Monodic Temporal Resolution: the Expanding Domain Case

Anatoli Degtyarev[2], Michael Fisher[1], and Boris Konev[1]*

[1] Department of Computer Science, University of Liverpool, Liverpool L69 7ZF, U.K.
{M.Fisher, B.Konev}@csc.liv.ac.uk
[2] Department of Computer Science, King's College, Strand, London WC2R 2LS, U.K.
anatoli@dcs.kcl.ac.uk

1 Introduction

Temporal Logic has achieved a significant role in Computer Science, in particular, within the formal specification and verification of concurrent and distributed systems [13, 11, 10]. However, even though *first-order* temporal logics have been studied over a number of years and have been recognised as a concise and powerful formalism, most of the temporal logics used remain essentially propositional. The reason for this is that it is easy to show that first-order temporal logic is, in general, incomplete (i.e. not recursively-enumerable [14]). In fact, until recently, it has been difficult to find *any* non-trivial fragment of first-order temporal logic that has reasonable properties. A breakthrough by Hodkinson *et. al.* [9] showed that *monodic* fragments of first-order temporal logic could be complete, even decidable. (In spite of this, the addition of equality or function symbols leads to the loss of recursive enumerability [15, 4].)

In [3] we presented a resolution procedure for the monodic fragment with interpretations over *constant* domains, based on our work on clausal temporal resolution over a number of years [6, 8, 1, 2]. . In this paper, we provide a complete resolution calculus for monodic first-order temporal logic over expanding domains. The beauty of the result is that we could use exactly the same machinary, by restricting the form of *merged clauses*.

The structure of the paper is as follows. After a brief introduction into first-order temporal logic, we show that any monodic temporal problem can be transformed into a normal form (Section 3) and present the temporal resolution calculus (Section 4). The main focus, within this paper, is on establishing completeness results; hence the detailed definition of the formal underpinnings of the approach in Section 5.

2 First-Order Temporal Logic

First-Order (linear time) Temporal Logic, FOTL, is an extension of classical first-order logic with operators that deal with a linear and discrete model of time (isomorphic to $\mathbb{N}$, and the most commonly used model of time). The first-order temporal language is constructed in a standard way [7, 9] from: *predicate symbols* $P_0, P_1, \ldots$ each of which is

* On leave from Steklov Institute of Mathematics at St.Petersburg

of some fixed arity (null-ary predicate symbols are called *propositions*); *individual variables* $x_0, x_1, \ldots$; *individual constants* $c_0, c_1, \ldots$; *booleans operators* $\wedge$, $\neg$, $\vee$, $\Rightarrow$, $\equiv$, **true** ('true'), **false** ('false'); together with *temporal operators* $\Box$ ('always in the future'), $\Diamond$ ('sometime in the future'), $\bigcirc$ ('at the next moment'), U ('until'), and W ('weak until'). There are no function symbols and equality in our FOTL language.

For a given formula, ϕ, $const(\phi)$ denotes the set of constants occurring in ϕ. We write $\phi(x)$ to indicate that $\phi(x)$ *has at most one free variable* x (if not explicitly said otherwise).

We consider temporal models over *expanding domains*, and formulae in FOTL are interpreted in *first-order temporal structures* of the form $\mathfrak{M} = \langle D_n, I_n \rangle$, $n \in \mathbb{N}$, where every D_n is a non-empty set such that whenever $n < m$, $D_n \subseteq D_m$, and I_n is an interpretation of predicate and constant symbols over D_n. We require that the interpretation of constants is *rigid*: For every constant c and all moments of time $i, j \geq 0$, we have $I_i(c) = I_j(c)$.

A *(variable) assignment* $\mathfrak{a}$ is a function from the set of individual variables to $\cup_{n\in\mathbb{N}} D_n$. (This definition implies that variable assignments are rigid as well.) We denote the set of all assignments by $\mathfrak{V}$.

For every moment of time n, there is a corresponding *first-order* structure, $\mathfrak{M}_n = \langle D_n, I_n \rangle$; the corresponding set of variable assignments $\mathfrak{V}_n$ is a subset of the set of all assignments, $\mathfrak{V}_n = \{\mathfrak{a} \in \mathfrak{V} \mid \mathfrak{a}(x) \in D_n$ for every variable $x\}$; clearly, $\mathfrak{V}_n \subseteq \mathfrak{V}_m$ if $n < m$. Intuitively, FOTL formulae are interpreted in sequences of *worlds*, $\mathfrak{M}_0, \mathfrak{M}_1, \ldots$ with truth values in different worlds being connected by means of temporal operators.

The *truth* relation $\mathfrak{M}_n \models^{\mathfrak{a}} \phi$ in a structure $\mathfrak{M}$, *only for those assignments* $\mathfrak{a}$ *that satisfy the condition* $\mathfrak{a} \in \mathfrak{V}_n$, is defined inductively in the usual way under the following understanding of temporal operators:

$$
\begin{array}{ll}
\mathfrak{M}_n \models^{\mathfrak{a}} \bigcirc\phi & \text{iff } \mathfrak{M}_{n+1} \models^{\mathfrak{a}} \phi; \\
\mathfrak{M}_n \models^{\mathfrak{a}} \Diamond\phi & \text{iff there exists } m > n \text{ such that } \mathfrak{M}_m \models^{\mathfrak{a}} \phi; \\
\mathfrak{M}_n \models^{\mathfrak{a}} \Box\phi & \text{iff for all } m \geq n,\ \mathfrak{M}_m \models^{\mathfrak{a}} \phi; \\
\mathfrak{M}_n \models^{\mathfrak{a}} (\phi \,\mathsf{U}\, \psi) & \text{iff there exists } m \geq n, \text{ such that } \mathfrak{M}_m \models^{\mathfrak{a}} \psi, \\
 & \quad \text{and for all } i \in \mathbb{N},\ n \leq i < m \text{ implies } \mathfrak{M}_m \models^{\mathfrak{a}} \phi; \\
\mathfrak{M}_n \models^{\mathfrak{a}} (\phi \,\mathsf{W}\, \psi) & \text{iff } \mathfrak{M}_n \models^{\mathfrak{a}} (\phi \,\mathsf{U}\, \psi) \text{ or } \mathfrak{M}_n \models^{\mathfrak{a}} \Box\phi.
\end{array}
$$

$\mathfrak{M}$ is a *model* for a formula ϕ (or ϕ is *true* in $\mathfrak{M}$) if there exists an assignment $\mathfrak{a}$ such that $\mathfrak{M}_0 \models^{\mathfrak{a}} \phi$. A formula is *satisfiable* if it has a model. A formula is *valid* if it is satisfiable in any temporal structure under any assignment.

It is known that even "small" fragments of FOTL, such as the *two-variable monadic* fragment (all predicates are unary), are not recursively enumerable [12, 9].

An FOTL-formula ϕ is *monodic* if any subformulae of the form $\mathscr{T}\psi$, where $\mathscr{T}$ one of $\bigcirc$, $\Box$, $\Diamond$ (or $\psi_1 \mathscr{T} \psi_2$, where $\mathscr{T}$ is one of U, W), contains at most one free variable. The set of valid monodic formulae is known to be finitely axiomatisable [15].

3 Divided Separated Normal Form

Definition 1 (Temporal Step Clauses). *A* temporal step clause *is a formula either in the form $p \Rightarrow \bigcirc l$, where p is a proposition and l is a propositional literal, or* $\forall x(P(x) \Rightarrow$

$\bigcirc M(x))$*, where P is a unary predicate and M is a unary literal. We call a clause of the the first type an (original)* ground *step clause, and of the second type an (original)* non-ground *step clause.*

We are now interested in the satisfiability of *unconditional eventuality monodic temporal problems*.

Definition 2 (Monodic Temporal Problem). *An* unconditional eventuality monodic temporal problem*, P, consists of*

1. *a universal part, $\mathscr{U}$, given by a set of arbitrary closed first-order formulae;*
2. *an initial part, $\mathscr{I}$, again, a a set of arbitrary closed first-order formulae; and*
3. *a step part, $\mathscr{S}$, given by a set of (ground and non-ground) temporal step clauses.*
4. *an eventuality part, $\mathscr{E}$, given by a set of unconditional eventuality clauses of the form $\Diamond L(x)$ (a* non-ground *eventuality clause) and $\Diamond l$ (a* ground eventuality *clause), where l is a ground literal and $L(x)$ is a unary non-ground literal.*

Note that, in a monodic temporal problem, we do not allow two different temporal step clauses with the same left-hand sides. In what follows we will not distinguish between a finite set of formulae $\mathscr{X}$ and the conjunction $\bigwedge \mathscr{X}$ of formulae within it. To each unconditional eventuality monodic temporal problem, we associate the formula

$$\mathscr{I} \wedge \Box \mathscr{U} \wedge \Box \forall x \mathscr{S} \wedge \Box \forall x \mathscr{E}.$$

Now, when we talk about particular properties of temporal problems (e.g., satisfiability, validity, logical consequences etc) we mean properties of the associated formula.

Theorem 1 (Transformation). *Every monodic first-order temporal formula can be reduced, in a satisfiability equivalence preserving way, to an unconditional eventuality monodic temporal problem with at most a linear increase in the size of the problem.*

4 Temporal Resolution for the Monodic Expanding Domain Case

Definition 3 (E-Derived Step Clauses). *Let P be a monodic temporal problem, and let*

$$P_{i_1}(x) \Rightarrow \bigcirc M_{i_1}(x), \ldots, P_{i_k}(x) \Rightarrow \bigcirc M_{i_k}(x)$$

be a subset of the set of its original non-ground step clauses. Then

$$\exists x (P_{i_1}(x) \wedge \ldots \wedge P_{i_k}(x)) \Rightarrow \bigcirc \exists x (M_{i_1}(x) \wedge \ldots \wedge M_{i_k}(x)),$$
$$P_{i_j}(c) \Rightarrow \bigcirc M_{i_j}(c)$$

are e-derived *step clauses, where c is a constant occurring in P.*

Note 1. The difference between this definition and the definition in [3] is that we do not consider

$$\forall x (P_{i_1}(x) \vee \ldots \vee P_{i_k}(x)) \Rightarrow \bigcirc \forall x (M_{i_1}(x) \vee \ldots \vee M_{i_k}(x))$$

(which is not a logical consequence of the original step clauses in the expanding domain case) as an e-derived step clause any more.

Definition 4 (Merged E-Derived Step Clauses). *Let*

$$\{\Phi_1 \Rightarrow \bigcirc\Psi_1, \ldots, \Phi_n \Rightarrow \bigcirc\Psi_n\}$$

be a set of e-derived step clauses or original ground *step clauses. Then*

$$\bigwedge_{i=1}^{n} \Phi_i \Rightarrow \bigcirc \bigwedge_{i=1}^{n} \Psi_i$$

is called a merged e-derived step clause.

Definition 5 (Full Merged Step Clauses). *Let $\mathscr{A} \Rightarrow \bigcirc\mathscr{B}$ be a merged e-derived step clause, possibly including the* degenerate *clause* **true** $\Rightarrow \bigcirc$**true**[3]*, let*

$$P_1(x) \Rightarrow \bigcirc M_1(x), \ldots, P_k(x) \Rightarrow \bigcirc M_k(x)$$

be original step clauses, and let $A(x) \leftrightharpoons \bigwedge_{i=1}^{k} P_i(x)$, $B(x) \leftrightharpoons \bigwedge_{i=1}^{k} M_i(x)$.Then

$$\forall x(\mathscr{A} \wedge A(x) \Rightarrow \bigcirc(\mathscr{B} \wedge B(x)))$$

is called a full merged step clause. *(In the case $k = 0$, the conjunctions $A(x)$, $B(x)$ are empty, i.e., their truth value is* **true***, and the merged step clause is just a merged e-derived step clause.)*

Definition 6 (Constant Flooding). *Let P be a monodic temporal problem, and let $\mathfrak{C}$ be the set of constants occurring in P. Now, denote by P^c the set $\mathsf{P} \cup \{\Diamond L(c) \mid \Diamond L(x) \in \mathscr{E}, c \in \mathfrak{C}\}$. If $\mathsf{P} = \mathsf{P}^c$, then we say that the temporal problem is* constant flooded [4].

The inference system we use consists of the following inference rules. (Recall that the premises and conclusion of these rules are (implicitly) closed under the $\Box$ operator.)

Inference Rules. In what follows, $\mathscr{A} \Rightarrow \bigcirc\mathscr{B}$ and $\mathscr{A}_i \Rightarrow \bigcirc\mathscr{B}_i$ denote merged e-derived step clauses, $\forall x(\mathscr{A} \wedge A(x) \Rightarrow \bigcirc(\mathscr{B} \wedge B(x)))$ and $\forall x(\mathscr{A}_i \wedge A_i(x) \Rightarrow \bigcirc(\mathscr{B}_i \wedge B_i(x)))$ denote full merged step clauses, and $\mathscr{U}$ denotes the (current) universal part of the problem.

- *Step resolution rule w.r.t. $\mathscr{U}$:* $\dfrac{\mathscr{A} \Rightarrow \bigcirc\mathscr{B}}{\neg\mathscr{A}}\ (\bigcirc^{\mathscr{U}}_{res})$, where $\mathscr{U} \cup \{\mathscr{B}\} \vdash \bot$.
- *Initial termination rule w.r.t. $\mathscr{U}$:* The contradiction $\bot$ is derived and the derivation is (successfully) terminated if $\mathscr{U} \cup \mathscr{I} \vdash \bot$.
- *Eventuality resolution rule w.r.t. $\mathscr{U}$:*

$$\frac{\begin{array}{lc} \forall x(\mathscr{A}_1 \wedge A_1(x) \Rightarrow \bigcirc(\mathscr{B}_1 \wedge B_1(x))) & \\ \quad\ldots & \Diamond L(x) \\ \forall x(\mathscr{A}_n \wedge A_n(x) \Rightarrow \bigcirc(\mathscr{B}_n \wedge B_n(x))) & \end{array}}{\forall x \bigwedge_{i=1}^{n} (\neg\mathscr{A}_i \vee \neg A_i(x))}\ (\Diamond^{\mathscr{U}}_{res}),$$

[3] This was (implicitly) not allowed in [3], but there is no reason for that: The clause is a logical consequence of the original step clauses; the eventuality resolution rule is still sound; and the completeness result holds.

[4] Strictly speaking, P^c is not in DSNF: we have to rename ground eventualities by propositions. Rather than 'flooding', we could have introduced special inference rules to deal with constants.

where $\forall x(\mathscr{A}_i \wedge A_i(x) \Rightarrow \bigcirc\mathscr{B}_i \wedge B_i(x))$ are full merged step clauses such that for all $i \in \{1,\dots,n\}$, the *loop* side conditions

$$\forall x(\mathscr{U} \wedge \mathscr{B}_i \wedge B_i(x) \Rightarrow \neg L(x)) \quad \text{and} \quad \forall x(\mathscr{U} \wedge \mathscr{B}_i \wedge B_i(x) \Rightarrow \bigvee_{j=1}^{n} (\mathscr{A}_j \wedge A_j(x))$$

are both valid.
The set of merged step clauses, satisfying the loop side conditions, is called a *loop in* $\Diamond L(x)$ and the formula $\bigvee_{j=1}^{n} (\mathscr{A}_j(x) \wedge A_j(x))$ is called a *loop formula*.

- *Eventuality termination rule w.r.t.* $\mathscr{U}$*:* The contradiction $\perp$ is derived and the derivation is (successfully) terminated if $\mathscr{U} \vdash \forall x \neg L(x)$, where $\Diamond L(x) \in \mathscr{E}$.
- *Ground eventuality resolution rule w.r.t.* $\mathscr{U}$*:*

$$\frac{\mathscr{A}_1 \Rightarrow \bigcirc\mathscr{B}_1, \quad \dots, \quad \mathscr{A}_n \Rightarrow \bigcirc\mathscr{B}_n \qquad \Diamond l}{(\bigwedge_{i=1}^{n} \neg\mathscr{A}_i)} \ (\Diamond^{\mathscr{U}}_{res}),$$

where $\mathscr{A}_i \Rightarrow \bigcirc\mathscr{B}_i$ are merged e-derived step clauses such that the *loop* side conditions

$$\mathscr{U} \wedge \mathscr{B}_i \vdash \neg l \quad \text{and} \quad \mathscr{U} \wedge \mathscr{B}_i \vdash \bigvee_{j=1}^{n} \mathscr{A}_j \quad \text{for all} \quad i \in \{1,\dots,n\}$$

are satisfied. *Ground loop* and *ground loop formula* are defined similarly to the case above.
- *Ground eventuality termination rule w.r.t.* $\mathscr{U}$*:*
The contradiction $\perp$ is derived and the derivation is (successfully) terminated if $\mathscr{U} \vdash \neg l$, where $\Diamond l \in \mathscr{E}$.

Successful termination means that the given problem is unsatisfiable.

Note 2. The proof procedure does not change the $\mathscr{I}$, $\mathscr{S}$ or $\mathscr{E}$ parts of the temporal problem; the conclusion of an inference rule is added to $\mathscr{U}$ which is extended step by step.

In the following, we assume that we are given a procedure that checks side conditions of all the inference rules. Thus, our procedure is a decision procedure when the side condition checks are decidable, and is a semi-decision procedure otherwise.

Theorem 2 (Soundness). *From consideration of the models, it straightforwardly follows that the temporal resolution rules preserve satisfiability.*

Theorem 3 (Completeness of Temporal Resolution). *If* P *is an unsatisfiable unconditional eventuality constant flooded monodic temporal problem, then the temporal resolution procedure will derive a contradiction when applied to it.*

4.1 Loop search

The notion of a full merged step clause is quite involved and the search for appropriate merging of simpler clauses is computationally hard. Finding *sets* of such full merged clauses needed for the temporal resolution rule is even more difficult. In [3] we have introduced a search algorithm that finds a *loop formula* (cf. page 5) — a disjunction of the left-hand sides of full merged step clauses that together with an eventuality literal form the premises for the temporal resolution rule. The algorithm is based on a Dixon's loop search algorithm for the propositional case [5]. It can be readily checked that this algorithm can be also employed in the expanding domain case.

5 Completeness of Monodic Resolution Over Expanding Domains

In order to prove completeness of the temporal resolution method, we introduce additional concepts (some of which were already defined in [1]). Let $\mathsf{P} = \langle \mathscr{U}, \mathscr{I}, \mathscr{S} \rangle \cup \mathscr{E}$ be an unconditional eventuality monodic temporal problem. Let $\{P_1, \dots, P_N\}$ and $\{p_1, \dots, p_n\}$, $N, n \geq 0$, be the sets of all (monadic) predicate symbols and all propositional symbols, respectively, occurring in $\mathscr{S} \cup \mathscr{E}$. Let Δ be the set of all mappings from $\{1, \dots, N\}$ to $\{0,1\}$, and Θ be the set of all mappings from $\{1, \dots, n\}$ to $\{0,1\}$. An element $\delta \in \Delta$ ($\theta \in \Theta$) is represented by the sequence $[\delta(1), \dots, \delta(N)] \in \{0,1\}^N$ ($[\theta(1), \dots, \theta(n)] \in \{0,1\}^n$). Let us call elements of Δ and Θ predicate and propositional *colours*, respectively. Let Γ be a subset of Δ, θ be an element of Θ, and ρ be a map from the set of constants of P to Γ. A triple (Γ, θ, ρ) is called a *colour scheme*, and ρ is called a *constant distribution*. If a predicate $P_i(x)$ from $\mathscr{S} \cup \mathscr{E}$ "occurs" in a predicate colour γ (i.e., $\gamma(i) = 1$), we also write $P_i(x) \in \gamma$; and if $\gamma(i) = 0$, we also write $P(x) \notin \gamma$ or $\neg P(x) \in \gamma$. The same convention is used for propositional colours and constant distributions.

For every colour scheme $\mathscr{C} = \langle \Gamma, \theta, \rho \rangle$ let us construct the formulae $\mathscr{F}_\mathscr{C}$, $\mathscr{A}_\mathscr{C}$, $\mathscr{B}_\mathscr{C}$ in the following way. For every $\gamma \in \Gamma$ and for every θ, introduce the conjunctions:

$$F_\gamma(x) = \bigwedge_{\gamma(i)=1 \& i \leq N} P_i(x) \wedge \bigwedge_{\gamma(i)=0 \& i \leq N} \neg P_i(x), \qquad F_\theta = \bigwedge_{\theta(i)=1 \& i \leq n} p_i \wedge \bigwedge_{\theta(i)=0 \& i \leq n} \neg p_i.$$

Let us define two sets of indexes

$J_\gamma = \{i,\ 1 \leq i \leq N \mid \gamma(i) = 1$ and $P_i(x) \Rightarrow \bigcirc M_i(x)$ belongs to $\mathscr{S}$ for some $M_i\}$ and

$J_\theta = \{j,\ 1 \leq i \leq n \mid \theta(j) = 1$ and $p_j \Rightarrow \bigcirc m_i$ belongs to $\mathscr{S}$ for some $m_i\}$.

(Recall that there are no two different step clauses with the same left-hand side.) Let

$A_\gamma(x) = \bigwedge_{i \in J_\gamma} P_i(x), \quad B_\gamma(x) = \bigwedge_{i \in J_\gamma} M_i(x), \quad A_\theta = \bigwedge_{i \in J_\theta} p_i, \quad B_\theta = \bigwedge_{i \in J_\theta} m_i.$

Now $\mathscr{F}_\mathscr{C}$, $\mathscr{A}_\mathscr{C}$, $\mathscr{B}_\mathscr{C}$ are of the following forms[5]:

$$\begin{aligned}
\mathscr{F}_\mathscr{C} &= \bigwedge_{\gamma \in \Gamma} \exists x F_\gamma(x) \wedge F_\theta \wedge \bigwedge_{c \in C} F_{\rho(c)}(c) \wedge \forall x \bigvee_{\gamma \in \Gamma} F_\gamma(x), \\
\mathscr{A}_\mathscr{C} &= \bigwedge_{\gamma \in \Gamma} \exists x A_\gamma(x) \wedge A_\theta \wedge \bigwedge_{c \in C} A_{\rho(c)}(c), \\
\mathscr{B}_\mathscr{C} &= \bigwedge_{\gamma \in \Gamma} \exists x B_\gamma(x) \wedge B_\theta \wedge \bigwedge_{c \in C} B_{\rho(c)}(c).
\end{aligned}$$

We can consider the formula $\mathscr{F}_\mathscr{C}$ as a 'categorical' formula specification of the quotient structure given by a colour scheme. In turn, the formula $\mathscr{A}_\mathscr{C}$ represents the part of this

[5] The difference with the constant domain case is that a universally quantified part is omitted from $\mathscr{A}$ and $\mathscr{B}$.

specification which is 'responsible' just for 'transferring' temporal requirements from the current world (quotient structure) to its immediate successors.

Definition 7 (Canonical merged e-derived step clauses). *Let P be a first-order temporal problem, $\mathscr{C}$ be a colour scheme for P. Then the clause*

$$(\Box)(\mathscr{A}_{\mathscr{C}} \Rightarrow \bigcirc\mathscr{B}_{\mathscr{C}}),$$

is called a canonical merged e-derived step clause *for P.*

If all the sets J_γ, for all $\gamma \in \Gamma$, and J_θ are empty, the clause $(\mathscr{A}_{\mathscr{C}} \Rightarrow \bigcirc\mathscr{B}_{\mathscr{C}})$ degenerates *to* $(\mathbf{true} \Rightarrow \bigcirc\mathbf{true})$. *If a conjunction $A_\gamma(x)$, $\gamma \in \Gamma$, is empty, that is its truth value is* **true***, then the formula $\forall x \bigvee_{\gamma\in\Gamma} A_\gamma(x)$ (or $\forall x \bigvee_{\gamma\in\Gamma} B_\gamma(x)$) disappears from $\mathscr{A}_{\mathscr{C}}$ (or from $\mathscr{B}_{\mathscr{C}}$ respectively). In the propositional case, the clause $(\mathscr{A}_{\mathscr{C}} \Rightarrow \bigcirc\mathscr{B}_{\mathscr{C}})$ reduces to $(A_\theta \Rightarrow \bigcirc B_\theta)$.*

Definition 8 (Canonical merged step clause). *Let $\mathscr{C}$ be a colour scheme, $\mathscr{A}_{\mathscr{C}} \Rightarrow \bigcirc\mathscr{B}_{\mathscr{C}}$ be a canonical merged e-derived step clause, and $\gamma \in \mathscr{C}$.*

$$\forall x(\mathscr{A}_{\mathscr{C}} \wedge A_\gamma(x) \Rightarrow \bigcirc(\mathscr{B}_{\mathscr{C}} \wedge B_\gamma(x)))$$

is called a canonical merged step clause. *If the set J_γ is empty, the truth value of the conjunctions $A_\gamma(x)$, $B_\gamma(x)$ is* **true***, and the canonical merged step clause is just a canonical merged e-derived step clause.*

Now, given a temporal problem $\mathsf{P} = \langle \mathscr{U}, \mathscr{I}, \mathscr{S} \rangle \cup \mathscr{E}$ we define a finite directed graph G as follows. Every node of G is a colour scheme $\mathscr{C}$ for P such that $\mathscr{U} \cup \mathscr{F}_{\mathscr{C}}$ is satisfiable. For each node $\mathscr{C} = (\Gamma, \theta, \rho)$, there is an edge in G to $\mathscr{C}' = (\Gamma', \theta', \rho')$, if $\mathscr{U} \wedge \mathscr{F}_{\mathscr{C}'} \wedge \mathscr{B}_{\mathscr{C}}$ is satisfiable. They are the only edges originating from $\mathscr{C}$.
A node $\mathscr{C}$ is designated as an *initial* node of G if $\mathscr{I} \wedge \mathscr{U} \wedge \mathscr{F}_{\mathscr{C}}$ is satisfiable.
The *behaviour graph* H of P is the subgraph of G induced by the set of all nodes reachable from the initial nodes.

Definition 9 (Path; Path Segment). *A* path*, π, through a behaviour graph, H, is a function from $\mathbb{N}$ to the vertexes of the graph such that for any $i \geq 0$ there is an edge $\langle \pi(i), \pi(i+1) \rangle$ in H. In the similar way, we define a* path segment *as a function from $[m,n]$, $m < n$, to the vertexes of H with the same property.*

Lemma 1. *Let $P_1 = \langle \mathscr{U}_1, \mathscr{I}, \mathscr{S} \rangle \cup \mathscr{E}$ and $P_2 = \langle \mathscr{U}_2, \mathscr{I}, \mathscr{S} \rangle \cup \mathscr{E}$ be two problems such that $\mathscr{U}_1 \subseteq \mathscr{U}_2$. Then the behaviour graph H_2 of P_2 is a subgraph of the behaviour graph H_1 of P_1.*

Proof Satisfiability of $\mathscr{U}_2$ implies satisfiability of $\mathscr{U}_1$. ❑

Definition 10 (Suitability). *For $\mathscr{C} = (\Gamma, \theta, \rho)$ and $\mathscr{C}' = (\Gamma', \theta', \rho')$, let $(\mathscr{C}, \mathscr{C}')$ be an ordered pair of colour schemes for a temporal problem P. An ordered pair of predicate colours (γ, γ') where $\gamma \in \Gamma$, $\gamma' \in \Gamma'$ is called* suitable *if the formula $\mathscr{U} \wedge \exists x(F_{\gamma'}(x) \wedge B_\gamma(x))$ is satisfiable. Similarly, an ordered pair of propositional colours (θ, θ') is suitable if $\mathscr{U} \wedge F_{\theta'} \wedge B_\theta$ is satisfiable, and an ordered pair of constant distributions (ρ, ρ') is suitable if, for every $c \in C$, the pair $(\rho(c), \rho'(c))$ is suitable.*

Note that the satisfiability of $\exists x(F_{\gamma'}(x) \wedge B_\gamma(x))$ implies $\vdash \forall x(F_{\gamma'}(x) \Rightarrow B_\gamma(x))$ as the conjunction $F_{\gamma'}(x)$ contains a valuation at x of *all* predicates occurring in $B_\gamma(x)$.

Lemma 2. *Let H be the behaviour graph for the problem $\mathsf{P} = \langle \mathscr{U}, \mathscr{I}, \mathscr{S}, \mathscr{E} \rangle$ with an edge from a node $\mathscr{C} = (\Gamma, \theta, \rho)$ to a node $\mathscr{C}' = (\Gamma', \theta', \rho')$. Then*

- *for every $\gamma \in \Gamma$ there exists a $\gamma' \in \Gamma'$ such that the pair (γ, γ') is suitable;*
- *the pair of propositional colours (θ, θ') is suitable;*
- *the pair of constant distributions (ρ, ρ') is suitable.*

Proof It can be seen that the proof from [1], Lemma 3 works for our expanding domain case as well. The missing condition 2 of Lemma 3 does not hold in our case. ❑

Let H be the behaviour graph for a temporal problem $\mathsf{P} = \langle \mathscr{U}, \mathscr{I}, \mathscr{S} \rangle \cup \mathscr{E}$ and $\pi = \mathscr{C}_0, \ldots, \mathscr{C}_n, \ldots$ be a path in H where $\mathscr{C}_i = (\Gamma_i, \theta_i, \rho_i)$. Let $\mathscr{G}_0 = \mathscr{I} \cup \{\mathscr{F}_{\mathscr{C}_o}\}$ and $\mathscr{G}_n = \mathscr{F}_{\mathscr{C}_n} \wedge \mathscr{B}_{\mathscr{C}_{n-1}}$ for $n \geq 1$. According to the definition of a behaviour graph, the set $\mathscr{U} \cup \{\mathscr{G}_n\}$ is satisfiable for every $n \geq 0$. From classical model theory, since the language $\mathscr{L}$ is countable and does not contain equality, the following lemma holds.

Lemma 3. *Let κ be a cardinal, $\kappa \geq \aleph_0$. For every $n \geq 0$, if the set $\mathscr{U} \cup \{\mathscr{G}_n\}$ is satisfiable then there exists an $\mathscr{L}$-model $\mathfrak{M}_n = \langle D_n, I_n \rangle$ of $\mathscr{U} \cup \{\mathscr{G}_n\}$ such that for every $\gamma \in \Gamma_n$ the set $D_{(n,\gamma)} = \{a \in D \mid \mathfrak{M}_n \models F_\gamma(a)\}$ is of cardinality κ.*

Definition 11 (Run/E-Run). *Let π be a path through a behaviour graph H of a temporal problem P. By a* run *in π we mean a function mapping its domain, $dom(r) = \{n \in \mathbb{N} \mid n \geq n_0\}$ for some $n_0 \in \mathbb{N}$, to $\bigcup_{i \in \mathbb{N}} \Gamma_i$ with such that for every $n \in dom(r)$, $r(n) \in \Gamma_n$ and the pair $(r(n), r(n+1))$ is suitable. In the similar way, we define a* run segment *as a function from $[m,n]$, $m < n$, to $\bigcup_{i \in \mathbb{N}} \Gamma_i$ with the same property.*
A run r is called an e-run *if $\forall i \in dom(r) \forall \Diamond L(x) \in \mathscr{E} \exists j > i(L(x) \in r(j))$.*

Let π be a path, the set of all runs in π is denoted by $\mathscr{R}(\pi)$, and the set of all e-runs in π is denoted by $\mathscr{R}_e(\pi)$. If π is clear, we may omit it.

Theorem 4 (Existence of a model). *Let $\mathsf{P} =$ be a temporal problem. Let H be the behaviour graph of P, let $\mathscr{C}$ and $\mathscr{C}'$ be nodes of H such that $\mathscr{C} = (\Gamma, \theta, \rho)$ and $\mathscr{C}' = (\Gamma', \theta', \rho')$. If both the set of initial nodes of H is non-empty and the following conditions hold*[6]

$$\forall \mathscr{C} \forall \gamma \in \Gamma \; \forall \Diamond L(x) \in \mathscr{E} \; \exists \mathscr{C}' \; \exists \gamma' \in \Gamma' \; \left((\mathscr{C}, \gamma) \to^+ (\mathscr{C}', \gamma') \wedge L(x) \in \gamma'\right), \quad (1)$$

$$\forall c \in \mathscr{L} \; \forall \mathscr{C} \forall \Diamond L(x) \in \mathscr{E} \qquad \exists \mathscr{C}' \; \left(\mathscr{C} \to^+ \mathscr{C}' \wedge L(x) \in \rho'(c)\right), \quad (2)$$

$$\forall \mathscr{C} \forall \Diamond l \in \mathscr{E} \qquad \exists \mathscr{C}' \; \left(\mathscr{C} \to^+ \mathscr{C}' \wedge l \in \theta'\right), \quad (3)$$

then P has a model.

Proof [of Theorem 4] The proof relies on the following lemma (whose proof follows).

[6] Where $(\mathscr{C}, \gamma) \to^+ (\mathscr{C}', \gamma')$ denotes that there exists a path π from $\mathscr{C}$ to $\mathscr{C}'$ such that γ and γ' belong to a run in π ($\mathscr{C} \to^+ \mathscr{C}'$ denotes that there exists a path from $\mathscr{C}$ to $\mathscr{C}'$).

Lemma 4. *Under the conditions of Theorem 4, there exists a path π through H where:*

(a) $\pi(0)$ *is an initial node of* H*;*
(b) *for every colour scheme* $\mathscr{C} = \pi(i)$, $i \geq 0$, *and every ground eventuality literal* $\Diamond l \in \mathscr{E}$ *there exists a colour scheme* $\mathscr{C}' = \pi(j)$, $j > i$, *such that* $l \in \theta'$*;*
(c) *for every colour scheme* $\mathscr{C} = \pi(i)$, $i \geq 0$ *and every predicate colour* γ *from the colour scheme there exists an e-run* $r \in \mathscr{R}_e(\pi)$ *such that* $r(i) = \gamma$*; and*
(d) *for every constant* $c \in \mathscr{L}$, *the function* $r_c(n)$ *defined by* $r_c(n) = \rho_n(c)$, *where* ρ_n *is the constant distribution from* $\pi(n)$, *is an e-run in* π.

Following [9, 1] take a cardinal $\kappa \geq \aleph_0$ exceeding the cardinality of the set $\mathscr{R}_e$. For every $n \in \mathbb{N}$ we define a domain $D_n = \{\langle r, \xi\rangle \mid r \in \mathscr{R}_e, n \in dom(r), \xi < \kappa\}$. It follows from Lemma 2 that $D_n \subseteq D_m$ if $n < m$.

Then for every $n \in \mathbb{N}$ we have

$D_n = \bigcup_{\gamma \in \Gamma_n} D_{(n,\gamma)}$, where $D_{(n,\gamma)} = \{\langle r,\xi\rangle \in D_n \mid r(n) = \gamma\}$ and $\left|D_{(n,\gamma)}\right| = \kappa$.

Hence, by Lemma 3, for every $n \in \mathbb{N}$ there exists an $\mathscr{L}$-structure $\mathfrak{M}_n = \langle D_n, I_n\rangle$ satisfying $\mathscr{U} \cup \{\mathscr{G}_n\}$ such that $D_{(n,\gamma)} = \{\langle r,\xi\rangle \in D_n \mid \mathfrak{M}_n \models F_\gamma(\langle r,\xi\rangle)\}$. Moreover, we can suppose that $c^{I_n} = \langle r_c, 0\rangle$ for every constant $c \in \mathscr{L}$. A potential first order temporal model is $\mathfrak{M} = \langle D_n, I_n\rangle$ for all $n \in \mathbb{N}$. To be convinced of this we have to check validity of step and eventuality clauses. (Recall that satisfiability of $\mathscr{I}$ in $\mathfrak{M}_0$ is implied by satisfiability of $\mathscr{G}_0$ in $\mathfrak{M}_0$.)

Let $\Box \forall x(P_i(x) \Rightarrow \bigcirc R_i(x))$ be an arbitrary step clause; we show that it is true in $\mathfrak{M}$. Namely, we show that for every $n \geq 0$ and every $\langle r,\xi\rangle \in D_n$, if $\mathfrak{M}_n \models P_i(\langle r,\xi\rangle)$ then $\mathfrak{M}_{n+1} \models R_i(\langle r,\xi\rangle)$. Suppose $r(n) = \gamma \in \Gamma_n$ and $r(n+1) = \gamma' \in \Gamma'$, where (γ,γ') is a suitable pair in accordance with the definition of a run. It follows that $\langle r,\xi\rangle \in D_{(n,\gamma)}$ and $\langle r,\xi\rangle \in D_{(n+1,\gamma')}$, in other words $\mathfrak{M}_n \models F_\gamma(\langle r,\xi\rangle)$ and $\mathfrak{M}_{n+1} \models F_{\gamma'}(\langle r,\xi\rangle)$. Since $\mathfrak{M}_n \models P_i(\langle r,\xi\rangle)$ then $\gamma(i) = 1$. It follows that $R_i(x)$ is a conjunctive member of $B_\gamma(x)$. Since the pair (γ,γ') is suitable, it follows that the conjunction $\exists x(F_{\gamma'}(x) \wedge B_\gamma(x))$ is satisfiable and, moreover, $\vdash \forall x(F_{\gamma'}(x) \Rightarrow B_\gamma(x))$. Together with $\mathfrak{M}_{n+1} \models F_{\gamma'}(\langle r,\xi\rangle)$ this implies that $\mathfrak{M}_{n+1} \models R_i(\langle r,\xi\rangle)$. Propositional step clauses are treated in a similar way.

Let $(\Box \forall x)\Diamond L(x)$ be an arbitrary eventuality clause. We show that for every $n \geq 0$ and every $\langle r,\xi\rangle \in D_n$, there exists $m > n$ such that $\mathfrak{M}_m \models L(\langle r,\xi\rangle)$. Since r is an e-run, there exists $\mathscr{C}' = \pi(m)$ for some $m > n$ such that $r(m) = \gamma' \in \Gamma'$ and $L(x) \in \gamma'$. It follows that $\langle r,\xi\rangle \in D_{(m,\gamma')}$, that is $\mathfrak{M}_m \models F_{\gamma'}(\langle r,\xi\rangle)$. In particular, $\mathfrak{M}_m \models L(\langle r,\xi\rangle)$. Propositional eventuality clauses are considered in a similar way. ❑

Note 3. For *constant flooded* problems condition 3 of Theorem 4 implies condition 2.

Note 4. The theorem still holds a place if a subgraph H' of H is taken instead of the complete behaviour graph H. In this case the relation $\rightarrow^+$ denotes the existence of a path through H', respectively.

Proof [of Lemma 4] We construct a path, π, through the behaviour graph, H, satisfying properties (a), (b), and (d) in exactly the same way as in the proof for constant domains.

The only difference is in the way how we prove condition (c). We assume denotation from that proof.

Let $\mathscr{C} = \pi(i)$ and $\gamma \in \Gamma_{\mathscr{C}}$. Then there exists $\gamma'' \in \mathscr{C}_n$ such that $(\mathscr{C}, \gamma) \to^{+} (\mathscr{C}_n, \gamma'')$. Since for every $\gamma'' \in \mathscr{C}_n$ there exists $\gamma''' \in \mathscr{C}_n^{(\gamma_{s_n}, L_k)}$ such that all eventualities are satisfied on the run-segment from γ'' to γ''' and there exists $\gamma^{(4)} \in \mathscr{C}_n$, $(\mathscr{C}_n^{(\gamma_{s_n}, L_k)}, \gamma''') \to^{+} (\mathscr{C}_n, \gamma^{(4)})$, then there is an e-run, r, such that $r(i) = \gamma$, i.e., property (c) holds[7]. ❑

Proof [of the completeness theorem; Theorem 3] It can be readily checked that the proof for the constant domain case works here. ❑

References

1. A. Degtyarev and M. Fisher. Towards first-order temporal resolution. In *KI 2001, Proceedings*, volume 2174 of *Lecture Notes in Computer Science*, pages 18–32. Springer, 2001.
2. A. Degtyarev, M. Fisher, and B. Konev. A simplified clausal resolution procedure for propositional linear-time temporal logic. In *Tableaux 2002, Proceedings*, volume 2381 of *Lecture Notes in Computer Science*, pages 85–99. Springer, 2002.
3. A. Degtyarev, M Fisher, and B. Konev. Monodic temporal resolution. Submitted. Available as Technical report ULCS-03-001 from `http://www.csc.liv.ac.uk/research/techreports`, 2003.
4. A. Degtyarev, M. Fisher, and A. Lisitsa. Equality and monodic first-order temporal logic. *Studia Logica*, 72(2):147–156, 2002.
5. C. Dixon. Search strategies for resolution in temporal logics. In *Proc. CADE-13*, volume 1104 of *LNAI*, pages 673–687. Springer, 1996.
6. M. Fisher. A resolution method for temporal logic. In J. Myopoulos and R. Reiter, editors, *Proc. IJCAI'91*, pages 99–104. Morgan Kaufman, 1991.
7. M. Fisher. A normal form for first-order temporal formulae. In *Proc. CADE-11*, volume 607 of *LNCS*, pages 370–376. Springer, 1992.
8. M. Fisher, C. Dixon, and M. Peim. Clausal temporal resolution. *ACM Transactions on Computational Logic*, 2(1):12–56, 2001.
9. I. Hodkinson, F. Wolter, and M. Zakharyaschev. Decidable fragments of first-order temporal logics. *Annals of Pure and Applied Logic*, 106:85–134, 2000.
10. G. J. Holzmann. The model checker Spin. *IEEE Trans. on Software Engineering*, 23(5):279–295, 1997.
11. Z. Manna and A. Pnueli. *The Temporal Logic of Reactive and Concurrent Systems: Specification*. Springer, 1992.
12. S. Merz. Decidability and incompleteness results for first-order temporal logic of linear time. *Journal of Applied Non-Classical Logics*, 2:139–156, 1992.
13. A Pnueli. The Temporal Logic of Programs. In *Proceedings of the Eighteenth Symposium on the Foundations of Computer Science*, 1977.
14. A. Szalas and L. Holenderski. Incompleteness of First-Order Temporal Logic with Until. *Theoretical Computer Science*, 57:317–325, 1988.
15. F. Wolter and M. Zakharyaschev. Axiomatizing the monodic fragment of first-order temporal logic. To appear in *Annals of Pure and Applied logic*, 2001.

[7] We do not assume any more that the e-run starts at $\mathscr{C}_0$.