Monodic Temporal Resolution

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Abstract. First-order temporal logic is a concise and powerful notation, with many potential applications in both Computer Science and Artificial Intelligence. While the full logic is highly complex, recent work on *monodic* first-order temporal logics have identified important enumerable and even decidable fragments. In this paper we present the first resolution-based calculus for monodic first-order temporal logic. Although the main focus of the paper is on establishing completeness results, we also consider implementation issues and define a basic loop-search algorithm that may be used to guide the temporal resolution system.

1 Introduction

Temporal Logic has achieved a significant role in Computer Science, in particular, within the formal specification and verification of concurrent and distributed systems [13, 11, 9]. However, even though *first-order* temporal logics have been studied over a number of years and have been recognised as a concise and powerful formalism, most of the temporal logics used remain essentially propositional. The reason for this is that it is easy to show that first-order temporal logic is, in general, incomplete (i.e. not recursively-enumerable [14]). In fact, until recently, it has been difficult to find *any* non-trivial fragment of first-order temporal logic that has reasonable properties. A breakthrough by Hodkinson *et. al.* [8] showed that *monodic* fragments of first-order temporal logic could be complete, even decidable. (In spite of this, the addition of equality or function symbols leads to the loss of recursive enumerability [15, 3].)

The definition of the monodic fragment holds great promise for increasing the power of logic-based formal methods. However, there were, until now, no practical proof techniques for monodic fragments of first-order temporal logics. A general framework, which provides conditions to yield a tableau-based procedure for decidable monodic fragments, and a number of its instantiations, has been presented in [10]. In this paper, we provide a complete resolution calculus for monodic first-order temporal logic, based on our work on clausal temporal resolution over a number of years [5, 7, 1, 2].

The structure of the paper is as follows. After a brief introduction into first-order temporal logic, we show that any monodic temporal problem can be transformed into a normal form (Section 3) and present the temporal resolution calculus (Section 4). The main focus, within this paper, is on establishing completeness results; hence the detailed

⋆ On leave from Steklov Institute of Mathematics at St.Petersburg definition of the formal underpinnings of the approach in Section 5. In Section 6, we consider implementation problems and define a basic *loop search* algorithm that can be used, as in the propositional case, to guide the clausal temporal resolution system.

2 First-Order Temporal Logic

First-Order (linear time) Temporal Logic, FOTL, is an extension of classical first-order logic with operators that deal with a linear and discrete model of time (isomorphic to $\mathbb{N}$, and the most commonly used model of time). The first-order temporal language is constructed in a standard way [6, 8] from: *predicate symbols* $P_0, P_1, \ldots$ each of which is of some fixed arity (null-ary predicate symbols are called *propositions*); *individual variables* $x_0, x_1, \ldots$; *individual constants* $c_0, c_1, \ldots$; *booleans operators* $\wedge$, $\neg$, $\vee$, $\Rightarrow$, $\equiv$ **true** ('true'), **false** ('false'); together with *temporal operators*, such as[3] $\Box$ ('always in the future'), $\Diamond$ ('sometime in the future'), and $\bigcirc$ ('at the next moment'). There are no function symbols and equality in our FOTL language.

Formulae in FOTL are interpreted in *first-order temporal structures* of the form $\mathfrak{M} = \langle D, I \rangle$, where D is a non-empty set, the *domain* of $\mathfrak{M}$, and I is a function associating with every moment of time, $n \in \mathbb{N}$, an interpretation of predicate and constant symbols over D. A *(variable) assignment* $\mathfrak{a}$ over D is a function from the set of individual variables to D. We require that the interpretation of variables and constants is *rigid*, e.g., neither assignments nor interpretations of constants depend on time. For every moment of time n, there is a corresponding *first-order* structure, $\mathfrak{M}_n = \langle D, I_n \rangle$, where $I_n = I(n)$. Intuitively, FOTL formulae are interpreted in sequences of *worlds*, $\mathfrak{M}_0, \mathfrak{M}_1, \ldots$ with truth values in different worlds being connected by means of temporal operators.

The *truth* relation $\mathfrak{M}_n \models^{\mathfrak{a}} \phi$ in a structure $\mathfrak{M}$, for an assignment $\mathfrak{a}$, is defined inductively in the usual way under the following understanding of temporal operators:

$$\begin{aligned}
&\mathfrak{M}_n \models^{\mathfrak{a}} \bigcirc\phi \text{ iff } \mathfrak{M}_{n+1} \models^{\mathfrak{a}} \phi;\\
&\mathfrak{M}_n \models^{\mathfrak{a}} \Diamond\phi \text{ iff there exists } m > n \text{ such that } \mathfrak{M}_m \models^{\mathfrak{a}} \phi;\\
&\mathfrak{M}_n \models^{\mathfrak{a}} \Box\phi \text{ iff for all } m \geq n,\ \mathfrak{M}_m \models^{\mathfrak{a}} \phi.
\end{aligned}$$

$\mathfrak{M}$ is a *model* for a formula ϕ (or ϕ is *true* in $\mathfrak{M}$) if there exists an assignment $\mathfrak{a}$ such that $\mathfrak{M}_0 \models^{\mathfrak{a}} \phi$. A formula is *satisfiable* if it has a model. A formula is *valid* if it is satisfiable in any temporal structure under any assignment.

It is known that even "small" fragments of FOTL, such as the *two-variable monadic* fragment, are not recursively enumerable [12, 8]. An FOTL-formula ϕ is *monodic* if any subformulae of the form $\mathscr{T}\psi$, where $\mathscr{T}$ is a temporal operator, contains at most one free variable. The set of valid monodic formulae is known to be finitely axiomatisable [15].

3 Divided Separated Normal Form

Definition 1 (Temporal Step Clauses). *A* temporal step clause *is a formula either in the form $p \Rightarrow \bigcirc l$, where p is a proposition and l is a propositional literal, or $\forall x(P(x) \Rightarrow$*

[3] W.r.t. satisfiability, other temporal operators can be represented using these operators [6, 1].

$\bigcirc M(x))$*, where* P *is a unary predicate and* M *is a unary literal. We call a clause of the the first type an (original)* ground *step clause, and of the second type an (original)* non-ground *step clause.*

Definition 2 (Monodic Temporal Specification). *A* Monodic Temporal Specification*,* SP*, is a triple consisting of:*

1. *a universal part,* $\mathscr{U}$*, given by a set of arbitrary closed first-order formulae;*
2. *an initial part,* $\mathscr{I}$*, again, a a set of arbitrary closed first-order formulae; and*
3. *a step part,* $\mathscr{S}$*, given by a set of (ground and non-ground) temporal step clauses.*

Note that, in a monodic temporal specification, we do not allow two different temporal step rules with the same left-hand sides. We are now interested in the satisfiability of *unconditional eventuality monodic temporal problems*.

Definition 3 (Monodic Temporal Problem). *An* unconditional eventuality monodic temporal problem*,* P*, consists of a monodic temporal specification* SP *with*

4. *an eventuality part,* $\mathscr{E}$*, given by a set of unconditional eventuality clauses of the form* $\Diamond L(x)$ *(a* non-ground *eventuality clause) and* $\Diamond l$ *(a* ground eventuality *clause), where* l *is a ground literal and* $L(x)$ *is a unary non-ground literal.*

This combination is denoted $\mathsf{P} = \mathsf{SP} \cup \mathscr{E}$.

In what follows we will not distinguish between a finite set of formulae $\mathscr{X}$ and the conjunction $\bigwedge \mathscr{X}$ of formulae within it. To each unconditional eventuality monodic temporal problem, we associate the formula

$$\mathscr{I} \wedge \Box\mathscr{U} \wedge \Box\forall x\mathscr{S} \wedge \Box\forall x\mathscr{E}.$$

Now, when we talk about particular properties of temporal problems (e.g., satisfiability, validity, logical consequences etc) we mean properties of the associated formula. Analogously for temporal specifications.

In [1], it was noted that any monodic FOTL formula can be reduced to a normal form where, in addition to the parts above, *conditional* eventuality clauses of the form

$$P(x) \Rightarrow \Diamond L(x) \text{ and } p \Rightarrow \Diamond l$$

are allowed. The reduction is based on using a renaming technique to substitute non-atomic subformulae and replacing temporal operators by their fixpoint definitions. In [2], a reduction from conditional problems to unconditional ones for the propositional case is given. For the first-order case, satisfiability of $\Phi \cup \{\Box\forall x(P(x) \Rightarrow \Diamond L(x))\}$ is equivalent to satisfiability of

$$\Phi \cup \left\{ \begin{array}{l} \Box\forall x((P(x) \wedge \neg L(x)) \Rightarrow \mathit{waitforL}(x)), \\ \Box\forall x((\mathit{waitforL}(x) \wedge \bigcirc\neg L(x)) \Rightarrow \bigcirc\mathit{waitforL}(x)), \\ \Box\forall x\Diamond\neg\mathit{waitforL}(x) \end{array} \right\},$$

where $\mathit{waitforL}(x)$ is a new unary predicate symbol. (The second clause is translated into a step and a universal clauses.)

Theorem 1 (Transformation). *Every monodic first-order temporal formula can be reduced, in a satisfiability equivalence preserving way, to an unconditional eventuality monodic temporal problem with at most a linear increase in the size of the problem.*

Proof The proof consists of considering possible models and is a straightforward generalisation of Theorem 1 in [2] to the first-order case. ❑

Example 1. Let us consider the temporal formula $\exists x \Box \Diamond \forall y \forall z \exists u \Phi(x,y,z,u)$ where $\Phi(x,y,z,u)$ does not contain temporal operators. First, we rename the inmost temporal subformula by a new predicate,

$$\exists x \Box P_1(x) \wedge \Box \forall x [P_1(x) \Rightarrow \Diamond \forall y \forall z \exists u \Phi(x,y,z,u)].$$

Now, we rename the first '$\Box$'-formula and the formula under the '$\Diamond$' operator,

$$\exists x P_3(x) \wedge \Box \forall x [P_1(x) \Rightarrow \Diamond P_2(x)] \; \Box \forall x [P_2(x) \Rightarrow \forall y \forall z \exists u \Phi(x,y,z,u)] \wedge \Box \forall x [P_3(x) \Rightarrow \Box P_1(x)],$$

"unroll" the '$\Box$' operator

$$\begin{aligned} \exists x P_3(x) \;\wedge\; & \Box \forall x [P_1(x) \Rightarrow \Diamond P_2(x)] \wedge \Box \forall x [P_2(x) \Rightarrow \forall y \forall z \exists u \Phi(x,y,z,u)] \\ \wedge\; & \Box \forall x [P_3(x) \Rightarrow P_4(x)] \wedge \Box \forall x [P_4(x) \Rightarrow \bigcirc P_4(x)] \wedge \Box \forall x [P_4(x) \Rightarrow P_1(x)], \end{aligned}$$

and, finally, remove the condition on the eventuality

$$\begin{aligned} \exists x P_3(x) \;\wedge\; & \Box \forall x [P_2(x) \Rightarrow \forall y \forall z \exists u \Phi(x,y,z,u)] \wedge \Box \forall x [P_3(x) \Rightarrow P_4(x)] \\ \wedge\; & \Box \forall x [P_4(x) \Rightarrow \bigcirc P_4(x)] \wedge \Box \forall x [P_4(x) \Rightarrow P_1(x)] \\ \wedge\; & \Box \forall x [(P_1(x) \wedge \neg P_2(x)) \Rightarrow waitforP_2(x)] \\ \wedge\; & \Box \forall x [(waitforP_2(x) \wedge \bigcirc \neg P_2(x)) \Rightarrow \bigcirc waitforP_2(x)] \\ \wedge\; & \Box \forall x \Diamond \neg waitforP_2(x). \end{aligned}$$

The parts of this formula form the following monodic temporal problem (we also rename the complex $P_2(x) \vee waitforP_2(x)$ expression by $P_5(x)$):

$\mathcal{E} = \{ \Box \forall x \Diamond \neg waitforP_2(x) \}$, $\mathcal{I} = \{ \exists x P_3(x) \}$,

$$\mathcal{U} = \left\{ \begin{array}{l} \forall x (P_2(x) \Rightarrow \forall y \forall z \exists u \Phi(x,y,z,u)), \forall x (P_3(x) \Rightarrow P_4(x)), \quad \forall x (P_4(x) \Rightarrow P_1(x)), \\ \forall x ((P_1(x) \wedge \neg P_2(x)) \Rightarrow waitforP_2(x)), \quad \forall x (P_5(x) \Rightarrow P_2(x) \vee waitforP_2(x)) \end{array} \right\},$$

$\mathcal{S} = \{ P_4(x) \Rightarrow \bigcirc P_4(x), waitforP_2(x) \Rightarrow \bigcirc P_5(x) \}$.

4 Temporal Resolution for Monodic Non-Ground Case

As in the propositional case [5, 2], our calculus works with *merged step clauses*, but here the notion of merged step clauses is much more complex. This is, of course, because of first-order nature of the problem and the fact that skolemization is not allowed under temporal operators. First, we provide some required definitions.

Definition 4 (Derived Step Clauses). *Let* SP *be a monodic temporal specification, and let*

$$P_{i_1}(x) \Rightarrow \bigcirc M_{i_1}(x), \ldots, P_{i_k}(x) \Rightarrow \bigcirc M_{i_k}(x)$$

be a subset of the set of its original non-ground step clauses. Then

$$\forall x(P_{i_1}(x) \vee \ldots \vee P_{i_k}(x)) \Rightarrow \bigcirc \forall x(M_{i_1}(x) \vee \ldots \vee M_{i_k}(x)),$$
$$\exists x(P_{i_1}(x) \wedge \ldots \wedge P_{i_k}(x)) \Rightarrow \bigcirc \exists x(M_{i_1}(x) \wedge \ldots \wedge M_{i_k}(x)),$$
$$P_{i_j}(c) \Rightarrow \bigcirc M_{i_j}(c)$$

are derived *step clauses, where c is a constant occurring in* P.

A derived step clause is a logical consequence of its premises obtained by "dividing" and bounding left-hand and right-hand sides.

Definition 5 (Merged Derived Step Clauses). *Let* $\{\Phi_1 \Rightarrow \bigcirc\Psi_1, \ldots, \Phi_n \Rightarrow \bigcirc\Psi_n\}$ *be a set of derived step clauses or original* ground *step clauses. Then*

$$\bigwedge_{i=1}^{n} \Phi_i \Rightarrow \bigcirc \bigwedge_{i=1}^{n} \Psi_i$$

is called a merged derived step clause. *Note that the left-hand and right-hand sides of any merged derived step clause are closed formulae.*

Definition 6 (Full Merged Step Clauses). *Let* $\mathscr{A} \Rightarrow \bigcirc\mathscr{B}$ *be a merged derived step clause,* $P_1(x) \Rightarrow \bigcirc M_1(x), \ldots, P_k(x) \Rightarrow \bigcirc M_k(x)$, *be original step clauses, and* $A(x) \risingdotseq \bigwedge_{i=1}^{k} P_i(x)$, $B(x) \risingdotseq \bigwedge_{i=1}^{k} M_i(x)$. *Then* $\forall x(\mathscr{A} \wedge A(x) \Rightarrow \bigcirc(\mathscr{B} \wedge B(x)))$ *is called a* full merged step clause. *(In the case where* $k = 0$, *the conjunctions* $A(x)$, $B(x)$ *are empty, i.e., their truth value is* **true**, *and the merged step clause is just a merged derived step clause.*

Definition 7 (Constant Flooding). *Let* P *be a monodic temporal problem, and let* $\mathfrak{C}$ *be the set of constants occurring in* P. *Now, denote by* P^c *the set* $\mathsf{P} \cup \{\Diamond L(c) \mid \Diamond L(x) \in \mathscr{E}, c \in \mathfrak{C}\}$. *If* $\mathsf{P} = \mathsf{P}^c$, *then we say that the temporal problem is* constant flooded [4].

The inference system we use consists of the following inference rules. (Recall that the premises and conclusion of these rules are (implicitly) closed under the $\Box$ operator.)

Inference Rules. In what follows, $\mathscr{A} \Rightarrow \bigcirc\mathscr{B}$ and $\mathscr{A}_i \Rightarrow \bigcirc\mathscr{B}_i$ denote merged derived step clauses, $\forall x(\mathscr{A} \wedge A(x) \Rightarrow \bigcirc(\mathscr{B} \wedge B(x)))$ and $\forall x(\mathscr{A}_i \wedge A_i(x) \Rightarrow \bigcirc(\mathscr{B}_i \wedge B_i(x)))$ denote full merged step clauses, and $\mathscr{U}$ denotes the (current) universal part of the problem.

- *Step resolution rule w.r.t.* $\mathscr{U}$: $\dfrac{\mathscr{A} \Rightarrow \bigcirc\mathscr{B}}{\neg\mathscr{A}}$ $(\bigcirc^{\mathscr{U}}_{res})$, where $\mathscr{U} \cup \{\mathscr{B}\} \vdash \perp$.
- *Initial termination rule w.r.t.* $\mathscr{U}$ *:* The contradiction $\perp$ is derived and the derivation is (successfully) terminated if $\mathscr{U} \cup \mathscr{I} \vdash \perp$.
- *Eventuality resolution rule w.r.t.* $\mathscr{U}$ *:*

$$\frac{\begin{array}{ll} \forall x(\mathscr{A}_1 \wedge A_1(x) \Rightarrow \bigcirc(\mathscr{B}_1 \wedge B_1(x))) & \\ \ldots & \Diamond L(x) \\ \forall x(\mathscr{A}_n \wedge A_n(x) \Rightarrow \bigcirc(\mathscr{B}_n \wedge B_n(x))) & \end{array}}{\forall x \bigwedge_{i=1}^{n} (\neg\mathscr{A}_i \vee \neg A_i(x))} \; (\Diamond^{\mathscr{U}}_{res}),$$

[4] Strictly speaking, P^c is not in DSNF: we have to rename ground eventualities by propositions. Rather than 'flooding', we could have introduced special inference rules to deal with constants.

where $\forall x(\mathscr{A}_i \wedge A_i(x) \Rightarrow \bigcirc \mathscr{B}_i \wedge B_i(x))$ are full merged step clauses such that for all $i \in \{1,\ldots,n\}$, the *loop* side conditions

$$\forall x(\mathscr{U} \wedge \mathscr{B}_i \wedge B_i(x) \Rightarrow \neg L(x)) \quad \text{and} \quad \forall x(\mathscr{U} \wedge \mathscr{B}_i \wedge B_i(x) \Rightarrow \bigvee_{j=1}^{n} (\mathscr{A}_j \wedge A_j(x))$$

are both valid.

The set of merged step clauses, satisfying the loop side conditions, is called a *loop in* $\Diamond L(x)$ and the formula $\bigvee\limits_{j=1}^{n} (\mathscr{A}_j(x) \wedge A_j(x))$ is called a *loop formula*.

- *Eventuality termination rule w.r.t.* $\mathscr{U}$*:* The contradiction $\perp$ is derived and the derivation is (successfully) terminated if $\mathscr{U} \vdash \forall x \neg L(x)$, where $\Diamond L(x) \in \mathscr{E}$.
- *Ground eventuality resolution rule w.r.t.* $\mathscr{U}$*:*

$$\frac{\mathscr{A}_1 \Rightarrow \bigcirc \mathscr{B}_1, \quad \ldots, \quad \mathscr{A}_n \Rightarrow \bigcirc \mathscr{B}_n \qquad \Diamond l}{(\bigwedge\limits_{i=1}^{n} \neg \mathscr{A}_i)} \ (\Diamond^{\mathscr{U}}_{res}),$$

  where $\mathscr{A}_i \Rightarrow \bigcirc \mathscr{B}_i$ are merged grounded step clauses such that the *loop* side conditions

$$\mathscr{U} \wedge \mathscr{B}_i \vdash \neg l \quad \text{and} \quad \mathscr{U} \wedge \mathscr{B}_i \vdash \bigvee_{j=1}^{n} \mathscr{A}_j \quad \text{for all} \quad i \in \{1,\ldots,n\}$$

  are satisfied. *Ground loop* and *ground loop formula* are defined similarly to the case above.
- *Ground eventuality termination rule w.r.t.* $\mathscr{U}$*:*
  The contradiction $\perp$ is derived and the derivation is (successfully) terminated if $\mathscr{U} \vdash \neg l$, where $\Diamond l \in \mathscr{E}$.

Successful termination means that the given problem is unsatisfiable.

Note 1. The proof procedure does not change the $\mathscr{I}$, $\mathscr{S}$ or $\mathscr{E}$ parts of the temporal problem; the conclusion of an inference rule is added to $\mathscr{U}$ which is extended step by step.

Note 2. The *eventuality resolution rule* above can be thought of as two separate rules: an induction rule to extract a formula of the form $\forall x(P(x) \Rightarrow \bigcirc \Box \neg L(x))$ and a resolution rule to resolve this with $\forall y \Diamond L(y)$, i.e.

- *Induction rule w.r.t.* $\mathscr{U}$*:*

$$\frac{\forall x(\mathscr{A}_1 \wedge A_1(x) \Rightarrow \bigcirc(\mathscr{B}_1 \wedge B_1(x)))\ldots\forall x(\mathscr{A}_n \wedge A_n(x) \Rightarrow \bigcirc(\mathscr{B}_n \wedge B_n(x)))}{\forall x(\bigvee\limits_{i=1}^{n} (\mathscr{A}_i \wedge A_i(x)) \Rightarrow \bigcirc \Box \neg L(x))} \ (ind^{\mathscr{U}}),$$

  (with the same side conditions as the eventuality resolution rule above).
  The formula $\bigvee\limits_{i=1}^{n} (\mathscr{B}_i \wedge B_i(x))$ can be considered as an *invariant formula* since, within the loop detected, this formula is always true.

– *Pure eventuality resolution:*

$$\frac{\forall x(\bigvee_{i=1}^{n}(\mathscr{A}_i \wedge A_i(x)) \Rightarrow \bigcirc \Box \neg L(x)) \qquad \Diamond L(x)}{\forall x \bigwedge_{i=1}^{n}(\neg \mathscr{A}_i \vee \neg A_i(x))}\ (\Diamond_{res}).$$

The *ground eventuality resolution rule* can be split into two parts in the similar way.

We state the following results and prove the completeness result later. In the following, we assume that we are given a procedure that checks side conditions of all the inference rules. Thus, our procedure is a decision procedure when the side condition checks are decidable, and is a semi-decision procedure otherwise.

Theorem 2 (Soundness). *From consideration of the models, it straightforwardly follows that the temporal resolution rules preserve satisfiability.*

Theorem 3 (Completeness of Temporal Resolution). *If $P = SP \cup \mathscr{E}$ is an unsatisfiable unconditional eventuality constant flooded monodic temporal problem, then the temporal resolution procedure will derive a contradiction when applied to it.*

Example 2. Let us consider an unsatisfiable temporal problem given by

$$\mathscr{I} = \{\, i1.\ \exists x Q(x) \,\}, \qquad \mathscr{U} = \left\{ \begin{array}{ll} u1. & \Box \exists x(P_1(x) \wedge P_2(x)) \\ u2. & \Box \forall x(Q(x) \wedge \exists y(\neg P_1(y) \wedge \neg P_2(y)) \Rightarrow L(x)) \end{array} \right\},$$

$$\mathscr{E} = \{\, e1.\ \Box \forall x \Diamond \neg L(x) \,\},\ \mathscr{S} = \left\{ \begin{array}{ll} s1. & \Box \forall x(P_1(x) \Rightarrow \bigcirc \neg P_1(x)) \\ s2. & \Box \forall x(P_2(x) \Rightarrow \bigcirc \neg P_2(x)) \\ s3. & \Box \forall x(Q(x) \Rightarrow \bigcirc Q(x)) \end{array} \right\}$$

and apply temporal resolution to this. First, we produce the following derived step clause from $s1$ and $s2$:

$$g1.\ \exists y(P_1(y) \wedge P_2(y)) \Rightarrow \bigcirc \exists y(\neg P_1(y) \wedge \neg P_2(y)).$$

Then merge $g1$ and $s3$ to give

$$m1.\ \ \Box \forall x(\exists y(P_1(y) \wedge P_2(y)) \wedge Q(x) \Rightarrow \bigcirc(\exists y(\neg P_1(y) \wedge \neg P_2(y)) \wedge Q(x)).$$

It can be immediately checked that the loop side conditions are valid for $m1$, i.e.,

$$\begin{array}{ll} \exists y(\neg P_1(y) \wedge \neg P_2(y)) \wedge Q(x) \Rightarrow L(x) & \text{(see } u2\text{)}, \\ \exists y(\neg P_1(y) \wedge \neg P_2(y)) \wedge Q(x) \Rightarrow \exists y(P_1(y) \wedge P_2(y)) \wedge Q(x) & \text{(see } u1\text{)}. \end{array}$$

We apply the eventuality resolution rule to $e1$ and $m1$ and derive a new universal clause

$$nu1.\ \ \Box \forall x(\neg(\exists y(P_1(y) \wedge P_2(y))) \vee \neg Q(x))$$

which contradicts clauses $u1$ and $i1$ (the initial termination rule is applied).

5 Completeness of Temporal Resolution

In order to prove completeness of the temporal resolution method, we introduce additional concepts (some of which were already defined in [1]). Let $\mathsf{P} = \langle \mathscr{U}, \mathscr{I}, \mathscr{S} \rangle \cup \mathscr{E}$ be an unconditional eventuality monodic temporal problem. Let $\{P_1, \dots, P_N\}$ and $\{p_1, \dots, p_n\}$, $N, n \geq 0$, be the sets of all (monadic) predicate symbols and all propositional symbols, respectively, occurring in $\mathscr{S} \cup \mathscr{E}$. Let Δ be the set of all mappings from $\{1, \dots, N\}$ to $\{0, 1\}$, and Θ be the set of all mappings from $\{1, \dots, n\}$ to $\{0, 1\}$. An element $\delta \in \Delta$ ($\theta \in \Theta$) is represented by the sequence $[\delta(1), \dots, \delta(N)] \in \{0, 1\}^N$ ($[\theta(1), \dots, \theta(n)] \in \{0, 1\}^n$). Let us call elements of Δ and Θ predicate and propositional *colours*, respectively. Let Γ be a subset of Δ, θ be an element of Θ, and ρ be a map from the set of constants of P to Γ. A triple (Γ, θ, ρ) is called a *colour scheme*, and ρ is called a *constant distribution*. If a predicate $P_i(x)$ from $\mathscr{S} \cup \mathscr{E}$ "occurs" in a predicate colour γ (i.e., $\gamma(i) = 1$), we also write $P_i(x) \in \gamma$; and if $\gamma(i) = 0$, we also write $P(x) \notin \gamma$ or $\neg P(x) \in \gamma$. The same convention is used for propositional colours and constant distributions.

For every colour scheme $\mathscr{C} = \langle \Gamma, \theta, \rho \rangle$ let us construct the formulae $\mathscr{F}_\mathscr{C}$, $\mathscr{A}_\mathscr{C}$, $\mathscr{B}_\mathscr{C}$ in the following way. For every $\gamma \in \Gamma$ and for every θ, introduce the conjunctions:

$$F_\gamma(x) = \bigwedge_{\gamma(i)=1 \& i \leq N} P_i(x) \wedge \bigwedge_{\gamma(i)=0 \& i \leq N} \neg P_i(x), \qquad F_\theta = \bigwedge_{\theta(i)=1 \& i \leq n} p_i \wedge \bigwedge_{\theta(i)=0 \& i \leq n} \neg p_i.$$

Let us define two sets of indexes

$J_\gamma = \{i,\ 1 \leq i \leq N \mid \gamma(i) = 1$ and $P_i(x) \Rightarrow \bigcirc M_i(x)$ belongs to $\mathscr{S}$ for some $M_i\}$ and

$J_\theta = \{j,\ 1 \leq i \leq n \mid \theta(j) = 1$ and $p_j \Rightarrow \bigcirc m_i$ belongs to $\mathscr{S}$ for some $m_i\}$.

(Recall that there are no two different step clauses with the same left-hand side.) Let $A_\gamma(x) = \bigwedge_{i \in J_\gamma} P_i(x)$, $\quad B_\gamma(x) = \bigwedge_{i \in J_\gamma} M_i(x)$, $\quad A_\theta = \bigwedge_{i \in J_\theta} p_i$, $\quad B_\theta = \bigwedge_{i \in J_\theta} m_i$.

Now $\mathscr{F}_\mathscr{C}$, $\mathscr{A}_\mathscr{C}$, $\mathscr{B}_\mathscr{C}$ are of the following forms:

$$\begin{aligned}
\mathscr{F}_\mathscr{C} &= \bigwedge_{\gamma \in \Gamma} \exists x F_\gamma(x) \wedge F_\theta \wedge \bigwedge_{c \in C} F_{\rho(c)}(c) \wedge \forall x \bigvee_{\gamma \in \Gamma} F_\gamma(x), \\
\mathscr{A}_\mathscr{C} &= \bigwedge_{\gamma \in \Gamma} \exists x A_\gamma(x) \wedge A_\theta \wedge \bigwedge_{c \in C} A_{\rho(c)}(c) \wedge \forall x \bigvee_{\gamma \in \Gamma} A_\gamma(x), \\
\mathscr{B}_\mathscr{C} &= \bigwedge_{\gamma \in \Gamma} \exists x B_\gamma(x) \wedge B_\theta \wedge \bigwedge_{c \in C} B_{\rho(c)}(c) \wedge \forall x \bigvee_{\gamma \in \Gamma} B_\gamma(x).
\end{aligned}$$

We can consider the formula $\mathscr{F}_\mathscr{C}$ as a 'categorical' formula specification of the quotient structure given by a colour scheme. In turn, the formula $\mathscr{A}_\mathscr{C}$ represents the part of this specification which is 'responsible' just for 'transferring' temporal requirements from the current world (quotient structure) to its immediate successors.

Definition 8 (Canonical merged derived step clauses). *Let P be a first-order temporal problem, $\mathscr{C}$ be a colour scheme for P. Then the clause*

$$(\square)(\mathscr{A}_\mathscr{C} \Rightarrow \bigcirc \mathscr{B}_\mathscr{C}),$$

is called a canonical merged derived step clause *for P.*

If all the sets J_γ, for all $\gamma \in \Gamma$, and J_θ are empty, the clause $(\mathscr{A}_\mathscr{C} \Rightarrow \bigcirc \mathscr{B}_\mathscr{C})$ degenerates *to* (**true** $\Rightarrow \bigcirc$**true**). *If a conjunction $A_\gamma(x)$, $\gamma \in \Gamma$, is empty, that is its truth*

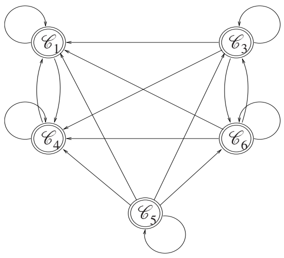

Fig. 1. Behaviour graph for the problem $\mathscr{I} = \emptyset$, $\mathscr{U} = \{l \Rightarrow \exists x P(x)\}$, $\mathscr{S} = \{P(x) \Rightarrow \bigcirc P(x)\}$, $\mathscr{E} = \{\Diamond \neg P(x), \Diamond l\}$ (Example 3).

value is **true***, then the formula* $\forall x \bigvee_{\gamma \in \Gamma} A_\gamma(x)$ *(or* $\forall x \bigvee_{\gamma \in \Gamma} B_\gamma(x)$*) disappears from* $\mathscr{A}_\mathscr{C}$ *(or from* $\mathscr{B}_\mathscr{C}$ *respectively). In the propositional case, the clause* $(\mathscr{A}_\mathscr{C} \Rightarrow \bigcirc \mathscr{B}_\mathscr{C})$ *reduces to* $(A_\theta \Rightarrow \bigcirc B_\theta)$.

Definition 9 (Canonical merged step clause). *Let* $\mathscr{C}$ *be a colour scheme,* $\mathscr{A}_\mathscr{C} \Rightarrow \bigcirc \mathscr{B}_\mathscr{C}$ *be a canonical merged derived step clause, and* $\gamma \in \mathscr{C}$.

$$\forall x (\mathscr{A}_\mathscr{C} \wedge A_\gamma(x) \Rightarrow \bigcirc (\mathscr{B}_\mathscr{C} \wedge B_\gamma(x)))$$

is called a canonical merged step clause. *If the set* J_γ *is empty, the truth value of the conjunctions* $A_\gamma(x)$, $B_\gamma(x)$ *is* **true***, and the canonical merged step clause is just a canonical merged derived step clause.*

Now, given a temporal problem $\mathsf{P} = \langle \mathscr{U}, \mathscr{I}, \mathscr{S} \rangle \cup \mathscr{E}$ we define a finite directed graph G as follows. Every node of G is a colour scheme $\mathscr{C}$ for P such that $\mathscr{U} \cup \mathscr{F}_\mathscr{C}$ is satisfiable. For each node $\mathscr{C} = (\Gamma, \theta, \rho)$, there is an edge in G to $\mathscr{C}' = (\Gamma', \theta', \rho')$, if $\mathscr{U} \wedge \mathscr{F}_{\mathscr{C}'} \wedge \mathscr{B}_\mathscr{C}$ is satisfiable. They are the only edges originating from $\mathscr{C}$.
A node $\mathscr{C}$ is designated as an *initial* node of G if $\mathscr{I} \wedge \mathscr{U} \wedge \mathscr{F}_\mathscr{C}$ is satisfiable.
The *behaviour graph* H of P is the subgraph of G induced by the set of all nodes reachable from the initial nodes.

Example 3. Consider a monodic temporal problem, P, given by
$\mathscr{I} = \emptyset$, $\mathscr{U} = \{l \Rightarrow \exists x P(x)\}$, $\mathscr{S} = \{P(x) \Rightarrow \bigcirc P(x)\}$, $\mathscr{E} = \{\Diamond \neg P(x), \Diamond l\}$.
For this problem, there exist two predicate colours, $\gamma_1 = [1]$ and $\gamma_2 = [0]$; two propositional colours $\theta_1 = [1]$ and $\theta_2 = [0]$; and six colour schemes, $\mathscr{C}_1 = (\{\gamma_1\}, \theta_1)$, $\mathscr{C}_2 = (\{\gamma_2\}, \theta_1)$, $\mathscr{C}_3 = (\{\gamma_1, \gamma_2\}, \theta_1)$. $\mathscr{C}_4 = (\{\gamma_1\}, \theta_2)$, $\mathscr{C}_5 = (\{\gamma_2\}, \theta_2)$, $\mathscr{C}_6 = (\{\gamma_1, \gamma_2\}, \theta_2)$. Now, we construct the formulae:

$\mathscr{F}_{\mathscr{C}_1} = \exists x P(x) \wedge \forall x P(x) \wedge l$	$\mathscr{A}_{\mathscr{C}_1} = \exists x P(x) \wedge \forall x P(x)$	$\mathscr{B}_{\mathscr{C}_1} = \exists x P(x) \wedge \forall x P(x)$
$\mathscr{F}_{\mathscr{C}_2} = \exists x \neg P(x) \wedge \forall x \neg P(x) \wedge l$	$\mathscr{A}_{\mathscr{C}_2} = \mathbf{true}$	$\mathscr{B}_{\mathscr{C}_2} = \mathbf{true}$
$\mathscr{F}_{\mathscr{C}_3} = \exists x P(x) \wedge \exists x \neg P(x) \wedge l$	$\mathscr{A}_{\mathscr{C}_3} = \exists x P(x)$	$\mathscr{B}_{\mathscr{C}_3} = \exists x P(x)$
$\mathscr{F}_{\mathscr{C}_4} = \exists x P(x) \wedge \forall x P(x) \wedge \neg l$	$\mathscr{A}_{\mathscr{C}_4} = \exists x P(x) \wedge \forall x P(x)$	$\mathscr{B}_{\mathscr{C}_4} = \exists x P(x) \wedge \forall x P(x)$
$\mathscr{F}_{\mathscr{C}_5} = \exists x \neg P(x) \wedge \forall x \neg P(x) \wedge \neg l$	$\mathscr{A}_{\mathscr{C}_5} = \mathbf{true}$	$\mathscr{B}_{\mathscr{C}_5} = \mathbf{true}$
$\mathscr{F}_{\mathscr{C}_6} = \exists x P(x) \wedge \exists x \neg P(x) \wedge \neg l$	$\mathscr{A}_{\mathscr{C}_6} = \exists x P(x)$	$\mathscr{B}_{\mathscr{C}_6} = \exists x P(x)$

Note that $\mathscr{F}_{\mathscr{C}_2} \wedge \mathscr{U} \models \perp$. The behaviour graph for P, given in Fig. 1, consists of five vertexes; all nodes are initial.

Definition 10 (Path; Path Segment). *A* path*, π, through a behaviour graph, H, is a function from $\mathbb{N}$ to the vertexes of the graph such that for any $i \geq 0$ there is an edge $\langle \pi(i), \pi(i+1) \rangle$ in H. In the similar way, we define a* path segment *as a function from $[m,n]$, $m < n$, to the vertexes of H with the same property.*

Lemma 1. *Let $P_1 = \langle \mathscr{U}_1, \mathscr{I}, \mathscr{S} \rangle \cup \mathscr{E}$ and $P_2 = \langle \mathscr{U}_2, \mathscr{I}, \mathscr{S} \rangle \cup \mathscr{E}$ be two problems such that $\mathscr{U}_1 \subseteq \mathscr{U}_2$. Then the behaviour graph H_2 of P_2 is a subgraph of the behaviour graph H_1 of P_1.*

Proof Satisfiability of $\mathscr{U}_2$ implies satisfiability of $\mathscr{U}_1$. ❑

Definition 11 (Suitability). *For $\mathscr{C} = (\Gamma, \theta, \rho)$ and $\mathscr{C}' = (\Gamma', \theta', \rho')$, let $(\mathscr{C}, \mathscr{C}')$ be an ordered pair of colour schemes for a temporal problem P. An ordered pair of predicate colours (γ, γ') where $\gamma \in \Gamma$, $\gamma' \in \Gamma'$ is called* suitable *if the formula $\mathscr{U} \wedge \exists x (F_{\gamma'}(x) \wedge B_\gamma(x))$ is satisfiable. Similarly, an ordered pair of propositional colours (θ, θ') is suitable if $\mathscr{U} \wedge F_{\theta'} \wedge B_\theta$ is satisfiable, and an ordered pair of constant distributions (ρ, ρ') is suitable if, for every $c \in C$, the pair $(\rho(c), \rho'(c))$ is suitable.*

Note that the satisfiability of $\exists x (F_{\gamma'}(x) \wedge B_\gamma(x))$ implies $\vdash \forall x (F_{\gamma'}(x) \Rightarrow B_\gamma(x))$ as the conjunction $F_{\gamma'}(x)$ contains a valuation at x of *all* predicates occurring in $B_\gamma(x)$.

Lemma 2. *Let H be the behaviour graph for the specification $SP = \langle \mathscr{U}, \mathscr{I}, \mathscr{S} \rangle$ with an edge from a node $\mathscr{C} = (\Gamma, \theta, \rho)$ to a node $\mathscr{C}' = (\Gamma', \theta', \rho')$. Then*

- *for every $\gamma \in \Gamma$ there exists a $\gamma' \in \Gamma'$ such that the pair (γ, γ') is suitable;*
- *for every $\gamma' \in \Gamma'$ there exists a $\gamma \in \Gamma$ such that the pair (γ, γ') is suitable;*
- *the pair of propositional colours (θ, θ') is suitable;*
- *the pair of constant distributions (ρ, ρ') is suitable.*

Proof See [1]. ❑

Let H be the behaviour graph for a temporal problem $\mathsf{P} = \langle \mathscr{U}, \mathscr{I}, \mathscr{S} \rangle \cup \mathscr{E}$ and $\pi = \mathscr{C}_0, \ldots, \mathscr{C}_n, \ldots$ be a path in H where $\mathscr{C}_i = (\Gamma_i, \theta_i, \rho_i)$. Let $\mathscr{G}_0 = \mathscr{I} \cup \{\mathscr{F}_{\mathscr{C}_o}\}$ and $\mathscr{G}_n = \mathscr{F}_{\mathscr{C}_n} \wedge \mathscr{B}_{\mathscr{C}_{n-1}}$ for $n \geq 1$. According to the definition of a behaviour graph, the set $\mathscr{U} \cup \{\mathscr{G}_n\}$ is satisfiable for every $n \geq 0$. From classical model theory, since the language $\mathscr{L}$ is countable and does not contain equality, the following lemma holds.

Lemma 3. *Let κ be a cardinal, $\kappa \geq \aleph_0$. For every $n \geq 0$, if the set $\mathscr{U} \cup \{\mathscr{G}_n\}$ is satisfiable then there exists an $\mathscr{L}$-model $\mathfrak{M}_n = \langle D, I_n \rangle$ of $\mathscr{U} \cup \{\mathscr{G}_n\}$ such that for every $\gamma \in \Gamma_n$ the set $D_{(n,\gamma)} = \{a \in D \mid \mathfrak{M}_n \models F_\gamma(a)\}$ is of cardinality κ.*

Definition 12 (Run/E-Run). *Let π be a path through a behaviour graph H of a temporal problem P. By a* run *in π we mean a function from $\mathbb{N}$ to $\bigcup_{i \in \mathbb{N}} \Gamma_i$ such that for every $n \in \mathbb{N}$, $r(n) \in \Gamma_n$ and the pair $(r(n), r(n+1))$ is suitable. In the similar way, we define a* run segment *as a function from $[m,n]$, $m < n$, to $\bigcup_{i \in \mathbb{N}} \Gamma_i$ with the same property.*
A run r is called an e-run *if $\forall i \geq 0 \forall \Diamond L(x) \in \mathscr{E} \exists j > i (L(x) \in r(j))$.*

Let π be a path, the set of all runs in π is denoted by $\mathscr{R}(\pi)$, and the set of all e-runs in π is denoted by $\mathscr{R}_e(\pi)$. If π is clear, we may omit it.

Theorem 4 (Existence of a model). *Let $\mathsf{P} = \mathsf{SP} \cup \mathscr{E}$ be a temporal problem. Let H be the behaviour graph of P, let $\mathscr{C}$ and $\mathscr{C}'$ be nodes of H such that $\mathscr{C} = (\Gamma, \theta, \rho)$ and $\mathscr{C}' = (\Gamma', \theta', \rho')$. If both the set of initial nodes of H is non-empty and the following conditions hold*[5]

$$\forall\gamma \in \Gamma\ \forall\mathscr{C}\forall\Diamond L(x) \in \mathscr{E}\ \exists\gamma' \in \Gamma'\ \exists\mathscr{C}'\ \left((\mathscr{C},\gamma) \to^{+} \left(\mathscr{C}',\gamma'\right) \wedge L(x) \in \gamma'\right), \quad (1)$$

$$\forall c \in \mathscr{L}\ \forall\mathscr{C}\forall\Diamond L(x) \in \mathscr{E}\qquad \exists\mathscr{C}'\ \left(\mathscr{C} \to^{+} \mathscr{C}' \wedge L(x) \in \rho'(c)\right), \quad (2)$$

$$\forall\mathscr{C}\forall\Diamond l \in \mathscr{E}\qquad \exists\mathscr{C}'\ \left(\mathscr{C} \to^{+} \mathscr{C}' \wedge l \in \theta'\right), \quad (3)$$

then P has a model.

Proof [of Theorem 4] The proof relies on the following lemma (whose proof follows).

Lemma 4. *Under the conditions of Theorem 4, there exists a path π through H where:*

(a) $\pi(0)$ is an initial node of H;
(b) for every colour scheme $\mathscr{C} = \pi(i)$, $i \geq 0$, and every ground eventuality literal $\Diamond l \in \mathscr{E}$ there exists a colour scheme $\mathscr{C}' = \pi(j)$, $j > i$, such that $l \in \theta'$;
(c) for every colour scheme $\mathscr{C} = \pi(i)$, $i \geq 0$ and every predicate colour γ from the colour scheme there exists an e-run $r \in \mathscr{R}_e(\pi)$ such that $r(i) = \gamma$; and
(d) for every constant $c \in \mathscr{L}$, the function $r_c(n)$ defined by $r_c(n) = \rho_n(c)$, where ρ_n is the constant distribution from $\pi(n)$, is an e-run in π.

Following [8, 1] take a cardinal $\kappa \geq \aleph_0$ exceeding the cardinality of the set $\mathscr{R}_e$. Let us define a domain $D = \{\langle r,\xi\rangle \mid r \in \mathscr{R}_e, \xi < \kappa\}$. Then for every $n \in \mathbb{N}$ we have

$$D = \bigcup_{\gamma\in\Gamma_n} D_{(n,\gamma)}, \text{ where } D_{(n,\gamma)} = \{\langle r,\xi\rangle \mid r(n) = \gamma\} \text{ and } \left|D_{(n,\gamma)}\right| = \kappa.$$

Hence, by Lemma 3, for every $n \in \mathbb{N}$ there exists an $\mathscr{L}$-structure $\mathfrak{M}_n = \langle D, I_n\rangle$ satisfying $\mathscr{U} \cup \{\mathscr{G}_n\}$ such that $D_{(n,\gamma)} = \{\langle r,\xi\rangle \in D \mid \mathfrak{M}_n \models F_\gamma(\langle r,\xi\rangle)\}$. Moreover, we can suppose that $c^{I_n} = \langle r_c, 0\rangle$ for every constant $c \in \mathscr{L}$. A potential first order temporal model is $\mathfrak{M} = \langle D, I\rangle$, where $I(n) = I_n$ for all $n \in \mathbb{N}$. To be convinced of this we have to check validity of step and eventuality clauses. (Recall that satisfiability of $\mathscr{I}$ in $\mathfrak{M}_0$ is implied by satisfiability of $\mathscr{G}_0$ in $\mathfrak{M}_0$.)

Let $\Box\forall x(P_i(x) \Rightarrow \bigcirc R_i(x))$ be an arbitrary step clause; we show that it is true in $\mathfrak{M}$. Namely, we show that for every $n \geq 0$ and every $\langle r,\xi\rangle \in D$, if $\mathfrak{M}_n \models P_i(\langle r,\xi\rangle)$ then $\mathfrak{M}_{n+1} \models R_i(\langle r,\xi\rangle)$. Suppose $r(n) = \gamma \in \Gamma_n$ and $r(n+1) = \gamma' \in \Gamma'$, where (γ,γ') is a suitable pair in accordance with the definition of a run. It follows that $\langle r,\xi\rangle \in D_{(n,\gamma)}$ and $\langle r,\xi\rangle \in D_{(n+1,\gamma')}$, in other words $\mathfrak{M}_n \models F_\gamma(\langle r,\xi\rangle)$ and $\mathfrak{M}_{n+1} \models F_{\gamma'}(\langle r,\xi\rangle)$. Since $\mathfrak{M}_n \models P_i(\langle r,\xi\rangle)$ then $\gamma(i) = 1$. It follows that $R_i(x)$ is a conjunctive member of $B_\gamma(x)$. Since the pair (γ,γ') is suitable, it follows that the conjunction $\exists x(F_{\gamma'}(x) \wedge B_\gamma(x))$ is satisfiable and, moreover, $\vdash \forall x(F_{\gamma'}(x) \Rightarrow B_\gamma(x))$. Together with $\mathfrak{M}_{n+1} \models F_{\gamma'}(\langle r,\xi\rangle)$ this implies that $\mathfrak{M}_{n+1} \models R_i(\langle r,\xi\rangle)$. Propositional step clauses are treated in a similar way.

[5] Where $(\mathscr{C},\gamma) \to^{+} (\mathscr{C}',\gamma')$ denotes that there exists a path π from $\mathscr{C}$ to $\mathscr{C}'$ such that γ and γ' belong to a run in π ($\mathscr{C} \to^{+} \mathscr{C}'$ denotes that there exists a path from $\mathscr{C}$ to $\mathscr{C}'$).

Let $(\Box\forall x)\Diamond L(x)$ be an arbitrary eventuality clause. We show that for every $n \geq 0$ and every $\langle r,\xi\rangle \in D$, $r \in \mathscr{R}_e, \xi < \kappa$, there exists $m > n$ such that $\mathfrak{M}_m \models L(\langle r,\xi\rangle)$. Since r is an e-run, there exists $\mathscr{C}' = \pi(m)$ for some $m > n$ such that $r(m) = \gamma' \in \Gamma'$ and $L(x) \in \gamma'$. It follows that $\langle r,\xi\rangle \in D_{(m,\gamma')}$, that is $\mathfrak{M}_m \models F_{\gamma'}(\langle r,\xi\rangle)$. In particular, $\mathfrak{M}_m \models L(\langle r,\xi\rangle)$. Propositional eventuality clauses are considered in a similar way. ❑

Note 3. For *constant flooded* problems condition 3 of Theorem 4 implies condition 2.

Note 4. The theorem still holds a place if a subgraph H' of H is taken instead of the complete behaviour graph H. In this case the relation $\to^+$ denotes the existence of a path through H', respectively.

Proof [of Lemma 4] Let $\Diamond L_1(x),\ldots,\Diamond L_k(x)$ be all non-ground eventuality literals from $\mathscr{E}$; $\Diamond l_1,\ldots,\Diamond l_p$ be all ground eventuality literals from $\mathscr{E}$; and $c_1,\ldots,c_q$ be all constants of P. Let $\mathscr{C}_0$ be an initial node of H. We construct the path π as follows. Let $\{\gamma_1,\ldots,\gamma_{s_0}\}$ be all predicate colours from $\Gamma_{\mathscr{C}_0}$. By condition (1) there exists a node $\mathscr{C}_0^{(\gamma_1,L_1)}$ and a predicate colour $\gamma_1^{(1)} \in \Gamma_{\mathscr{C}_0^{(\gamma_1,L_1)}}$ such that $(\mathscr{C}_0,\gamma_1) \to^+ (\mathscr{C}_0^{(\gamma_1,L_1)},\gamma_1^{(1)})$ and $L_1(x) \in \gamma_1^{(1)}$. In the same way, there exists a node $\mathscr{C}_0^{(\gamma_1,L_2)}$ and a predicate colour $\gamma_1^{(2)} \in \Gamma_{\mathscr{C}_0^{(\gamma_1,L_2)}}$ such that $(\mathscr{C}_0^{(\gamma_1,L_1)},\gamma_1^{(1)}) \to^+ (\mathscr{C}_0^{(\gamma_1,L_2)},\gamma_1^{(2)})$ and $L_2(x) \in \gamma_1^{(2)}$. And so on. Finally, there exists a node $\mathscr{C}_0^{(\gamma_1,L_k)}$ and a predicate colour $\gamma_1^{(k)} \in \Gamma_{\mathscr{C}_0^{(\gamma_1,L_k)}}$ such that $(\mathscr{C}_0^{(\gamma_1,L_{k-1})},\gamma_1^{(k-1)}) \to^+ (\mathscr{C}_0^{(\gamma_1,L_k)},\gamma_1^{(k)})$ and $L_k(x) \in \gamma_1^{(k)}$. Clearly, $\gamma_1,\ldots,\gamma_1^{(1)},\ldots,\gamma_1^{(2)},\ldots,\gamma_1^{(k)}$ forms a segment of a run and every non-ground eventuality is satisfied along this segment.

Now, let $\gamma_2^{(0)}$ be any successor of γ_2 in $\Gamma_{\mathscr{C}_0^{(\gamma_1,L_k)}}$. As above, there exists a sequence of nodes $\mathscr{C}_0^{(\gamma_2,L_1)},\ldots,\mathscr{C}_0^{(\gamma_2,L_k)}$ and a sequence of predicate colours $\gamma_2^{(1)} \in \Gamma_{\mathscr{C}_0^{(\gamma_2,L_1)}},\ldots,\gamma_2^{(k)} \in \Gamma_{\mathscr{C}_0^{(\gamma_2,L_k)}}$ such that $\gamma_2,\ldots,\gamma_2^{(0)},\ldots,\gamma_2^{(1)},\ldots,\gamma_2^{(k)}$ forms a segment of a run and every non-ground eventuality is satisfied along this segment. And so on. At certain point we construct a segment of a path from $\mathscr{C}_0$ to a node $\mathscr{C}_0^{(\gamma_{s_0},L_k)}$ such that for every $\gamma \in \mathscr{C}_0$ there exists $\gamma' \in \mathscr{C}_0^{(\gamma_{s_0},L_k)}$ such that all eventualities are satisfied on the run-segment from γ to γ'.

In a similar way we can construct a node $\mathscr{C}_0^{(c_1,L_1)}$ such that $\mathscr{C}_0^{(\gamma_{s_0},L_k)} \to^+ \mathscr{C}_0^{(c_1,L_1)}$ and $L_1(x) \in \rho_{\mathscr{C}_0^{(c_1,L_1)}}(c_1)$. And so on. Then we can construct a node $\mathscr{C}_0^{(l_1)}$ such that $\mathscr{C}_0^{(c_q,L_k)} \to^+ \mathscr{C}_0^{(l_1)}$ and $l_1 \in \theta_{\mathscr{C}_0^{(l_1)}}$. And so on.

Finally, we construct a node $\mathscr{C}_0' = \mathscr{C}_0^{(l_p)}$ such that $\mathscr{C}_0 \to^+ \mathscr{C}_0'$ and on this path segment all conditions of the theorem hold for $\mathscr{C} = \mathscr{C}_0$. Let us denote this path segment as λ_0, and let $\mathscr{C}_1$ be any successor of $\mathscr{C}_0'$.

By analogy, we can construct a node $\mathscr{C}_1'$ and a path segment λ_1 from $\mathscr{C}_1$ to $\mathscr{C}_1'$ such that all conditions of the theorem hold for $\mathscr{C} = \mathscr{C}_1$. An so forth. Eventually, we construct a sequence $\mathscr{C}_0, \mathscr{C}_1, \ldots, \mathscr{C}_j$ such that there exists n, $0 \leq n < j$ and $\mathscr{C}_n = \mathscr{C}_j$ because there are only finitely many different colour schemes. Let $\pi_1 = \lambda_0, \ldots, \lambda_{n-1}$, $\pi_2 = \lambda_n, \ldots \lambda_{j-1}$. Now, we define our path π as $\pi_1(\pi_2)^*$. Properties (a) and (b) evidently hold on π.

Let $\mathscr{C} = \pi(i)$ and $\gamma \in \Gamma_{\mathscr{C}}$. Clearly, there exist $\gamma' \in \mathscr{C}_0$ and $\gamma'' \in \mathscr{C}_n$ such that $(\mathscr{C}_0, \gamma') \to^+ (\mathscr{C}, \gamma)$ and $(\mathscr{C}, \gamma) \to^+ (\mathscr{C}_n, \gamma'')$. Since for every $\gamma'' \in \mathscr{C}_n$ there exists $\gamma''' \in \mathscr{C}_n^{(\gamma_n, L_k)}$ such that all eventualities are satisfied on the run-segment from γ'' to γ''' and there exists $\gamma^{(4)} \in \mathscr{C}_n$, $(\mathscr{C}_n^{(\gamma_n, L_k)}, \gamma''') \to^+ (\mathscr{C}_n, \gamma^{(4)})$, then there is an e-run, r, such that $r(i) = \gamma$, i.e., property (c) holds.

Note that, for every constant c of P the sequence $r_c(n)$ is a run in π. By construction, for every $\Diamond L(x) \in \mathscr{E}$ there is a node $\mathscr{C}_n^{(c,L)}$ in π_2 such that $L(x) \in \rho_{\mathscr{C}_n^{(c,L)}}(c)$. Therefore, $r_c(n)$ is an e-run in π and property (d) holds. ❑

Proof [of the completeness theorem; Theorem 3] The proof proceeds by induction on the number of nodes in the behaviour graph H for $\mathsf{P} = \mathsf{SP} \cup \mathscr{E}$, which is finite. Let $\mathsf{SP} = \langle \mathscr{U}, \mathscr{I}, \mathscr{S} \rangle$. If H is empty then the set $\mathscr{U} \cup \mathscr{I}$ is unsatisfiable. In this case the derivation is successfully terminated by the initial termination rule.

Now suppose H is not empty. Let $\mathscr{C}$ be a node of H which has no successors. In this case the set $\mathscr{U} \cup \mathscr{B}_{\mathscr{C}}$ is unsatisfiable. Indeed, suppose $\mathscr{U} \cup \{\mathscr{B}_{\mathscr{C}}\}$ is true in a model $\langle D', I' \rangle$. Then we can define a colour scheme $\mathscr{C}'$ such that $\langle D', I' \rangle \models \mathscr{F}_{\mathscr{C}'}$. (Indeed, for every $a \in D'$ let $\gamma_{(a)}$ be a map from $\{1, \ldots, N\}$ to $\{0, 1\}$ such that $\gamma_{(a)}(i) = 1$ if, and only if, $\mathfrak{M} \models P_i(a)$ for every $1 \leq i \leq N$. Similarly, let θ be a map from $\{1, \ldots, n\}$ to $\{0, 1\}$ such that $\theta(j) = 1$ if, and only if, $\mathfrak{M} \models p_i$ for every $1 \leq j \leq n$. Define Γ as $\{\gamma_{(a)} \mid a \in D'\}$, and $\rho(c)$ as $\gamma_{(c^{I'})}$.) As $\mathscr{B}_{\mathscr{C}} \wedge \mathscr{F}_{\mathscr{C}'}$ is satisfiable, there exists an edge from the node $\mathscr{C}$ to the node $\mathscr{C}'$ in the contradiction with the choice of C as having no successor.

The conclusion of the step resolution rule, $\neg\mathscr{A}_{\mathscr{C}}$, is added to the set $\mathscr{U}$; this implies removing the node $\mathscr{C}$ from the behaviour graph because the set $\{\mathscr{F}_{\mathscr{C}}, \neg\mathscr{A}_{\mathscr{C}}\}$ is not satisfiable.
Next, we check the possibility where H is not empty and every node H has a successor. Ought to Note 3, we have two cases.

First condition of Theorem 4 does not hold. The negation of (1) gives the following:

$$\exists \mathscr{C}\ \exists \gamma \in \Gamma\ \exists \Diamond L(x) \in \mathscr{E}\ \forall \gamma' \in \Gamma'\ \forall \mathscr{C}'\ ((\mathscr{C}, \gamma) \to^+ (\mathscr{C}', \gamma') \Rightarrow L(x) \notin \gamma'). \qquad (4)$$

Let $\mathscr{C}_0$, γ_0, and $\Diamond L_0(x)$ be the node, colour and eventuality, respectively, determined by the existential quantifiers of (4). Let $\mathfrak{I}$ and $\mathfrak{J}_i, i \in \mathfrak{I}$ be finite nonempty sets of indexes such that $\{\mathscr{C}_i \mid i \in \mathfrak{I}\}$ is the set of *all* successors of $\mathscr{C}_0$ (possibly including $\mathscr{C}_0$ itself) and $\{\gamma_{i,j} \in \Gamma_i \mid i \in \mathfrak{I},\ j \in \mathfrak{J}_i,\ \gamma_0 \to^+ \gamma_{i,j}\}$ is the set of *all* predicate colours such that there exists a run going through γ_0 and the colour. (To unify notation, if $0 \notin \mathfrak{I}$, we define $\mathfrak{J}_0$ as $\{0\}$, and $\gamma_{0,0}$ as γ_0; and if $0 \in \mathfrak{I}$, we add the index of γ_0 to $\mathfrak{J}_0$. Therefore, $\mathfrak{J}_0$ is always defined and without loss of generality we may assume that $\gamma_{0,0} = \gamma_0$.)

Let $\mathscr{C}_{i_1},\ldots,\mathscr{C}_{i_k}$ be the set of all immediate successors of $\mathscr{C}_0$. To simplify the proof, we will represent canonical merged derived step clauses $\mathscr{A}_{\mathscr{C}_i} \Rightarrow \bigcirc\mathscr{B}_{\mathscr{C}_i}$ ($\mathscr{A}_{\mathscr{C}_{i_l}} \Rightarrow \bigcirc\mathscr{B}_{\mathscr{C}_{i_l}}$) simply as $\mathscr{A}_i \Rightarrow \bigcirc\mathscr{B}_i$ ($\mathscr{A}_{i_l} \Rightarrow \bigcirc\mathscr{B}_{i_l}$), and formulae $\mathscr{F}_{\mathscr{C}_i}$ ($\mathscr{F}_{\mathscr{C}_{i_l}}$) simply as $\mathscr{F}_i$ ($\mathscr{F}_{i_l}$).

Consider two cases depending on whether the canonical merged derived step clause $\mathscr{A}_0 \Rightarrow \bigcirc\mathscr{B}_0$ (or any of $\mathscr{A}_i \Rightarrow \bigcirc\mathscr{B}_i$, $i \in \mathfrak{I}$) degenerates or not.

1. Let $\mathscr{A}_0 = \mathscr{B}_0 = \textbf{true}$. It follows that $\mathscr{U} \vdash \forall x \neg L_0(x)$. Indeed, suppose $\mathscr{U} \cup \{\exists x L_0(x)\}$ has a model, $\langle D', I' \rangle$. Then we can construct a colour scheme $\mathscr{C}'$ such that $\langle D', I' \rangle \models \mathscr{F}_{\mathscr{C}'}$. Since $\mathscr{C}_{i_1},\ldots,\mathscr{C}_{i_k}$ is the set of all immediate successors of $\mathscr{C}_0$ and $\mathscr{B}_0 = \textbf{true}$, it holds that there exists $j, 1 \leq j \leq k$, such that $\mathscr{C}_{i_j} = \mathscr{C}'$. Since $B_{\gamma_0}(x) = \textbf{true}$, every pair (γ_0, γ'), where $\gamma' \in \Gamma'$, is suitable; hence $\neg L_0(x) \in \gamma'$ for every $\gamma' \in \Gamma'$, and $\mathscr{F}_{\mathscr{C}'} \vdash \forall x \neg L_0(x)$ leading to a contradiction. Therefore, $\mathscr{U} \vdash \forall x \neg L_0(x)$ and the eventuality termination rule can be applied. The same holds if any one of $\mathscr{A}_i \Rightarrow \bigcirc\mathscr{B}_i$ degenerates.
2. Let none of the $\mathscr{A}_i \Rightarrow \bigcirc\mathscr{B}_i$ degenerate. We are going to prove that the eventuality resolution rule can be applied. First, we have to check the side conditions for such an application.
   (a) $\forall x(\mathscr{U} \wedge \mathscr{B}_i \wedge B_{\gamma_{i,j}}(x) \Rightarrow \neg L_0(x))$ for all $i \in \mathfrak{I} \cup \{0\}$, $j \in \mathfrak{I}_i$.
   Consider the case when $i = j = 0$ (for other indexes the arguments are similar). We show that
   $$\forall x(\mathscr{U} \wedge \mathscr{B}_0 \wedge B_{\gamma_0}(x) \Rightarrow \bigvee_{l \in \{1,\ldots,k\},\, \gamma' \in \Gamma_{i_l},\, \gamma \to \gamma'} F_{\gamma'}(x))$$
   is valid (it follows, in particular, that $\forall x(\mathscr{U} \wedge \mathscr{B}_0 \wedge B_{\gamma_0}(x) \Rightarrow \neg L_0(x))$ is valid). Suppose $\langle D', I' \rangle$ is a model for
   $$\exists x(\mathscr{U} \wedge \mathscr{B}_0 \wedge B_{\gamma_0}(x) \wedge \bigwedge_{l \in \{1,\ldots,k\},\, \gamma' \in \Gamma_{i_l},\, \gamma \to \gamma'} \neg F_{\gamma'}(x)).$$
   Then there exists a colour scheme $\mathscr{C}'$ such that $\langle D', I' \rangle \models \mathscr{F}_{\mathscr{C}'}$. Since $\langle D', I' \rangle \models \mathscr{B}_0 \wedge \mathscr{F}_{\mathscr{C}'}$, we conclude that $\mathscr{C}'$ is among $\mathscr{C}_{i_1},\ldots,\mathscr{C}_{i_k}$. Note that $\langle D', I' \rangle \models \mathscr{F}_{\mathscr{C}'}$ follows. In particular $\langle D', I' \rangle \models \forall x \bigvee_{\gamma'' \in \Gamma'} F_{\gamma''}(x)$ and, hence, $\langle D', I' \rangle \models \forall x(B_{\gamma_0}(x) \Rightarrow \bigvee_{\gamma'' \in \Gamma'} F_{\gamma''}(x))$. Together with the fact that $\langle D', I' \rangle \models \exists x(B_{\gamma_0}(x) \wedge F_{\gamma''}(x))$ implies $\gamma_0 \to \gamma''$, we have $\langle D', I' \rangle \models \forall x(B_{\gamma_0}(x) \Rightarrow \bigvee_{\gamma'' \in \Gamma',\, \gamma_0 \to \gamma''} F_{\gamma''}(x))$. This contradicts the choice of the structure $\langle D', I' \rangle$.
   (b) $\forall x(\mathscr{U} \wedge \mathscr{B}_i \wedge B_{\gamma_{i,j}}(x) \Rightarrow \bigvee_{k \in \mathfrak{I} \cup \{0\},\, l \in \mathfrak{I}_k} (\mathscr{A}_k \wedge A_{\gamma_{k,l}}(x)))$ for all $i \in \mathfrak{I} \cup \{0\}$, $j \in \mathfrak{I}_i$.
   Again, consider the case $i = j = 0$. Suppose
   $$\mathscr{U} \wedge \mathscr{B}_0 \wedge \exists x(B_{\gamma_0}(x) \wedge \bigwedge_{k \in \mathfrak{I} \cup \{0\},\, l \in \mathfrak{I}_k} (\neg(\mathscr{A}_k \wedge A_{\gamma_{k,l}}(x))))$$

is satisfied in a structure $\langle D', I' \rangle$. Let $\mathscr{C}'$ be a colour scheme such that $\langle D', I' \rangle \models \mathscr{F}_{\mathscr{C}'}$. By arguments similar to the ones given above, there is a node $\mathscr{C}_{i_l}$, $1 \leq l \leq k$, which is an immediate successor of $\mathscr{C}_0$, such that $\mathscr{C}_{i_l} = \mathscr{C}'$, and hence $\langle D', I' \rangle \models \mathscr{A}'$. It suffices to note that

$$\langle D', I' \rangle \models \forall x (B_{\gamma_0}(x) \Rightarrow \bigvee_{\gamma'' \in \Gamma', \, \gamma_0 \to \gamma''} A_{\gamma''}(x)).$$

(As in the case 2(a) above, $\langle D', I' \rangle \models \forall x (B_{\gamma_0}(x) \Rightarrow \bigvee_{\gamma'' \in \Gamma', \, \gamma_0 \to \gamma''} F_{\gamma''}(x))$, and for all $\gamma'' \in \Gamma'$, the formula $\forall x (F_{\gamma''}(x) \Rightarrow A_{\gamma''}(x))$ is valid.)

After applying the eventuality resolution rule we add to $\mathscr{U}$ its conclusion:

$$\forall x \bigwedge_{i \in \mathfrak{I} \cup \{0\}, \, j \in \mathfrak{I}_i} (\neg \mathscr{A}_i \vee \neg A_{\gamma_{i,j}}(x)).$$

Then, the node $\mathscr{C}_0$ will be removed from the behaviour graph (recall that $\mathscr{F}_0 \vdash \mathscr{A}_0 \wedge \exists x A_{\gamma_0}(x)$).

Third condition of Theorem 4 does not hold. This case was already considered in [1]. We sketch here the proof. The negation of (2) gives the following:

$$\exists \mathscr{C} \; \exists \Diamond l \in \mathscr{E} \; \forall \mathscr{C}' \; (\mathscr{C} \to^+ \mathscr{C}' \Rightarrow l \notin \theta') \tag{5}$$

Let $\mathscr{C}_0$, and l_0 be the node and eventuality determined by the existential quantifiers of (5). Let $\mathfrak{I}$ be a finite nonempty set of indexes, $\{\mathscr{C}_i \mid i \in \mathfrak{I}\}$ be the set of all successors of $\mathscr{C}_0$ (possibly including $\mathscr{C}_0$ itself). As in the previous case, one can show that

- If any of $\mathscr{A}_i \Rightarrow \bigcirc \mathscr{B}_i$ (where $i \in \mathfrak{I}$) degenerates then $\mathscr{U} \vdash \neg l$, and the ground eventuality termination rule can be applied.
- If none of the canonical merged derived step clauses degenerate then the following conditions hold
  - for all $i \in \mathfrak{I} \cup \{0\}$ $\quad \mathscr{U} \cup \mathscr{B}_i \vdash l_0$
  - for all $i \in \mathfrak{I} \cup \{0\}$ $\quad \mathscr{U} \cup \mathscr{B}_i \vdash \bigvee_{j \in \mathfrak{I} \cup \{0\}} \mathscr{A}_j$

  and so the ground eventuality resolution rule can be applied. ❑

Example 4 (example 3 contd.). We illustrate the proof of Theorem 3 on the temporal problem introduced in Example 3. The behaviour graph of the problem is not empty; every node has a successor. It is not hard to see that the first condition of Theorem 4 does not hold and, for example, if for $\mathscr{C}_0 = \mathscr{C}_1$, $\gamma_0 = \gamma_1$, and $L_0(x) = \neg P(x)$ then

$$\forall \mathscr{C}' \forall \gamma' \in \Gamma' ((\mathscr{C}_0, \gamma_0) \to^+ (\mathscr{C}', \gamma') \Rightarrow L_0(x) \notin \gamma')$$

holds. The set of all (and all immediate) successors of $\mathscr{C}_1$ is $\{\mathscr{C}_1, \mathscr{C}_4\}$. Note that the canonical full merged step clauses corresponding to $\mathscr{C}_1$ and $\mathscr{C}_4$ are identical, and none of them degenerates. For $i \in \{1, 4\}$, the loop side conditions,

$$\forall x ((\underbrace{(l \Rightarrow \exists x P(x))}_{\mathscr{U}_i} \wedge \underbrace{(\exists x P(x) \wedge \forall x P(x))}_{\mathscr{B}_i} \wedge \underbrace{P(x)}_{B_{\gamma_1}(x)}) \Rightarrow P(x))$$

and

$$\forall x(((\underbrace{(l \Rightarrow \exists x P(x))}_{\mathscr{U}_i} \wedge \underbrace{(\exists x P(x) \wedge \forall x P(x))}_{\mathscr{B}_i} \wedge \underbrace{P(x)}_{B_{\gamma_1}(x)}) \Rightarrow \bigvee_{j \in \{1,4\}} (\underbrace{\exists x P(x) \wedge \forall x P(x)}_{\mathscr{A}_j} \wedge \underbrace{P(x)}_{A_{\gamma_1}(x)}))$$

hold. Therefore, we can apply the eventuality resolution rule whose conclusion,

$$\forall x(\bigwedge_{j \in \{1,4\}} (\neg(\exists x P(x) \wedge \forall x P(x))) \wedge \neg P(x)),$$

can be simplified to $\exists x \neg P(x)$. After the conclusion of the rule is added to $\mathscr{U}$, nodes $\mathscr{C}_1$ and $\mathscr{C}_4$ and edges leading to and from them are deleted from the behaviour graph.

For the temporal problem with the new universal part, again the first condition of Theorem 4 does not hold and, for example, if $\mathscr{C}_0 = \mathscr{C}_3$, $\gamma_0 = \gamma_1$, and $L_0(x) = \neg P(x)$, then

$$\forall \mathscr{C}' \forall \gamma' \in \Gamma'((\mathscr{C}_0, \gamma_0) \to^+ (\mathscr{C}', \gamma') \Rightarrow L_0(x) \notin \gamma')$$

holds. (Note that γ_2 is never a successor of γ_1.) The set of all (and all immediate) successors of $\mathscr{C}_3$ is $\{\mathscr{C}_3, \mathscr{C}_6\}$. The canonical full merged step clauses corresponding to $\mathscr{C}_3$ and $\mathscr{C}_6$ are identical, and none of them degenerates. In a similar way, the loop side conditions hold and the conclusion of the eventuality resolution rule simplifies to $\forall x \neg P(x)$. This time, nodes $\mathscr{C}_3$ and $\mathscr{C}_6$ are deleted from the behaviour graph.

For the new problem, the third condition of Theorem 4 does not hold, and if $\mathscr{C}_0 = \mathscr{C}_5$, $l_0 = l$ then

$$\forall \mathscr{C}'(\mathscr{C}_0 \to^+ \mathscr{C}' \Rightarrow l_0 \notin \theta')$$

holds. As the canonical full merged step clause degenerates (and $\mathscr{U} \vdash \neg l$), the ground eventuality termination rule can be applied.

Note that if, in the beginning, instead of $\mathscr{C}_1$ we had selected $\mathscr{C}_3$ (or $\mathscr{C}_6$) as $\mathscr{C}_0$, nodes $\mathscr{C}_1$, $\mathscr{C}_3$, $\mathscr{C}_4$, and $\mathscr{C}_6$ would be deleted after the first application of the eventuality resolution rule.

6 Loop Search Algorithm

The notion of a full merged step clause is quite involved and the search for appropriate merging of simpler clauses is computationally hard. Finding *sets* of such full merged clauses needed for the temporal resolution rule is even more difficult. In this section we present a search algorithm that finds a *loop formula* (cf. page 6) — a disjunction of the left-hand sides of full merged step clauses that together with an eventuality literal form the premises for the temporal resolution rule. The algorithm is based on a Dixon's loop search algorithm for the propositional case [4]. For the sake of space, in what follows we consider non-ground eventualities only. The algorithm and the proof of its properties for the ground case can be obtained by considering merged derived step clauses instead of the general case and by deleting the parameter "x" and quantifiers from the text below.

Breadth First Search algorithm.

Input A temporal specification SP and an eventuality clause $\Diamond L(x) \in \mathscr{E}$.
Output A formula $H(x)$ with at most one free variable.
Method

1. Let $H_0(x) = \mathbf{true}$; $N_0 = \emptyset$; $i = 0$.
2. Let $N_{i+1} = \{\forall x(\mathscr{A}_j^{(i+1)}(x) \Rightarrow \bigcirc\mathscr{B}_j^{(i+1)}(x))\}_{j=1}^{k}$ be the set of *all* full merged step clauses such that for every $j \in \{1 \dots k\}$, $\forall x(\mathscr{U} \wedge \mathscr{B}_j^{(i+1)}(x) \Rightarrow (\neg L(x) \wedge H_i(x)))$ holds. (The set N_{i+1} possibly includes the degenerate clause $\mathbf{true} \Rightarrow \bigcirc\mathbf{true}$ in the case $\mathscr{U} \Rightarrow \forall x(\neg L(x) \wedge H_i(x))$.)
3. If $N_{i+1} = \emptyset$, let $H_{i+1}(x) = \mathbf{false}$; else let $H_{i+1}(x) = \bigvee\limits_{i=1}^{k}(\mathscr{A}_j^{(i+1)}(x))$.
4. If $N_{i+1} = \emptyset$ return **false**.
5. If $\forall x(H_i(x) \Rightarrow H_{i+1}(x))$ return $H_{i+1}(x)$.
6. $i = i + 1$; goto 2.

We are going to show now that the algorithm terminates (Lemma 6), its output is a loop formula (Lemma 7), and temporal resolution is complete if we consider only the loops generated by the algorithm (Theorem 5).

Lemma 5. *For the formulae $H_i(x)$, $i \geq 0$, constructed by the BFS algorithm, the following holds: $\forall x(H_{i+1}(x) \Rightarrow H_i(x))$.*

Proof By induction. In the base case $i = 0$, we have $H_0(x) \equiv \mathbf{true}$ and obviously $\forall x(H_1(x) \Rightarrow \mathbf{true})$. The induction hypothesis is that $\forall x(H_i(x) \Rightarrow H_{i-1}(x))$. In the induction step let $N_{i+1} \neq \emptyset$ (otherwise, $H_{i+1}(x) \equiv \mathbf{false}$ and evidently $\forall x(\mathbf{false} \Rightarrow H_i(x))$ holds). Let $N_{i+1} = \{\forall x(\mathscr{A}_j^{(i+1)}(x) \Rightarrow \bigcirc\mathscr{B}_j^{(i+1)}(x))\}_{j=1}^{k}$. For every $j \in \{1 \dots k\}$ we have $\forall x(\mathscr{B}_j^{(i+1)}(x) \Rightarrow (\neg L(x) \wedge H_i(x)))$. By the induction hypothesis, $\forall x(H_i(x) \Rightarrow H_{i-1}(x))$ and, therefore, $\forall x(\mathscr{B}_j^{(i+1)}(x) \Rightarrow (\neg L(x) \wedge H_{i-1}(x)))$, that is, $N_{i+1} \subset N_i$. It follows that $\forall x(H_{i+1}(x) \Rightarrow H_i(x))$. ❑

Lemma 6. *The BFS algorithm terminates.*

Proof There are only finitely many different $H_i(x)$. Therefore, either there exists k such that $H_k(x) \equiv \mathbf{false}$, or there exist $l, m : l < m$ such that $\forall x(H_l(x) \equiv H_m(x))$. In the former case, the algorithm terminates at line 4 while, in the latter case, by Lemma 5 we have $\forall x(H_{m-1}(x) \Rightarrow H_l(x))$, that is $\forall x(H_{m-1}(x) \Rightarrow H_m(x))$, and the algorithm terminates at line 5. ❑

Lemma 7. *Let $H(x)$ be a formula produced by the BFS algorithm. Then $\forall x(\mathscr{U} \wedge H(x) \Rightarrow \bigcirc\Box\neg L(x))$.*

Proof If $H(x) = \mathbf{false}$, the lemma holds. Otherwise, consider the last computed set N_{i+1} (that is, $H(x) = H_{i+1}(x)$). Let $N_{i+1} = \{\forall x(\mathscr{A}_j^{(i+1)}(x) \Rightarrow \bigcirc\mathscr{B}_j^{(i+1)}(x))\}_{j=1}^{k}$. Note that for all $j \in \{1 \dots k\}$, it holds $\forall x(\mathscr{U} \wedge \mathscr{B}_j^{(i+1)}(x) \Rightarrow \neg L(x))$ and, since $\forall x(H_i(x) \Rightarrow H_{i+1}(x))$, we also have $\forall x(\mathscr{U} \wedge \mathscr{B}_j^{(i+1)}(x) \Rightarrow H_{i+1}(x))$, that is, N_{i+1} is a loop and $H_{i+1}(x)$ is its loop formula. ❑

Lemma 8. *Let $\mathsf{P} = \mathsf{SP} \cup \mathscr{E}$ be a monodic temporal problem, $\mathscr{L}$ be a loop in $\Diamond L(x) \in \mathscr{E}$, and $\mathbf{L}(x)$ be its loop formula. Then for the formula $H(x)$, produced by the BFS algorithm on SP and $\Diamond L(x)$, the following holds: $\forall x(\mathbf{L}(x) \Rightarrow H(x))$.*

Proof We show by induction that for all sets of full merged step clauses N_{i+1}, constructed by the algorithm, $\mathscr{L} \subset N_{i+1}$ holds. In the base case $i = 0$, $H_0(x) \equiv$ **true** and for every full merged step clause $\forall x(\mathscr{A}(x) \Rightarrow \mathscr{B}(x)) \in \mathscr{L}$, we have $\forall x(\mathscr{U} \wedge \mathscr{B}(x) \Rightarrow (\neg L(x) \wedge \mathbf{true}))$; therefore, $\mathscr{L} \subset N_1$.

Our induction hypothesis is that $\mathscr{L} \subset N_i$, i.e. $N_i = \mathscr{L} \cup N_i'$. Then $H_i(x) = \mathbf{L}(x) \vee H_i'(x)$. Let $\forall x(\mathscr{A}(x) \Rightarrow \mathscr{B}(x))$ be any full merged step clause from $\mathscr{L}$. By the definition of a loop, $\forall x(\mathscr{U} \wedge \mathscr{B}(x) \Rightarrow (\neg L(x) \wedge \mathbf{L}(x)))$, hence, $\forall x(\mathscr{U} \wedge \mathscr{B}(x) \Rightarrow ((\neg L(x) \wedge \mathbf{L}(x)) \vee (\neg L(x) \wedge H_i'(x))))$, that is, $\forall x(\mathscr{U} \wedge \mathscr{B}(x) \Rightarrow (\neg L(x) \wedge H_i(x)))$. Since the set N_{i+1} consists of all full merged step clauses, $\forall x(\mathscr{A}_j^{(i+1)}(x) \Rightarrow \bigcirc \mathscr{B}_j^{(i+1)}(x))$, such that $\forall x(\mathscr{U} \wedge \mathscr{B}_j^{(i+1)}(x) \Rightarrow (\neg L(x) \wedge H_i(x)))$ holds, we have $\forall x(\mathscr{A}(x) \Rightarrow \mathscr{B}(x)) \in N_{i+1}$. As $\forall x(\mathscr{A}(x) \Rightarrow \mathscr{B}(x))$ is an arbitrary full merged step clause from $\mathscr{L}$, it means that $\mathscr{L} \subset N_{i+1}$.

It follows that $\forall x(\mathbf{L}(x) \Rightarrow H(x))$. ❑

The proof of the completeness theorem goes by showing that there exists an eventuality $\Diamond L(x) \in \mathscr{E}$ and a loop $\mathscr{L} = \{\forall x(\mathscr{A}_i(x) \Rightarrow \bigcirc \mathscr{B}_i(x))\}_{i=1}^k$ such that the application of the eventuality resolution rule to $\Diamond L(x)$ and $\mathscr{L}$ leads to the deletion of some nodes from the eventuality graph. A node $\mathscr{C}$ is deleted from the graph if the categorical formula, $\mathscr{F}_\mathscr{C}$, together with the universal part, $\mathscr{U}$, is satisfiable, but $\mathscr{F}_\mathscr{C} \wedge \forall x \neg \bigvee_{j=1}^k \mathscr{A}_j(x) \wedge \mathscr{U}$ is unsatisfiable.

Theorem 5. *Temporal resolution is complete if we restrict ourselves to loops found by the BFS algorithm.*

Proof Let $H(x)$ be the output of the BFS algorithm, let $\mathbf{L}(x) \leftrightharpoons \bigvee_{j=1}^k \mathscr{A}_j(x)$. By Lemma 8, $\forall x(L(x) \Rightarrow H(x))$ holds; therefore, $H(x)$ is not **false**. From the proof of Lemma 7 it follows that the last computed set N_{i+1} (that is, $H(x) = H_{i+1}(x)$) is a loop in $\Diamond L(x)$ and $H(x)$ is its loop formula. Since $\forall x(\mathbf{L}(x) \Rightarrow H(x))$, the formula $\mathscr{F}_\mathscr{C} \wedge \forall x \neg H(x) \wedge \mathscr{U}$ is unsatisfiable as well and the application of the eventuality resolution rule to $\Diamond L(x)$ and N_{i+1} leads to deletion of at least the same nodes from the eventuality graph. ❑

Note 5. The need to include *all* full merged step clauses satisfying some particular conditions into N_{i+1} might lead to quite extensive computations. Note however that due to the trivial fact that if $\forall x(A(x) \Rightarrow B(x))$ then $\forall x((A(x) \vee B(x)) \equiv B(x))$, we can restrict the choice to only those full merged step clauses whose left-hand sides do not imply the left-hand side of any other clause in N_{i+1} yielding a formula $H_{i+1}'(x)$ equivalent to the original formula $H_{i+1}(x)$.

Example 5. Let us consider an unsatisfiable monodic temporal problem, P, given by $\mathscr{I} = \{\exists x A(x)\}$, $\mathscr{U} = \{\forall x(B(x) \Rightarrow A(x) \wedge \neg L(x))\}$, $\mathscr{S} = \{A(x) \Rightarrow \bigcirc B(x)\}$, $\mathscr{E} = \{\Diamond L(x)\}$ and apply the BFS algorithm to its temporal specification and $\Diamond L(x)$.

The set of all full merged step clauses, N_1, whose right-hand sides imply $\neg L(x)$, is:

$$(\forall y A(y)) \Rightarrow \bigcirc(\forall y B(y)), \tag{6}$$
$$(A(x) \wedge \forall y A(y)) \Rightarrow \bigcirc(B(x) \wedge \forall y B(y)), \tag{7}$$
$$(A(x) \wedge \exists y A(y)) \Rightarrow \bigcirc(B(x) \wedge \exists y B(y)). \tag{8}$$

Note that $\forall x(\forall y A(y) \Rightarrow A(x) \wedge \forall y A(y))$ and $\forall x(A(x) \wedge \forall y A(y)) \Rightarrow A(x) \wedge \exists y A(y))$; therefore, clauses (6) and (7) can be deleted from N_1 yielding

$$N_1' = \{(A(x) \wedge \exists y A(y)) \Rightarrow \bigcirc(B(x) \wedge \exists y B(y))\} \quad \text{and} \quad H_1'(x) = (A(x) \wedge \exists y A(y)).$$

The set off all full merged step clauses N_2 whose right-hand sides imply $L(x) \wedge H_1'(x)$ coincides with N_1 and the output of the algorithm is $H_2'(x) \equiv H_1'(x)$. The conclusion of the eventuality resolution rule, $\forall x \neg A(x) \vee \neg\exists y A(y)$, simplified to $\forall x \neg A(x)$, contradicts the initial part of the problem.

Note that all full merged step clauses from N_1 are loops in $\Diamond L(x)$, but both conclusions of the eventuality resolution rule, applied to the loops (6) and (7), can be simplified to $\exists x \neg A(x)$ which does not contradict the initial part.

7 Conclusion and future work

In this paper, we have introduced a resolution-based calculus for the monodic fragment of first-order temporal logic. We have shown that the calculus is sound and complete and considered some problems of implementation. We have suggested an algorithm that "guides" the search for loops in order to avoid unnecessary enumeration of all possibilities. We are going to refine also the step resolution rule in a way similar to the original temporal resolution method for the propositional case [5] that could serve as a basis for a practical implementation.

An alternative tableaux-based approach [10] also utilises similar "separation" ideas dividing the proof search procedure into *temporal* and *first-order* parts. Note that the method in [10] requires the first-order component to give a finite representation of *all possible* first-order models; whereas our method requires from it just an *yes/no* answer (to test side conditions of the rules of temporal resolution). Our procedure is a decision procedure when the side condition checks are decidable, and is a semi-decision procedure otherwise.

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