Qualitative Coalitional Reasoning with Preferences

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March 26, 2004

Abstract

Qualitative coalitional games (QCGs) are a variation of conventional coalitional games in which each coalition may choose to cooperate in a number of different ways, with different choices resulting in potentially different sets of goals being achieved; each agent is associated with a set of goals, the intuition being that an agent is "satisfied" if any of its goals are achieved, but is indifferent between them. In this paper, we extend the framework of QCGs to incorporate *preferences* that agents have over their goals. In addition to establishing some basic properties of QCGs with Preferences (QCGPs), we investigate and characterise the complexity of six natural decision problems associated with QCGPs. For example, we prove that the problem of establishing Pareto optimality of a goal set with respect to some coalition is co-NP-complete. We end with some brief conclusions and a discussion of related work.

1 Introduction

Coalitional games are abstract models of cooperative interaction between self-interested agents, and as such, they have been widely studied in the game theory literature [7, pp.255–312]. The success of such models has led to them being adapted for use within the multiagent systems community, where they have proved to be of great value in understanding the nature of coalitions and coalition formation [12, 13, 11]. Conventional coalitional games (with transferable payoff) assign to each potential coalition a numeric value, corresponding to utility that could be distributed among coalition members if the coalition chose to cooperate. Given such models, solution concepts such as the core attempt to predict which coalitions might form, by considering the potential payoff an agent could get from joining different coalitions: a coalition is viewed as being "stable" if the members of that coalition have no incentive to defect and join any other coalition [7, pp.258–259].

In [14], we introduced a variation of coalitional games called *Qualitative Coalitional Games* (QCGs). In a QCG, each agent is assumed to have a set of goals: an agent is "satisfied" with any outcome that accomplishes one of its goals, but is indifferent about *which* of its goals should be achieved – all are considered equally good (individual rational). Each potential coalition is then modelled as having a set of choices available, intutitively corresponding to the different ways in which they could choose to cooperate. Associated with each choice is a set of goals, which would be achieved if the coalition chose to cooperate in this way. QCGs seem an appropriate abstract framework within which to reason about goal-oriented multiagent systems, where numeric utility values are either inappropriate or else impossible to derive. In [14], we formulated and investigated the computational complexity of a wide range of solution concepts associated with QCGs: for example, we demonstrated that showing the core of a QCG is non-empty (i.e., that a coalition have a choice from which there is no incentive for any member of the coalition to deviate) is D^p-complete.

The aim of the present paper is to extend the basic framework of QCGs, by allowing for agents that have preferences over goals. We begin by formulating *Qualitative Coalitional Games with Preferences* (QCGPs), defining some concepts (such as the core and stable sets) associated with QCGPs, and establishing some fundamental properties of these concepts. The remainder of the paper is then largely taken up with an investigation into the computational complexity of six decision problems associated with QCGPs. For example, we prove that the problem of establishing Pareto optimality of a goal set with respect to some coalition is co-NP-complete. We end with a brief discussion of related work and some conclusions.

Notational Conventions

We use the symbols $\top$ and $\bot$ to indicate the Boolean constants **true** and **false**. In general, upper case Greek letters – Φ, Ψ, etc – are used for propositional formulae. For a propositional formula $\Phi(x_1, \ldots, x_n)$ defined over the variables $X_n = \langle x_1, \ldots, x_n \rangle$, given $Z \subseteq X_n$, $\Phi[Z]$ is the result of evaluating Φ under the instantiation $x_i := \top$ if $x_i \in Z$, $x_i := \bot$ if $x_i \notin Z$. Thus, $\Phi[Z]$ is equivalent to the value of $\Phi(\zeta_1, \ldots, \zeta_n)$ where the tuple $\zeta = \langle \zeta_1, \ldots, \zeta_n \rangle$ describes the characteristic vector from $\langle \top, \bot \rangle^n$ for Z with respect to X_n.

In addition to the standard Boolean operations of conjunction ($\wedge$), disjunction ($\vee$), implication ($\Rightarrow$), and negation ($\neg$), some constructions will make of the binary *exclusive-or* operation which we denote by $\oplus$.

2 Qualitative Coalitional Games

Qualitative Coalitional Games (QCGs) were introduced in Wooldridge and Dunne [14] as a model for describing scenarios in which an agent, having some set of goals of which it would like to realise at least one, may have to form a coalition with other agents in order to bring this about. The formulation has similarities with the concept of classical coalitional games as described in, e.g. [7].

In the formal setting of [14]:

Definition 1 *A* qualitative coalitional game *(*QCG*) is an* $n+3$*-tuple:*

$$\Gamma = \langle G, Ag, G_1, \ldots, G_n, \Psi \rangle$$

where $G = \{g_1, \ldots, g_m\}$ *is a (finite) set of* possible goals*;* $Ag = \{1, \ldots, n\}$ *is a set of* agents*;* $G_i \subseteq G$ *is the set of* acceptable goals *for the agent* $i \in Ag$*, the intended interpretation being that any of the goals in* G_i *would satisfy* i*, however* i *is indifferent between the members of* G_i*.*

$\Psi(Ag, G)$ *is a well-formed propositional formula defined over the variables* $Ag = \langle a_1, \ldots, a_n \rangle$ *and* $G = \langle g_1, \ldots, g_m \rangle$*. The propositional logic function function described by* $\Psi(Ag, G)$ *defines the* coalitional capability function*. For every coalition* $C \subseteq Ag$ *this determines a set* $V(C)$ *of* choices *(subsets of* G*) for that coalition, i.e. when* $G' \subseteq G$ *belongs to* $V(C)$ *then the instantiation* $[C, G']$ *is such that* $\Psi[C, G'] = \top$*. Thus if* $G' \in V(C)$*, then one of the choices available to* C *is to bring about* all *the states in* G' *simultaneously.*

We say a set of goals G' satisfies *agent* i *if* $G' \cap G_i \neq \emptyset$*. Also, we say that* G' *is* feasible *for coalition* C *if* $G' \in V(C)$*, i.e.* $\Psi[C, G'] = \top$*.*

In [14] the computational complexity of a number of decision problems having a given GCG as part of their instance, is considered. Among these (together with their complexity classification) are,

SUCCESSFUL COALITION: (SC)
Instance: QCG $\langle G, Ag, G_1, \ldots, G_n, \Psi \rangle$ and coalition $C \subseteq Ag$.
Answer: "Yes" if $\exists G' \subseteq G$ s.t. $\forall i \in C$, $G_i \cap G' \neq \emptyset$. and $\Psi[C, G'] = \top$
Complexity: NP–complete.

MINIMAL COALITION: (MC)
Instance: QCG $\langle G, Ag, G_1, \ldots, G_n, \Psi \rangle$ and coalition $C \subseteq Ag$.
Answer: "Yes" if $\forall C' \subset C$, $\forall G' \subseteq G$, should $\forall i \in C'$ $G' \cap G_i \neq \emptyset$. then $\Psi[C', G'] = \bot$
Complexity: co-NP–complete.

CORE NON-EMPTY: (CNE)
Instance: QCG $\langle G, Ag, G_1, \ldots, G_n, \Psi \rangle$ and coalition $C \subseteq Ag$.
Answer: "Yes" if C is both successful and minimal.
Complexity: D^p–complete, D^p being the class of languages L formed by the intersection of languages $L_1 \in$ NP and $L_2 \in$ co $-$ NP, cf. [8]

TRIVIAL GAME: (TG)
Instance: QCG $\langle G, Ag, G_1, \ldots, G_n, \Psi\rangle$.
Answer: "Yes" if every coalition is successful.
Complexity: Π_2^p–complete.

GLOBAL UNATTAINABILITY: (GU)
Instance: QCG $\langle G, Ag, G_1, \ldots, G_n, \Psi\rangle$
Answer: "Yes" if there exists $G' \subseteq G$ such that for every coalition $C \subseteq Ag$ should $(\forall i \in C\ G_i \cap G' \neq \emptyset)$ then $\Psi[C, G'] = \perp$.
Complexity: Σ_2^p–complete.

INCOMPLETE GAME: (IG)
Instance: QCG $\langle G, Ag, G_1, \ldots, G_n, \Psi\rangle$
Answer: "Yes" if every $C \subseteq Ag$ is successful *but* there is some $G' \subseteq G$ such that for every coalition $C \subseteq Ag$ should $(\forall i \in C\ G_i \cap G' \neq \emptyset)$ then $\Psi[C, G'] = \perp$.
Complexity: D_2^p–complete, D_2^p being the class of languages L expressible as the intersection of languages $L_1 \in \Sigma_2^p$ and $L_2 \in \Pi_2^p$.

For further background on computational complexity theory and the classes mentioned, the reader is referred to [5, 9].

2.1 Preferences of Agents

In the setting described above if it is the case that G' satisfies and if feasible for some coalition C, i.e. $\forall i \in C, G_i \cap G \neq \emptyset$ and $\Psi[C, G'] = \top$, the presumption is that any agent in C is satisfied that it may realise any of the goals in $G_i \cap G'$ and thus is unconcerned about the goals in G_i/G' that would not be satisfied.

In 'practice' one would not expect indifference of the form "as long as one of my goals can be satisfied I don't care which", e.g. a speculator with two goals representing g_1 (make a non-zero profit) and g_2 (make a profit of at least \$100) would prefer to be part of a coalition in which g_2 can be satisfied instead of one which only guarantees g_1. Our aim in the present paper is to develop the framework for QCGs by incorporating models of agents having preferences regarding the goals they would like to be satisfied.

Definition 2 *A* Qualitative Coalitional Game with Preferences QCGP *is defined by a* $2n+3$*-tuple*

$$\Gamma^{\rhd} = \langle G, Ag, G_1, \ldots, G_n, \Psi, \rhd_1, \ldots, \rhd_n\rangle$$

in which $\langle G, Ag, G_1, \ldots, G_n, \Psi\rangle$ *is a* QCG *as in Definition 1 and* $\rhd_i$ *is a binary* preference relation *over* G_i*, so that* $g_r \rhd_i g_s$ *indicates that* i *would rather have its goal* g_r *satisfied than its goal* g_s.

We assume that $\triangleright_i$ is a partial order and is presented as a directed acyclic graph $D(G_i, R)$ so that forming the transitive closure of this graph's adjacency matrix will yield a matrix D^* for which $D^*[r,s] = \top$ if and only if $g_r \triangleright_i g_s$.

Given a QCGP, $\Gamma^{\triangleright}$ we can extend the concept of an agent's preferences to that of a *coalition*'s. We present two definitions and discuss how these are related.

For $C \subseteq Ag$ define the set of subsets of G, $\mathcal{X}(C)$, by

$$\mathcal{X}(C) \;=\; \{\; G' \subseteq G : (\bigwedge_{i \in C} (G_i \cap G \neq \emptyset) \wedge \Psi[C, G']) = \top \}$$

and let $\mathcal{X}^{\Gamma^{\triangleright}} = \cup_{C \subseteq Ag} \mathcal{X}(C)$.

Definition 3 *For $C \subseteq Ag$ and G', $G'' \in \mathcal{X}^{\Gamma^{\triangleright}}$, we say that C* strongly prefers *the goal set G' to G'', written $G' \sqsupset_C G''$ if*

1. $G' \in \mathcal{X}(C)$.
2. $\forall i \in C,\ \exists r_i \in G' \cap G_i\ \forall s_i \in G'' \cap G_i\ r_i \triangleright_i s_i$.

We say that C weakly prefers *the goal set G' to G'', written $G' \succ_C G''$ if*

1. $G' \in \mathcal{X}(C)$.
2. $\forall i \in C,\ \forall s_i \in G'' \cap G_i\ \exists r_i \in G' \cap G_i\ r_i \triangleright_i s_i$.

If $G' \sqsupset_C G''$ this indicates that for each $i \in C$ there is a *single goal*, $r_i \in G' \cap G_i$ that i ranks higher than *every one* of the goals, s_i, that it could have satisfied within G''.

If $G' \succ_C G''$ this indicates that for each $i \in C$ no matter which goal $s_i \in G'' \cap G_i$, i may have the possibility of realising, it can identify some goal $r_i \in G' \cap G_i$ that it would rather achieve.

We note that the goal set G'' is *not* required to be in the set $\mathcal{X}(C)$, although as a member of $\mathcal{X}^{\Gamma^{\triangleright}}$ it must belong to $\mathcal{X}(S)$ for at least one coalition $S \subseteq Ag$.

In total our formulations of G' being preferred to G'' by C capture the idea that G' is a set of a goals that are feasible for and satisfy C, i.e. $G' \in \mathcal{X}(C)$, whereas for each member i of C, G'' either fails to satisfy i at all or any goal $s_i \in G_i$ that can be realised within G'' can be outranked by a goal $r_i \in G' \cap G_i$: the difference between *strong* and *weak* preference is that in the former case at least one r_i must be present in $G' \cap G_i$ that is preferred to *every* goal $s_i \in G'' \cap G$, whereas in the latter case different r_i can be used depending on the goal chosen within G''.

Before proceeding with some further properties of these relationships we extend our preference concepts to apply to *sets* of coalitions. We are primarily interest in the case $\mathcal{R} = 2^C$ for $C \subseteq Ag$, however, the definition below is given in terms of arbitrary (non-empty) sets $\mathcal{R} \subseteq 2^{Ag}$. To avoid excessive repetition we use the relational symbol $\triangleright$ to indicate either $\sqsupset$ or $\succ$.

Definition 4 *Given a* set $\mathcal{R}$ *of coalitions*[1] *i.e.* $\mathcal{R} \subseteq 2^{Ag}$ *the binary relationship* $\rhd_{\mathcal{R}}$ *over* $\mathcal{X}(\mathcal{R}) = \cup_{C \in \mathcal{R}} \mathcal{X}(C)$ *is defined as* $G' \rhd_{\mathcal{R}} G''$ *if for some coalition* $C \in \mathcal{R}$ *it holds that* $G' \rhd_C G''$.

If $G' \rhd_{\mathcal{R}} G''$ *we say that* G' *strongly/weakly* dominates G'' *with respect to* $\mathcal{R}$*, noting that* $G' \rhd_{\mathcal{R}} G''$ *implies there is some* $C \in \mathcal{R}$ *for which* G' *strongly/weakly dominates* G'' *with respect to* C.

The following Lemma summarises some elementary properties of the relations $\succ_C$ and $\sqsupset_C$.

Lemma 1

a. *If* $G' \sqsupset_{\mathcal{R}} G''$ *then* $G' \succ_{\mathcal{R}} G''$*, however the converse does not always hold.*

b. *If* $\mathcal{R} = \{C\}$*, i.e. contains exactly one coalition, then* $\rhd_{\mathcal{R}}$ *induces a partial order over* $\mathcal{X}(C)$.

c. *If* $\mathcal{R}$ *satisfies the property that for every pair* $\{C, D\}$ *of coalitions in* $\mathcal{R}$*,* $C \cap D \neq \emptyset$ *then for all* $\{G', G''\} \in \mathcal{X}(\mathcal{R})$ *at most one of* $G' \rhd_{\mathcal{R}} G''$ *and* $G'' \rhd_{\mathcal{R}} G'$ *hold, i.e.* $\rhd_{\mathcal{R}}$ *is asymmetric.*

Proof: (a) That $G' \sqsupset_{\mathcal{R}} G''$ implies $G' \succ_{\mathcal{R}} G''$ is immediate from the definitions. To see that the converse may fail to hold, consider the QCGP, $\Gamma^{\rhd}$ in which

$$\begin{array}{l} Ag = \{a_1\} \\ G = G_1 = \{g_1, g_2, g_3, g_4\} \\ V(\emptyset) = \emptyset \\ V(\{a_1\}) = \{\{g_1, g_3\},\ \{g_2, g_4\}\} \\ \rhd_1 = \{g_1 \rhd_1 g_2,\ g_3 \rhd_1 g_4\} \end{array}$$

Then $\{g_1, g_3\} \succ_{\{a_1\}} \{g_2, g_4\}$ since g_1 outranks g_2 and g_3 outranks g_4. It is not the case, however, that $\{g_1, g_3\} \sqsupset_{\{a_1\}} \{g_2, g_4\}$ since g_1 is not strictly preferred to g_4 and g_3 is not strictly preferred to g_2.

(b) Consider the set $\mathcal{X}(C)$ and (distinct) goal sets G', G'' in this. The relationship $\rhd_{\mathcal{R}}$ is identical to $\rhd_C$. To see that $\sqsupset_C$ induces a partial order over $\mathcal{X}(C)$ it suffices to show $\sqsupset_C$ is asymmetric and transitive (that $\sqsupset_C$ is irreflexive is obvious). Suppose, then that $G' \sqsupset_C G''$. By definition for each $i \in C$ there is some $r_i \in G' \cap G_i$ such that for every $s_i \in G'' \cap G_i$ we have $r_i \rhd_i s_i$. From this we cannot have $G'' \sqsupset_C G'$ hence $\sqsupset_C$ is asymmetric. To see that it is also transitive consider any $G^{(p)}$, $G^{(q)}$, and $G^{(r)}$ in $\mathcal{X}(C)$ for which $G^{(p)} \sqsupset_C G^{(q)}$ and $G^{(q)} \sqsupset_C G^{(r)}$. From the first of these, for each $i \in C$ we find $p_i \in G^{(p)} \cap G_i$ for which $p_i \rhd_i q_i$ for any $q_i \in G^{(q)} \cap G_i$. From the second for each $i \in C$ we find $q_i \in G^{(q)} \cap G_i$ for which $q_i \rhd_i r_i$ for every $r_i \in G^{(r)} \cap G_i$. Since the

[1] We note the distinction, $\mathcal{R} = \emptyset$ and $\mathcal{R} = \{\emptyset\}$, the former being the set of *no coalitions*, the latter being the set containing *only* the empty coalition. The first is of little interest and to avoid repetition it will be assumed subsequently that $\mathcal{R} \subseteq 2^{Ag}$ is always *non-empty*.

relation $\triangleright_i$ defines a partial order over G_i it follows that $p_i \triangleright_i r_i$ and hence $G^{(p)} \sqsupset_C G^{(r)}$ as required.

For the case, $\succ$, it is again immediate that $G' \succ_C G'$ never holds. To prove asymmetry choose any G', G'' in $\mathcal{X}(C)$ and assume, to the contrary, that both $G' \succ_C G''$ and $G'' \succ_C G'$. Let $i \in C$ and suppose that $G' \cap G_i = \{r_1, \ldots, r_p\}$ and $G'' \cap G_i = \{s_1, \ldots, s_q\}$. From the assumption that $G' \succ_C G''$ there is some $r \in G' \cap G_i$, without loss of generality r_1, for which $r_1 \triangleright_i s_1$. In the same way, via $G'' \succ_C G'$ there is some $s \in G'' \cap G_i$, with $s \triangleright_i r_1 \triangleright_i s_1$: s cannot be s_1 ($\triangleright_i$ is a partial order) so, again without loss of generality assume it is s_2. Continuing in this way we identify a sequence of distinct r_j and s_j which satisfy

$$s_j \ \triangleright_i \ r_{j-1} \ \triangleright_i \ s_{j-1} \ \ldots \ s_2 \ \triangleright_i \ r_1 \ \triangleright_i \ s_1$$

This sequence, however, cannot be continued indefinitely since $G' \cap G_i$ and $G'' \cap G_i$ are finite sets, and thus the assumption that both $G' \succ_C G''$ and $G'' \succ_C G'$ would contradict the fact that $\triangleright_i$ is a partial order. Finally that $\succ_C$ is transitive is a straightforward argument similar to that employed to demonstrate that $\sqsupset_C$ is transitive.

For (c), consider any set of coalitions with the stated property. Suppose the contrary and that for some $G', G'' \in \mathcal{X}(\mathcal{R})$ we have both $G' \triangleright_{\mathcal{R}} G''$ and $G'' \triangleright_{\mathcal{R}} G'$. Then there must be coalitions C and D in $\mathcal{R}$ for which $G' \triangleright_C G''$ and $G'' \triangleright_D G'$. The condition on $\mathcal{R}$, however, indicates that $C \cap D \neq \emptyset$. Using $D \neq \emptyset$ and the arguments proving $\triangleright_D$ to be asymmetric above now establish the result claimed. ∎

From Lemma 1(b), the following subsets of $\mathcal{X}(C)$ are well-defined for any coalition.

Definition 5 *For any $C \subseteq Ag$, the* maximal strongly preferred goal sets with respect to C*, denoted $\mu^{\sqsupset}(C)$ are defined through*

$$\mu^{\sqsupset}(C) = \{G' \in \mathcal{X}(C) : \forall G'' \in \mathcal{X}(C) \text{ it is not the case that } G'' \sqsupset_C G'\}$$

For any $C \subseteq Ag$, the maximal weakly preferred goal sets with respect to C*, denoted $\mu^{\succ}(C)$ are defined through*

$$\mu^{\succ}(C) = \{G' \in \mathcal{X}(C) : \forall G'' \in \mathcal{X}(C) \text{ it is not the case that } G'' \succ_C G'\}$$

In the event of $\mu^{\sqsupset}(C) = \mu^{\succ}(C)$ we write simply $\mu(C)$.

We note from this definition that $G' \in \mathcal{X}(C)/\mu^{\triangleright}(C)$ implies there is some $G'' \in \mu^{\triangleright}(C)$ such that $G'' \triangleright_C G'$.

3 The Core and Stability

In this section we introduce some solution concepts in respect of the preferred goal sets that a (set of) coalitions might wish to realise.

Definition 6 *Let $\Gamma^{\rhd}$ be a* QCG $G = \langle G, Ag, G_1, \ldots, G_n, \Psi\rangle$ *with preference relations* $\langle \rhd_1, \ldots, \rhd_n\rangle$*. For a coalition $C \subseteq Ag$, the* strong core *of C, denoted $\kappa^{\sqsupset}(C)$ is the set*

$$\{G' \in \mu^{\sqsupset}(C) : \forall\, C' \subset C,\ \forall\, G'' \in \mathcal{X}(C') \textit{ it is } \text{not } \textit{the case that } G'' \sqsupset_{C'} G'\}$$

For a coalition $C \subseteq Ag$, the weak core *of C, denoted $\kappa^{\succ}(C)$ is the set*

$$\{G' \in \mu^{\succ}(C) : \forall\, C' \subset C,\ \forall\, G'' \in \mathcal{X}(C') \textit{ it is } \text{not } \textit{the case that } G'' \succ_{C'} G'\}$$

Again, if it is the case that $\mu^{\sqsupset}(D) = \mu^{\succ}(D)$ for all $D \in 2^C$ we refer simply to the core *of C denoting this $\kappa(C)$.*

Definition 7 *For $\mathcal{R} \subseteq 2^{Ag}$ and $\mathcal{Y} \subseteq \mathcal{X}(\mathcal{R})$, we say that $\mathcal{Y}$ is* internally stable with respect to the set of coalitions $\mathcal{R}$ *if*

$$\forall G',\ G'' \in \mathcal{Y} \textit{ there is no coalition } C \in \mathcal{R} \textit{ for which } G' \rhd_C G'' \textit{ or } G'' \rhd_C G'.$$

The set $\mathcal{Y}$ is externally stable with respect to the set of coalitions $\mathcal{R}$ *if*

$$\forall G' \in \mathcal{X}(\mathcal{R})/\mathcal{Y}\ \exists G'' \in \mathcal{Y} \textit{ and } C \in \mathcal{R} \textit{ such that } G'' \rhd_C G'$$

The set $\mathcal{Y}$ is stable with respect to the set of coalitions $\mathcal{R}$ *if it is both internally and externally stable with respect to $\mathcal{R}$.*

Definition 8 *For $\mathcal{R} \subseteq 2^{Ag}$ and $\mathcal{Y} \subseteq \mathcal{X}(\mathcal{R})$, we say that a subset $\mathcal{Y}$ of $\mathcal{X}(\mathcal{R})$ is an* admissible goal set with respect to $\mathcal{R}$ *if $\mathcal{Y}$ is internally stable with respect to $\mathcal{R}$ and for every $G^p \in \mathcal{X}(\mathcal{R})/\mathcal{Y}$, if there is a coalition $C \in \mathcal{R}$ for which $G^p \rhd_C G^q$ for $G^q \in \mathcal{Y}$ then there is some $G^r \in \mathcal{Y}$ and coalition $D \in \mathcal{R}$ for which $G^r \rhd_D G^p$. An admissible goal set $\mathcal{Y}$ is maximal with respect to $\mathcal{R}$ if no strict superset of $\mathcal{Y}$ is admissible.*

Notice that every *stable set* with respect to $\mathcal{R}$ is also a *maximal* admissible goal set with respect to $\mathcal{R}$, however, the converse does not always hold, e.g. consider $\mathcal{R} = \{C, D, E\}$ whose members are pairwise disjoint; furthermore suppose we have three goal sets $G^{(C)} \in \mathcal{X}(C)$, $G^{(D)} \in \mathcal{X}(D)$ and $G^{(E)} \in \mathcal{X}(E)$. It may be the case that

$$G^{(C)} \rhd_C G^{(D)} \rhd_D G^{(E)} \rhd_E G^{(C)}$$

If $\mathcal{X}(\mathcal{R}) = \{G^{(C)}, G^{(D)}, G^{(E)}\}$ then the maximal admissible subset is the empty set, however, this is not externally stable: in fact this system has no stable set. This possibility motivates the following

Definition 9 *The set of coalitions $\mathcal{R} \subseteq 2^{Ag}$ is* coherent *in the* QCGP $\Gamma^{\rhd}$*, if every $\mathcal{Y} \subseteq \mathcal{X}(\mathcal{R})$ that defines a maximal admissible set with respect to $\mathcal{R}$ is a stable set with respect to $\mathcal{R}$.*

The sets $\mathcal{X}(\mathcal{R})$ and concepts of core and stability introduced in Definitions 6, 7 are analogous to the idea of *imputation*, core, and stability in classical coalitional games, cf. [7, Section 14.2].

As an aside we note that the definition of the core of a succesful coalition differs from the (non-preference model) view of [14] in which "core" of a coalition, C, is non-empty if and only if $\mathcal{X}(C) \neq \emptyset$ *and* for all strict subsets C' of C $\mathcal{X}(C') = \emptyset$. Suppose, however, we allow definitions of $\rhd_C$ to be *independent* of the preference relations $\langle \rhd_1, \ldots, \rhd_n \rangle$: then defining $\kappa(C)$ with respect to these we can capture the interpretation of [14] as shown in the following easy lemma.

Lemma 2 *Let* $\Gamma^{\rhd}$ *be a* QCGP *and define a preference relation for* coalitions $C \subseteq Ag$ *by*

$$G' \rhd_C G'' \textit{ if } G' \in \mathcal{X}(C) \textit{ and } \exists C' \supset C \textit{ for which } G'' \in \mathcal{X}(C')$$

so that $\kappa(C)$ *– the core of* C *– is*

$$\{G' \in \mathcal{X}(C) : \forall\, C' \subset C,\ \forall\, G'' \in \mathcal{X}(C') \textit{ it is } \text{not } \textit{the case that } G'' \rhd_{C'} G'\}$$

Then $\kappa(C) \neq \emptyset$ *if and only if* C *is both minimal and successful.*

Proof: If $\kappa(C) \neq \emptyset$ then it is certainly the case that C is successful since $\kappa(C) \subseteq \mathcal{X}(C)$. It must also be the case, however, that C is minimal: for otherwise we have some $C' \subset C$ with $\mathcal{X}(C') \neq \emptyset$ so that for any $G'' \in \mathcal{X}(C')$ we have $G'' \rhd_{C'} G'$ for every $G' \in \mathcal{X}(C)$ contradicting $\kappa(C) \neq \emptyset$. Similarly if C is both successful and minimal then the former yields $\mathcal{X}(C) \neq \emptyset$ while the latter indicates that for every $C' \subset C$, we have $\mathcal{X}(C') = \emptyset$ hence $\kappa(C) = \mathcal{X}(C) \neq \emptyset$ as required. ∎

We note that we could restrict notions of stability and admissibility to the case where the underlying set of coalitions $\mathcal{R}$ is simply 2^{Ag}, i.e. the set of all possible coalitions. There are, however, some disadvantages of this. Although it is certainly the case any set $\mathcal{Y} \subseteq \mathcal{X}^{\Gamma^{\rhd}}$ that is internally stable with respect to 2^{Ag} will also have this property with respect to any subset of the set of coalitions

$$\mathcal{R} = \{C : \mathcal{Y} \cap \mathcal{X}(C) \neq \emptyset\}$$

it may be the case that we wish to regard some goal sets as internally stable (with respect to a given set of coalitions $\mathcal{R}$) that could not be considered as such in terms of the set 2^{Ag}, i.e. if $\mathcal{Y}$ is internally stable for 2^C with some $C \subset Ag$, it does not necessarily have this property with respect to any $D \not\subseteq C$. Similarly, as regards our definitions of external stability and admissibility, using only $\mathcal{R} = 2^{Ag}$, will view some $\mathcal{Y}$ is externally stable (or admissible) that do not have this property with respect to a given coalition C or even 2^C, e.g. we may have $\mathcal{Y} \subseteq \mathcal{X}(2^C)$ but with some $G'' \in \mathcal{X}(2^C)/\mathcal{Y}$ for which no goal set $G' \in \mathcal{Y}$ and coalition $D \in 2^C$ gives $G' \rhd_D G''$ (hence $\mathcal{Y}$ is not externally stable with respect to the set of coalitions 2^C) even though $G' \rhd_{2^{Ag}} G'$: the latter preference being exhibited by some $D \not\subseteq C$.

In total the choice of $\mathcal{R} = 2^{Ag}$ may be too restrictive sensibly to consider *internal* stability, but rather too general to use as a basis for *external* stability.

We introduce some further notation prior to proving some basic properties of these structures. For $\mathcal{R} \subseteq 2^{Ag}$ we define the sets

$$\begin{array}{lll} \iota^{\triangleright}(\mathcal{R}) & =_{def} & \{\mathcal{Y} \subseteq \mathcal{X}(\mathcal{R}) : \mathcal{Y} \text{ is internally stable w.r.t. } \mathcal{R}\} \\ \eta^{\triangleright}(\mathcal{R}) & =_{def} & \{\mathcal{Y} \subseteq \mathcal{X}(\mathcal{R}) : \mathcal{Y} \text{ is externally stable w.r.t. } \mathcal{R}\} \\ \sigma^{\triangleright}(\mathcal{R}) & =_{def} & \iota^{\triangleright}(\mathcal{R}) \cap \eta^{\triangleright}(\mathcal{R}) \\ \delta^{\triangleright}(\mathcal{R}) & =_{def} & \{\mathcal{Y} \subseteq \mathcal{X}(\mathcal{R}) : \mathcal{Y} \text{ is maximally admissible w.r.t. } \mathcal{R}\} \end{array}$$

Thus, $\sigma^{\triangleright}(\mathcal{R})$ defines the set of all stable goal sets with respect to $\mathcal{R}$.

3.1 Basic Properties of Stable Sets and the Core

The relationships between these sets are summarised in the next lemma.

Lemma 3 *Given a* QCGP *and any* $\mathcal{R} \subseteq 2^{Ag}$*:*

a. $\iota^{\succ}(\mathcal{R}) \subseteq \iota^{\sqsupset}(\mathcal{R})$

b. $\eta^{\sqsupset}(\mathcal{R}) \subseteq \eta^{\succ}(\mathcal{R})$

c. $\iota^{\succ}(\mathcal{R}) \cap \eta^{\sqsupset}(\mathcal{R}) \subseteq \sigma^{\triangleright}(\mathcal{R}) \subseteq \iota^{\sqsupset}(\mathcal{R}) \cap \eta^{\succ}(\mathcal{R})$

Proof: The proofs of (a) and (b) follow directly from Lemma 1(a): if $\mathcal{Y} \in \iota^{\succ}(\mathcal{R})$ then for any G', $G'' \in \mathcal{Y}$ and $D \in \mathcal{R}$, by definition, neither $G' \succ_D G''$ nor $G'' \succ_D G'$ hold, thus neither $G' \sqsupset_D G''$ nor $G'' \sqsupset_D G'$, i.e. $\mathcal{Y} \in \iota^{\sqsupset}(\mathcal{R})$. Similarly $\mathcal{Y} \in \eta^{\sqsupset}(\mathcal{R})$ indicates that for each $G' \in \mathcal{X}(\mathcal{R})/\mathcal{Y}$ we have some $G'' \in \mathcal{Y}$ and $D \in \mathcal{R}$ for which $G'' \sqsupset_D G'$, which gives $G'' \succ_D G'$ and thus $\mathcal{Y} \in \eta^{\succ}(\mathcal{R})$.

Finally, (c) is immediate from (a), (b), and the definition of $\sigma^{\triangleright}(\mathcal{R})$. ∎

Regarding $\kappa^{\triangleright}(C)$ for $C \subseteq Ag$, we observe that parts (b–d) of our next result can be seen as an analogous result to [7, Proposition 279.2]

Lemma 4 *For any* $C \subseteq Ag$*,*

a. $\kappa^{\succ}(C) \subseteq \kappa^{\sqsupset}(C) \subseteq \mu^{\sqsupset}(C) \subseteq \mu^{\succ}(C)$.

b. For every $\mathcal{Y} \in \sigma^{\triangleright}(2^C)$, $\kappa^{\triangleright}(C) \subseteq \mathcal{Y}$.

c. If $\mathcal{Y}$ *and* $\mathcal{Z}$ *are distinct sets in* $\sigma^{\triangleright}(2^C)$ *then* $\mathcal{Y} \not\subseteq \mathcal{Z}$ *and* $\mathcal{Z} \not\subseteq \mathcal{Y}$.

d. If $\kappa^{\triangleright}(C) \in \sigma^{\triangleright}(2^C)$ *then* $\sigma^{\triangleright}(2^C) = \{\kappa^{\triangleright}(C)\}$.

Proof: For (a), $\mu^{\sqsupset}(C) \subseteq \mu^{\succ}(C)$ is immediate from Lemma 1(a) while $\kappa^{\sqsupset}(C)$ is a subset of $\mu^{\sqsupset}(C)$ by definition. So it remains to establish $\kappa^{\succ}(C) \subseteq \kappa^{\sqsupset}(C)$. Suppose the contrary and that $G' \in \kappa^{\succ}(C)$ but $G' \notin \kappa^{\sqsupset}(C)$. From the first of these we must have $G' \in \mathcal{X}(C)$ and from the second there must be some $D \subseteq C$ and $G'' \subseteq G$ for which $G'' \sqsupset_D G'$. From Lemma 1(a), however, $G'' \sqsupset_D G'$ would imply $G'' \succ_D G'$ and this contradicts $G' \in \kappa^{\succ}(C)$.

For (b) consider any $\mathcal{Y} \in \sigma^{\rhd}(2^C)$, i.e. stable set with respect to the set of coalitions formed by all subsets of C. Suppose there is some goal set $G' \in \kappa^{\rhd}(C)$ for which $G' \notin \mathcal{Y}$. The set $\mathcal{Y}$ is externally stable with respect to $\mathcal{R} = 2^C$ and thus contains some goal set G'' for which $G'' \rhd_{\mathcal{R}} G'$. We cannot have $G'' \in \mathcal{X}(C)$ as this would contradict $\kappa^{\rhd}(C) \subseteq \mu^{\rhd}(C)$. It follows that $G'' \in \mathcal{X}(D)$ for some $D \subset C$ and $G'' \rhd_D G'$: this, however, contradicts the assumption that $G' \in \kappa^{\rhd}(C)$.

Part (c) follows easily from internal and external stability: were it the case that $\mathcal{Y} \subset \mathcal{Z}$ for distinct $\mathcal{Y}$ and $\mathcal{Z}$ in $\sigma^{\rhd}(2^C)$ then since $\mathcal{Y} \in \eta^{\rhd}(2^C)$ any goal set $G' \in \mathcal{Z}/\mathcal{Y}$. is dominated by some goal set in $\mathcal{Y}$ with respect to 2^C. This, however, would indicate that $\mathcal{Z} \notin \iota(2^C)$, i.e. $\mathcal{Z}$ is not internally stable.

For (d), suppose that $\kappa^{\rhd}(C) \in \sigma^{\rhd}(2^C)$ and consider any $\mathcal{Y} \in \sigma^{\rhd}(2^C)$. From part (b) we have $\kappa^{\rhd}(C) \subseteq \mathcal{Y}$; from (b) we cannot have $\kappa^{\rhd}(C) \subset \mathcal{Y}$ and hence $\kappa^{\rhd}(C) = \mathcal{Y}$ which suffices to prove $\sigma^{\rhd}(2^C) = \{\kappa^{\rhd}(C)\}$. ∎

Lemma 5 $\sigma^{\rhd}(\{C\}) = \{\mu^{\rhd}(C)\}$.

Proof: That $\mu^{\rhd}(C)$ is a stable set with respect to C is immediate from the definition of $\mu^{\rhd}(C)$: if $\{G', G''\} \in \mu^{\rhd}(C)$ then certainly neither $G' \rhd_C G''$ nor $G'' \rhd_C G'$ hold and, thus, $\mu^{\rhd}(C)$ is internally stable with respect to C. In addition if we consider any $G' \in \mathcal{X}(C)/\mu^{\rhd}(C)$ then since $G' \notin \mu^{\rhd}(C)$ there must be some $G'' \in \mu^{\rhd}(C)$ for which $G'' \rhd_C G'$ establishing that $\mu^{\rhd}(C)$ is externally stable with respect to C. To see that $\mu^{\rhd}(C)$ is the unique stable set with respect to C, it suffices to observe that from external stability $\mu^{\rhd}(C) \subseteq \mathcal{Y}$ for any $\mathcal{Y} \in \sigma^{\rhd}(\{C\})$, which suffices to ensure uniqueness. ∎

We note that Lemma 5 indicates that *every* QCGP has *at least one subset of* 2^G which is stable. This set may, of course, simply be the empty set: in the event $C \subseteq Ag$ having no satisfying and feasible goal set, i.e. $\mathcal{X}(C) = \emptyset$, then $\mu^{\rhd}(C) = \emptyset$ which is stable. Since $\mu^{\rhd}(C)$ is the unique maximal admissible set with respect to C, it follows that in every QCGP the set of coalitions $\mathcal{R} = \{C\}$ is *coherent* for every $C \subseteq Ag$. Another interesting and easily proven consequence of Lemma 5 is the following,

Theorem 1 *Let* $\Gamma^{\rhd}$ *be a* QCGP $\Gamma^{\rhd} = \langle G, Ag, G_1, \ldots, G_n, \Psi, \rhd_1, \ldots, \rhd_n \rangle$. *There exists* $\mathcal{R} \subseteq 2^{Ag}$ *for which* $\sigma^{\rhd}(\mathcal{R}) \neq \{\emptyset\}$ *if and only if there exists some coalition* $C \subseteq Ag$ *for which* $\mathcal{X}(C) \neq \emptyset$.

Informally, Theorem 1 asserts that there is a *non-empty* stable set of goal sets within the QCGP $\Gamma^{\rhd}$ if and and only some coalition has a feasible and satisfying goal set.

Proof: Suppose $C \subseteq Ag$ is such that $\mathcal{X}(C) \neq \emptyset$. From Lemma 1(b), the set $\mu^{\rhd}(C)$ is well-defined and non-empty. From Lemma 5, $\mu^{\rhd}(C)$ is a stable set with respect to C. Thus if for some $C \subseteq Ag$, it holds that $\mathcal{X}(C) \neq \emptyset$ then we identify $\mathcal{R} \subseteq 2^{Ag}$, i.e. $\mathcal{R} = \{C\}$, for which $\sigma^{\rhd}(\mathcal{R}) \neq \emptyset$.

For the reverse direction suppose that $\mathcal{R} \subseteq 2^{Ag}$ is such that $\sigma^{\rhd}(\mathcal{R}) \neq \{\emptyset\}$ and let $\mathcal{Y}$ be a stable set with respect to the set of coalitions $\mathcal{R}$. By definition

$$\mathcal{Y} \subseteq \mathcal{X}(\mathcal{R}) \;=\; \bigcup_{C \in \mathcal{R}} \mathcal{X}(C)$$

and thus, $\mathcal{Y} \neq \emptyset$ implies that for some $C \subseteq Ag$, $\mathcal{X}(C) \neq \emptyset$. ∎

One of the main points of interest regarding Theorem 1 is that it allows the computational complexity of the following decision problem to be determined exactly.

> <u>NON-TRIVIAL STABLE SET</u>: (NTSS)
> *Instance*: QCGP $\langle G, Ag, G_1, \ldots, G_n, \Psi, \rhd_1, \ldots, \rhd_n \rangle$
> *Answer*: "Yes" if there exists $\mathcal{R} \subseteq 2^{Ag}$ and $\mathcal{Y} \subseteq \mathcal{X}(\mathcal{R})$ such that $\mathcal{Y} \neq \emptyset$ and $\mathcal{Y} \in \sigma^{\rhd}(\mathcal{R})$.

Corollary 1 NTSS *is* NP*–complete.*

Proof: From Theorem 1 an instance $\Gamma^{\rhd} = \langle G, \rhd_1, \ldots, \rhd_n \rangle$ of NTSS is accepted if and only if some coalition is successful, i.e. NTSS is equivalent to deciding if G is accepted as instance of the problem NON-EMPTY GAME (the complement of the problem EMPTY GAME which accepts instances G of QCGs for which no coalition succeeds). The decision problem EMPTY GAME was shown to be co-NP–complete in [14], and thus its complement decision problem – NON-EMPTY GAME – is NP–complete. This suffices to deduce that NTSS is also NP–complete. ∎

While NP–hardness of NTSS is unsurprising, it is, perhaps, less obvious that the problem belongs to NP. Theorem 1, however, provides a decision method in NP that obviates any requirement to consider sets of coalitions $\mathcal{R} \subseteq 2^{Ag}$ and sets of goal sets $\mathcal{Y} \subseteq 2^{G}$ whose size is superpolynomial in $n + m$ by characterising the existence of non-empty stable sets in terms of the existence of successful coalitions.

3.2 Optimal Goal Sets

The maximal preferred sets - $\mu^{\rhd}(C)$ - provide one approach to defining what is meant by a goal set being optimal for a coalition C: if $G' \in \mu^{\rhd}(C)$ then there is no choice $G'' \in \mathcal{X}(C)$ that will result in *every* member of C being able to *strictly improve* upon the goals is realise within G'. Another widely studied concept of optimality, particularly within the context of multiagent resource/task allocation settings is that of *Pareto Optimality*. Formally

Definition 10 *A goal set $G' \in \mathcal{X}(C)$ is* Pareto Optimal *(with respect to $\sqsupset_C$) if for all other $G'' \in \mathcal{X}(C)$, should it be the case that for some i in C there is a goal $r_i \in G'' \cap G_i$ which is strictly preferred to every $s_i \in G' \cap G_i$, then there is some $s_j \in G' \cap G_j$ which is strictly preferred to every $r_j \in G'' \cap G_j$.*

A goal set $G' \in \mathcal{X}(C)$ is Pareto Optimal *(with respect to $\succ_C$) if for all other $G'' \in \mathcal{X}(C)$, should it be the case that for some $i \in C$ for every $s_i \in G' \cap G_i$ there is some $r_i \in G'' \cap G_i$ for which $r_i \rhd_i s_i$, then there is some $j \in C$ in which: for every $r_j \in G'' \cap G_j$ there exists $s_j \in G' \cap G_i$ with $s_j \rhd_j r_j$.*

Thus G' is a Pareto Optimal goal set for a coalition, C, if it is feasible for and satisfies each member of C, but for any other goal set, G'' that is feasible and satisfies C, if some agent can realise a more preferred goal with G'' this will be at the expense of another agent having to accept a goal that it prefers less to its optimal goals within G'. We note the distinction between G' being Pareto Optimal and a maximal preferred goal set (with either $\succ$ or $\sqsupset$ as the underlying preference relations). As counterparts to the sets $\mu^{\rhd}$ let

$$\pi^{\rhd}(C) \;=\; \{G' \in \mathcal{X}(C) : G' \text{ is Pareto Optimal with respect to } \rhd_C\}$$

It is *not necessarily* the case that $\pi^{\rhd}(C) = \mu^{\rhd}(C)$, although it is easily shown via the respective definitions that: for every QCGP and coalition $C \subseteq Ag$, $\pi^{\rhd}(C) \subseteq \mu^{\rhd}(C)$. Consider, however, the following

Example 1 *Let $\Gamma^{\rhd}$ be a* QCGP *with $Ag = \{a_1, a_2\}$, $G = \{g_1, g_2, g_3\}$, $G_1 = \{g_1, g_2\}$, $G_2 = \{g_3\}$, $\rhd_1 = \{g_1 \rhd_1 g_2\}$, $\rhd_2 = \emptyset$, and $\Psi(Ag, G) = a_1 a_2 g_3 (g_1 \vee g_2)$.*

For this QCGP *it is easily checked that*

$$\mathcal{X}(C) = \begin{cases} \{\{g_1, g_3\},\ \{g_2, g_3\}\} & \text{if} \quad C = \{a_1, a_2\} \\ \emptyset & \text{if} \quad C \neq \{a_1, a_2\} \end{cases}$$

Furthermore,

$$\mu^{\sqsupset}(\{a_1, a_2\}) = \mu^{\succ}(\{a_1, a_2\}) = \mathcal{X}(\{a_1, a_2\}) = \{\{g_1, g_3\},\ \{g_2, g_3\}\}$$

On the other hand, the Pareto Optimal sets are

$$\pi^{\sqsupset}(\{a_1, a_2\}) = \pi^{\succ}(\{a_1, a_2\}) = \{\{g_1, g_3\}\}$$

In both cases strict subsets of $\mu^{\rhd}(\{a_1, a_2\})$ which also contains $\{g_2, g_3\}$. The goal set $\{g_1, g_3\}$ strictly improves the goal that can be realised by a_1 ($g_1 \rhd_1 g_2$) but does not leave a_2 less *satisfied than before. Thus for this example $\pi(\{a_1, a_2\}) \subset \mu(\{a_1, a_2\})$.*

As an aside we note that the behaviour illustrated in Example 1, indicates a significant difference between the *qualitative* model of goal achievement and preference discussed in this paper, and the widely studied *quantitative* concepts of resource/task allocations in multiagent settings as considered, for example in [3, 4, 6, 10]. In these models a collection of *resources*, $\mathcal{R}$ replaces the set of *goals* G. Each agent, $a_i \in Ag$, associates a numerical value, $u_i(S)$ with every subset S of $\mathcal{R}$.[2] Given some partition $\Pi = \langle P_1, \ldots, P_n \rangle$ of $\mathcal{R}$ among Ag, a number of concepts of what it means for Π to be *optimal* have been considered. Two such are *maximum social welfare* – a partition, Π which maximises $w(\Pi) = \sum_{i=1}^{n} u_i(P_i)$ – and *Pareto Optimality*: a partition, $\langle P_1, \ldots, P_n \rangle$ for which: given any alternative partition $\langle Q_1, \ldots, Q_n \rangle$ of $\mathcal{R}$ among Ag, should it be the case that for some a_i, $u_i(Q_i) > u_i(P_i)$ then for some a_j we have $u_j(Q_j) < u_j(P_j)$. Denoting by $\mu(\mathcal{R})$ and $\pi(\mathcal{R})$ the set of partitions of $\mathcal{R}$ which maximise social welfare, respectively are Pareto Optimal, it is easy to demonstrate for any choice of utility functions that $\mu(\mathcal{R}) \subseteq \pi(\mathcal{R})$, i.e. any allocation that optimises social welfare is Pareto Optimal. In contrast, it is not difficult to construct ("natural") utility functions for which $\mu(\mathcal{R}) \subset \pi(\mathcal{R})$, i.e. there are Pareto optimal allocations that do not maximise social welfare. Thus for the qualitative concepts of goal set and Pareto optimality we have $\pi(C) \subseteq \mu(C)$ and cases with $\pi(C) \subset \mu(C)$, whereas for the quantitative view of welfare and Pareto optimality we have $\mu(\mathcal{R}) \subseteq \pi(\mathcal{R})$ and cases for which $\mu(\mathcal{R}) \subset \pi(\mathcal{R})$.

4 Decision Problems for QCGPs

We turn now to the three decision problems listed below. Although, in principle one could define distinct variants of these in terms of the two different preference relationships – $\sqsupset$ and $\succ$ – this turns out, in fact, to be unnecessary. For the complexity classifications that are proved subsequently all of the lower bound results, i.e. hardness proofs, construct instances for which $G' \sqsupset_C G''$ if and only if $G' \succ_C G''$ for every coalition C. For upper bounds arguments, i.e. membership of a given class it is easily verified given a QCGP, coalition C, and goal sets G', G'' that the tests $G' \succ_C G''$ and $G' \sqsupset_C G''$ can both be accomplished easily. The one significant exception arises in the result proved in Theorem 4 a $\mathcal{C}$–completeness result for which establishing membership in $\mathcal{C}$ is a non-trivial argument involving differing constructions dependent on the exact preference relation employed.

CORE MEMBERSHIP: (CM)
Instance: QCGP $\langle G, Ag, G_1, \ldots, G_n, \Psi, \rhd_1, \ldots, \rhd_n \rangle$, coalition $C \subseteq Ag$, goal set $G' \subseteq G$.
Answer: "Yes" if $G' \in \kappa^{\rhd}(C)$.

MAXIMAL GOAL SET: (MGS)
Instance: QCGP $\langle G, Ag, G_1, \ldots, G_n, \Psi, \rhd_1, \ldots, \rhd_n \rangle$, coalition $C \subseteq Ag$, goal set $G' \subseteq G$.
Answer: "Yes" if $G' \in \mu^{\rhd}(C)$.

[2] Issues of concise representation formalisms for such *utility functions* are discussed in [3, pp. 5–6].

PARETO OPTIMAL GOAL SET: (PO)
Instance: QCGP $\langle G, Ag, G_1, \ldots, G_n, \Psi, \rhd_1, \ldots, \rhd_n\rangle$, coalition $C \subseteq Ag$, goal set $G' \subseteq G$.
Answer: "Yes" if $G' \in \pi^{\rhd}(C)$.

Theorem 2 CM *is co-*NP*–complete.*

Proof: Membership in co-NP holds since $G' \in \kappa^{\rhd}(C)$ if and only if

$$\forall G'',\ D \subseteq C\ (G' \in \mathcal{X}(C) \text{ and it is not the case that } G'' \rhd_D G')$$

That is if G' satisfies and is feasible for C and is not dominated by any other goal set G'' with respect to any subset of C. We note that if $G' \notin \mu^{\rhd}(C)$ (and hence cannot belong to the core) then it will be dominated by some $G'' \in \mu^{\rhd}(C)$ with respect to C. By quantifying over all subsets of C, i.e. not simply strict subsets, this case is detected.

To complete the proof we use a reduction from UNSAT. Let $\Phi(x_1, \ldots, x_n)$ be an instance of UNSAT. We form an instance $\langle \Gamma^{\rhd}_{\Phi}, C, G'\rangle$ of CM for which $G' \in \kappa^{\rhd}(C)$ if and only if $\Phi(x_1, \ldots, x_n)$ is unsatisfiable.

The QCGP $\Gamma^{\rhd}_{\Phi}$ has $Ag = \{a_1, \ldots, a_n, a_{n+1}\}$, $G_i = \{g_i^{\top}, g_i^{\perp}, g_i^{\min}\}$ so that $G = \cup_{i=1}^{n+1} G_i$. The coalitional capability function $\Psi(Ag, G)$ is given as

$$\left(\bigwedge_{i=1}^{n+1} a_i g_i^{\min}(\neg g_i^{\top})(\neg g_i^{\perp})\right) \ \vee\ (\neg a_{n+1})\Phi(a_1(g_1^{\top} \vee \neg g_1^{\perp}), \ldots, a_n(g_n^{\top} \vee \neg g_n^{\perp}))$$

The preference relation $\rhd_i$ contains exactly two elements: $\{g_i^{\top} \rhd_i g_i^{\min},\ g_i^{\perp} \rhd_i g_i^{\min}\}$. Finally we set $C = Ag$ and $G' = \cup_{i=1}^{n+1}\{g_i^{\min}\}$ to form the instance $\langle \Gamma^{\rhd}_{\Phi}, C, G'\rangle$ of CM.

We note that $G' \sqsupset_C G''$ if and only if $G' \succ_C G''$ for all $C \subseteq Ag$, and hence $\kappa^{\succ}(C) = \kappa^{\sqsupset}(C)$ for every coalition C. We claim that

$$G' = \bigcup_{i=1}^{n+1}\{g_i^{\min}\}\ \in\ \kappa(Ag) \text{ if and only if } \Phi(x_1, \ldots, x_n) \text{ is unsatisfiable}$$

First suppose that $G' \in \kappa(Ag)$. It is certainly the case that $G' \in \mu(Ag)$, since $\mathcal{X}(Ag)$ contains exactly one set, i.e. G'. From the assumption that $G' \in \kappa(Ag)$ it must be the case that no G'' dominates G' with respect to any $D \subset Ag$. Certainly this is the case when $a_{n+1} \in D \subset Ag$ since for any such D, $\Psi[D, G''] = \perp$. For the case $D = Ag/\{a_{n+1}\}$, $\Psi[D, G'']$ is equivalent to

$$\Phi(g_1^{\top} \vee \neg g_1^{\perp}, \ldots, g_n^{\top} \vee \neg g_n^{\perp})[G'']$$

which must assume the value $\perp$ for every possible subset of $\cup_{i=1}^{n}\{g_i^{\top}, g_i^{\perp}\}$ that satisfies each $i \in \{a_1, \ldots, a_n\}$. For each instantiation of $\langle x_1, \ldots, x_n\rangle$

$$\alpha = \langle \alpha_1, \ldots, a_n\rangle\ \in\ \langle \top, \perp\rangle^n$$

we can choose exactly one of $\{g_i^\top, g_i^\perp\}$ so that $\alpha_i = g_i^\top \vee \neg g_i^\perp$ and since any such choice satisfies a_i (using a goal that it strictly prefers to $g_i^{\min}$) it follows that $\Phi(\alpha) = \perp$ for every instantiation α, i.e. Φ is unsatisfiable.

On the other hand, suppose that $\Phi(x_1, \ldots, x_n)$ is unsatisfiable. We show that $G' \in \kappa(Ag)$. We have already noted that $\mathcal{X}(Ag) = \{G'\}$, and so G' could only fail to be in $\kappa(Ag)$ were there some strict subset D of Ag and $G'' \in \mathcal{X}(D)$ for which $G'' \rhd_D G'$. So consider any $D \subset Ag$ and $G'' \subseteq G$. If $a_{n+1} \in D$ then $\Psi[D, G''] = \perp$ so that G'' does not dominate G' with respect to D. If $a_{n+1} \notin D$, then $\Psi[D, G'']$ equals $\Phi(\beta_1, \ldots, \beta_n)$ where β_i is the value of $a_i(g_i^\top \vee \neg g_i^\perp)$. From our assumption, however, Φ is unsatisfiable and thus $\Phi(\beta_1, \ldots, \beta_n) = \perp$. We deduce that if Φ is unsatisfiable then $G' \in \kappa(Ag)$, completing the proof the CM is co-NP–complete. ∎

Corollary 2

a) MGS *is co-*NP*–complete.*

b) PO *is co-*NP*–complete.*

Proof: For (a) membership is immediate from the fact that $\langle \Gamma^{\rhd}, C, G' \rangle$ is accepted as an instance of MGS if and only if

$$\forall G'' \ (G' \in \mathcal{X}(C) \text{ and if } G'' \in \mathcal{X}(C) \text{ then it is not the case that } G'' \rhd_C G')$$

a test easily accomplished by a co-NP algorithm. For (b) membership is established from the relation $\langle \Gamma^{\rhd}, C, G' \rangle$ is accepted as an instance of PO if and only if: $G' \in \mathcal{X}(C)$ and for all $G'' \subseteq G$ for which $G'' \in \mathcal{X}(C)$:

$$\begin{array}{l}(\exists i \in C \text{ and } s_i \in G'' \cap G_i \text{ with } s_i \rhd_i r_i,\ \forall r_i \in G' \cap G_i) \quad \Rightarrow \\ (\exists j \in C \text{ and } r_j \in G' \cap G_j \text{ with } r_j \rhd_j s_j,\ \forall s_j \in G'' \cap G_j)\end{array}$$

An identical reduction from UNSAT serves to prove co-NP–hardness in both cases. We use a similar reduction from instances $\Phi(x_1, \ldots, x_n)$ of UNSAT as that of Theorem 2 but with $Ag = \{a_1, \ldots, a_n\}$, G_i, $\rhd_i$ as before and $\Psi(Ag, G)$ in $\Gamma^{\rhd}_{\Phi}$ given by

$$\left(\bigwedge_{i=1}^{n} a_i g_i^{\min} (\neg g_i^\top)(\neg g_i^\perp) \right) \ \vee \ \Phi(a_1(g_1^\top \vee \neg g_1^\perp), \ldots, a_n(g_n^\top \vee \neg g_n^\perp))$$

We set $C = Ag$ and $G' = \cup_{i=1}^{n} \{g_i^{\min}\}$. It is clearly the case that $G' \in \mathcal{X}(Ag)$. Furthermore by a similar argument to that of Theorem 2 $\mathcal{X}(Ag) = \{G'\}$ if and only if $\Phi(x_1, \ldots, x_n)$ is unsatisfiable. As we noted earlier $\mathcal{X}(C)$ containing a single goal set indicates that this set is maximal, i.e. in $\mu(C)$. Equally G' is Pareto Optimal if and only if $\Phi(x_1, \ldots, x_n)$ is unsatisfiable. ∎

We note that via near identical constructions to those of Theorem 2 and Corollary 2 it is a trivial matter to show that given $\langle \Gamma^{\rhd}, C \rangle$ the problem of deciding $\mathcal{X}(C) = \mu(C)$

is co-NP–complete: use the construction of Corollary 2 together with the observation that for this construction $|\mathcal{X}(Ag)| = 1$, (thence giving $\mathcal{X}(C) = \mu(C)$), if and only Φ is unsatisfiable.

Our observations earlier that the sets of Pareto Optimal goal sets for a given coalition C may be a *strict subset* of the set of maximally preferred coalitions for C, motivates the following decision problem

<u>MAXIMAL ONLY GOAL SET</u>: (MOGS)
Instance: QCGP $\langle G, Ag, G_1, \ldots, G_n, \Psi, \rhd_1, \ldots, \rhd_n \rangle$, $C \subseteq Ag$, $G' \subseteq G$
Answer: "Yes" if $G' \in \mu^{\rhd}(C)/\pi^{\rhd}(C)$.

Theorem 3 MOGS *is* D^p*–complete.*

Proof: For membership in D^p it suffices to observe that

$$L = \{\langle \Gamma^{\rhd}, C, G' \rangle : G' \in \mu^{\rhd}(C)/\pi^{\rhd}(C)\}$$

is formed by $L_1 \cap L_2$ where

$$L_1 = \{\langle \Gamma^{\rhd}, C, G' \rangle : G' \notin \pi^{\rhd}(C)\}$$

$$L_2 = \{\langle \Gamma^{\rhd}, C, G' \rangle : G' \in \mu^{\rhd}(C)\}$$

Since, $L_1 \in$ NP and $L_2 \in$ CO-NP, from Corollary 2, we deduce that MOGS $\in$ D^p.

To complete the proof we show that MOGS is D^p-hard, employing a reduction from SAT-UNSAT to do so. Instances of this comprise a pair $\langle \Phi_1(x_1, \ldots, x_n), \Phi_2(x_1, ..., x_n) \rangle$ of propositional formulae. An instance is accepted if $\Phi_1(x_1, \ldots, x_n)$ is satisfiable *and* $\Phi_2(x_1, \ldots, x_n)$ is unsatisfiable. D^p–completeness of SAT-UNSAT was proved in [8].

Given $\langle \Phi_1(x_1, \ldots, x_n), \Phi_2(x_1, \ldots, x_n) \rangle$ an instance of SAT-UNSAT we form the instance $\Gamma^{\rhd}_{\langle \Phi_1, \Phi_2 \rangle}$ as follows. Fix $Ag = \{a_1, \ldots, a_{n+1}\}$; $G_i = \{g_i^{\top}, g_i^{\perp}, g_i^{\min}\}$, with the preference relation

$$\rhd_i = \{g_i^{\top} \rhd_i g_i^{\min},\ g_i^{\perp} \rhd_i g_i^{\min}\}$$

As in earlier reductions we have $G' \sqsupset_C G''$ if and only if $G' \succ_C G''$.

The capability function, $\Psi(Ag, G)$ is defined as,

$$\left(\bigwedge_{i=1}^{n+1} a_i g_i^{\min} (\neg g_i^{\top})(\neg g_i^{\perp}) \right) \ \vee\ \Psi_1(Ag, G)\ \vee\ \Psi_2(Ag, G)$$

where $\Psi_1(Ag, G)$ is

$$a_{n+1} g_{n+1}^{\min} (\neg g_{n+1}^{\perp})(\neg g_{n+1}^{\top}) \Phi_1(a_1(g_1^{\top} \vee \neg g_1^{\perp}), \ldots, a_n(g_n^{\top} \vee \neg g_n^{\perp}))$$

and $\Psi_2(Ag, G)$ is

$$a_{n+1} g_{n+1}^{\top} (\neg g_{n+1}^{\perp})(\neg g_{n+1}^{\min}) \Phi_2(a_1(g_1^{\top} \vee \neg g_1^{\perp}), \ldots, a_n(g_n^{\top} \vee \neg g_n^{\perp}))$$

Finally we set $C = Ag$ and $G' = \cup_{i=1}^{n+1}\{g_i^{\min}\}$ to form the instance of MOGS. We claim that $G' \in \mu(Ag)/\pi(Ag)$ if and only if Φ_1 is satisfiable and Φ_2 is unsatisfiable.

Suppose that $G' \in \mu(Ag)/\pi(Ag)$. It is easily seen that $G' \in \mathcal{X}(Ag)$ and from the assumption that $G' \in \mu(Ag)$ it must the case that given any $G'' \subseteq \cup_{i=1}^{n+1}\{g_i^{\top}, g_i^{\min}\}$ for which $G_i \cap G'' \neq \emptyset$ for each i, we have $G'' \notin \mathcal{X}(Ag)$. Consider any such set containing $g_{n+1}^{\top}$ but not containing either $g_{n+1}^{\min}$ or $g_{n+1}^{\perp}$. Then $\Psi[Ag, G'']$ is equivalent to the value of

$$\Phi_2(g_1^{\top} \vee \neg g_1^{\perp}, \ldots, g_n^{\top} \vee \neg g_n^{\perp})[G'']$$

which we know to be $\perp$. It follows that $G' \in \mu(Ag)$ implies that Φ_2 is unsatisfiable. We have assumed, in addition, that $G' \notin \pi(Ag)$ and that there is therefore some goal set G'' which for at least one $a_i \in Ag$ attains a more preferred goal for a_i without reducing the preference of any other a_i. Furthermore, from $G' \in \mu(Ag)$, it cannot be the case that G'' improves the goal attained for *every* a_i: since then $G'' \rhd_{Ag} G'$ contradicting $G' \in \mu(Ag)$. Consider the maximal goal attained by a_{n+1} in G''. This cannot be $g_{n+1}^{\perp}$ since $\Psi[Ag, G''] = \perp$ if $g_{n+1}^{\perp} \in G''$. Similarly it cannot be $g_{n+1}^{\top}$, for again $\Psi[Ag, G''] = \perp$ by virtue of the fact the $G' \in \mu(Ag)$ indicates Φ_2 is unsatisfiable. It follows that a_{n+1} can only achieve $g_{n+1}^{\min}$ within any $G'' \in \mathcal{X}(Ag)$, in particular in the G'' that improves the goal attained by some a_i, we have $g_{n+1}^{\min} \in G''$. For such G'', $\Psi[Ag, G'']$ reduces to,

$$\Phi_1(g_1^{\top} \vee \neg g_1^{\perp}, \ldots, g_n^{\top} \vee \neg g_n^{\perp})[G'']$$

which evaluates to $\top$ since $G'' \notin \pi(Ag)$. In this way from G'' we construct a satisfying assignment for Φ_1.

For the reverse direction suppose that Φ_1 is satisfiable and Φ_2 unsatisfiable. We show that this implies $G' \in \mu(C)/\pi(C)$. Let $\langle \alpha_1, \ldots, \alpha_n \rangle$ be a satisfying instantiation for Φ_1. It is easily checked that the goal set

$$G'' = \{g_{n+1}^{\min}\} \cup \bigcup_{i=1}^{n} \{g_i^{\alpha_i}\}$$

is in $\mathcal{X}(Ag)$ and indicates that $G' \notin \pi(Ag)$. Similarly, if we consider any G'' for which it is potentially the case that $G'' \rhd_{Ag} G'$, we see that $g_{n+1}^{\top}$ must belong to G'', and now from the assumption that Φ_2 is unsatisfiable $G'' \notin \mathcal{X}(Ag)$ and hence $G' \in \mu(Ag)$.

This completes the argument that $\langle \Phi_1, \Phi_2 \rangle$ is a positive instance of SAT-UNSAT if and only if $\langle \Gamma^{\rhd}_{\langle \Phi_1, \Phi_2 \rangle}, Ag, G' \rangle$ is a positive instance of MOGS and the proof that MOGS is D^p-complete. ∎

To conclude this section we consider the following problem,

CORE COMPLETENESS: (CC)
Instance: QCGP $\langle G, Ag, G_1, \ldots, G_n, \Psi, \rhd_1, \ldots, \rhd_n \rangle$, $C \subseteq Ag$
Answer: "Yes" if $\kappa^{\rhd}(C) = \mathcal{X}(C)$ and $\mathcal{X}(C) \neq \emptyset$.

Thus, CC is concerned with the question of whether C is successful *and* every feasible goal set that satisfies each member of C is in the core.

We prove in Theorem 4 below that CC is D^p–complete. While a single reduction (from SAT-UNSAT) suffices to establishes D^p–hardness for the complexity of deciding either $\kappa^{\succ}(C) = \mathcal{X}(C)$ and $\kappa^{\sqsupset}(C) = \mathcal{X}(C)$ different approaches are required to show that these variants can be decided by D^p algorithms.

To this end we introduce the following definition.

Definition 11 *Given a* QCGP

$$\Gamma^{\triangleright} = \langle G, Ag, G_1, \ldots, G_n, \Psi, \triangleright_1, \ldots, \triangleright_n \rangle$$

the propositional formulae $\Xi_i^{\succ}(G,H)$ *and* $\Xi_i^{\sqsupset}(G,H)$ *whose propositional variables are*

$$\begin{aligned} G &= \{g_1, g_2, \ldots, g_m\} \\ H &= \{h_1, h_2, \ldots, h_m\} \end{aligned}$$

are defined for each $a_i \in Ag$ *as*

$$\Xi_i^{\succ}(G,H) = \bigwedge_{r \,:\, g_r \in G_i} \left(\neg g_r \;\vee \bigvee_{s \,:\, g_s \triangleright_i g_r} h_s \right)$$

$$\Xi_i^{\sqsupset}(G,H) = \bigvee_{r \,:\, g_r \in G_i} \left(h_r \;\wedge \bigwedge_{s \,:\, \neg(g_r \triangleright_i g_s)} (\neg g_s) \right)$$

The following technical lemma is central to our subsequent argument that $\mathrm{CC} \in \mathrm{D}^p$.

Lemma 6 *Let* $\Gamma^{\triangleright}$ *be a* QCGP, C, $D \subseteq Ag$ *and* G', G'' *be subsets of* G. *Furthermore let* $\Psi'(B,H)$ *be a copy of the formula* $\Psi(Ag,G)$ *in which the variable* $a_i \in Ag$ *is renamed* b_i *and the variable* $g_j \in G$ *renamed* h_j. *Finally, let* $\Theta(Ag,B,G,H)$ *be the formula*

$$\bigwedge_{i=1}^{n} \left(a_i \Rightarrow \bigvee_{g \in G_i} g \right) \Psi(Ag,G) \bigwedge_{i=1}^{n} \left(b_i \Rightarrow \bigvee_{h \in H_i} h \right) \Psi'(B,H) \bigwedge_{i=1}^{n} (\neg a_i \Rightarrow \neg b_i)$$

a. $D \subseteq C$, $G' \in \mathcal{X}(C)$ *and* $G'' \succ_D G'$ *if and only if* $\Delta^{\succ}[C,D,G',G''] = \top$, *where* $\Delta^{\succ}(Ag,B,G,H)$ *is*

$$\Theta(Ag,B,G,H) \bigwedge_{i=1}^{n} (\, \neg b_i \;\vee \Xi_i^{\succ}(G,H)\,)$$

with C *used to instantiate the variables* Ag, D *to instantiate* B, G' *for* G, *and* G'' *for* H, *i.e.* $a_i = \top$ *if and only if* $a_i \in C$, $b_i = \top$ *if and only if* $a_i \in D$, $g_j = \top$ *if and only if* $g_j \in G'$, *and* $h_j = \top$ *if and only if* $g_j \in G''$.

b. $D \subseteq C$*,* $G' \in \mathcal{X}(C)$ *and* $G'' \sqsupset_D G'$ *if and only if* $\Delta^{\sqsupset}[C, D, G', G''] = \top$*, where* $\Delta^{\sqsupset}(Ag, B, G, H)$ *is*

$$\Theta(Ag, B, G, H) \bigwedge_{i=1}^{n} (\ \neg b_i \ \vee \Xi_i^{\sqsupset}(G, H)\ )$$

with the instantiation of $\langle Ag, B, G, H\rangle$ *by* $\langle C, D, G', G''\rangle$ *as before.*

Proof: Let $C, D \subseteq Ag$ and G', G'' be subsets of G. The instantiations of the variables Ag and B induced by C and D are such that $\Theta[C, D, G', G''] = \top$ if and only if $G' \in \mathcal{X}(C)$, $G'' \in \mathcal{X}(D)$ and $D \subseteq C$. To see this it suffices to observe that $G' \in \mathcal{X}(C)$ if and only if,

$$\left(\bigwedge_{i=1}^{n} \left(a_i \Rightarrow \bigvee_{g \in G_i} g\right) \Psi(Ag, G)\right) [C, G'] = \top$$

and, similarly, $G'' \in \mathcal{X}(D)$ if and only if

$$\left(\bigwedge_{i=1}^{n} \left(b_i \Rightarrow \bigvee_{h \in H_i} h\right) \Psi'(B, H)\right) [D, G''] = \top$$

In addition $D \subseteq C$ if and only if

$$\left(\bigwedge_{i=1}^{n} (\neg a_i \Rightarrow \neg b_i)\right) [C, D] = \top$$

Since the conjunction of these expressions is equivalent to $\Theta[C, D, G', G'']$ it follows that $\Theta[C, D, G', G''] = \top$ if and only if $G' \in \mathcal{X}(C)$, $G'' \in \mathcal{X}(D)$ and $D \subseteq C$ as claimed.

It remains to deal with the conditions governing $G'' \rhd_D G'$.

a. Assuming $D \subseteq C$, $G' \in \mathcal{X}(C)$ and $G'' \in \mathcal{X}(D)$ we need to show that $G'' \succ_D G'$ if and only if

$$\left(\bigwedge_{i=1}^{n} (\ \neg b_i \ \vee \Xi_i^{\succ}(G, H)\ )\right) [C, D, G', G''] = \top$$

This holds if and only if for each $a_i \in D$, i.e. for which $b_i = \top$, the term $\Xi_i^{\succ}[G', G'']$ takes the value $\top$. We have, $\Xi_i^{\succ}[G', G'']$ is equivalent to

$$\bigwedge_{r\ :\ g_r \in G_i} \left(\neg g_r \ \vee \bigvee_{s\ :\ g_s \rhd_i g_r} h_s\right) [G', G'']$$

which evaluates to $\top$ if and only if: for each $g_r \in G' \cap G_i$ there is some $g_s \in G'' \cap G_i$ for which $g_s \rhd_i g_r$. This, however, is exactly the definition of $G'' \succ_D G'$ given the preconditions $D \subseteq C$, and $G'' \in \mathcal{X}(D)$. Hence, we deduce that

$$\left(\Theta(Ag, B, G, H) \bigwedge_{i=1}^{n} (\ \neg b_i \ \vee \Xi_i^{\succ}(G, H)\ )\right) [C, D, G', G''] = \top$$

if and only if $D \subseteq C$, $G' \in \mathcal{X}(C)$ and $G'' \succ_D G'$.

b. Again, assuming $D \subseteq C$, $G' \in \mathcal{X}(C)$ and $G'' \in \mathcal{X}(D)$ we need to show that $G'' \sqsupset_D G'$ if and only if

$$\left(\bigwedge_{i=1}^{n} (\ \neg b_i \ \vee \Xi_i^{\sqsupset}(G,H)\ )\right)[C,D,G',G''] = \top$$

i.e., $\Xi_i^{\sqsupset}[G',G''] = \top$ for each $a_i \in D$. From the definition of $\Xi_i^{\sqsupset}$,

$$\left(\bigvee_{r\ :\ g_r \in G_i} \left(h_r \ \wedge \bigwedge_{s\ :\ \neg(g_r \rhd_i g_s)} (\neg g_s)\right)\right)[G',G'']$$

evaluates to $\top$ if and only if for some $g_r \in G'' \cap G_i$ there are *no* elements $g_s \in G' \cap G_i$ for which either $g_s \rhd_i g_r$ or *i is indifferent* between g_r and g_s. In other words $G'' \cap G_i$ contains some goal g_r which i prefers to every goal $g_s \in G' \cap G_i$. This, given the assumptions $G' \in \mathcal{X}(C)$, $G'' \in \mathcal{X}(D)$ and $D \subseteq C$, defines exactly the condition $G'' \sqsupset_D G'$. We deduce that

$$\left(\Theta(Ag,B,G,H) \bigwedge_{i=1}^{n} (\ \neg b_i \ \vee \Xi_i^{\sqsupset}(G,H)\ )\right)[C,D,G',G''] = \top$$

if and only if $D \subseteq C$, $G' \in \mathcal{X}(C)$ and $G'' \sqsupset_D G'$.

∎

Theorem 4 CC *is* D^p*-complete.*

Proof: Since the proof of membership is non-trivial, we first show that CC is D^p-hard again employing a reduction from the problem SAT-UNSAT.

Let $\langle \Phi_1(x_1,\ldots,x_n), \Phi_2(x_1,\ldots,x_n)\rangle$ be an instance of SAT-UNSAT. The instance $\langle \Gamma^{\rhd}_{\langle \Phi_1,\Phi_2\rangle}, C\rangle$ of CC has $Ag = \{a_1,\ldots,a_n,a_{n+1}\}$, $G_i = \{g_i^{\top}, g_i^{\perp}, g_i^{\max}\}$, and $G = \cup_{i=1}^{n+1} G_i$. The preference relationship is $\rhd_i = \{g_i^{\max} \rhd_i g_i^{\perp},\ g_i^{\max} \rhd_i g_i^{\top}\}$. We define $\Psi(Ag,G)$ as

$$\left(\bigwedge_{i=1}^{n+1} a_i \ \Rightarrow\ \Psi_1(Ag,G)\right) \wedge \left(\bigvee_{i=1}^{n+1} \neg a_i \ \Rightarrow \Psi_2(Ag,G)\right)$$

in which $\Psi_1(Ag,G)$ is

$$\Phi_1((g_1^{\top} \vee \neg g_1^{\perp}),\ldots,(g_n^{\top} \vee \neg g_n^{\perp})) \wedge g_{n+1}^{\top} \wedge \bigwedge_{i=1}^{n+1} (\neg g_i^{\max})(g_i^{\top} \oplus g_i^{\perp})$$

and $\Psi_2(Ag,G)$ is

$$\Phi_2(a_1,\ldots,a_n) \wedge \bigwedge_{i=1}^{n+1} (a_i \Rightarrow g_i^{\max})$$

Finally we set $C = Ag$ to form the instance of CC. We first observe that since the goals $g_i^\top$ and $g_i^\perp$ are preferred equally by a_i if $G' \subseteq G$ is such that $G' \in \mathcal{X}(Ag)$ then $G' \in \mu(Ag)$. We claim that $\kappa^\triangleright(Ag) = \mathcal{X}(Ag) \neq \emptyset$ for this instance if and only if Φ_1 is satisfiable and Φ_2 is unsatisfiable.

Suppose the first of these holds. Noting that for each i and any $G' \in \mathcal{X}(Ag)$ it must be the case that $|G_i \cap G'| = 1$, let

$$G' = \{g_1^{\alpha_1}, g_2^{\alpha_2}, \ldots, g_{n+1}^\top\} \in \mathcal{X}(Ag) = \mu(Ag)$$

where $\alpha_i \in \{\top, \perp\}$. It is easy to check that, if β_i is the value taken by $g_i^\top \vee \neg g_i^\perp$ under G' then $\beta_i = \alpha_i$. Since we have assumed that $G' \in \mathcal{X}(Ag)$ it follows that $\Psi[Ag, G']$ evaluates to $\top$. Inspecting the definition of $\Psi(Ag, G)$ we see that $\Psi[Ag, G']$ take the value of $\Phi_1(\alpha_1, \alpha_2, \ldots, \alpha_n)$, and thus we deduce that if $\mathcal{X}(Ag) \neq \emptyset$ then Φ_1 is satisfiable.

The assumption that $\langle \Gamma^\triangleright_{\langle \Phi_1, \Phi_2 \rangle}, Ag \rangle$ is a positive instance of CC, indicates that any element G' of $\mathcal{X}(Ag)$ must also belong to $\kappa(Ag)$. We now argue that this implies Φ_2 is unsatisfiable. Consider any strict subset C of Ag. For such coalitions and goal set $G' \subseteq G$, the formula $\Psi(Ag, G)$ is equivalent to the expression

$$\left(\Phi_2(a_1, \ldots, a_n) \wedge \bigwedge_{i=1}^{n+1} (a_i \Rightarrow g_i^{\max}) \right) [C, G']$$

If we consider any $C \subset Ag$ (including $C = \emptyset$) it is certainly the case that the goal set $G'' = \cup_{i=1}^{n+1} \{g_i^{\max}\}$ satisfies each $a_i \in C$ with a goal that is strictly preferred to $g_i^{\alpha_i} \in G'$, Thus, $\Psi[C, G'']$ takes the value $\Phi_2(\beta_1, \ldots, \beta_n)$ where $\beta_i = \top$ if and only if $a_i \in C \subseteq Ag/\{a_{n+1}\}$: the assumption that $G' \in \kappa(Ag)$ now indicates that Φ_2 must be unsatisfiable.

In total if $\langle \Gamma^\triangleright_{\langle \Phi_1, \Phi_2 \rangle}, Ag \rangle$ is a accepted as an instance of CC then $\langle \Phi_1, \Phi_2 \rangle$ is accepted as an instance of SAT-UNSAT.

For the reverse direction, suppose that Φ_1 is satisfiable and Φ_2 unsatisfiable. Let $\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle$ be any satisfying assignment of Φ_1. The goal set $\{g_1^{\alpha_1}, \ldots, g_n^{\alpha_n}, g_{n+1}^\top\}$ is easily seen to belong to $\mathcal{X}(Ag)$, which is thereby non-empty. Since Φ_2 is unsatisfiable, for any $C \subset Ag$, and goal set $G' \subseteq G$, the formula $\Psi(Ag, G)$ will reduce to

$$\left(\Phi_2(a_1, \ldots, a_n) \wedge \bigwedge_{i=1}^{n+1} (a_i \Rightarrow g_i^{\max}) \right) [C, G']$$

which is equivalent to $\perp$. In summary, if Φ_1 is satisfiable and Φ_2 unsatisfiable then $\mathcal{X}(Ag) \neq \emptyset$ and every feasible and satisfying goal set for Ag belongs to $\kappa(Ag)$.

To complete the proof we show that $\text{CC} \in \text{D}^p$. Given

$$\langle \langle G, Ag, G_1, \ldots, G_n, \Psi, \triangleright_1, \ldots, \triangleright_n \rangle, C \rangle$$

as an instance of CC, let α_C be the instantiation $a_i = \top$ if and only if $a_i \in C$ and $\Psi^{(C)}(G)$ be the propositional formula defined via

$$\Psi^{(C)}(G) = \Psi(\alpha_C, G) \wedge \bigwedge_{a_i \in C} \left(\bigvee_{g \in G_i} g \right)$$

It is clearly the case that the formula $\Psi^{(C)}(G)$ can be constructed in time which is polynomial in the size of the instance of CC, and that, furthermore, $\mathcal{X}(C) \neq \emptyset$ if and only if $\Psi^{(C)}(G)$ is satisfiable.

Now consider the formula $\Psi^{(pref)}(B, G, H)$ in which $B = \{b_1, \ldots, b_n\}$ and $H = \{h_1, h_2, \ldots, h_m\}$ (i.e. a renaming of the propositional variables in $Ag \cup G$). This formula is defined as,

$$\neg\Psi^{(C)}(G) \ \vee \ \left(\bigvee_{i \notin C} b_i \right) \ \vee \ \neg\Delta^{\triangleright}(\alpha_C, B, G, H)$$

Consider under what conditions $\Psi^{(pref)}[D, G', G'']$ evaluates to $\bot$: if the subset $G' \notin \mathcal{X}(C)$ then $\Psi^{(pref)}[D, G', G''] = \top$ irrespective of the choice of $D \subseteq Ag$ or $G'' \subseteq G$; if $D \not\subseteq C$ then, again, $\Psi^{(pref)}[D, G', G''] = \top$ irrespective of G' and G'', since the term $(\vee_{i \notin C} b_i)[D]$ evaluates to $\top$.

These leave the case $G' \in \mathcal{X}(C)$ *and* $D \subseteq C$, in which event $\Psi^{(pref)}[D, G', G''] = \bot$ if and only if

$$\Delta^{\triangleright}[C, D, G', G''] = \top$$

From Lemma 6(a,b) $\Delta^{\triangleright}[C, D, G', G''] = \top$ if and only if $D \subseteq C$, $G' \in \mathcal{X}(C)$ and $G'' \triangleright_D G'$, and thus $G' \notin \kappa^{\triangleright}(C)$ since G'' and $D \subseteq C$ attest to its elimination from the core.

This gives the following decision procedure in D^p: test if $\Psi^{(C)}(G)$ is *satisfiable* – guaranteeing $\mathcal{X}(C) \neq \emptyset$ if this test succeeds; then test that $\neg(\Psi^{(pref)}(\alpha_C, B, G, H))$ is *unsatisfiable*, i.e. that $\Psi^{(pref)}(\alpha_C, B, G, H)$ is a propositional tautology. If this test is positive then $\kappa^{\triangleright}(C) = \mathcal{X}(C)$: for if $[D, G', G'']$ were a falsifying instantiation of $\Psi^{(pref)}(\alpha_C, B, G, H)$, i.e. a satisfying instantiation of $\neg\Psi^{(pref)}(\alpha_C, B, G, H)$, then from our analysis of the previous paragraph, $D \subseteq C$ and $G' \in \mathcal{X}(C)$ and $G'' \triangleright_D G'$ so that G'' eliminates G' from $\kappa^{\triangleright}(C)$. ∎

5 Conclusions

We have introduced Qualitative Coalitional Games with Preferences, a variation of Qualitative Coalitional Games in which agents are assumed to have preferences over goals. We defined a number of solution concepts for such games, established some properties of these solution concepts, and investigated their computational complexity.

Probably closest to our work is that of Conitzer and Sandholm, who investigated the complexity of determining non-emptiness of the core in a subclass of conventional coalitional games [2]. By assuming superadditivity, they were able to derive a succinct representation of characteristic functions, and proved that determining non-emptiness of the core assuming this representation was NP-complete, irrespective of whether or not utility was transferable. Bilbao and colleagues derived a number of complexity results for other subclasses of cooperative games by interpreting these games over combinatorial structures of various kinds (e.g., minimal spanning trees) [1].

A number of approaches to coalition structure generation and related problems have been described in the literature [12, 13, 11]. For example, Shehory and Kraus developed algorithms for coalition structure formation in which agents were modelled as having different capabilities, and were assumed to benevolently desire some overall task to be accomplished, where this task had some complex structure [12, 13]. Sandholm and colleagues developed algorithms to find optimal coalition structures within some given ratio bound k of optimal [11].

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