Extremal Behaviour in Multiagent Contract Negotiation

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Abstract

We consider models of contract negotiation for settings in which a set of agents exchange resources in order to optimise their individual holdings. The schemes discussed relate to well-known negotiation protocols proposed in earlier work and we consider a number of alternative notions of "contract rationality" covering both quantitative measures, e.g. cooperative and individual rationality and more qualitative forms, e.g. Pigou-Dalton transfers. While it is known that imposing particular rationality and structural restrictions on the form of contracts may render these unable to realise every rational deal, in this paper we address the issue of how many deals may be involved in restricted rational contracts to implement a particular deal when it *is* possible to do so. We construct examples showing the optimal path length may be exponential (in the number of resources m). We further show that k agent rational negotiations may achieve in a single deal, exchanges that require exponentially many $k-1$ agent rational negotiations.

1. Introduction

Mechanisms for negotiating allocation of resources within a group of agents form an important body of work within the study of multiagent systems. Typical abstract models derive from game-theoretic perspectives in economics and among the issues that have been addressed are strategies that agents may use to negotiate, e.g. [10, 12, 14], and protocols for negotiation in agent societies, e.g. [3, 4, 5, 11].

The setting we are concerned with is encapsulated in the following definition.

Definition 1 *A* resource allocation setting *is defined by a triple* $\langle \mathcal{A}, \mathcal{R}, \mathcal{U} \rangle$ *where*

$$\mathcal{A} = \{A_1, A_2, \ldots, A_n\} \quad ; \quad \mathcal{R} = \{r_1, r_2, \ldots, r_m\}$$

are, respectively, a set of (at least two) agents and a collection of (non-shareable) resources. A utility function*, u, is a mapping from subsets of $\mathcal{R}$ to rational values. Each agent $A_i \in \mathcal{A}$ has associated with it a particular utility function u_i, so that $\mathcal{U}$ is $\langle u_1, u_2, \ldots, u_n \rangle$. An* allocation *$P$ of $\mathcal{R}$ to $\mathcal{A}$ is a partition $\langle P_1, P_2, \ldots, P_n \rangle$ of $\mathcal{R}$. The value $u_i(P_i)$ is called the* utility *of the resources assigned to A_i.*

Within the very general context of Definition 1, a number of issues arise stemming from the observation that it is unlikely that some *initial allocation* will be seen as satisfactory either with respect to the views of all agents in the system or with respect to divers global considerations. Thus, by proposing changes to the initial assignment individual agents seek to negotiate a 'better' allocation. This scenario raises two immediate questions: how to evaluate a given partition and thus have a basis for forming improved or optimal allocations; and what restrictions, if any, should be imposed on the form that proposed exchanges (*deals*) may take. With respect to the first of these, several notions of value have been mooted.

Some of the more widely studied of these in multiagent settings, e.g. [6, 7, 8, 10, 13, 14] are reviewed in the following,

Definition 2 *Let $\langle \mathcal{A}, \mathcal{R}, \mathcal{U} \rangle$ be a resource allocation setting. For an allocation P of $\mathcal{R}$ among $\mathcal{A}$,*

a. *The* utilitarian social welfare *of P, denoted $\sigma_u(P)$ is the value $\sum_{i=1}^{n} u_i(P_i)$.*

b. *The* egalitarian social welfare *of P, denoted $\sigma_e(P)$ is the value $\min_{1 \leq i \leq n} u_i(P_i)$.*

c. *The* elitist (or selfish) social welfare *of P, denoted $\sigma_s(P)$ is the value $\max_{1 \leq i \leq n} u_i(P_i)$.*

Thus agents may seek to negotiate allocations that maximise any single one, or indeed, some combination of these.

In maximising σ_u, say, the focus is on optimising a quantitative criterion defined in terms of the allocation. There are, however, also, *qualitative* criteria that may be employed to decide whether a given allocation can be improved.

Definition 3 *For a resource allocation setting $\langle \mathcal{A}, \mathcal{R}, \mathcal{U} \rangle$ and allocation P of $\mathcal{R}$ among $\mathcal{A}$,*

a. *P is* Pareto Optimal *if for all allocations $Q = \langle Q_1, \ldots, Q_n \rangle$ should it be the case that for some i, $u_i(Q_i) > u_i(P_i)$ then there is some j for which $u_j(Q_j) < u_j(P_j)$, i.e. if P is Pareto Optimal then any allocation, Q, that strictly increases the utility enjoyed by some agent i, does so at the cost of some other agent j being worse off.*

b. *P is* envy free *if for each distinct pair, i and j, $u_i(P_i) \geq u_i(P_j)$, i.e. in an envy-free allocation, A_i values what it has been given at least as highly as it would value the resources granted to any other agent by the allocation.*

Turning to our second question – the restrictions that may be imposed on the form that proposed contracts may take – we, first, recall the concept of a *deal.*

Definition 4 *Let $\langle \mathcal{A}, \mathcal{R}, \mathcal{U} \rangle$ be a resource allocation setting. A* deal *is a pair $\langle P, Q \rangle$ where $P = \langle P_1, \ldots, P_n \rangle$ and $Q = \langle Q_1, \ldots, Q_n \rangle$ are distinct partitions of $\mathcal{R}$. The effect of implementing the deal $\langle P, Q \rangle$ is that the allocation of resources specified by P is replaced with that specified by Q. Following the notation of [8] for a deal $\delta = \langle P, Q \rangle$, we use $\mathcal{A}^\delta$ to indicate the subset of $\mathcal{A}$ involved in the deal, i.e. $A_k \in \mathcal{A}^\delta$ if and only if $P_k \neq Q_k$.*

From the standpoint of an individual agent, A_i, a given deal $\langle P, Q \rangle$ may have three different outcomes: $u_i(P_i) < u_i(Q_i)$, i.e. A_i values the allocation Q_i as superior to P_i; $u_i(P_i) = u_i(Q_i)$, i.e. A_i is indifferent between P_i and Q_i; and $u_i(P_i) > u_i(Q_i)$, i.e. A_i is worse off after the exchange. It is also the case that from any given *initial* allocation P_0, for each of the concepts of optimality presented in Definitions 2 and 3, there is an allocation P_{opt} that attains the particular optimality criterion, and thus if P_0 is sub-optimal then the single deal $\langle P_0, P_{opt} \rangle$ will achieve an optimal allocation in a single round of negotiation.[1] There are, however, a number of reasons whereby negotiation mechanisms that simply identify and propose such deals $\langle P_0, P_{opt} \rangle$ could be seen as "unreasonable".

1. We have assumed here that the agents involved share a common view of the concept of optimality involved.

The total number of possible allocations is n^m, and so for moderately large numbers of resources (m) there are too many feasibly to enumerate (even when $n = 2$). In addition it may not be possible to implement the optimising contract in a *single* transaction even if only two agents are involved: the environment in which the trading process is implemented may not be suited to handling transactions in which large numbers of resources are involved; similarly the protocol used for negotiation and contract description may not allow arbitrarily large numbers of resources to be dealt with. In addition, where global optima such as utilitarian social welfare are to be maximised, there is the question of what incentive there is for any agent to accept a deal $\langle P, Q\rangle$ under which they are left with a less valuable resource holding. The standard approach to this latter question is to introduce the notion of a *pay-off* function, i.e. in order for A_i to accept an exchange $\langle P, Q\rangle$ under which $u_i(Q_i) < u_i(P_i)$ A_i receives some payment sufficient to compensate for the resulting loss in utility. Of course such compensation must be made by other agents in the system who in providing it do not wish to pay in excess of any gain in resource. In defining notions of 'pay-off' the interpretation is that in any transaction each agent A_i makes a payment, π_i: if $\pi_i < 0$ then A_i is given $-\pi_i$ in return for accepting a contract; if $\pi_i > 0$ then A_i contributes π_i to the amount to be distributed among those agents whose pay-off is negative.

Formally, such a notion of 'sensible transfer' could be captured by the concept of *individual rationality.*

Definition 5 *A deal $\langle P, Q\rangle$ is said to be* individually rational *(IR) if there is a* pay-off vector $\pi = \langle \pi_1, \pi_2, \ldots, \pi_n\rangle$ *satisfying,*

a. $\sum_{i=1}^{n} \pi_i = 0.$

b. $u_i(Q_i) - u_i(P_i) > \pi_i$, *for each agent A_i,* except that π_i *may be* 0 *if* $P_i = Q_i$, *i.e. should the deal $\langle P, Q\rangle$ leave the agent A_i with no change in its resource then it is not* required *that A_i be rewarded (have $\pi_i < 0$).*

A result linking utilitarian social welfare and individual rationality, which we make use of subsequently, is given in recent work of Endriss *et al.*[7, Lemma 1]

Fact 1 *A deal $\langle P, Q\rangle$ is IR if and only if $\sigma_u(Q) > \sigma_u(P)$.*

A number of alternative notions of "rationality" of interest within so-called *money free* settings (so that compensatory payment for a loss in utility is not an option) have been considered in recent work of Endriss and Maudet [8].

Definition 6 *Let $\delta = \langle P, Q\rangle$ be a deal.*

a. δ *is* cooperatively rational *if for every agent, A_i, $u_i(Q_i) \geq u_i(P_i)$ and there is at least one agent, A_j, for whom $u_j(Q_j) > u_j(P_j)$.*

b. δ *is* equitable *if* $\min_{i\in\mathcal{A}^\delta} u_i(Q_i) > \min_{i\in\mathcal{A}^\delta} u_i(P_i)$.

c. δ *is a* (Simple) Pigou-Dalton transfer *if* $\mathcal{A}^\delta = \{i, j\}$, $u_i(P_i) + u_j(P_j) = u_i(Q_i) + u_j(Q_j)$ *and* $|u_i(Q_i) - u_j(Q_j)| < |u_i(P_i) - u_j(P_j)|$ *(where $|\ldots|$ is absolute value).*

Individual rationality is one condition that we may wish to be met with respect to "acceptable" deals $\langle P, Q\rangle$ and is prescribed in terms of the *outcome* of a deal rather than in terms of its *structure*, i.e. an individually rational deal may still involve a complex exchange of resources among a number of agents.

An important contribution to the analysis of deals which are required to satisfy given structural criteria is presented in Sandholm [13] and is based around Smith's *Contract Net* model from [15].

The *contract net* (in its most general instantiation) for scenarios of m resources distributed among n agents, is the complete directed graph with n^m vertices (each of which is associated with a distinct allocation). Thus representing a possible deal $\langle P, Q\rangle$ as an edge directed from the vertex labelled with P to that labelled Q.

In practice, it is unrealistic to consider this complete graph structure, and instead one considers the substructure resulting by including only those directed edges that correspond to particular contract types. Thus, from [13] the following classes of contract type are introduced.

Definition 7 *([13]) Let $\delta = \langle P, Q\rangle$ be a deal involving an allocation of $\mathcal{R}$ among $\mathcal{A}$.*

a. δ is a one contract *(O-contract) if*

- *O1.* $\mathcal{A}^\delta = \{i, j\}$.
- *O2. There is a unique resource $r \in P_i \cup P_j$ for which $Q_i = P_i \cup \{r\}$ and $Q_j = P_j - \{r\}$ (with $r \in P_j$) or $Q_j = P_j \cup \{r\}$ and $Q_i = P_i - \{r\}$ (with $r \in P_i$)*

b. δ is a cluster contract *(C-contract) if*

- *C1.* $\mathcal{A}^\delta = \{i, j\}$.
- *C2. There is a unique* set *$S \subseteq P_i$ or $S \subseteq \cup P_j$ with $|S| > 1$ and $Q_i = P_i \cup S$ and $Q_j = P_j - S$ (with $S \subseteq P_j$) or $Q_j = P_j \cup S$ and $Q_i = P_i - S$ (with $S \subseteq P_i$)*

c. δ is a swap contract *(S-contract) if*

- *S1.* $\mathcal{A}^\delta = \{i, j\}$.
- *S2. There are unique resources $r_i \in P_i$ and $r_j \in P_j$ for which $Q_i = P_i - \{r_i\} \cup \{r_j\}$ and $Q_j = P_j - \{r_j\} \cup \{r_i\}$.*

d. δ is a multiagent contract *(M-contract) if*

- *M1.* $|\mathcal{A}^\delta| \geq 3$.
- *M2.* $\cup_{j \in \mathcal{A}^\delta} Q_j = \cup_{j \in \mathcal{A}^\delta} P_j$.

Thus, O-contracts involve the transfer of *exactly one* resource from a particular agent's allocation to another; C-contracts the transfer of *at least two* resources; S-contracts an exchange of single resources between two allocations; and M-contracts a reallocation of the combined holding of *at least three* agents among themselves.

In describing the process by which a particular deal can be effected, an important idea is that of a *contract path*: in informal terms this is a *directed path* between distinct allocations in the underlying contract net graph. This leads to the following

Definition 8 *Let* $\delta = \langle P, Q \rangle$ *be a deal. A* contract path realising δ *is a sequence of allocations*

$$\Delta = \langle P^{(1)},\ P^{(2)}\ ,\ldots,\ P^{(t-1)},\ P^{(t)} \rangle$$

in which $P = P^{(1)}$ *and* $P^{(t)} = Q$*. The* length *of* Δ*, denoted* $|\Delta|$ *is* $t-1$*, i.e. the number of* deals *in* Δ*. For* Φ *a predicate over distinct pairs of allocations, a contract path realising* $\langle P, Q \rangle$ – $\langle P^{(1)},\ P^{(2)}\ ,\ldots,\ P^{(t-1)},\ P^{(t)} \rangle$ – *is a* Φ*-path if for each* $1 \leq i < t$*,* $\Phi(P^{(i)}, P^{(i+1)})$ *holds.*

We say that Φ *is* complete *if any deal* δ *may be realised by a* Φ*-path. We, further, say that* Φ *is* complete with respect to Ψ deals *(where* Ψ *is a predicate over distinct pairs of allocations) if any deal* δ *for which* $\Psi(\delta)$ *holds may be realised by a* Φ*-path.*

The main interest in earlier studies of these ideas has been in areas such as identifying necessary and/or sufficient conditions on deals to be complete with respect to particular criteria, e.g. [13]; and in establishing "convergence" and termination properties, e.g. [7, 8] consider deal types that will eventually result in allocations that are Pareto optimal.

In [13], Sandholm examines how restricting deals to contracts of particular types, e.g. with $\Phi(P, Q) = \top$ if and only if $\langle P, Q \rangle$ is an O-contract, may affect the existence of paths to realise deals. Of particular interest, from the viewpoint of heuristics for exploring the contract net graph, are cases where $\Phi(P, Q) = \top$ if and only if the deal $\langle P, Q \rangle$ is individually rational. For the classes of contract given in Definition 7 the following are known:

Theorem 1

a. *O-contracts are complete.*

b. *IR O-contracts are* not *complete with respect to IR deals.*

c. *For each $T \in \{C, S, M\}$ T-contracts are not complete.*

d. *For each $T \subseteq \{O, C, S, M\}$ IR T-contracts are not complete with respect to IR deals.*

One aspect that appears to have been largely neglected, however, is that of complexity and algorithmic issues. Some preliminary investigation of such questions is presented in Dunne *et al.* [6] whose results are, briefly, summarised in

Theorem 2 *Even in the case* $n = 2$ *and with monotone utility functions, i.e.* $S \subseteq T$ *implies that* $u(S) \leq u(T)$*, the following hold.*

a. *The problem of deciding if an allocation maximises σ_u is* CO-NP*–complete.*

b. *The problem of deciding if there is an allocation, P, with $\sigma_u(P) > 2$ is* NP*–complete.*

c. *The problem of deciding if an allocation is Pareto Optimal is* CO-NP*–complete.*

d. *The problem of deciding if an IR O-contract path exists to realise the IR deal $\langle P, Q \rangle$ is* NP*–hard.*

A weaker version of (b) has been given in Kraus [10] and we note that for the formal definitions of the decision problems reviewed in Theorem 2 the set of utility functions $\langle u_1, u_2, \ldots, u_n\rangle$ form part of the instance. The representation of these, however, is *not* as an explicit enumeration of "subset/value" pairs (which would necessitate instances whose length was exponential in m) but rather through the medium of "straight-line programs". In consequence the hardness reductions (from 3-SAT or 3-UNSAT) construct instances for which the utility functions are concisely encoded.

Our aim in the present article is to develop some further results regarding complexity considerations in restricting contract types. In particular we consider the *length* of contract paths. Thus in his proof of Theorem 1 (b), Sandholm observes that when an IR O-contract path exists for a given IR deal, it may be the case that its length exceeds m, i.e. some agent passes a resource to another and then accepts the same resource at a later stage. We show that there are instances for which the shortest IR O-contract path may have length exponential in m.[2]

It will be useful to consider a variation on the classes given in Definition 7 motivated by the fact that contracts which allow any arbitrary reallocation of resources involving exactly two agents are not covered by any of these: M-contracts involve at least three agents; O and C-contracts, the transfer of a subset of one agent's resource holding to another without receiving any new resources; and S-contracts an exchange of one single resource with another. We thus introduce a class which will be referred to as $M(k)$-contracts.

Definition 9 *For a value $k \geq 2$, the deal $\delta = \langle P, Q\rangle$ is an $M(k)$-*contract *if $2 \leq |\mathcal{A}^\delta| \leq k$ and $\cup_{i\in\mathcal{A}^\delta}\ Q_i \ = \ \cup_{i\in\mathcal{A}^\delta}\ P_i$.*

In the next section we will be interested in lower bounds on the value of the following function: we introduce this in general terms to avoid unneccesary subsequent repetition.

Definition 10 *Let $\langle\mathcal{A},\mathcal{R},\mathcal{U}\rangle$ be a resource allocation setting. Additionally let Φ and Ψ be two predicates on deals. For a Ψ-deal $\delta = \langle P, Q\rangle$ the partial function $\rho(\delta, \langle\mathcal{A},\mathcal{R},\mathcal{U}\rangle, \Phi)$ is the length of the* shortest *Φ-contract path realising $\langle P, Q\rangle$ if such a path exists (and is undefined if no such path is possible). The partial function $\rho^{opt}(\langle\mathcal{A},\mathcal{R},\mathcal{U}\rangle, \Phi, \Psi)$ is*

$$\rho^{opt}(\langle\mathcal{A},\mathcal{R},\mathcal{U}\rangle, \Phi, \Psi) \ = \ \max_{\Psi\text{-}deals\ \delta} \ \rho(\delta, \langle\mathcal{A},\mathcal{R},\mathcal{U}\rangle, \Phi)$$

Finally, the partial function $\rho^{\max}(n, m, \Phi, \Psi)$ is

$$\rho^{\max}(n, m, \Phi, \Psi) \ = \ \max_{\mathcal{U}=\langle u_1,u_2,\ldots,u_n\rangle,\delta} \ \rho^{opt}(\langle\mathcal{A}_n,\mathcal{R}_m,\mathcal{U}\rangle, \Phi, \Psi)$$

where consideration is restricted to those Ψ-deals $\delta = \langle P, Q\rangle$ for which a realising Φ-path exists.

We note the qualification that ρ^{opt} and thence $\rho^{\max}$ are defined only for Ψ-deals that *are* capable of being realised by Φ-paths, and thus do not consider cases for which no appropriate contract path exists. We also observe that the maximisation (for ρ^{opt}) is over Ψ-deals within

2. [13] gives an upper bound on the length of such paths which is also exponential in m, but does not explicitly state any lower bound other than that already referred to.

a *specific instance* of a resource allocation setting, i.e. ρ^{opt} is parameterised by $\langle \mathcal{A}_n, \mathcal{R}_m, \mathcal{U}\rangle$. On the other hand $\rho^{\max}$ is concerned with the behaviour of ρ^{opt} as a function of n and m and ranges over *all* n-tuples of utility functions $\langle u : 2^{\mathcal{R}} \rightarrow \mathbf{Q}\rangle^n$. It is not immediately apparent whether this function is computable,[3] however, as n and m are fixed, if there is at least one Φ-path, Δ realising some Ψ-deal it is certainly well-defined since $2 \leq |\Delta| \leq n^m$.

The main results of this article are presented in the next section. We consider two general classes of contract: O-contract paths under various rationality conditions; and, similarly, $M(k)$-contracts for arbitrary values of $k \geq 2$. Our results are concerned with the construction of resource allocation settings $\langle \mathcal{A}, \mathcal{R}_m, \mathcal{U}\rangle$ for which given some rationality requirement, e.g. that deals be individually rational, there is some deal $\langle P, Q\rangle$ that satisfies the rationality condition, can be realised by an O-contract path (respectively, $M(k)$-contract path), but with the number of deals required by such paths being exponential in m. In the resource allocation settings constructed for demonstrating these properties with $M(k)$-contract paths, the constructed deal $\langle P, Q\rangle$ is realisable with a *single* $M(k+1)$-contract but unrealisable by any $M(k-1)$-contract path. Conclusions and some directions for further work are presented in the final section.

2. Lower Bounds on Path Length

Our first result clarifies one issue in the presentation of [13, Proposition 2].

Theorem 3 *Let $\Phi(P, Q)$ be the predicate which holds whenever $\langle P, Q\rangle$ is an IR O-contract and $\Psi(P, Q)$ that which holds whenever $\langle P, Q\rangle$ is IR. For $m \geq 7$*

$$\rho^{\max}(2, m, \Phi, \Psi) \;\geq\; \left(\frac{77}{256}\right)\, 2^m \;-\; 2$$

Proof. Given $\mathcal{R} = \{r_1, r_2, \ldots, r_m\}$ any subset S of $\mathcal{R}$ can be described by a binary word, $\beta(S)$, of length m, i.e. $\beta(S) = b_1 b_2 \ldots b_m$ with $b_i = 1$ if and only if $r_i \in S$. For $n = 2$ the contract net graph induced by IR O-contracts can be seen as a subgraph, $\mathcal{G}_m$ of the m-dimensional hypercube graph $\mathcal{H}_m$ with any directed path from $\beta(P)$ to $\beta(Q)$ in $\mathcal{G}_m$ corresponding to a possible IR O-contract path from the allocation $\langle P, \mathcal{R} - P\rangle$ to the allocation $\langle Q, \mathcal{R} - Q\rangle$. Consider a path in $\mathcal{H}_m$

$$\mathcal{C} = \langle C_1, C_2, \ldots, C_t\rangle$$

for which the following property[4] holds:

$$\forall\, 1 \leq i < j \leq t \;(j \geq i+2 \;\Rightarrow\; (\beta(C_i) \text{ and } \beta(C_j) \text{ differ in at least 2 positions}) \qquad \text{(SC)}$$

e.g. if $m = 4$ then

$$\emptyset, \{r_1\},\; \{r_1, r_3\}, \{r_1, r_2, r_3\}, \{r_2, r_3\}, \{r_2, r_3, r_4\}, \{r_2, r_4\}, \{r_1, r_2, r_4\}$$

3. If we consider the function $\rho^{\max}_{\Phi,\Psi}(n, m)$, i.e. $\rho^{\max}$ for a fixed setting of $\langle \Phi, \Psi\rangle$, then subject to various conditions on Φ and Ψ this can be shown to be partial recursive. Since, at present, the sufficiency conditions include some rather artificial elements, in the absence of a more general characterisation we omit the proof of this result.
4. This defines the so-called "snake-in-the-box" codes introduced in [9].

is such a path as it gives rise to the sequence $\langle 0000, 1000, 1010, 1110, 0110, 0111, 0101, 1101\rangle$.

Choose $\mathcal{C}^{(m)}$ to be a *longest* such path with this property that could be formed in $\mathcal{H}_m$, letting $\langle P_1, P_2, \ldots, P_t\rangle$ be the sequence of allocations to A_1 matching this, so that

$$\Delta_m = \langle\langle P_1, \mathcal{R}/P_1\rangle, \langle P_2, \mathcal{R}/P_2\rangle, \ldots, \langle P_t, \mathcal{R}/P_t\rangle\rangle$$

We now define the utility functions u_1 and u_2 so that

$$u_1(S) + u_2(\mathcal{R}/S) = \begin{cases} k & \text{if} \quad S = P_k \\ 0 & \text{if} \quad S \notin \{P_1, P_2, \ldots, P_t\} \end{cases}$$

With this choice, the contract path Δ_m describes the *unique* IR O-contract path realising the IR deal $\langle\langle P_1, \mathcal{R}/P_1\rangle, \langle P_t, \mathcal{R}/P_t\rangle\rangle$: that Δ_m is an IR O-contract path is immediate from Fact 1, since

$$\sigma_u(\langle P_{i+1}, \mathcal{R}/P_{i+1}\rangle) = i + 1 > i = \sigma_u(\langle P_i, \mathcal{R}/P_i\rangle)$$

That it is unique follows from the fact that for all $1 \le i \le t$ and $i + 2 \le j \le t$, the deal $\langle\langle P_i, \mathcal{R}/P_i\rangle, \langle P_j, \mathcal{R}/P_j\rangle\rangle$ is not an O-contract (hence there are no "short-cuts" possible), and for each P_i there is exactly one IR O-contract that can follow the allocation $\langle P_i, \mathcal{R}/P_i\rangle$, i.e. $\langle P_{i+1}, \mathcal{R}/P_{i+1}\rangle$.[5]

From the preceding argument it follows that any lower bound on the length of $\mathcal{C}^{(m)}$, i.e. a sequence of sets satisfying the condition (SC), is a lower bound on $\rho^{\max}(2, m, \Phi, \Psi)$. The corresponding sequences of binary words – paths in $\mathcal{H}_m$ – were originally studied by Kautz [9] in the context of coding theory and the lower bound on their length of $(77/256)2^m - 2$ established by Abbott and Katchalski [1]. □

Corollary 1 *For each of the cases below,*

a. $\Phi(\delta) \Leftrightarrow$ *δ is a cooperatively rational O-contract.*
$\Psi(\delta) \Leftrightarrow$ *δ is cooperatively rational.*

b. $\Phi(\delta) \Leftrightarrow$ *δ is an equitable O-contract.*
$\Psi(\delta) \Leftrightarrow$ *δ is equitable.*

c. $\Phi(\delta) \Leftrightarrow$ *δ is a (Simple) Pigou-Dalton O-contract.*
$\Psi(\delta) \Leftrightarrow$ *δ is a Simple Pigou-Dalton transfer.*

$$\rho^{\max}(2, m, \Phi, \Psi) \ \ge \ \left(\frac{77}{256}\right) 2^m \ - \ 2$$

Proof. We employ exactly the same sequence of allocations, Δ_m described in the proof of Theorem 3 but modify the utility functions $\langle u_1, u_2\rangle$ for each case.

5. In our example with $m = 4$, the sequence $\langle 0000, 1000, 1001, 1101\rangle$, although defining an O-contract path gives rise to a deal which is not IR, namely that corresponding to $\langle 1000, 1001\rangle$.

a. Choose $\langle u_1, u_2 \rangle$ with $u_2(S) = 0$ for all $S \subseteq \mathcal{R}$ and

$$u_1(S) = \begin{cases} k & \text{if} \quad S = P_k \\ 0 & \text{if} \quad S \notin \{P_1, \ldots, P_t\} \end{cases}$$

The resulting O-contract path is cooperatively rational: the utility enjoyed by A_2 remains constant while that enjoyed by A_1 increases by 1 with each deal. Any deviation from this contract path (employing an alternative O-contract) will result in a loss of utility for A_1.

b. Choose $\langle u_1, u_2 \rangle$ with $u_2(S) = u_1(\mathcal{R}/S)$ and

$$u_1(S) = \begin{cases} k & \text{if} \quad S = P_k \\ 0 & \text{if} \quad S \notin \{P_1, \ldots, P_t\} \end{cases}$$

The O-contract path is equitable: both A_1 and A_2 increase their respective utility values by 1 with each deal. Again, any O-contract deviating from this will result in both agents losing some utility.

c. Choose $\langle u_1, u_2 \rangle$ as

$$u_1(S) = \begin{cases} k & \text{if} \quad S = P_k \\ 0 & \text{if} \quad S \notin \{P_1, \ldots, P_t\} \end{cases} \quad ; \quad u_2(S) = \begin{cases} 2^m - k & \text{if} \quad \mathcal{R}/S = P_k \\ 2^m & \text{if} \quad \mathcal{R}/S \notin \{P_1, \ldots, P_t\} \end{cases}$$

To see that the O-contract path consists of Simple Pigou-Dalton transfers, it suffices to note that $u_1(P_i) + u_2(\mathcal{R}/P_i) = 2^m$ for each $1 \leq i \leq t$. In addition, $|u_2(\mathcal{R}/P_{i+1}) - u_1(P_{i+1})| = 2^m - 2i - 2$ which is strictly less than $|u_2(\mathcal{R}/P_i) - u_1(P_i)| = 2^m - 2i$. Finally, any O-contract $\langle P, Q \rangle$ which deviates from this sequence will not be a Simple Pigou-Dalton transfer since

$$|u_2(Q_2) - u_1(Q_1)| = 2^m \; > \; |u_2(P_2) - u_1(P_1)|$$

which violates one of the conditions required of Pigou-Dalton transfers. □

Corollary 2 *For each of the choices of $\langle \Phi, \Psi \rangle$ considered in Theorem 3 and Corollary 1, and all $n \geq 2$,*

$$\rho^{\max}(n, m, \Phi, \Psi) \; \geq \; \left(\frac{77}{256}\right) 2^m \; - \; 2$$

Proof. Fix allocations in which A_1 is given P_1, A_2 allocated $\mathcal{R}_m/P_1$, and A_j assigned $\emptyset$ for each $3 \leq j \leq n$. Using identical utility functions $\langle u_1, u_2 \rangle$ as in each of the previous cases, we employ for u_j: $u_j(\emptyset) = 1$, $u_j(S) = 0$ whenever $S \neq \emptyset$ ($\langle \Phi, \Psi \rangle$ as in Theorem 3); $u_j(S) = 0$ for all S (Corollary 1(a)); $u_j(\emptyset) = 2^m$, $u_j(S) = 0$ whenever $S \neq \emptyset$ (Corollary 1(b)); and, finally, $u_j(S) = 2^m$ for all S, (Corollary 1(c)). Considering a realisation of the Ψ deal $\langle \langle P_1, \mathcal{R}_m/P_1 \rangle, \langle P_t, \mathcal{R}_m/P_t \rangle \rangle$ the only Φ-contract path admissible is the path Δ_m defined in the related proofs. This gives the lower bound stated. □

We note, at this point, some other consequences of Corollary 1 with respect to [8, Theorems 1, 3], which state

Fact 2

a. *Any sequence of cooperatively rational deals will eventually give Pareto optimality.*

b. *Any sequence of equitable deals will eventually give an allocation maximising* σ_e.

The sequence of cooperatively rational deals in Corollary 1(a) terminates in the Pareto Optimal allocation $\langle P_t, \mathcal{R}/P_t\rangle$: the allocation for A_2 always has utility 0 and there is no allocation to A_1 whose utility can exceed t. Similarly, the sequence of equitable deals in Corollary 1(b) terminates in the allocation $\langle P_t, \mathcal{R}/P_t\rangle$, for which $\sigma_e(\langle P_t, \mathcal{R}/P_t\rangle) = t$ the maximum that can be attained for the instance defined. In both cases, however, the optima are reached by sequences of exponentially many (in m) deals: thus, although Fact 2 guarantees convergence of particular deal sequences to optimal states, it may be the case, as illustrated in Corollary 1(a–b) that the process of convergence takes considerable time.

We now turn to similar issues with respect to $M(k)$-contracts, recalling that in one respect these offer a form of deal that does not fit into the classification of Definition 7 since $M(k)$-contracts permit *two* agents to exchange resources without the deal involved being an OCS-contract. In one regard, however, $M(k)$-contracts are not as general as multiagent agent contracts (M-contracts) since a preset bound (k) is specified for the number of agents involved.

Our main result on $M(k)$-contract paths is the following development of Theorem 3.

Theorem 4 *Let $\Phi_k(P,Q)$ be the predicate which holds whenever $\langle P,Q\rangle$ is an IR $M(k)$-contract and $\Psi(P,Q)$ that which holds whenever $\langle P,Q\rangle$ is IR. For all $k \geq 3$, there is a resource allocation setting $\langle \mathcal{A},\mathcal{R},\mathcal{U}\rangle$ and an IR deal $\delta = \langle P,Q\rangle$ for which,*

$$\begin{array}{lcll} \rho(\delta, \langle \mathcal{A},\mathcal{R},\mathcal{U}\rangle, \Phi_k) & = & 1 & (a) \\ \rho(\delta, \langle \mathcal{A},\mathcal{R},\mathcal{U}\rangle, \Phi_{k-1}) & \geq & 2^{2m/k(k-1)} - 1 & (b) \end{array}$$

Proof. Given the notational complexities that arise in the construction we first give an outline of its structure and illustrate specific details with respect to an example for $k = 4$ (using $m = 18 = 6 \times 3$). In total we describe an IR $M(k-1)$-contract path

$$\Delta \ = \ \langle P^{(1)}, P^{(2)}, \ldots, P^{(t)}\rangle$$

that realises the IR $M(k)$ deal $\langle P^{(1)}, P^{(t)}\rangle$. The main idea is to exploit one well-known property of the s-dimensional hypercube network (which we denote $\mathcal{H}_s$). Interpreted as a directed graph, $\mathcal{H}_s$ has 2^s vertices each of which is identified with a distinct s-bit label. Using $\underline{v} = v_1 v_2 \ldots v_s$ to denote the label associated with the vertex v, the edges of $\mathcal{H}_s$ are formed by

$$\{\ \langle v,w\rangle,\ \langle w,v\rangle\ :\ \underline{v} \text{ and } \underline{w} \text{ differ in } \textit{exactly one} \text{ bit position}\}$$

The property of $\mathcal{H}_s$ we exploit is: for every $s \geq 0$, $\mathcal{H}_s$ contains a *Hamiltonian cycle*, i.e. a simple directed cycle formed using only the edges of $\mathcal{H}_s$ and containing all 2^s vertices, see e.g. any standard text on Coding Theory such as [2]. In order to simplify the presentation we employ a setting in which there are exactly k agents, $\mathcal{A} = \{A_0, A_1, \ldots, A_{k-1}\}$. In the contract path Δ each pair of agents $\{A_i, A_j\}$ will have a set of s resources – which we denote

$\mathcal{R}^{\{i,j\}}$ – shared between them. In any $M(k-1)$-deal, $\delta \in \Delta$, for which $\{A_i, A_j\} \subseteq \mathcal{A}^{\delta}$, *exactly one* of the resources in $\mathcal{R}^{\{i,j\}}$ will change between A_i and A_j. Now, suppose

$$\mathcal{S}^{(v)} = \underline{v}^{(0)}, \underline{v}^{(1)}, \ldots, \underline{v}^{(i)}, \ldots, \underline{v}^{(2^s-1)}, \underline{v}^{(0)}$$

is a Hamiltonian cycle in the hypercube $\mathcal{H}_s$ and

$$\mathcal{S}^{(w)} = \underline{w}^{(0)}, \underline{w}^{(1)}, \ldots, \underline{w}^{(i)}, \ldots, \underline{w}^{(2^s-1)}, \underline{w}^{(0)}$$

the Hamiltonian cycle in which $\underline{w}^{(i)}$ is obtained by complementing each bit in $\underline{v}^{(i)}$. We may interpret the s-bit label $\underline{v} = v_1 v_2 \ldots v_s$ as describing a particular subset of $\mathcal{R}^{\{i,j\}}$, i.e. that subset in which $r_k^{\{i,j\}}$ occurs if and only if $v_k = 1$. Similarly from any subset of $\mathcal{R}^{\{i,j\}}$ we may define a unique s-bit word. Now suppose that $P_i^{(d)}$ is the allocation held by A_i in the allocation $P^{(d)}$ of Δ. The deal $\delta = \langle P^{(d)}, P^{(d+1)} \rangle$ will affect $P_i^{(d)} \cap \mathcal{R}^{\{i,j\}}$ in the following way: if $i \notin \mathcal{A}^{\delta}$ or $j \notin \mathcal{A}^{\delta}$ then $P_i^{(d+1)} \cap \mathcal{R}^{\{i,j\}} = P_i^{(d)} \cap \mathcal{R}^{\{i,j\}}$ and $P_j^{(d+1)} \cap \mathcal{R}^{\{i,j\}} = P_j^{(d)} \cap \mathcal{R}^{\{i,j\}}$. Otherwise we have $\{i,j\} \subseteq \mathcal{A}^{\delta}$ and the (complementary) holdings $P_i^{(d)} \cap \mathcal{R}^{\{i,j\}}$ and $P_j^{(d)} \cap \mathcal{R}^{\{i,j\}}$ define (complementary) s-bit labels of vertices in $\mathcal{H}_s$: if these correspond to places $\langle \underline{v}^{(t)}, \underline{w}^{(t)} \rangle$ in the Hamiltonian cycle, then in $P_i^{(d+1)}$ and $P_j^{(d+1)}$ the s-bit labels defined from $P_i^{(d+1)} \cap \mathcal{R}^{\{i,j\}}$ and $P_j^{(d+1)} \cap \mathcal{R}^{\{i,j\}}$ produce the s-bit labels $\underline{v}_{t+1}$ and $\underline{w}_{t+1}$, i.e. the vertices that succeed $\underline{v}_t$ and $\underline{w}_t$ in the Hamiltonian cycles. In total, for each j, A_i initially holds either the subset of $\mathcal{R}^{\{i,j\}}$ that maps to $\underline{v}^{(0)}$ or that maps to $\underline{w}^{(0)}$ and, at the conclusion of the $M(k-1)$-path, holds the subset that maps to $\underline{v}^{(2^s-1)}$ (or $\underline{w}^{(2^s-1)}$). The final detail is that the progression through the Hamiltonian cycles is conducted over a series of *rounds* each round comprising k $M(k-1)$-deals.

We now complete the formal description of the resource allocation setting $\langle \mathcal{A}, \mathcal{R}, \mathcal{U} \rangle$ and the $M(k-1)$-contract path employed.

Fix $\mathcal{A} = \{A_0, A_1, \ldots, A_{k-1}\}$. $\mathcal{R}$ consists of $\binom{k}{2}$ pairwise disjoint sets of s resources

$$\mathcal{R}^{\{i,j\}} = \{r_1^{\{i,j\}}, r_2^{\{i,j\}}, \ldots, r_s^{\{i,j\}}\}$$

For $k = 4$ and $s = 3$ these yield $\mathcal{A} = \{A_0, A_1, A_2, A_3\}$ and

$$\begin{array}{rcl}
\mathcal{R}^{\{0,1\}} & = & \{r_1^{\{0,1\}}, r_2^{\{0,1\}}, r_3^{\{0,1\}}\} \\
\mathcal{R}^{\{0,2\}} & = & \{r_1^{\{0,2\}}, r_2^{\{0,2\}}, r_3^{\{0,2\}}\} \\
\mathcal{R}^{\{0,3\}} & = & \{r_1^{\{0,3\}}, r_2^{\{0,3\}}, r_3^{\{0,3\}}\} \\
\mathcal{R}^{\{1,2\}} & = & \{r_1^{\{1,2\}}, r_2^{\{1,2\}}, r_3^{\{1,2\}}\} \\
\mathcal{R}^{\{1,3\}} & = & \{r_1^{\{1,3\}}, r_2^{\{1,3\}}, r_3^{\{1,3\}}\} \\
\mathcal{R}^{\{2,3\}} & = & \{r_1^{\{2,3\}}, r_2^{\{2,3\}}, r_3^{\{2,3\}}\}
\end{array}$$

We use three ordering structures in defining the $M(k-1)$-contract path.

a. $C_k = \langle X_0, X_1, \ldots, X_{k-1} \rangle$ the ordering of all $(k-1)$ element subsets of $\mathcal{A}$ in which $X_i = \mathcal{A}/\{A_i\}$.

b.

$$\mathcal{S}^{(v)} = \underline{v}^{(0)}, \underline{v}^{(1)}, \ldots, \underline{v}^{(i)}, \ldots, \underline{v}^{(2^s-1)}, \underline{v}^{(0)}$$

a Hamiltonian cycle in $\mathcal{H}_s$, where without loss of generality, $\underline{v}^{(0)} = 111\ldots11$.

c.

$$\mathcal{S}^{(w)} = \underline{w}^{(0)}, \underline{w}^{(1)}, \ldots, \underline{w}^{(i)}, \ldots, \underline{w}^{(2^s-1)}, \underline{w}^{(0)}$$

the complementary Hamiltonian cycle to this, so that $\underline{w}^{(0)} = 000\ldots00$.

Thus for $k = 4$ and $s = 3$ we obtain

$$\begin{array}{lrcl} \text{a.} & C_4 & = & \langle\{1,2,3\},\{0,2,3\},\{0,1,3\},\{0,1,2\}\rangle \\ \text{b.} & \mathcal{S}^{(v)} & = & \langle 111,110,010,011,001,000,100,101\rangle \\ \text{c.} & \mathcal{S}^{(w)} & = & \langle 000,001,101,100,110,111,011,010\rangle \end{array}$$

We can now describe the $M(k-1)$-contract path.

$$\Delta \;=\; \langle P^{(1)}, P^{(2)}, \ldots, P^{(t)}\rangle$$

Initial Allocation: $P^{(1)}$.

Define the $k \times k$ Boolean matrix, $B = [b_{i,j}]$ (with $0 \le i, j \le k-1$) by

$$b_{i,j} \;=\; \begin{cases} \bot & \text{if} \quad i = j \\ \neg b_{j,i} & \text{if} \quad i > j \\ \neg b_{i,j-1} & \text{if} \quad i < j \end{cases}$$

We then have for each $1 \le i \le k$,

$$P_i^{(1)} \;=\; \bigcup_{j=0}^{i-1} \{\; R^{\{j,i\}} \;:\; b_{i,j} = \top\} \;\cup\; \bigcup_{j=i+1}^{k-1} \{\; R^{\{i,j\}} \;:\; b_{i,j} = \top\}$$

Thus, in our example,

$$B \;=\; \begin{bmatrix} \bot & \top & \bot & \top \\ \bot & \bot & \top & \bot \\ \top & \bot & \bot & \top \\ \bot & \top & \bot & \top \end{bmatrix}$$

Yielding the starting allocation

$$\begin{array}{lclcl} P_0^{(1)} & = & \mathcal{R}^{\{0,1\}} \cup \mathcal{R}^{\{0,3\}} & = & \langle 111,000,111\rangle \\ P_1^{(1)} & = & \mathcal{R}^{\{1,2\}} & = & \langle 000,111,000\rangle \\ P_2^{(1)} & = & \mathcal{R}^{\{0,2\}} \cup \mathcal{R}^{\{2,3\}} & = & \langle 111,000,111\rangle \\ P_3^{(1)} & = & \mathcal{R}^{\{1,3\}} & = & \langle 000,111,000\rangle \end{array}$$

The final column in $P_i^{(1)}$ indicating the 3-bit labels characterising each of the subsets of $\mathcal{R}^{\{i,j\}}$ for the three values that j can assume.

Rounds: The initial allocation is changed over a series of *rounds*

$$Q^1, Q^2, \ldots, Q^z$$

each of which involves exactly k distinct $M(k-1)$-contracts. We use $Q^{x,p}$ to indicate the allocation resulting after stage p in round x where $0 \leq p \leq k-1$ with $P^{(1)}$ the allocation $Q^{0,k-1}$. We note the following:

a. $Q^{1,0}$ is obtained from $P^{(1)}$, i.e. $P^{(2)} = Q^{1,0}$.

b. $Q^{x,0}$ is obtained from $Q^{x-1,k-1}$ (when $x > 1$).

c. $Q^{x,p}$ is obtained from $Q^{x,p-1}$ (when $p > 0$).

Our final item of notation is that of the *cube position of i with respect to j in an allocation* P, denoted $\chi(i,j,P)$. Letting $\underline{u}$ be the s-bit string describing $P_i \cap \mathcal{R}^{\{i,j\}}$ in some allocation P, $\chi(i,j,P)$ is the *index* of $\underline{u}$ in the Hamiltonian cycle $S^{(v)}$ (when $\mathcal{R}^{\{i,j\}} \subseteq P_i^{(1)}$) or the Hamiltonian cycle $S^{(w)}$ (when $\mathcal{R}^{\{i,j\}} \cap P_i^{(1)} = \emptyset$). When $P = Q^{x,p}$ for some allocation in the sequence under construction we employ the notation $\chi(i,j,x,p)$, noting that one invariant of our path will be $\chi(i,j,x,p) = \chi(j,i,x,p)$, a property that certainly holds true of $P^{(1)} = Q^{0,k-1}$ since $\chi(i,j,0,k-1) = \chi(j,i,0,k-1) = 0$.

The sequence of allocations in Δ is built as follows. Since $Q^{1,0}$ is the immediate successor of the initial allocation $Q^{0,k-1}$, it suffices to describe how $Q^{x,p}$ is formed from $Q^{x,p-1}$ (when $p > 0$) and $Q^{x+1,0}$ from $Q^{x,k-1}$. Let $Q^{y,q}$ be the allocation to be formed from $Q^{x,p}$. The deal $\delta = \langle Q^{x,p}, Q^{y,q} \rangle$ will be an $M(k-1)$ contract in which $\mathcal{A}^\delta = X_q = \mathcal{A}/\{A_q\}$. For each pair $\{i,j\} \subseteq \mathcal{A}^\delta$ we have $\chi(i,j,x,p) = \chi(j,i,x,p)$ in the allocation $Q^{x,p}$. In moving to $Q^{y,q}$ exactly one element of $\mathcal{R}^{\{i,j\}}$ is reallocated between A_i and A_j in such a way that in $Q^{y,q}$, $\chi(i,j,y,q) = \chi(i,j,x,p)+1$, since A_i and A_j are tracing complementary Hamiltonian cycles with respect to $\mathcal{R}^{\{i,j\}}$, this ensures that $\chi(j,i,y,q) = \chi(j,i,x,p)+1$, thereby maintaining the invariant property. The tables below illustrates this process for our example

		A_0			A_1			A_2			A_3		
x	p	01	02	03	10	12	13	20	21	23	30	31	32
0	3	111	000	111	000	111	000	111	000	111	000	111	000
1	0	111	000	111	000	110	001	111	001	110	000	110	001
1	1	111	001	110	000	110	001	110	001	010	001	110	101
1	2	110	001	010	001	110	101	110	001	010	101	010	101
1	3	010	101	010	101	010	101	010	101	010	101	010	101
2	0	010	101	011	101	011	100	010	100	011	101	011	100
2	1	010	100	001	101	011	100	011	100	001	100	011	110
2	2	011	100	001	100	011	110	011	100	001	110	001	110
2	3	001	110	001	110	001	110	001	110	001	110	001	110
$\vdots$	$\vdots$		$\vdots$			$\vdots$			$\vdots$			$\vdots$	

Successive Resource Holdings in $Q^{x,p}$ ($k=4$, $s=3$)

		A_0			A_1			A_2			A_3		
x	p	01	02	03	10	12	13	20	21	23	30	31	32
0	3	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	1	1	0	1	1	0	1	1
1	1	0	1	1	0	1	1	1	1	2	1	1	2
1	2	1	1	2	1	1	2	1	1	2	2	2	2
1	3	2	2	2	2	2	2	2	2	2	2	2	2
2	0	2	2	2	2	3	3	2	3	3	2	3	3
2	1	2	3	3	2	3	3	3	3	4	3	3	4
2	2	3	3	4	3	3	4	3	3	4	4	4	4
2	3	4	4	4	4	4	4	4	4	4	4	4	4
$\vdots$	$\vdots$		$\vdots$			$\vdots$			$\vdots$			$\vdots$	

Cube Positions $\chi(i,j,x,p)$ **(**$k=4$**,** $s=3$**)**

It is certainly the case that this process of applying successive rounds of k deals could be continued, however, we wish to do this only so long as it is not possible to go from some allocation in the sequence $P^{(r)}$ to another $P^{(r+d)}$ for some $d \geq 2$ via an $M(k-1)$-deal.

Now if $Q^{x,p}$ and $Q^{y,q}$ are distinct allocations generated by the process above then the deal $\delta = \langle Q^{x,p}, Q^{y,q} \rangle$ is an $M(k-1)$-contract if and only if for some A_i, $Q_i^{x,p} = Q_i^{y,q}$. It follows that if $\langle P^{(r)}, P^{(r+d)} \rangle$ is an $M(k-1)$-contract for some $d > 1$, then for some i and all $j \neq i$, $P_i^{(r+d)} \cap \mathcal{R}^{\{i,j\}} = P_i^{(r)} \cap \mathcal{R}^{\{i,j\}}$.

To determine the minimum value of $d > 1$ with which $P_i^{(r+d)} = P_i^{(r)}$, we observe that without loss of generality we need consider only the case $r = i = 0$, i.e. we determine the minimum number of changes before $P_0^{(1)}$ reappears. First note that in each round, Q^x, if $\chi(0,j,x-1,k-1) = p$ then $\chi(0,j,x,k-1) = p+k-2$, i.e. each round advances the cube position $k-2$ places: $\chi(0,j,x-1,k-1) = \chi(0,j,x,0)$ and $\chi(0,j,x,j) = \chi(0,j,x,j-1)$. We can also observe that $P_0^{(1)} = Q_0^{0,k-1} \neq Q_0^{x,p}$ for any p with $0 < p < k-1$, since

$$\chi(0,1,x,p) = \chi(0,2,x,p) = \ldots = \chi(0,k-1,x,p)$$

only in the cases $p = 0$ and $p = k-1$. It follows that our value $d > 1$ must be of the form qk where q must be such that $q(k-2)$ is an *exact multiple* of 2^s. From this observation we see that,

$$\min\{\ d > 1 : P_0^{(1)} = P_0^{(1+d)}\ \} \ = \ \min\{\ qk \ : \ q(k-2) \text{ is a multiple of } 2^s\}$$

Now, if k is *odd* then $q = 2^s$ is the minimal such value, so that $d = k2^s$. If k is even then it may be uniquely written in the form $z2^l + 2$ where z is *odd* so giving q as 1 (if $l \geq s$) or 2^{s-l} (if $l \leq s$), so that these give $d = k$ and $d = z2^s + 2^{s-l+1}$, e.g. for $k = 4$ and $s = 3$, we get $k = 1 \times 2^1 + 2$ so that $d = 2^3 + 2^{3-1+1} = 16$ and in our example $P_0^{(1)} = P_0^{(17)}$ may be easily verified. In total,

$$d \geq \begin{cases} k2^s & \text{if} \quad k \text{ is odd} \\ k & \text{if} \quad k = z2^l + 2,\ z \text{ is odd, and } l \geq s \\ 2^s & \text{if} \quad k = z2^l + 2,\ z \text{ is odd and } l \leq s \end{cases}$$

All of which immediately give $d \geq 2^s$ (in the second case $k \geq 2^s$, so the inequality holds trivially), and thus we can continue the chain of $M(k-1)$ contracts for at least 2^s moves. Recalling that $m = s\begin{pmatrix} k \\ 2 \end{pmatrix}$, this gives the length of the $M(k-1)$-contract path

$$\Delta \;=\; \langle P^{(1)}, P^{(2)}, \ldots, P^{(t)} \rangle$$

written in terms of m and k as at least

$$2^{m/\begin{pmatrix} k \\ 2 \end{pmatrix}} - 1 \;=\; 2^{\frac{2m}{k(k-1)}} \;-\; 1$$

It remains to define appropriate utility functions $\mathcal{U} = \langle u_0, \ldots, u_{k-1} \rangle$ in order to ensure that Δ is the unique IR $M(k-1)$-contract path realising the IR $M(k)$-deal $\langle P_1, P_t \rangle$. In defining $\mathcal{U}$ it will be convenient to denote Δ as the path

$$\Delta = \langle Q^{0,k-1}, Q^{1,0}, Q^{1,1}, \ldots, Q^{1,k-1}, \ldots, Q^{x,p}, \ldots, Q^{r,k-1} \rangle$$

and, since $rk \geq 2^s$, we may without loss of generality, focus on the first 2^s allocations in this contract path.

Recalling that $\chi(i,j,x,p)$ is the index of the s-bit label $\underline{u}$ corresponding to $Q_i^{x,p} \cap \mathcal{R}^{\{i,j\}}$ in the relevant Hamiltonian cycle – i.e. $\mathcal{S}^{(v)}$ if $\mathcal{R}^{\{i,j\}} \subseteq Q_i^{0,k}$, $\mathcal{S}^{(w)}$ if $Q_i^{0,k-1} \cap \mathcal{R}^{\{i,j\}} = \emptyset$ – we note the following properties of the sequence of allocations defined by Δ that hold for each distinct i and j.

P1. $\forall\ x,p\ \chi(i,j,x,p) = \chi(j,i,x,p)$

P2. If $Q^{y,q}$ is the immediate successor of $Q^{x,p}$ in Δ then $\chi(i,j,y,q) \leq \chi(i,j,x,p) + 1$ with equality if and only if $q \notin \{i,j\}$.

P3. $\forall\ i',j'$ with $0 \leq i',j' \leq k-1$, $\chi(i,j,x,k-1) = \chi(i',j',x,k-1)$.

The first two properties have already been established in our description of Δ. The third follows from the observation that within each round Q^x, each cube position is advanced by exactly $k-2$ in progressing from $Q^{x-1,0}$ to $Q^{x,k-1}$.

The utility function u_i is now given, for $S \subseteq \mathcal{R}_m$, by

$$u_i(S) = \begin{cases} \sum_{j \neq i} \chi(i,j,x,p) & \text{if } S = Q_i^{x,p} \text{ for some } 0 \leq x \leq r,\ 0 \leq p \leq k-1 \\ -2^{km} & \text{otherwise} \end{cases}$$

We claim that, with these choices,

$$\Delta = \langle Q^{0,k-1}, Q^{1,0}, Q^{1,1}, \ldots, Q^{1,k-1}, \ldots, Q^{x,p}, \ldots, Q^{r,k-1} \rangle$$

is the unique IR $M(k-1)$-contract path realising the IR $M(k)$-deal $\langle Q^{0,k-1}, Q^{r,k-1} \rangle$. Certainly, Δ *is* an IR $M(k-1)$-contract path: each deal $\delta = \langle Q^{x,p}, Q^{y,q} \rangle$ on this path has $|\mathcal{A}^\delta| = k-1$ and since for each agent A_i in $\mathcal{A}^\delta = \mathcal{A}/\{A_q\}$ the utility of $Q_i^{y,q}$ has increased by exactly $k-2$, i.e. each cube position of i with respect to j whenever $q \notin \{i,j\}$ has increased, it follows that $\sigma_u(Q^{y,q}) > \sigma_u(Q^{x,p})$ and hence $\langle Q^{x,p}, Q^{y,q} \rangle$ is IR via Fact 1.

We now show that Δ is the *unique* IR $M(k-1)$-contract path continuation of $Q^{0,k-1}$ Suppose $\delta = \langle Q^{x,p}, P\rangle$ is a deal that deviates from the contract path Δ (having followed it through to the allocation $Q^{x,p}$). Certainly both of the following must hold of P: for each i, $P_i \subseteq \cup_{j\neq i}\mathcal{R}^{\{i,j\}}$; and there is a k-tuple of pairs $\langle (x_0,p_0),\ldots,(x_{k-1},p_{k-1})\rangle$ with which $P_i = Q_i^{x_i,p_i}$, for if either fail to be the case for some i, then $u_i(P_i) = -2^{km}$ with the consequent effect that $\sigma_u(P) < 0$ and thence not IR. Now, if $Q^{y,q}$ is the allocation that would succeed $Q^{x,p}$ in Δ then $P \neq Q^{y,q}$, and thus for at least one agent, $Q_i^{x_i,p_i} \neq Q_i^{y,q}$. It cannot be the case that $Q_i^{x_i,p_i}$ corresponds to to an allocation occurring *strictly later* than $Q_i^{y,q}$ in Δ since such allocations could not be realised by an $M(k-1)$-contract. In addition, since $P_i = Q_i^{x_i,p_i}$ it must be the case that $|\mathcal{A}^\delta| = k-1$ since exactly $k-1$ cube positions in the holding of A_i must change. It follows that there are only two possibilities for (y_i,p_i): P_i reverts to the the allocation immediately preceding $Q_i^{x,p}$ or advances to the holding $Q_i^{y,q}$. It now suffices to observe that a deal in which some agents satisfy the first of these while the remainder proceed in accordance with the second either does not give rise to a valid allocation or cannot be realised by an $M(k-1)$-contract. On the other hand if P corresponds to the allocation preceding $Q^{x,p}$ then δ is not IR. We deduce, therefore, that the *only* IR $M(k-1)$ deal that is consistent with $Q^{x,p}$ is that prescribed by $Q^{y,q}$. □

We observe, as an aside, that the $M(k)$-deal described in the proof of Theorem 4 cannot be realised by an $M(k-2)$-contract path.

We obtain a similar development of Corollary 1 in

Corollary 3 *For all $k \geq 3$ and each of the cases below,*

a. $\Phi_k(\delta) \Leftrightarrow$ *δ is a cooperatively rational $M(k)$-contract.*
$\Psi(\delta) \Leftrightarrow$ *δ is cooperatively rational.*

b. $\Phi_k(\delta) \Leftrightarrow$ *δ is an equitable $M(k)$-contract.*
$\Psi(\delta) \Leftrightarrow$ *δ is equitable.*

there is a resource allocation setting $\langle \mathcal{A},\mathcal{R},\mathcal{U}\rangle$ and a Ψ-deal $\delta = \langle P, Q\rangle$ for which

$$\begin{array}{lccc}
\rho(\delta,\langle \mathcal{A},\mathcal{R},\mathcal{U}\rangle,\Phi_k) & = & 1 & (a)\\
\rho(\delta,\langle \mathcal{A},\mathcal{R},\mathcal{U}\rangle,\Phi_{k-1}) & \geq & 2^{2m/k(k-1)}-1 & (b)\\
\rho(\delta,\langle \mathcal{A},\mathcal{R},\mathcal{U}\rangle,\Phi_{k-2}) & & \textit{is undefined} & (c)
\end{array}$$

Proof. As with the proof of Corollary 1 in relation to Theorem 3, in each case we employ the contract path from the proof of Theorem 4, varying the definition of $\mathcal{U} = \langle u_1, u_2, \ldots, u_k\rangle$ in order to establish each result. Thus let

$$\begin{array}{lcl}
\Delta_m & = & \langle P^{(1)}, P^{(2)}, \ldots, P^{(r)}, \ldots, P^{(t)}\rangle\\
 & = & \langle Q^{0,k-1}, Q^{1,0}, \ldots, Q^{x,p}, \ldots, Q^{z,r}\rangle
\end{array}$$

be the $M(k-1)$-contract path realising the $M(k)$-deal $\langle P^{(1)}, P^{(t)}\rangle$ described in the proof of Theorem 4, this path having length $t \geq 2^{2m/k(k-1)}-1$.

a. The utility functions $\mathcal{U} = \langle u_0, \ldots, u_{k-1}\rangle$ of Theorem 4 ensure that $\langle P^{(1)}, P^{(t)}\rangle$ is cooperatively rational and that Δ_m is a cooperatively rational $M(k-1)$-contract path realising $\langle P^{(1)}, P^{(t)}\rangle$: the utility held by A_i never decreases in value and there is at least one agent (in fact exactly $k-1$) whose utility increases in value. Furthermore Δ_m is the unique cooperatively rational $M(k-1)$-contract path realising $\langle P^{(1)}, P^{(t)}\rangle$ since, by the same argument used in Theorem 4, any deviation will result in some agent suffering a loss of utility.

b. Set the utility functions $\mathcal{U} = \langle u_0, \ldots, u_{k-1}\rangle$ as,

$$u_i(S) = \begin{cases} -1 & \text{if} \quad S \neq Q_i^{x,p} \text{ for any } Q^{x,p} \in \Delta_m \\ xk^2 + k - i & \text{if} \quad S = Q_i^{x,k-1} \\ (x-1)k^2 + k + p & \text{if} \quad S = Q_0^{x,p},\ p < k-1 \text{ and } i = 0 \\ (x-1)k^2 + k - i + p + 1 & \text{if} \quad S = Q_i^{x,p},\ p < i-1 \text{ and } i \neq 0. \\ xk^2 + 1 & \text{if} \quad S = Q_i^{x,i-1} = Q_i^{x,i} \text{ and } i \neq 0. \\ xk^2 + 1 + p - i & \text{if} \quad S = Q_i^{x,p},\ p > i \text{ and } i \neq 0 \end{cases}$$

To see that these choices admit Δ_m as an equitable $M(k-1)$-contract path realising the equitable deal $\langle Q^{0,k-1}, Q^{z,r}\rangle$, we first note that

$$\min_{0 \leq i \leq k-1} \{u_i(Q_i^{z,r}\} \ > \ 1 \ = \ \min_{0 \leq i \leq k-1} \{u_i(Q_i^{0,k-1}\}$$

thus, $\langle Q^{0,k-1}, Q^{z,r}\rangle$ is indeed equitable. Consider any deal $\delta = \langle Q^{x,p}, Q^{y,q}\rangle$ occuring within Δ_m. It suffices to show that

$$\min_{0 \leq i \leq k-1} \{u_i(Q_i^{x,p}\} \ \neq \ u_q(Q_q^{x,p})$$

since $A_q \notin \mathcal{A}^\delta$, and for all other agents $u_i(Q_i^{y,q}) > u_i(Q_i^{x,p})$. We have two possibilities: $q = 0$ (in which case $p = k-1$ and $y = x+1$); $q > 0$ (in which case $p = q-1$). Consider the first of these: $u_0(Q_0^{x,k-1}) = xk^2 + k$, however,

$$\min\{u_i(Q_i^{x,k-1})\} = xk^2 + 1 = u_{k-1}(Q_{k-1}^{x,k-1})$$

and hence every deal $\langle Q^{x,k-1}, Q^{x+1,0}\rangle$ forming part of Δ_m is equitable.

In the remaining case, $u_q(Q_q^{x,q-1}) = xk^2 + 1$ and

$$\begin{aligned} \min\{u_i(Q_i^{x,q-1})\} &\leq u_0(Q_0^{x,q-1}) \\ &= (x-1)k^2 + k + q - 1 \\ &< xk^2 - (k^2 - 2k + 1) \\ &= xk^2 - (k-1)^2 \\ &< xk^2 + 1 \\ &= u_q(Q_q^{x,q-1}) \end{aligned}$$

and thus the remaining deals $\langle Q^{x,q-1}, Q^{x,q}\rangle$ within Δ_m are equitable. By a similar argument to that employed in Theorem 4 it follows that Δ_m is the unique equitable $M(k-1)$-contract path realising $\langle Q^{0,k-1}, Q^{z,r}\rangle$.

□

3. Conclusions and Further Work

Our aim in this article has been to develop the earlier studies of Sandholm [13] concerning the scope and limits of particular 'practical' contract forms. While [13] has established that insisting on individual rationality in addition to the structural restriction prescribed by O-contracts leads to scenarios which are incomplete (in the sense that there are individually rational deals that cannot be realised by individually rational O-contracts) our focus has been with respect to deals which can be realised by restricted contract paths, with the intention of determining to what extent the combination of structural and rationality conditions increases the number of deals required. We have shown that, using a number of natural definitions of 'rationality', for settings involving m resources, *rational* O-contract paths of length $\Omega(2^m)$ are needed, whereas without the rationality restriction on individual deals, at most m O-contracts suffice to realise any deal. We have also considered a class of contracts – $M(k)$-contracts – that were not examined in [13], establishing for these cases that, when particular rationality conditions are imposed, $M(k-1)$-contract paths of length $\Omega(2^{2m/k^2})$ are needed to realise a deal that can be achieved by a single $M(k)$-contract.

We note that our analyses have primarily been focused on worst-case lower bounds on path length when appropriate paths exist, and as such there are several questions of practical interest that merit further discussion. It may be noted that the path structures and associated utility functions are rather artificial, being directed to attaining a path of a specific length meeting a given rationality criterion. While these give some indication of the potential cost of requiring each individual deal in a contract path to satisfy a rationality constraint, it would be of interest to gain some insight into what the 'average' path length is likely to be. In attempting to address this question, however, a number of challenging issues arise. The most immediate of these concerns, of course, the notion of modelling an 'average' utility function given our definitions of rationality in terms of the value agents attach to their resource holdings. In principle an average analysis of scenarios involving exactly *two* agents could be carried out in purely graph-theoretic terms: for any of the rationality measures considered it is always possible to fix utility functions under which an allocation $(P, \mathcal{R}/P)$ is 'better than' $(Q, \mathcal{R}/Q)$ and thus from any *acyclic* directed sub-graph of the complete contract-net graph for m resources allocated among two agents one may define utility functions $\langle u_1, u_2 \rangle$ for which $(P, \mathcal{R}/P)$ improves upon $(Q, \mathcal{R}/Q)$ if and only if there is a directed *path* from the vertex labelled $(Q, \mathcal{R}/Q)$ to that labelled $(P, \mathcal{R}/P)$. With this approach, an estimation of the average rational O-contract path length reduces to a purely combinatorial question, albeit one which appears highly non-trivial in nature. It is unclear, however, whether this reduction to a graph-theoretic analysis obviating the need for consideration of literal utility functions, can be extended beyond settings involving exactly two agents. The difficulty arising with three or more agents is that our utility functions are *context free*, i.e. given an allocation $\langle X, Y, Z \rangle$ to three agents, $u_1(X)$ is unchanged should $Y \cup Z$ be redistributed among A_2 and A_3. As a result it is possible to form (acyclic) directed subgraphs of the complete three agent contract-net graph on m resources for which there is *no choice* of utility functions with $\langle X_2, Y_2, Z_2 \rangle$ improving upon $\langle X_1, Y_1, Z_1 \rangle$ if and only if there is a directed path from $\langle X_1, Y_1, Z_1 \rangle$ to $\langle X_2, Y_2, Z_2 \rangle$ in the graph.

As one final set of issues that may merit further study we raise the following. In our constructions, the individual deals on a contract path must satisfy both a *structural* condition (be an O-contract or involve at most k agents), and a *rationality* constraint. Focusing on O-contracts we have the following extremes: from [13], at most m O-contracts suffice to realise any rational deal; from our results above, $\Omega(2^m)$ *rational* O-contracts are needed to realise some rational deals. Suppose we relax the condition that every single deal on an O-contract path be rational? Thus, if we go to the extent of allowing up to m 'irrational' O-contracts, then any rational deal can be realised efficiently. It would be of some interest to examine issues such as the effect of allowing a *constant* number, t, of irrational deals and questions such as whether there are situations in which t irrational contracts yield a 'short' contract path but $t-1$ force one of exponential length. Of particular interest, from an application viewpoint, is the following: define a $(\gamma(m), O)$-path as an O-contract path containing at most $\gamma(m)$ O-contracts which are not individually rational. We know that if $\gamma(m) = 0$ then individually rational $(0, O)$-contracts are not complete with respect to individually rational deals; similarly if $\gamma(m) = m$ then (m, O)-contracts *are* complete with respect to individually rational deals. A question of some interest would be to establish if there is some $\gamma(m) = o(m)$ for which $(\gamma(m), O)$-contracts are complete with respect to individually rational deals *and* with the maximum length of such a contract path bounded by a polynomial function of m.

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