ON POINTED ENRICHMENTS AND ILLEGAL COMPOSITIONS

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Abstract. This note gives a brief and relatively basic description of "adding an illegal point" to deal with forbidden compositions. The main result is Theorem 4.2, although Lemma 5.2 seems to have very broad applications. The point of the theorem is that 2-categories are not necessary to study reasonable examples of "precategories" with standard categorical techniques, despite this being the most common approach in the literature.

1. Symmetric Monoidal Categories

Recall that a cartesian closed category $\mathcal{C}$ is one with the property that left (and right) multiplication by an object C is a left adjoint. Here the term multiplication means the categorical product, so I mean the functor $B \mapsto C \times B$ which is an endofunctor of $\mathcal{C}$. The right adjoint is generally referred to as an exponential and often written as such. Hopefully, such categories are familiar as they are equivalent to λ-calculi.

However, it seems reasonable to consider such a situation where we do not want multiplication to be the categorical product. For example, in the category of groups, a homomorphism $R \times S \to T$ has no particular signifigance. Instead, we care about bilinear maps of this type, i.e. those which descend to homomorphisms $R \otimes S \to T$. This example is highly useful as a base case and the notation of $\otimes$ has become used for the general case.

The categorical product has a number of good properties which we might like to impose on our multiplication. For example, there is a unique isomorphism (generally called a switch map) $B \times C \approx C \times B$ directly from the universal property of the categorical product. Thus, we are lead to the notion of a symmetric monoidal category, where there is a multiplication which is functorial in each variable and comes equipped with a (possibly choice of) natural isomorphism like a switch map (the symmetric part). Symmetric monoidal closed categories also satisfy the additional condition that multiplication is a left adjoint. The right adjoint is often still called an exponential object but is also commonly called a cotensor, while the multiplication is called a tensor. The notation for cotensors varies rather irregularly throughout the literature, but I'll write $\mathrm{Hom}(C, ?)$ for the right adjoint to $? \otimes C$.

The most basic example of a symmetric monoidal category is just any cartesian closed category, such as the category Sets. Other examples include Abelian groups under the usual tensor product over the integers, chain complexes of R-modules under graded R-tensor product, and our most interesting example here of pointed sets, Sets_*, under the smash product $\wedge$.

Notice the categorical product of pointed sets is simply the cartesian product with the ordered pair consisting of both basepoints as the basepoint. However, the

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mapping set of POINTED maps does not act as a right adjoint to this product. The problem is that the basepoints go wrong. Essentially, any ordered pair with a basepoint in the first component must correspond to the constant map on the basepoint. In other words, we would like to consider the subset consisting of ordered pairs with a basepoint in either component, often called the wedge and written $B \bigvee C$, and collapse it to a point. This leads us to define $B \wedge C$ as the quotient set $(B \times C)/(B \bigvee C)$ with the collapse point as baepoint. In this case, the operation $? \wedge C$ has the pointed mapping set as a right adjoint. In other words, this is a symmetric monoidal closed structure on Sets_*.

2. Enriched Categories

The general approach here is that Sets acts as the "ground field" of ordinary category theory while any symmetric monoidal closed category (especially Sets_*) could be used instead. This leads to the notion of an enriched category, which Lawvere once claimed should be the proper foundation for modeling real world systems in their true complexity. Here the usual notation is $\mathcal{V}$ for the symmetric monoidal closed category and $\mathcal{V}$-Cat for the category of all small $\mathcal{V}$-enriched categories. There are two good standard references for this general theory, the second volume of Borceux's Handbook of Categorical Algebra, Chapter 6 and Kelley's Introduction to Enriched Category Theory (in that order of difficulty). However, part of my point is that with pointed sets these references are only necessary if you want to do something quite unusual. In general, the constructions in Cat_* (our shorthand for Sets_*-Cat) start out almost the same as those for ordinary categories and then one takes an appropriate quotient or subobject to deal with the basepoints. For a general $\mathcal{V}$ things can become quite complicated, but our examples should be quite simple.

We say a category $\mathcal{C}$ is enriched over $\mathcal{V}$ if the morphism sets, $\hom_{\mathcal{C}}(B, C)$ are actually the "underlying sets" of objects $\mathcal{C}(B, C) \in \mathcal{V}$ and the composition law is really a morphism $\mathcal{C}(C, D) \otimes \mathcal{C}(B, C) \to \mathcal{C}(B, D)$ in $\mathcal{V}$. For example, a category enriched over topological spaces (under the Cartesian product) means that we have chosen topologies on each of the morphism sets and that the composition law induces a continuous map under these topologies. If we enrich over the category of Abelian groups, we get the notion of an additive category (an Abelian category is more restrictive), where there is an Abelian group structure on each morphism set and the composition law is a group homomorphism. We call a functor enriched if the map it induces on morphism sets (taking morphisms to morphisms) actually lifts to a morphism in $\mathcal{V}$ of morphism objects

$$\mathcal{C}(B, C) \to \mathcal{D}(F(B), F(C)).$$

Unfortunately, the notion of enriched natural transformation can become quite complicated, unless the target category of the two functors happens to be the category $\mathcal{V}$ itself, as with enriched sheaf or presheaf categories (the topic of part of my thesis).

To see why this might be useful, recall that a monoid is (equivalent to) a category with one object. By definition, a monoid is a set equipped with a unital, associative multiplication. We translate this into a category by taking a unique object (simply a placeholder) and designating the set as the morphisms of this object. The unit becomes the identity and the multiplication becomes the associative composition law

$$\hom_{\mathcal{C}}(*,*) \times \hom_{\mathcal{C}}(*,*) \to \hom_{\mathcal{C}}(*,*).$$

What if we take an additive category with one object? This should correspond to an Abelian group G (replacing the set of maps) equipped with a group homomorphism

$$G \otimes_{\mathbb{Z}} G \to G$$

which is simply an unusual way of specifying that G is actually a ring. In short, an additive category with one object is just a ring. This coincides with the general philosophy of category theory that, by working in a complex world, one achieves simple descriptions of common mathematical objects.

A slightly fancier version of this same argument is the notion of an operad which has become wildly popular in physics. This is a collection of groups indexed by the natural numbers that act on one another in a prescribed way involving symmetric groups as well. However, we can build a complicated symmetric monoidal closed structure on the category of graded groups, called the circle product, which interelates the various entries of the graded group. Now an operad becomes simply a category with one object enriched over graded groups in this complicated structure.

3. Enriching over Sets$_*$

In our case, enriching over Sets$_*$ just means we have chosen a basepoint for each morphism set and that the composition law descends to the smash product. In other words, any composition with the basepoint on either side produces a basepoint. Just about all of the standard constructions of category theory are available in Sets$_*$-Cat, including limits, pushouts, etc.

One strange thing to notice in general is that there is also a new type of adjoint called an enriched adjoint. The idea is that a standard adjunction simply implies a certain natural isomorphism of morphism sets, but need not extend to isomorphisms as objects of $\mathcal{V}$. Thus, we have a stronger notion of adjoints (hence of limits) which in our case just means the bijections must preserve the basepoints as well, so must themselves be enriched functors. So for us, the fancy new notion does not actually introduce anything new other than the fact that we should only consider adjoints where both functors are pointed (short for enriched over pointed sets).

4. Strict Precategories

I would like to make precise the method of replacing precategories with categories enriched over pointed sets, which is embodied in Theorem 4.2 below. To this end, I need to give a definition of the types of precategories I can deal with, since the literature is somewhat muddled on these objects.

Definition 4.1. (1) A (small) strict precategory $\mathcal{C}$ will consist of two sets, a set of objects and a set of morphism, together with an assignment of source and target object for each morphism. We will also require the existence of a restricted composition law which is associative in the strict sense. That is, there should be a map of sets $M_{B,C,D} \to \hom_{\mathcal{C}}(B,D)$ for some subset $M_{B,C,D} \subset \hom_{\mathcal{C}}(C,D) \times \hom_{\mathcal{C}}(B,C)$ for each triple B,C,D which is associative in the sense that $(fg)h = f(gh)$ whenever both compositions exist and both must fail to exist if either does. In particular, there must exist identity maps which can compose with any other morphism on either side.

(2) A strict prefunctor will denote an assignment $F : \mathcal{C} \to \mathcal{D}$ between (small) strict precategories which preserves the two sets. We also require $F(1) = 1$ and $F(f)F(g) = F(fg)$ in the sense that they are equal when both compositions exist and both compositions fail to exist if either does.
(3) Define $sPre\,\mathrm{Cat}$ to be the category of small strict precategories and strict prefunctors. Notice, this is a category in the usual sense since composition of prefunctors is always defined by construction.

The symbol Cat_* will be used to indicate the category of small categories enriched over Sets_* under the smash product and enriched functors. It is not clear if the following is already known, but it seems useful in any case.

Theorem 4.2. *There is an (adjoint) equivalence of categories between $sPre\,\mathrm{Cat}$ and Cat_*.*

The proof of the theorem is by direct construction of the functors in question, whereupon the theorem is really just an observation. Since the constructions are the main part of the results, the proofs and statements of the intermediate results will be informal.

Lemma 4.3. *There is a functor $\Phi : sPre\,\mathrm{Cat} \to \mathrm{Cat}_*$ defined by adding basepoints to the morphisms sets.*

Given $\mathcal{C} \in sPre\,\mathrm{Cat}$, let the object set of $\Phi(\mathcal{C})$ simply be the object set of $\mathcal{C}$ itself. For each morphism set of $\Phi(\mathcal{C})$, we choose to add an illegal point $*$ which we denote by $\hom_{\mathcal{C}}(B, C)_+$ in line with standard notation for a disjoint basepoint in topology. Now, for each ordered pair in the complement of $M_{B,C,D}$ we define the composition to be $*$ in $\hom_{\mathcal{C}}(B, D)_+$. Since any pair with $*$ in either coordinate is not in $M_{B,C,D}$, we see this descends to a map from the smash product

$$\hom_{\mathcal{C}}(C, D)_+ \wedge \hom_{\mathcal{C}}(B, C)_+ \to \hom_{\mathcal{C}}(B, D)_+.$$

This new composition rule is now associative by the definition of a strict precategory.

For a strict prefunctor $F : \mathcal{C} \to \mathcal{C}'$, we define $\Phi(F)$ as F on objects and single morphisms and $*$ for any ordered pair in the complement of M as well as on each $*$. Since the identities were available and well-behaved in the precategories, this remains the case. To see that $\Phi(F)$ is a functor we must show it commutes with any composition. This splits into two observations: that F preserved existing compositions and that $\Phi(F)$ sends everything else to $*$. Finally, the fact that $\Phi(F)$ preserves $*$ by definition suffices to imply it is an enriched functor. It is immediate that $\Phi : sPre\,\mathrm{Cat} \to \mathrm{Cat}_*$ is then itself a functor, since the notions of composition of functors are consistent in the categories $sPre\,\mathrm{Cat}$ and Cat_*.

Lemma 4.4. *There is a functor $\Psi : \mathrm{Cat}_* \to sPre\,\mathrm{Cat}$ defined by removing basepoints and leaving such compositions undefined.*

Define the functor $\Psi : \mathrm{Cat}_* \to sPre\,\mathrm{Cat}$ on enriched categories as follows. For the objects of $\Psi(\mathcal{D})$, take the object set of $\mathcal{D}$ and for morphisms take the complement of the basepoint in each morphism set. The claim is that $\Psi(\mathcal{D})$ is then a strict precategory, with the restriction of the composition law of $\mathcal{D}$. In other words, we define $M_{B,C,D}$ to be the complement of the preimage of the basepoint under the composition map of $\mathcal{D}$

$$\hom_{\mathcal{D}}(C, D) \wedge \hom_{\mathcal{D}}(B, C) \to \hom_{\mathcal{D}}(B, D)$$

so that pairs in M have a composition in the complement of the basepoint in $\hom_{\mathcal{D}}(B, D)$, i.e. in $\hom_{\Psi(\mathcal{D})}(B, D)$. Since the composition law in $\mathcal{D}$ is associative, we see the result has the required property that $(fg)h = f(gh)$ whenever both compositions are defined and if one fails to exist, it must be the basepoint in $\mathcal{D}$ so that the other must also fail to exist because it also represented the basepoint in $\mathcal{D}$.

For an enriched functor $F : \mathcal{C} \to \mathcal{D}$ we simply take $\Psi(F)$ to be the restriction of the functor F. Notice that the enriched functoriality of F implies $F(fg)$ is the basepoint precisely when $F(f)F(g)$ is the basepoint, so the result is a strict prefunctor (identities are obvious). Once again, the consistency of composition implies $\Psi : \mathrm{Cat}_* \to sPre\,\mathrm{Cat}$ is then a functor as well.

Proof of Theorem 4.2. It should be clear that $\Psi\Phi$ is the identity functor from $sPre\,\mathrm{Cat}$ to itself by construction. To see $\Phi\Psi$ is the identity we simply notice that Ψ only forgets about compositions which go to the basepoint and the basepoint map itself. Since everything about the basepoint can be deduced from the enrichment, this does not actually lose any information. In other words, $\Phi\Psi$ is simply the process of renaming the basepoint as the illegal point $*$ which is (enriched) isomorphic to the original category. □

The theorem is actually quite satisfying as it says that any question about the category of strict precategories may be rephrased in terms of Cat_*. Thus, the functor Φ may be used to bring strict precategories into the world of true category theory without losing any important information visible in $sPre\,\mathrm{Cat}$.

In philosophical terms, we can take the morphisms whose composition we refuse to define and instead define them as an illegal morphism. Regardless, we have refused to give the same information in either description.

5. Collapsing Objects

Remark 5.1. This section is quite general, although the translation of the statement of the Lemma to the Sets_* case is all that we will actually use.

Let Mon_* denote the category of pointed monoids, i.e. the elements of Cat_* with a single object. More generally, let $\mathcal{V}$ be a symmetric monoidal closed category which is also bicomplete, i.e. has all set-indexed limits and colimits. Then the empty limit serves as a final object in $\mathcal{V}$ and we can view the category of objects under the final object, $\mathcal{V}_*$ as the analog of Sets_*. This category is equipped with a "smash product" and reduced coproduct of "wedge" which makes it symmetric monoidal closed and bicomplete as well. Let $\mathcal{V}_*$-Mon represent the subcategory of $\mathcal{V}_*$-Cat consisting of categories with a single object.

Lemma 5.2. *There is a collapse functor M from $\mathcal{V}_*$-Cat to $\mathcal{V}_*$-Mon and a natural transformation $id \to M$. Furthermore, there is an extension M' from $\mathcal{V}$-Cat to $\mathcal{V}_*$-Mon via a $\mathcal{V}$ version of Φ.*

The proof of the lemma is straightforward. Define the object of morphisms from the single object of $M(\mathcal{C})$ to itself as the wedge of the morphism objects of $\mathcal{C}$. Then the composition law is defined by applying the composition law of $\mathcal{C}$ wherever possible and collapsing all of the factors where the composition law of $\mathcal{C}$ does not apply via the zero map.

When we take $\mathcal{V}$ to be Sets we get $\mathcal{V}_* = \text{Sets}_*$ under the smash product. Thus, the target of M is Mon_*, the category of pointed monoids (hence the label M). Then the lemma reduces to the fact that we can define any previously undefined composition as $*$.

Suppose one has a morphism $F : \mathcal{C} \to \mathcal{D}$ in Sets_*-Cat. Then we can apply M to $\mathcal{D}$ and use the natural transformation of the lemma to extend to $F' : \mathcal{C} \to M(\mathcal{D})$ so the target is a pointed monoid. In many cases, the information encoded by F may be recovered from F'. Thus, for some purposes we can fairly readily restrict attention to overcategories where the target is a pointed monoid rather than an arbitrary pointed category.

6. Overview

In the denotational semantics for dialog games Peter McBurney and I are working on, we use morphisms in an element of Cat_* to represent locutions (the objects are simply place-holders of different world-lines if you like) and $*$ is the way to forbid certain combinations of locutions. This is somewhat interesting in the light of the theorem, as it says we could instead simply refuse to define any combination that we want to be illegal and the result would be logically equivalent.

Next, we think of the process of assigning a commitment to each locution as a functor (enriched over Sets_*). Thanks to Lemma 5.2, we can assume the target is a pointed monoid which is much more comfortable for most people.

If we think of the enriched analog of Lemma 5.2 over Abelian groups, we have a very nice result. Since every group has a zero and homomorphisms preserve it, we are already working over the pointed category. Then the Lemma says that every additive category has a ring associated to it, since an additive category with one object is just a ring. This construction is used in representation theory but I have not managed to convince anyone this view of it could be fruitful. It makes clear why there are a variety of important representations for that ring, as the representable objects in the original category should each give an irreducible projective representation. This is one of the applications I had in mind when thinking about these ideas.

The following conjecture is what originally led me to thinking about this program.

Conjecture 6.1. *The category used by Segal to propose a mathematical framework for Topological Quantum Field Theory should be a strict precategory (possibly topological as well) in this sense. Thus the problem of being unable to sew manifolds with different local structures may be avoided by using (a topological extension of)* Φ.

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