}}\top\}$-minimal) point x in $\mathfrak{M}$ with R-final (R^--final) and non-degenerate $C(x)$. Thus, $\mathfrak{F}$ is serial. Suppose $\{x\}$, $\{y\}$ are degenerate clusters with xRy. Lemma 5.4 gives formulas ψ_x and ψ_y defining $\{x\}$ and $\{y\}$ in $\mathfrak{M}$. As $\mathfrak{M}, x \models \Diamond_{\mathsf{F}}\psi_y$ and $\mathfrak{F} \models \Diamond_{\mathsf{F}}\psi_y \to \Diamond_{\mathsf{F}}\Diamond_{\mathsf{F}}\psi_y$, the formula $\Diamond_{\mathsf{F}}\psi_y \land \Diamond_{\mathsf{P}}\psi_x$ is satisfiable in $\mathfrak{M}$. Let z be $\{\Diamond_{\mathsf{F}}\psi_y \land \Diamond_{\mathsf{P}}\psi_x\}$-maximal in $\mathfrak{M}$. Then $xRzRy$. As $\mathfrak{M}, z \models \Diamond_{\mathsf{F}}(\Diamond_{\mathsf{F}}\psi_y \land \Diamond_{\mathsf{P}}\psi_x)$ by $\mathfrak{F} \models \Diamond_{\mathsf{F}}\psi_y \to \Diamond_{\mathsf{F}}\Diamond_{\mathsf{F}}\psi_y$, the cluster $C(z)$ is non-degenerate.

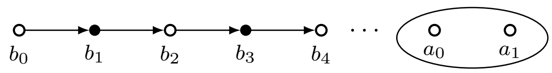
$\mathsf{Lin}_{\mathbb{R}}$: Non-degenerate $C(x) <_R C(y)$ cannot be $<_R$-consecutive because otherwise, by Lemma 5.4, $C(x)$ were ψ-maximal in $\mathfrak{M}$ for some formula ψ, and so $\mathfrak{M}, x \not\models \Box(\Box_{\mathsf{P}}\Diamond_{\mathsf{F}}\psi \to \Diamond_{\mathsf{F}}\Box_{\mathsf{P}}\Diamond_{\mathsf{F}}\psi) \to (\Box_{\mathsf{P}}\Diamond_{\mathsf{F}}\psi \to \Box_{\mathsf{F}}\Diamond_{\mathsf{F}}\psi)$, contrary to $\mathfrak{F} \models \mathsf{Lin}_{\mathbb{R}}$. Thus, there is z with $C(x) <_R C(z) <_R C(y)$. If z is irreflexive, we are done. Otherwise, by **(ref)**, there is some formula χ with $\Box_{\mathsf{F}}\chi \in t_{\mathfrak{M}}(y)$ and $\chi \notin t_{\mathfrak{M}}(z)$, and so $\mathfrak{M}, z \models \Diamond_{\mathsf{F}}\neg\chi$. Let z' be a $\Diamond_{\mathsf{F}}\neg\chi$-maximal point in $\mathfrak{M}$. Clearly, $C(x) <_R C(z') <_R C(y)$. If z' is irreflexive, we are done. Otherwise, we take the immediate R-successor z'' of z', which exists by Lemma 5.4. As $\mathfrak{M}, z'' \models \Box_{\mathsf{F}}\chi \land \neg\chi$, point z'' is irreflexive and $C(z'') <_R C(y)$.

$\mathsf{Lin}_{<\omega}$: If there existed a non-degenerate cluster $C(x)$ and a formula ψ with $(-\infty, C(x)) = \mathfrak{v}(\psi)$, then $\mathfrak{M}, x \not\models \Box_{\mathsf{P}}(\Box_{\mathsf{P}}\psi \to \psi) \to \Box_{\mathsf{P}}\psi$, contrary to $\mathfrak{F} \models \mathsf{Lin}_{<\omega}$.

$\mathsf{Lin}_{\mathbb{Z}}$: If there existed a non-degenerate cluster $C(x)$ and some formula ψ with $\emptyset \neq (C(x), +\infty) = \mathfrak{v}(\psi)$, then $\mathfrak{M}, x \not\models \Box_{\mathsf{F}}(\Box_{\mathsf{F}}\psi \to \psi) \to (\Diamond_{\mathsf{F}}\Box_{\mathsf{F}}\psi \to \Box_{\mathsf{F}}\psi)$, contrary to $\mathfrak{F} \models \mathsf{Lin}_{\mathbb{Z}}$. ⊣

Note that Lin and $\mathsf{Lin}_{\mathbb{Q}}$ are d-persistent while the other three logics are not [33].

Example 5.7. The descriptive frame $\mathfrak{F} = (W_2, R_{\circ\bullet}, \mathcal{P}_2)$ with $(W_2, R_{\circ\bullet})$ depicted below and $\mathcal{P}_2$ defined in Example 2.2 is serial, dense and Dedekind-complete, so $\mathfrak{F} \models \mathsf{Lin}_{\mathbb{R}}$.



It is readily seen, however, that $(W_2, R_{\circ\bullet}) \not\models \mathsf{Lin}_{\mathbb{R}}$, so $\mathsf{Lin}_{\mathbb{R}}$ is not d-persistent. ⊣

The notion of σ-block from Section 4.3 also needs a modification for temporal models. Namely, a set $\boldsymbol{b} \subseteq W$ is a *σ-block* in a temporal model $\mathfrak{M}$ based on $\mathfrak{F} = (W, R, \mathcal{P})$ if $\boldsymbol{b} = \boldsymbol{b}^\sigma_{\mathfrak{M}}(x)$, for some $x \in W$, where

$$\boldsymbol{b}^\sigma_{\mathfrak{M}}(x) = \big\{ y \in W \mid \Diamond_{\mathsf{X}} t^\sigma_{\mathfrak{M}}(y) \subseteq t^\sigma_{\mathfrak{M}}(x) \;\&\; \Diamond_{\mathsf{X}} t^\sigma_{\mathfrak{M}}(x) \subseteq t^\sigma_{\mathfrak{M}}(y), \text{ for } \mathsf{X} \in \{\mathsf{F}, \mathsf{P}\} \big\},$$

if both $\Diamond_{\mathsf{F}} t^\sigma_{\mathfrak{M}}(x) \subseteq t^\sigma_{\mathfrak{M}}(x)$ and $\Diamond_{\mathsf{P}} t^\sigma_{\mathfrak{M}}(x) \subseteq t^\sigma_{\mathfrak{M}}(x)$ hold; otherwise $\boldsymbol{b}^\sigma_{\mathfrak{M}}(x) = \{x\}$. Then we have the following temporal analogue of Lemma 4.14:

Lemma 5.8. *Suppose $\mathfrak{M}$ is a model based on a finitely $\mathfrak{M}$-generated temporal descriptive frame $\mathfrak{F} = (W, R, \mathcal{P})$. Then, for any σ-block $\boldsymbol{b}$ in $\mathfrak{M}$, there exist clusters $C^-_{\boldsymbol{b}}$ and $C^+_{\boldsymbol{b}}$ in $\mathfrak{F}$ such that the following hold*:

(*a*) $\boldsymbol{b} = \big[C^-_{\boldsymbol{b}}, C^+_{\boldsymbol{b}}\big]$;
(*b*) *if cluster $C^-_{\boldsymbol{b}}$ (cluster $C^+_{\boldsymbol{b}}$) is minimal (respectively, maximal) in $\mathfrak{M}$, then it is σ-minimal (respectively, σ-maximal) in $\mathfrak{M}$*;
(*c*) *if $\boldsymbol{b}$ is non-degenerate, then both $C^-_{\boldsymbol{b}}$ and $C^+_{\boldsymbol{b}}$ are non-degenerate*;
(*d*) *$\boldsymbol{b}$ is definable in $\mathfrak{M}$ iff $C^-_{\boldsymbol{b}}$ is not an R-limit cluster and $C^+_{\boldsymbol{b}}$ is not an R^--limit cluster*;
(*e*) $t^\sigma_{\mathfrak{M}}(\boldsymbol{b}) = t^\sigma_{\mathfrak{M}}\big(C^-_{\boldsymbol{b}}\big) = t^\sigma_{\mathfrak{M}}\big(C^+_{\boldsymbol{b}}\big)$.

Proof. This can be proved similarly to Lemma 4.14, using Lemmas 5.3, 5.4, and 5.1, in place of Lemmas 4.9, 4.10, and 2.4, respectively. ⊣

Given σ-bisimilar models $\mathfrak{M}_i$, $i = 1, 2$, based on finitely $\mathfrak{M}_i$-generated temporal frames, we can adapt Lemma 4.16 to the temporal setting to show that σ-blocks in $\mathfrak{M}_1$ and $\mathfrak{M}_2$ always come in σ-bisimilar pairs $\boldsymbol{b}$, $\boldsymbol{\beta}(\boldsymbol{b})$ (see (9)). Being equipped with these modifications, we show first how to extend the selection procedure from the proof of Theorem 3.4 to Lin, $\mathsf{Lin}_{\mathbb{Q}}$ and $\mathsf{Lin}_{\mathbb{R}}$.

Theorem 5.9. *Each $L \in \{\mathsf{Lin}, \mathsf{Lin}_{\mathbb{Q}}, \mathsf{Lin}_{\mathbb{R}}\}$ has the polysize bisimilar model property, and the IEP for L is* coNP*-complete.*

Proof. Suppose φ_1 and φ_2 have no interpolant in L, $\sigma = sig(\varphi_1) \cap sig(\varphi_2)$, and $\delta = sig(\varphi_1) \cup sig(\varphi_2)$. By Theorem 3.2, there are δ-models $\mathfrak{M}_i$, for $i = 1, 2$, based on $\mathfrak{M}_i$-generated temporal descriptive frames $\mathfrak{F}_i = (W_i, R_i, \mathcal{P}_i)$ for L with $\mathfrak{M}_1, x_1 \sim_\sigma \mathfrak{M}_2, x_2$, $\mathfrak{M}_1, x_1 \models \varphi_1$ and $\mathfrak{M}_2, x_2 \models \neg\varphi_2$. Let $\boldsymbol{\beta}$ be the largest σ-bisimulation between $\mathfrak{M}_1$ and $\mathfrak{M}_2$, that is, $y_1 \boldsymbol{\beta} y_2$ iff $t^\sigma_{\mathfrak{M}_1}(y_1) = t^\sigma_{\mathfrak{M}_2}(y_2)$, for all $y_i \in W_i$. We show that there exist such $\mathfrak{M}_i$ of polynomial size in $\max(|\varphi_1|, |\varphi_2|)$.

For any $i = 1, 2$ and $\tau \in sub(\varphi_i)$ satisfied in $\mathfrak{M}_i$, we take one $\{\tau\}$-maximal and one $\{\tau\}$-minimal point in $\mathfrak{M}_i$. Let M_i be the set of all selected points and let

$$T = \big\{ t^\sigma_{\mathfrak{M}_1}(x) \mid x \in \{x_1\} \cup M_1 \big\} \cup \big\{ t^\sigma_{\mathfrak{M}_2}(x) \mid x \in \{x_2\} \cup M_2 \big\}.$$

For each $t \in T$, we take a smallest set $S_i \subseteq W_i$ containing one t-maximal and one t-minimal point in $\mathfrak{M}_i$.

Let $W_i' = \{x_i\} \cup M_i \cup S_i$, $R_i' = R_i \restriction_{W_i'}$, $\mathfrak{F}_i' = (W_i', R_i')$, let $\mathfrak{M}_i'$ be the restriction of $\mathfrak{M}_i$ to $\mathfrak{F}_i'$, and let $x_1' \boldsymbol{\beta}' x_2'$ iff $t^\sigma_{\mathfrak{M}_1}(x_1') = t^\sigma_{\mathfrak{M}_2}(x_2')$, for all $x_1' \in W_1'$, $x_2' \in W_2'$. Following the proof of Lemma 3.3, we see that $\mathfrak{M}_1', x_1 \models \varphi_1$, $\mathfrak{M}_2', x_2 \models \neg\varphi_2$, and $\boldsymbol{\beta}'$ is a σ-bisimulation between $\mathfrak{M}_1'$ and $\mathfrak{M}_2'$ with $x_1 \boldsymbol{\beta}' x_2$. Clearly, $\mathfrak{F}_i' \models \mathsf{Lin}$ and the $\mathfrak{M}_i$ are of polynomial size in $\max(|\varphi_1|, |\varphi_2|)$.

For $L = \mathsf{Lin}_{\mathbb{Q}}$, we do not necessarily have $\mathfrak{F}_i' \models L$. To fix this, we add some extra points from W_i to W_i'. As $\mathfrak{F}_i \models \mathsf{Lin}_{\mathbb{Q}}$, the R- and R^--final clusters in $\mathfrak{F}_i$ are non-degenerate and, as observed in the selection procedure from Section 3, W_i' contains some points from these final clusters. Thus, $\mathfrak{F}_i' \not\models \mathsf{Lin}_{\mathbb{Q}}$ iff $\mathfrak{F}_i'$ contains an irreflexive point x with an immediate irreflexive R_i'-successor y. We call such pair x, y an *irr-defect in* $\mathfrak{F}_i'$. We are going to 'cure' one irr-defect after the other without introducing new irr-defects in either frame.

Given an irr-defect u_1, v_1 in $\mathfrak{F}_1'$, we find an R_1-reflexive z_1 with $u_1 R_1 z_1 R_1 v_1$, which exists by $\mathfrak{F}_1 \models \mathsf{Lin}_{\mathbb{Q}}$ and Lemma 5.6. Let $t = t^\sigma_{\mathfrak{M}_1}(z_1)$ and $\boldsymbol{b} = \boldsymbol{b}^\sigma_{\mathfrak{M}_1}(z_1)$. As $\Diamond_F t \subseteq t$ and $\Diamond_P t \subseteq t$, $\boldsymbol{b}$ is a non-degenerate σ-block in $\mathfrak{M}_1$. By Lemma 5.8, there are t-minimal and t-maximal points z_1^- and z_1^+ in the non-degenerate clusters $C^-_{\boldsymbol{b}}$ and $C^+_{\boldsymbol{b}}$. As $\boldsymbol{\beta}(\boldsymbol{b})$ is a non-degenerate σ-block in $\mathfrak{M}_2$ by Lemma 4.16, there are t-minimal and t-maximal points z_2^- and z_2^+ in the non-degenerate clusters $C^-_{\boldsymbol{\beta}(\boldsymbol{b})}$ and $C^+_{\boldsymbol{\beta}(\boldsymbol{b})}$. By adding z_1, z_1^-, z_1^+ to W_1' and z_2^-, z_2^+ to W_2' we cure the irr-defect u_1, v_1 without creating a new irr-defect in either frame. Let W_i'', $i = 1, 2$, be the sets we obtain after curing all irr-defects in both frames in this way, $R_i'' = R_i \restriction_{W_i''}$, $\mathfrak{F}_i'' = (W_i'', R_i'')$, let $\mathfrak{M}_i''$ the restriction of $\mathfrak{M}_i$ to $\mathfrak{F}_i''$, and $x_1' \boldsymbol{\beta}'' x_2'$ iff $t^\sigma_{\mathfrak{M}_1}(x_1') = t^\sigma_{\mathfrak{M}_2}(x_2')$, for all $x_1' \in W_1''$, $x_2' \in W_2''$. Then $\mathfrak{F}_i'' \models \mathsf{Lin}_{\mathbb{Q}}$, and

(minmax) for all $x \in W_1'' \cup W_2''$ and $i = 1, 2$, the set W_1'' contains $t^\sigma_{\mathfrak{M}_i}(x)$-minimal and $t^\sigma_{\mathfrak{M}_i}(x)$-maximal points in $\mathfrak{M}_1$, and W_2'' contains $t^\sigma_{\mathfrak{M}_i}(x)$-minimal and $t^\sigma_{\mathfrak{M}_i}(x)$-maximal points in $\mathfrak{M}_2$.

So it is readily seen (similarly to the proof of Lemma 3.3) that $\mathfrak{M}_1'', x_1 \models \varphi_1$, $\mathfrak{M}_2'', x_2 \models \neg\varphi_2$, and $\boldsymbol{\beta}''$ is a σ-bisimulation between $\mathfrak{M}_1''$ and $\mathfrak{M}_2''$ with $x_1 \boldsymbol{\beta}'' x_2$.

Finally, let $L = \mathsf{Lin}_{\mathbb{R}}$. Since $\mathsf{Lin}_{\mathbb{Q}} \subseteq \mathsf{Lin}_{\mathbb{R}}$, we first cure the irr-defects in the $\mathfrak{F}_i'$, $i = 1, 2$, as described above. Let $\mathfrak{F}_i''$, $i = 1, 2$, be the resulting serial and dense frames. Thus, $\mathfrak{F}_i'' \not\models L$ iff $\mathfrak{F}_i''$ contains two $<_{R_i''}$-consecutive non-degenerate clusters $C(x) \neq C(y)$. We call such x, y a *ref-defect in* $\mathfrak{F}_i''$. We show that the ref-defects can also be cured in a step-by-step manner without introducing new defects of either type, while maintaining **(minmax)**.

If u_1, v_1 is a ref-defect in $\mathfrak{F}_1''$, Lemma 5.6 provides an irreflexive $z_1 \in W_1$ with $u_1 R_1 z_1 R_1 v_1$. Let $t = t^\sigma_{\mathfrak{M}_1}(z_1)$. The insertion of extra points into W_1'' depends on whether u_1 and v_1 are in the same σ-block in $\mathfrak{M}_1$ or not.

Case 1: $u_1, v_1 \in \boldsymbol{b}$, for some σ-block $\boldsymbol{b}$ in $\mathfrak{M}_1$. By Lemma 5.8, $\boldsymbol{b}$ is non-degenerate, and there are t-minimal and t-maximal points z_1^- and z_1^+ in the non-degenerate clusters $C^-_{\boldsymbol{b}}$ and $C^+_{\boldsymbol{b}}$. By Lemma 4.16, $\boldsymbol{\beta}(\boldsymbol{b})$ is a non-degenerate σ-block in $\mathfrak{M}_2$, so there are t-minimal and t-maximal points z_2^- and z_2^+ in the non-degenerate clusters $C^-_{\boldsymbol{\beta}(\boldsymbol{b})}$ and $C^+_{\boldsymbol{\beta}(\boldsymbol{b})}$. By adding z_1, z_1^-, z_1^+ to W_1'' and z_2^-, z_2^+ to W_2'' we cure the ref-defect u_1, v_1 in $\mathfrak{F}_1''$ and maintain **(minmax)**. Also, as **(minmax)** held in $\mathfrak{F}_i''$, by Lemma 5.8 we already had some points from $C^-_{\boldsymbol{b}}$ and

$C^+_{\boldsymbol{b}}$ in W_1'' and some points from $C^-_{\boldsymbol{\beta}(\boldsymbol{b})}$ and $C^+_{\boldsymbol{\beta}(\boldsymbol{b})}$ in W_2''. So we did not create new defects in either frame, and the property **(minmax)** is maintained.

Case 2: $u_1 \in \boldsymbol{b}^{u_1}$, $v_1 \in \boldsymbol{b}^{v_1}$, for σ-blocks $\boldsymbol{b}^{u_1} \neq \boldsymbol{b}^{v_1}$ in $\mathfrak{M}_1$. By the definition of W_1'' and $C(u_1)$, $C(v_1)$ being $<_{R_1''}$-consecutive, $C(u_1) = C^+_{\boldsymbol{b}^{u_1}}$ and $C(v_1) = C^-_{\boldsymbol{b}^{v_1}}$, so $z_1 \notin \boldsymbol{b}^{u_1}$. We claim that there is an irreflexive $z \in W_1$ such that $u_1 R_1 z R_1 v_1$ and z is either $t^\sigma_{\mathfrak{M}_1}(z)$-maximal or $t^\sigma_{\mathfrak{M}_1}(z)$-minimal. Indeed, as $u_1 R_1 z_1$, we have $\Diamond_{\mathsf{F}} t^\sigma_{\mathfrak{M}_1}(z_1) \subseteq t^\sigma_{\mathfrak{M}_1}(u_1)$ and $\Diamond_{\mathsf{P}} t^\sigma_{\mathfrak{M}_1}(u_1) \subseteq t^\sigma_{\mathfrak{M}_1}(z_1)$. As $z_1 \notin \boldsymbol{b}^{u_1}$, there can be two cases: either (i) $\Diamond_{\mathsf{F}} t^\sigma_{\mathfrak{M}_1}(u_1) \not\subseteq t^\sigma_{\mathfrak{M}_1}(z_1)$ or (ii) $\Diamond_{\mathsf{P}} t^\sigma_{\mathfrak{M}_1}(u_1) \not\subseteq t^\sigma_{\mathfrak{M}_1}(z_1)$. In case (i), there is a σ-formula χ with $\mathfrak{M}_1, u_1 \models \Diamond_{\mathsf{F}}\chi$ but $\mathfrak{M}_1, z_1 \not\models \Diamond_{\mathsf{F}}\chi$. Take a $\{\Diamond_{\mathsf{F}}\chi\}$-maximal point z'. Clearly, $u_1 R_1 z' R_1 v_1$. If z' is irreflexive, we set $z = z'$ as it is $t^\sigma_{\mathfrak{M}_1}(z')$-maximal. Otherwise, Lemma 5.4 gives an immediate degenerate $<_{R_1}$-successor $C(z)$ of $C(z')$ such that z is $t^\sigma_{\mathfrak{M}_1}(z)$-maximal. In case (ii), there is a σ-formula χ with $\mathfrak{M}_1, u_1 \not\models \Diamond_{\mathsf{P}}\chi$ but $\mathfrak{M}_1, z_1 \models \chi$, and so $\mathfrak{M}_1, v_1 \models \Diamond_{\mathsf{P}}\chi$. Take a $\{\Diamond_{\mathsf{P}}\chi\}$-minimal point z'. Clearly, $u_1 R_1 z' R_1 v_1$. If z' is irreflexive, we set $z = z'$ as it is $t^\sigma_{\mathfrak{M}_1}(z')$-minimal. Otherwise, Lemma 5.4 gives an immediate degenerate $<_{R_1}$-predecessor $C(z)$ of $C(z')$ such that z is $t^\sigma_{\mathfrak{M}_1}(z)$-minimal.

Let $\boldsymbol{b} = \boldsymbol{b}^\sigma_{\mathfrak{M}_1}(z)$. Then $\boldsymbol{b}$ is a degenerate σ-block in $\mathfrak{M}_1$ by Lemma 5.8. By Lemma 9, $\boldsymbol{\beta}(\boldsymbol{b})$ is a degenerate σ-block in $\mathfrak{M}_2$ with $\boldsymbol{\beta}(\boldsymbol{b}^{u_1}) \prec_{\mathfrak{F}_2} \boldsymbol{\beta}(\boldsymbol{b}) \prec_{\mathfrak{F}_2} \boldsymbol{\beta}(\boldsymbol{b}^{v_1})$. Also, by **(minmax)** in $\mathfrak{F}_i''$, $C^+_{\boldsymbol{\beta}(\boldsymbol{b}^{u_1})}$ and $C^-_{\boldsymbol{\beta}(\boldsymbol{b}^{v_1})}$ are $<_{R_2''}$-consecutive non-degenerate clusters. Therefore, by adding z to W_1'' and z_2 with $C(z_2) = \boldsymbol{\beta}(\boldsymbol{b})$ to W_2'', we cured the ref-defect u_1, v_1 in $\mathfrak{F}_1''$ and we did not create new defects of either kind in either frame while maintaining **(minmax)**. So again it is readily seen (similarly to the proof of Lemma 3.3) that, after fixing all defects, we end up with a pair of models as required.

This establishes the polysize bisimilar model property of $L \in \{\mathsf{Lin}, \mathsf{Lin}_{\mathbb{Q}}, \mathsf{Lin}_{\mathbb{R}}\}$. We show that the IEP for L is in coNP using the description of finite frames for L in Lemma 5.6. $\dashv$

The finitary selection construction in the proof above does not work for logics $L \in \{\mathsf{Lin}_{<\omega}, \mathsf{Lin}_{\mathbb{Z}}\}$. In fact, these logics do not have the polysize bisimilar model property. However, below we show that they still have a kind of quasi-finite bisimilar model property similar to Definition 4.1 in the following sense. We can always witness the lack of an interpolant for φ_1, φ_2 in L by a pair of temporal models that are based on frames for L and assembled from $\mathcal{O}\big(\max(|\varphi_1|, |\varphi_2|)\big)$-many *atomic* descriptive frames of the forms $m^<$, Ⓚ, $\mathfrak{C}$(Ⓚ, •), $\mathfrak{C}$(•, Ⓚ, •), and $\mathfrak{C}$(•, Ⓚ), for $m, k = \mathcal{O}\big(\max(|\varphi_1|, |\varphi_2|)\big)$, $k > 0$ (see Example 5.2).

Given φ_i, $\mathfrak{M}_i$, x_i, for $i = 1, 2$, as above, let M_i, S_i, and $W_i' = \{x_i\} \cup M_i \cup S_i$, for $i = 1, 2$, be as defined in the proof of Theorem 5.9. As before, we call the points from W_i' *relevant in* $\mathfrak{M}_i$. A cluster or a σ-block in $\mathfrak{M}_i$ is *relevant* if it contains a relevant point in $\mathfrak{M}_i$. Given any pair $\boldsymbol{b}$, $\boldsymbol{\beta}(\boldsymbol{b})$ of σ-bisimilar σ-blocks in $\mathfrak{M}_1$ and $\mathfrak{M}_2$, we can now have the temporal analogue of Lemma 4.17, dealing not only with $S_1 \cap C^+_{\boldsymbol{b}}$ and $S_2 \cap C^+_{\boldsymbol{\beta}(\boldsymbol{b})}$ but also with $S_1 \cap C^-_{\boldsymbol{b}}$ and $S_2 \cap C^-_{\boldsymbol{\beta}(\boldsymbol{b})}$. We also have the following temporal analogues of (10) and (11):

(25) $\boldsymbol{b}$ is relevant in $\mathfrak{M}_1$ iff $\boldsymbol{\beta}(\boldsymbol{b})$ is relevant in $\mathfrak{M}_2$,

(26) if $\boldsymbol{b}$ and $\boldsymbol{\beta}(\boldsymbol{b})$ are relevant, then
$$|S_1 \cap C^-_{\boldsymbol{b}}| = |S_1 \cap C^+_{\boldsymbol{b}}| = |S_2 \cap C^-_{\boldsymbol{\beta}(\boldsymbol{b})}| = |S_2 \cap C^+_{\boldsymbol{\beta}(\boldsymbol{b})}| \neq 0.$$

THEOREM 5.10. *The IEPs for* $\mathsf{Lin}_{<\omega}$ *and* $\mathsf{Lin}_{\mathbb{Z}}$ *are both* coNP*-complete.*

PROOF. Let $\boldsymbol{b}_1^0, \dots, \boldsymbol{b}_1^N$ be all the relevant σ-blocks in $\mathfrak{M}_1$ ordered by $\prec_{\mathfrak{F}_1}$, for some $N = \mathcal{O}\big(\big(\max(|\varphi_1|, |\varphi_2|)\big)$. By (25) and Lemma 4.16, the $\prec_{\mathfrak{F}_2}$-ordered list of all relevant σ-blocks in $\mathfrak{M}_2$ is $\boldsymbol{b}_2^0, \dots, \boldsymbol{b}_2^N$, where $\boldsymbol{b}_2^j = \boldsymbol{\beta}(\boldsymbol{b}_1^j)$, for $j \le N$. By (26), for every $j \le N$ there is $k^j > 0$ with $k^j = |S_1 \cap C^-_{\boldsymbol{b}_1^j}| = |S_1 \cap C^+_{\boldsymbol{b}_1^j}| = |S_2 \cap C^-_{\boldsymbol{b}_2^j}| = |S_2 \cap C^+_{\boldsymbol{b}_2^j}|$. Also, by Lemma 4.16, $\boldsymbol{b}_1^j$ is degenerate iff $\boldsymbol{b}_2^j$ is degenerate, for $j \le N$.

Case $L = \mathsf{Lin}_{<\omega}$: By Lemmas 5.6 and 5.8, $\boldsymbol{b}_i^0$ and $\boldsymbol{b}_i^N$, $i = 1, 2$, are degenerate. By Lemmas 5.4, 5.6, 5.8, if $\boldsymbol{b}_i^j$ is non-degenerate, then $C^-_{\boldsymbol{b}_i^j}$ and $C^+_{\boldsymbol{b}_i^j}$ are R^-- and R-limit clusters, and $C \cap M_i = \emptyset$, for every non-degenerate cluster C in $\boldsymbol{b}_i^j$. (It can happen that x_i is in a non-degenerate cluster in $\boldsymbol{b}_i^j$ different from $C^-_{\boldsymbol{b}_i^j}$, $C^+_{\boldsymbol{b}_i^j}$.)

For all $i = 1, 2$ and $j \le N$, we let $m_i^j = \big|\big((\{x_i\} \cup M_i) \cap \boldsymbol{b}_i^j\big) \setminus (C^-_{\boldsymbol{b}_i^j} \cup C^+_{\boldsymbol{b}_i^j})\big|$ and define an atomic frame $\mathfrak{H}_i^j = (H_i^j, R_i^j, \mathcal{P}_i^j)$ by taking

$$\mathfrak{H}_i^j = \begin{cases} \bullet, & \text{if } \boldsymbol{b}_i^j \text{ is degenerate;} \\ \mathfrak{C}\big(\bullet, \textcircled{\scriptsize k^j}, \bullet\big), & \text{if } C^-_{\boldsymbol{b}_i^j} = C^+_{\boldsymbol{b}_i^j} \text{ is non-degenerate;} \\ \mathfrak{C}\big(\bullet, \textcircled{\scriptsize k^j}, \bullet\big) \lhd (m_i^j)^< \lhd \mathfrak{C}\big(\bullet, \textcircled{\scriptsize k^j}, \bullet\big), & \text{otherwise.} \end{cases}$$

Note that m_1^j and m_2^j might be different, and $(\{x_i\} \cup M_i) \cap \boldsymbol{b}_i^j = \emptyset$ (and so $m_i^j = 0$) can happen even when $C^-_{\boldsymbol{b}_i^j} \neq C^+_{\boldsymbol{b}_i^j}$. Let $\mathfrak{H}_i = (H_i, R'_i, \mathcal{P}'_i) = \mathfrak{H}_i^0 \lhd \dots \lhd \mathfrak{H}_i^N$. It is readily seen that $\mathfrak{H}_i$ is a frame for $\mathsf{Lin}_{<\omega}$, for $i = 1, 2$. Next, we define a 'parent' function $\mathsf{h}_i \colon H_i \to W'_i$ such that, for all $x \in H_i$,

(27) for all $j \le N$, if $x \in H_i^j$ then $\mathsf{h}_i(x) \in W'_i \cap \boldsymbol{b}_i^j$,

(28) for all $y \in H_i$, if xR'_iy then $\mathsf{h}_i(x) R_i \mathsf{h}_i(y)$,

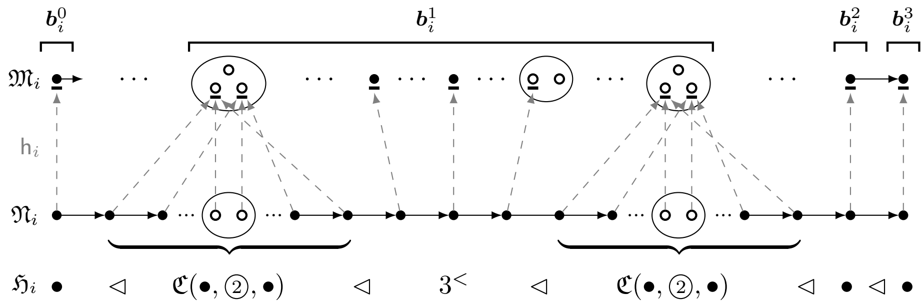
(29) for all $y \in M_i$, if $\mathsf{h}_i(x) R_i y$ then xR'_iz and $\mathsf{h}_i(z) = y$ for some z.

Finally, for $j \le N$, we define a model $\mathfrak{N}_i^j$ based on $\mathfrak{H}_i^j$ by taking, for all $x \in H_i^j$,

$$at_{\mathfrak{N}_i^j}(x) = at_{\mathfrak{M}_i}\big(\mathsf{h}_i(x)\big), \tag{30}$$

and let $\mathfrak{N}_i = \mathfrak{N}_i^0 \lhd \dots \lhd \mathfrak{N}_i^N$.

Instead of giving the general definitions of h_i and $\mathfrak{N}_i$, we illustrate the construction in the picture below, where $\mathfrak{M}_i$ has three degenerate σ-blocks $\boldsymbol{b}_i^0, \boldsymbol{b}_i^2$ and $\boldsymbol{b}_i^3$ and one non-definable non-degenerate σ-block $\boldsymbol{b}_i^1$; the relevant points in $\mathfrak{M}_i$ are underlined; $k^0 = k^2 = k^3 = 1$, $k^1 = 2$ and $m_i^1 = 3$.



It is readily seen that this way (27)–(29) hold and $\mathfrak{N}_i^j$ is based on $\mathfrak{H}_i^j$, for $j \le N$. Using (27)–(29), a proof similar to that of Lemma 4.24 (a) shows that each point x in $\mathfrak{N}_i$ makes true exactly the same formulas in $sub(\varphi_i)$ as its parent $\mathsf{h}_i(x)$ in $\mathfrak{M}_i$. It follows that $\mathfrak{N}_1, x_1' \models \varphi_1$ and $\mathfrak{N}_2, x_2' \models \neg\varphi_2$, where $x_i = \mathsf{h}_i(x_i')$.

Further, (27), (30) and Lemma 4.17 guarantee that each pair $(\mathfrak{N}_1^j, \mathfrak{N}_2^j)$, for $j \le N$, satisfies a condition similar to condition (d) in Definition 4.1: either

(d_1) both $\mathfrak{N}_1^j$ and $\mathfrak{N}_2^j$ are based on the same degenerate frame $\bullet$, or

(d_3) for every point y_1 in $\mathfrak{N}_1^j$, there is a point y_2 in each $\textcircled{k^j}$-cluster of $\mathfrak{N}_2^j$ with $at^\sigma_{\mathfrak{N}_1^j}(y_1) = at^\sigma_{\mathfrak{N}_2^j}(y_2)$, and for every y_2 in $\mathfrak{N}_2^j$, there is a point y_1 in each $\textcircled{k^j}$-cluster of $\mathfrak{N}_1^j$ with $at^\sigma_{\mathfrak{N}_1^j}(y_1) = at^\sigma_{\mathfrak{N}_2^j}(y_2)$.

Using these, a proof similar to that of Lemma 4.2 shows that $\mathfrak{N}_1^j$ and $\mathfrak{N}_2^j$ are globally σ-bisimilar for every $j \le N$, and so $\mathfrak{N}_1, x_1' \sim_\sigma \mathfrak{N}_2, x_2'$.

Case $L = \mathsf{Lin}_\mathbb{Z}$: While the definitions of $\mathfrak{H}_i^j$, for $0 < j < N$, are the same as above, for $j = 0, N$ we need new ones. Now, by Lemmas 5.6 and 5.8, $\boldsymbol{b}_i^0$ and $\boldsymbol{b}_i^N$ are non-degenerate, for $i = 1, 2$. Also, by Lemmas 5.4, 5.6, and 5.8, the R^--final cluster $C^-_{\boldsymbol{b}_i^0}$ in $\mathfrak{F}_i$ is an R^--limit cluster, and the R-final cluster $C^+_{\boldsymbol{b}_i^N}$ in $\mathfrak{F}_i$ is an R-limit cluster, for $i = 1, 2$. There are several cases. If $N = 0$ (that is, $\boldsymbol{b}_i^0 = W_i$) and $C^-_{\boldsymbol{b}_i^0} = C^+_{\boldsymbol{b}_i^0}$, then we let $\mathfrak{H}_i^0 = \textcircled{k^0}$. If $N = 0$ and $C^-_{\boldsymbol{b}_i^0} \neq C^+_{\boldsymbol{b}_i^0}$, then we let $\mathfrak{H}_i^0 = \mathfrak{C}\big(\textcircled{k^0}, \bullet\big) \lhd (m_i^0)^< \lhd \mathfrak{C}\big(\bullet, \textcircled{k^0}\big)$. If $N > 0$ then

$$\mathfrak{H}_i^0 = \begin{cases} \mathfrak{C}\big(\textcircled{k^0}, \bullet\big), & \text{if } C^-_{\boldsymbol{b}_i^0} = C^+_{\boldsymbol{b}_i^0}; \\ \mathfrak{C}\big(\textcircled{k^0}, \bullet\big) \lhd (m_i^0)^< \lhd \mathfrak{C}\big(\bullet, \textcircled{k^0}, \bullet\big), & \text{otherwise,} \end{cases}$$

and

$$\mathfrak{H}_i^N = \begin{cases} \mathfrak{C}\big(\bullet, \textcircled{k^N}\big), & \text{if } C^-_{\boldsymbol{b}_i^N} = C^+_{\boldsymbol{b}_i^N}; \\ \mathfrak{C}\big(\bullet, \textcircled{k^N}, \bullet\big) \lhd (m_i^N)^< \lhd \mathfrak{C}\big(\bullet, \textcircled{k^N}\big), & \text{otherwise.} \end{cases}$$

This way $\mathfrak{H}_i = \mathfrak{H}_i^0 \lhd \cdots \lhd \mathfrak{H}_i^N$ is a frame for $\mathsf{Lin}_\mathbb{Z}$, for $i = 1, 2$. Apart from these modifications, everything is similar to the $\mathsf{Lin}_{<\omega}$ case.

A coNP-algorithm deciding interpolant existence in $\mathsf{Lin}_{<\omega}$ or $\mathsf{Lin}_\mathbb{Z}$ is an obvious adaptation of the algorithm detailed in the proof of Theorem 4.6. ⊣

We conjecture that the IEP for every consistent finitely axiomatisable Priorean temporal logic is coNP-complete.

§6. Outlook and open problems. We have turned the lack of the CIP into a research question by asking whether deciding interpolant existence becomes harder than validity for modal logics without the CIP. As argued in [29, 31] for the closely related problem of separability of disjoint regular languages using a smaller language class (such as first-order definable languages), this question can be understood as a generalisation of satisfiability that provides new insights into the expressivity of the logic in question. We have shown that, in contrast to modal logics with nominals, the product modal logic $\mathsf{S5}\times\mathsf{S5}$, and the guarded and two-variable fragments of first-order logic, the complexity of deciding interpolant existence in finitely axiomatisable modal logics of linear frames is in coNP and, therefore, of the same complexity as validity. This appears to be the first general result about Craig interpolants for logics lacking the CIP. It gives rise to many further questions of which we mention only a few:

Q1: Is there a decidable modal logic above $\mathsf{K4}$ or GL with undecidable IEP?

Q2: Do all d-persistent (cofinal) subframe logics above $\mathsf{K4}$ have the finite bisimilar model property? Can one show a quasi-finite bisimilar model property for all (cofinal) subframe logics above $\mathsf{K4}$ and use it to prove that interpolant existence is decidable for all finitely axiomatisable ones?

Q3: What is the situation for propositional intermediate logics and intuitionistic or intermediate modal logics without the CIP?

Q4: Our proof is not constructive in the sense that is does not provide a non-trivial algorithm for computing interpolants if they exist (beyond exhaustive search) nor any upper bounds on their size. It would be of great interest to develop such algorithms.

Descriptive frames have been crucial for our proofs. It would therefore be interesting and in line with the modal logic tradition to characterise logics for which descriptive frames can be replaced by Kripke (or even finite) frames in Theorem 3.2. While d-persistence is clearly a sufficient condition, $\mathsf{Lin}_{\mathbb{R}}$ shows that it is not a necessary one; see Example 5.7. It is known, however, that $\mathsf{Lin}_{\mathbb{R}}$ is strongly complete [33], which suggests the conjecture that, in Theorem 3.2, descriptive frames for L can be replaced by Kripke frames iff L is strongly Kripke complete (in the sense that every L-consistent set of formulas is satisfiable in a Kripke frame for L). Note that a logic is strongly Kripke complete iff the corresponding variety of modal algebras is complex [15, 33].

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