TPI-Disputes and Proof by Clausal Tableaux

Paul E. Dunne

Dept. of Computer Science, University of Liverpool, Liverpool L69 7ZF, United Kingdom

Abstract

Two–Party Immediate Response Disputes (TPI-disputes) are a *dialogue game* formalism providing a sound and complete proof scheme for credulous reasoning in argument systems. In earlier work it has been shown that this dialogue protocol can be interpreted as a proof calculus with which to establish unsatisfiability of propositional formula presented in CNF, and that viewed thus, the number of 'moves' taken to resolve a dispute – i.e. prove that Φ is unsatisfiable – is bounded below by the number of lines in the shortest derivation of $\Rightarrow \neg\Phi$ in a CUT-free *Gentzen System*. In this note we prove an *upper* bound on the number of moves needed, showing this to be of the same order as the size of the smallest *clausal tableau* refutation of Φ. It thereby follows that TPI-disputes are 'polynomially equivalent' to these proof systems.

Key words: Argument System, Credulous Acceptance, Dialogue Game, Tableau Proof

1 Introduction

Argument Systems were introduced in the work of [3] and have since proved to be a powerful mechanism for representation and reasoning in non-monotonic logics. Central to such systems is the modelling of relationships between *arguments* via the concept of one argument *attacking* another. In such cases where the interaction between a finite number of argument positions is considered, the relevant argument system can be naturally presented as a directed graph whose vertices are associated with individual arguments and whose edges $\langle p, q\rangle$ reflect the fact that the argument p *attacks* the argument q. Within this basic framework, concepts of sets of argument being '*admissible*' – internally consistent and capable of rebutting external attacks – can be introduced so that varying notions of an argument being *acceptable* with respect to the system as a whole arise. Among the most widely studied of these notions is that of *credulous acceptance*, whereby an argument is accepted if it is an element of *at least one* admissible set.

Email address: ped@csc.liv.ac.uk (Paul E. Dunne).

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The problem of deciding given $\mathcal{H}(\mathcal{X}, \mathcal{A})$ and $x \in \mathcal{X}$ whether x is credulously accepted is NP–complete, [2], however, a number of approaches have been considered not only for the decision problem itself but also for determining admissible sets (particularly maximal or *preferred* sets), which provide certificates of credulous acceptability. In this article we are concerned with dialogue game approaches similar in spirit to the formal model propounded in [7,8], in particular that proposed by Vreesjwijk and Prakken in [12], called *Two-Party Immediate Response Disputes* (TPI-disputes). One important advantage of dialogue style mechanisms is that the reasoning process by which a given argument is defended becomes transparent: hence the game protocol provides a formal semantics yielding proofs of acceptability and non-acceptability. In [5] the concept of *dispute complexity* was introduced in order to study the efficiency of TPI-disputes in terms of the following question: given a dispute instance – i.e. an argument system $\mathcal{H}(\mathcal{X}, \mathcal{A})$ and argument $x \in \mathcal{X}$, how many *moves* may be needed to settle a dispute in both the cases when x *is* credulously accepted and when it is not? For the case that x is credulously accepted, [5] prove that the minimum number of moves required is the size of the smallest admissible set containing x and that, furthermore, a dispute instance attaining this bound exists (of course, *constructing* such a minimal dispute is another issue). For the case when an argument is inadmissible, i.e. not credulously accepted, TPI-disputes can require exponentially many (in $|\mathcal{X}|$) moves in order to reach this conclusion.

The present article is concerned with one issue that arises from the proof of the lower bound result in [5]. Two mechanisms are central to this proof. The first is a translation of arbitrary propositional formulae, Φ, in CNF to an argument system $\mathcal{H}_\Phi$ with the property that a specific argument (denoted Φ in [5]) of $\mathcal{H}_\Phi$ is admissible if and only if Φ is satisfiable. In consequence of this translation any sound and complete mechanism for credulous argumentation provides a proof calculus for demonstrating that $\neg\Phi$ is valid: the TPI-dispute protocol provides such a mechanism. The second element is a simulation of the progress of any TPI-dispute instance $\langle \mathcal{H}_\Phi, \Phi \rangle$ by a derivation of the sequent $\Rightarrow \cup_{i=1}^{m} \{\neg C_i\}$ (C_i being the component clauses of Φ) within a Gentzen System for which the CUT inference rule is not available. The number of steps in this derivation is of the same order as the number of moves in the simulated dispute. As a result of the simulation it is deduced that for any unsatisfiable Φ, the length of any TPI-dispute over $\langle \mathcal{H}_\Phi, \Phi \rangle$ is at least the length of the shortest CUT-free Gentzen proof of $\neg\Phi$. The exponential lower bound is then simply a matter of noting that there are instances $\neg\Phi$ whose shortest proof in such systems is exponential in the size of Φ, cf [10,11]. While [5] provides a lower bound on 'how good' a propositional proof calculus TPI-disputes provide, the question of upper bounds is left open. The present article treats this question. Specifically we show that for any unsatisfiable CNF-formula Φ, the number of moves required in a TPI-dispute over $\langle \mathcal{H}_\Phi, \Phi \rangle$ cannot be 'much larger' than the size of the smallest *clausal tableau refutation* of Φ. An immediate consequence of this result is that the propositional proof system afforded by TPI-disputes is polynomially equivalent – in sense of [1] – to CUT-free Gentzen

Systems and Clausal Tableaux, i.e. if $\langle \Pi_1, \Pi_2 \rangle$ are any two proof systems from $\{$Gentzen/CUT, Clausal Tableaux, TPI-dispute$\}$, then the length of the shortest validity proofs of $\neg\Phi$ for CNF-formulae Φ in Π_1 is at worst polynomially larger than the length of the shortest proof in the system Π_2. This follows from the equivalence of Clausal Tableaux and CUT-free Gentzen Systems, details of which may be found in [9, Chap. XI]

The remainder of this article is organised as follows. In the next section we introduce the main technical definitions used subsequently. In Section 3 we present our simulation of clausal tableau refutations by TPI-disputes. In the final Section we discuss some consequences and areas for further research.

2 Definitions

The basic definition below is derived from that given in [3].

Definition 1 *An* argument system *is a pair $\mathcal{H} = \langle \mathcal{X}, \mathcal{A} \rangle$, in which $\mathcal{X}$ is a set of* arguments *and $\mathcal{A} \subset \mathcal{X} \times \mathcal{X}$ is the* attack relationship *for $\mathcal{H}$. Unless otherwise stated, $\mathcal{X}$ is assumed to be* finite*, and $\mathcal{A}$ comprises a set of ordered pairs of* distinct *arguments. A pair $\langle x, y \rangle \in \mathcal{A}$ is referred to as 'x* is attacked by *y' or 'y* attacks *(or* is an attacker of*) x'.*

For R, S subsets of arguments in the system $\mathcal{H}(\langle \mathcal{X}, \mathcal{A} \rangle)$, we say that

- *a) $s \in S$ is* attacked *by R if there is some $r \in R$ such that $\langle r, s \rangle \in \mathcal{A}$.*
- *b) $x \in \mathcal{X}$ is* acceptable with respect to *S if for every $y \in \mathcal{X}$ that attacks x there is some $z \in S$ that attacks y.*
- *c) S is* conflict-free *if no argument in S is attacked by any other argument in S.*
- *d) A conflict-free set S is* admissible *if every argument in S is acceptable with respect to S.*
- *e) S is a* preferred extension *if it is a maximal (with respect to $\subseteq$) admissible set.*

An argument x is credulously accepted *is there is* some *preferred extension containing it.*

The decision problem *Credulous Acceptance* (CA) takes as an instance an argument system $\mathcal{H}(\mathcal{X}, \mathcal{A})$ and an argument $x \in \mathcal{X}$. The instance $\langle \mathcal{H}(\mathcal{X}, \mathcal{A}), x \rangle$ is accepted if x is credulously accepted in $\mathcal{H}$. We note that this is equivalent to deciding if there is an admissible subset, S of $\mathcal{X}$ that contains x.

Definition 2 *Let*

$$\Phi(Z_n) = \bigwedge_{i=1}^{m} C_i = \bigwedge_{i=1}^{m} \left(\bigvee_{j=1}^{i_k} y_{i,j} \right)$$

be a CNF*-formula with variables* $Z_n = \{z_1, \ldots, z_n\}$*, so that each* $y_{i,j}$ *is some literal* z *or* $\neg z$ *over a variable in this set. The argument system* $\mathcal{H}_\Phi(\mathcal{X}_\Phi, \mathcal{A}_\Phi)$ *is defined from* Φ *as follows. It has* $2n + m + 1$ *arguments,*

$$\mathcal{X} = \{z_i, \neg z_i \ : \ 1 \leq i \leq n\} \cup \{C_i \ : \ 1 \leq i \leq m\} \cup \{\Phi\}$$

and attacks, $\mathcal{A}_\Phi$ *comprising,*

$$\{\langle z_i, \neg z_i\rangle, \langle \neg z_i, z_i\rangle \ : \ 1 \leq i \leq n\} \quad \{\langle C_i, \Phi\rangle \ : \ 1 \leq i \leq m\}$$
$$\{\langle z_i, C_j\rangle \ : \ \text{if } z_i \text{ is a literal in } C_j\} \quad \{\langle \neg z_i, C_j\rangle \ : \ \text{if } \neg z_i \text{ is a literal in } C_j\}$$

We note this construction is similar to one used in [2], from which it is not difficult to show that following.

Fact 3 $\textsc{ca}(\mathcal{H}_\Phi(\mathcal{X}_\Phi, \mathcal{A}_\Phi), \Phi) \Leftrightarrow \Phi(Z_n)$ *is satisfiable.* □

We now present the rules and moves governing TPI-disputes. These were introduced in [12] and a formal model given in [5]. In order to simplify the description we do not present the full formalism as given in [5], concentrating instead on the somewhat simpler form this takes in its specific instantiation for instances $\langle \mathcal{H}_\Phi, \Phi\rangle$. This suffices for our purpose since we are concerned with bounds on the dispute length when Φ is *unsatisfiable*, i.e. $\neg\textsc{ca}(\mathcal{H}_\Phi, \Phi)$.

A TPI-*dispute* takes as an instance a pair $\langle \mathcal{H}_\Phi, \Phi\rangle$ and involves two players – the *Defender* (D) and *Challenger* (C) who alternate moves. A dispute, M, is described by a (finite) sequence of moves $M = \langle \mu_0, \mu_1, \ldots, \mu_k\rangle$, where even indexed moves are played by D and odd indexed moves by C. A dispute is *terminated* when the player whose turn it is has no permitted move available, in which case the opponent wins. A dispute $M_1 = \langle \mu_0^1, \ldots, \mu_r^1\rangle$ is a *possible continuation* of $M_2 = \langle \mu_0^2, \ldots, \mu_k^2\rangle$ if $r > k$, $\mu_i^1 = \mu_i^2$ for each $0 \leq i \leq k$, and M_1 defines a legal sequence of moves in the game. A dispute sequence, M is *ultimately terminating* if either M is terminated or every possible continuation of M is ultimately terminating, i.e. there is no *infinite* continuation of M.

The *state*, σ of a dispute M is a triple $\langle P, Q, R\rangle$ where P is a subset of literals over Z_n called the *current defence*; $Q \subseteq \{C_1, C_2, \ldots, C_m\}$ called the *current attacks* and $R \subset \wp(\{z_1, \ldots, z_n, \neg z_1, \ldots, \neg z_n\})$ called the *retracted defences*.

Letting $\sigma_k = \langle P_k, Q_k, R_k\rangle$ denote the state after k moves, the legal instantiations of a dispute are controlled as described in Table 1, in which for convenience we view a clause C_i of Φ both as an argument in $\mathcal{H}_\Phi$ (e.g. in moves by the Challenger) and as a set of literals. We recall that even indexed moves are made by the Defender and odd indexed moves by the Challenger.

k	μ_k	Pre-conditions	$\langle P_k, Q_k, R_k\rangle$
0	ASSERT(Φ)	$k = 0$	$\langle \emptyset, \emptyset, \emptyset\rangle$
$2r+1$	COUNTER(C_i)	$C_i \notin Q_{k-1}$ and $C_i \cap P_{k-1} = \emptyset$	$\langle P_{k-1}, Q_{k-1} \cup \{C_i\}, R_{k-1}\rangle$
$2r$ ($r \geq 1$)	COUNTER(y)	$y \in C_i = Q_{k-1}/Q_{k-2}$ and $\neg y \notin P_{k-1}$ and $\forall S \in R_{k-1}, P_{k-1} \cup \{y\} \not\subseteq S$	$\langle P_{k-1} \cup \{y\}, Q_{k-1}, R_{k-1}\rangle$
	RETRACT	COUNTER() unplayable and $P_{k-1} \neq \emptyset$	$\langle \emptyset, \emptyset, R_{k-1} \cup \{\{P_{k-1}\}\}\rangle$

Table 1
Rules and preconditions for moves in TPI-disputes over $\langle \mathcal{H}_\Phi, \Phi\rangle$

Thus a dispute over Φ in the system $\mathcal{H}_\Phi(\mathcal{X}_\Phi, \mathcal{A}_\Phi)$ starts with D asserting that the argument Φ is admissible, i.e. that $\Phi(Z_n)$ is satisfiable. The Challenger attacks this assertion by choosing any clause C_j so modelling the statement 'if C_j is $\perp$ under a given instantiation α then $\Phi(\alpha) = \perp$. The next move of D is to choose a literal, y, of C_j to counter-attack with the statement 'if $y = \top$ in any instantiation α then $C_j = \top$'. The dispute continues with C choosing alternative clauses subject to the condition that D has not already chosen literals which occur in these, while D repeatedly tries to select a literal, y, in the most recently chosen clause subject to the condition that D is not using $\neg y$ as an attack on an earlier clause chosen by C. If $\Phi(Z_n)$ is satisfiable, then D will be able to use the literals instantiated to $\top$ in a satisfying assignment as attacks on any clause put forward by C, thus after at most $\min\{n, m\}$ moves by the Challenger, no more legal moves will remain to attack Φ with – the Defender wins. Suppose, however, that $\Phi(Z_n)$ is *not* satisfiable. In this case, after an initial selection - Q_k - of clauses by C, the literals contained in P_k will be such that D cannot attack the last clause used by C since for each literal in this clause P_k will contain its negation. The RETRACT move by D reflects that fact that the subset of arguments $\{\Phi, P_k\}$ is not part of an admissible set. Since D wishes to find *some* admissible set containing Φ and its known that P_k cannot form part of such a set, the dispute continues as before, but with the additional constraint in force that D cannot construct a set of literals in P that have earlier been found to be collectively inadmissible with Φ.

It should be noted that, in general, there will be more than one legal sequence of moves associated with a given instance $\langle H_\Phi, \Phi\rangle$ depending on the choice of clause arguments invoked by C and literal selections made by D. The preceding paragraph has given an informal justification of the following result which follows from the proofs that TPI-disputes define a sound and complete proof theory for credulous reasoning given in [12,5].

Fact 4 *Let $\langle \mathcal{H}_\Phi, \Phi\rangle$ be a* TPI*-dispute instance defined from the* CNF*-formula $\Phi(Z_n)$.*

a) *If $M = \langle \mu_0, \ldots, \mu_k \rangle$ is any terminated dispute then $\Phi(Z_n)$ is satisfiable if and only if k is even, i.e. M describes a win for the Defender, establishing that Φ is credulously accepted in $\mathcal{H}_\Phi(\mathcal{X}_\Phi, \mathcal{A}_\Phi)$.*
b) *If $M_1 = \langle \mu_0^1, \ldots, \mu_k^1 \rangle$ and $M_2 = \langle \mu_0^2, \ldots, \mu_r^2 \rangle$ are* different *terminated disputes over Φ in $\mathcal{H}_\Phi$ then either both r and k are even or both of r and k are odd.*
c) *For any instance $\langle \mathcal{H}_\Phi, \Phi \rangle$ $\langle \text{ASSERT}(\Phi) \rangle$ is ultimately terminating.*

Our principal measure of interest for TPI-disputes is *dispute complexity*.

Definition 5 *Let $\langle \mathcal{H}(\mathcal{X}, \mathcal{A}), x \rangle$ be a* TPI-*dispute instance. The* dispute complexity *of x in $\mathcal{H}(\mathcal{X}, \mathcal{A})$ - denoted $\delta(\mathcal{H}, x)$ is,*

$$\min \{ k \ : \ \langle \mu_0, \ldots, \mu_k \rangle \textit{ is a terminated dispute over } x \textit{ in } \mathcal{H} \}$$

Thus the dispute complexity is the least number of *moves* in a TPI-dispute prior to one of the players having no legal move available.

For completeness, we review the lower bound result of [5]. The mechanism used therein exploits the *Gentzen* (or *Sequent*) *Calculus*, [6], with, however, one of its standard inference rules being unavailable. In its most general (propositional) form, the Gentzen Calculus prescribes rules for deriving *sequents* – $\Gamma \Rightarrow \Delta$ – where Γ, Δ are *sets* of propositional formulae (over a set of atomic propositional variables $\{x_1, x_2, x_3, \ldots\}$) built using some finite (complete) logical basis. A *proof* of the sequent $\Gamma \Rightarrow \Delta$, consists of a sequence of *derivation steps* each of which is either an axiom or follows by applying one of the rules to (at most) two previously derived sequents. In what follows we observe the convention of employing upper case Roman letters – $\{A, B, C, \ldots,\}$ – to denote propositional formulae, and upper case Greek letters – $\{\Gamma, \Delta, \ldots\}$ – to denote *sets* of such formulae. We use Γ, A to denote the set $\Gamma \cup \{A\}$.

Let $\mathcal{L}$ be the language of well-formed formulae using the basis $\{\wedge, \vee, \neg\}$ and propositional variables drawn from $\{z_1, z_2, z_3, \ldots\}$.

A *sequent* is an expression the form $\Gamma \Rightarrow \Delta$ where Γ, Δ are (finite) subsets of $\mathcal{L}$, i.e. sets of well-formed formulae. A *Gentzen System* is defined by a set $\mathcal{GS}$ of *axioms* and *inference rules*. A sequent $\Gamma \Rightarrow \Delta$ is *provable in the Gentzen System* $\mathcal{GS}$ (written $\vdash_{\mathcal{GS}} \Gamma \Rightarrow \Delta$), if there is a finite sequence of sequents,

$$S_1, S_2, \ldots, S_{k-1}, S_k, S_{k+1}, \ldots, S_t$$

for which S_t is the sequent $\Gamma \Rightarrow \Delta$ and for all k ($1 \leq k \leq t$), the sequent S_k is either an axiom of $\mathcal{GS}$ or there are sequents S_i, S_j (with $i, j < k$) and an inference rule r of $\mathcal{GS}$ such that S_k may be inferred from S_i and S_j as a consequence of the rule r. The *Proof Complexity* of a sequent S in the *Gentzen System* $\mathcal{GS}$ (denoted $\pi(S, \mathcal{GS})$ is defined for *provable* sequents, to be the least t such that S is derived by a sequence of t sequents.

A modification of the Gentzen system, $\mathcal{G}$ shown in Table 2, is used wherein A and B are members of $\mathcal{L}$, and Γ, Δ, etc. subsets of $\mathcal{L}$.

Axioms			
	$\{A\} \Rightarrow \{A\}$		
Rules			
$(\theta \Rightarrow)$	$\dfrac{\Gamma \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta}$	$\dfrac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}$	$(\Rightarrow \theta)$
(CUT)	$\dfrac{\Gamma \Rightarrow \Delta, A \;;\; \Gamma', A \Rightarrow \Delta'}{\Gamma \cup \Gamma' \Rightarrow \Delta \cup \Delta'}$		
$(\neg \Rightarrow)$	$\dfrac{\Gamma \Rightarrow \Delta, A}{\Gamma, \neg A \Rightarrow \Delta}$	$\dfrac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg A}$	$(\Rightarrow \neg)$
$(\wedge \Rightarrow)$	$\dfrac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta}$	$\dfrac{\Gamma \Rightarrow \Delta, A \;;\; \Gamma' \Rightarrow \Delta', B}{\Gamma \cup \Gamma' \Rightarrow \Delta \cup \Delta', A \wedge B}$	$(\Rightarrow \wedge)$
$(\vee \Rightarrow)$	$\dfrac{\Gamma, A \Rightarrow \Delta \;;\; \Gamma', B \Rightarrow \Delta'}{\Gamma \cup \Gamma', A \vee B \Rightarrow \Delta \cup \Delta'}$	$\dfrac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B}$	$(\Rightarrow \vee)$

Table 2
The Gentzen System $\mathcal{G}$

The Gentzen system considered in [5] is $\mathcal{G}/$CUT, i.e. that which allows all of the rules of the system $\mathcal{G}$ *except for* the CUT rule. [1]

Theorem 6 *([5]) Let*

$$\Phi(Z_n) = \bigwedge_{i=1}^{m} C_i \;=\; \bigwedge_{i=1}^{m} \left(\bigvee_{j=1}^{k_i} y_{i,j} \right)$$

be any unsatisfiable CNF*-formula, $\mathcal{H}_\Phi$ be the argument system defined from $\Phi(Z_n)$ as given in Defn. 2, and S_Φ the (provable) sequent,*

$$\emptyset \Rightarrow \bigcup_{i=1}^{m} \left\{ \neg \left(\bigvee_{j=1}^{k_i} y_{i,j} \right) \right\}.$$

[1] In fact, careful inspection of the simulation used by [5] shows that the inference rules governing $\wedge$ introduction are not used, thus the connective set employed is $\{\vee, \neg\}$.

Then,

$$\pi\left(S_\Phi, \mathcal{G}/\textsc{cut}\right) \leq \delta(\mathcal{H}_\Phi, \Phi) + 2n + m$$

Less formally, Theorem 6 states that the length of the shortest validity proof of $\neg\Phi$ (Φ being in CNF) within the CUT-free Gentzen system cannot be 'much greater than' the number of moves needed to form a successful rebuttal Φ in the argument system $\mathcal{H}_\Phi$.

The upper bound on dispute complexity we derive in the next section is based on a simulation of *clausal tableaux refutations.*

Definition 7 *Let $\Phi(Z_n) = \wedge_{i=1}^m C_i$ be an unsatisfiable* CNF*-formula with clause set $\{C_1, C_2, \ldots, C_m\}$. A* clausal tableau *for Φ is a tree $T(V, E)$ in which the non-leaf vertices, v, are associated with a clause $C(v)$ of Φ, in accordance with the following rules.*

On any path $\rho = v_0 \to v_1 \to \cdots \to v_r$ from the root (ρ) to a leaf v_r, each clause C_i of Φ labels at most one *of $\{v_0, v_1, \ldots, v_{r-1}\}$. If $\rho \in V$ is the root of T and $C(\rho) = \vee_{i=1}^k y_{\rho,i}$ the clause associated with ρ, then ρ has exactly k children – $\langle v_1, v_2, \ldots, v_k\rangle$ with the edge $\langle \rho, v_j\rangle$ labelled $y_{i,j}$. If $v \in V$ is a non-leaf vertex (other than the root) let U be the set of literals labelling edges on the (unique) path from ρ to v and $C(v) = \vee_{i=1}^s y_{v,i}$ be the clause associated with v. Again, v has exactly s children $\langle w_1, w_2, \ldots, w_s\rangle$ with the edges $\langle v, w_i\rangle$ labelled $y_{v,i}$. In this case, however, the vertex w_i is a leaf labelled $\perp$ if the literal $\neg y_{v,i} \in U$. A vertex is* closed *if every path from it leads to a leaf (labelled $\perp$). A clausal tableau is a* refutation *for Φ if its root is closed.*

The following is well-known and we state it without proof.

Fact 8 *A* CNF*-formula $\Phi(Z_n)$ is unsatisfiable if and only if $\Phi(Z_n)$ has a clausal tableau refutation in the sense of Definition 7.* $\Box$

The *size* of a clausal tableau $T(V, E)$ – denoted $\tau(T)$ – is the total number of internal vertices contained in it, i.e. excluding leaves. We note that while the measures of total number of vertices- $\nu(T)$ – or total number of leaf vertices – $\eta(T)$ – are also used as measures of tableau size, all three are easily related by the facts $\tau(T) = \nu(T) - \eta(T)$ and $\tau(T) \leq \eta(T) \leq (n-1)\tau(T)+1$: the latter inequalities holding since each internal vertex must have at least one child which will either be an internal vertex or a leaf; and since any internal vertex can have at most n children (since the associated clause contains at most n literals).

The *clausal tableau complexity* of an unsatisfiable CNF-formula $\Phi(Z_n)$, is

$$\tau(\Phi(Z_n)) \ =_{\text{def}} \ \min\{\tau(T) \ : \ T \text{ is a clausal tableau refutation of } \Phi(Z_n)\}$$

Finally, we have the following basic property concerning minimal size clausal tableaux.

Fact 9 *If $T(V,E)$ is a mimimal size clausal tableau refutation of $\Phi(Z_n)$ then any any path from the root of T to a leaf vertex, each distinct literal occurs at most once.* $\square$

In the next section we assume that this property holds of the clausal tableaux referred to in the statement and proofs of results.

3 Main Result

Recall that $\delta(\mathcal{H}_\Phi, \Phi)$ is the number of number of moves in the shortest TPI-dispute demonstrating that Φ is unsatisfiable, i.e. not credulously accepted in the argument system $\mathcal{H}_\Phi(\mathcal{X}_\Phi, \mathcal{A}_\Phi)$ of Defn 2. Without loss of generality we may assume that m (the number of clauses in Φ) is at least n (the number of propositional variables).

Theorem 10 *Let $\Phi(Z_n) = \wedge_{i=1}^{m} C_i$ be an unsatisfiable* CNF*-formula and $T(V,E)$ any clausal tableau refutation of $\Phi(Z_n)$, then $\delta(\mathcal{H}_\Phi, \Phi) \leq (2n+1)\tau(T)$.*

Proof: Let

$$\Phi(Z_n) = \bigwedge_{i=1}^{m} C_i \;=\; \bigwedge_{i=1}^{m} \left(\bigvee_{j=1}^{i_k} y_{i,j} \right)$$

be an unsatisfiable CNF-formula, $\mathcal{H}_\Phi(\mathcal{X}, \mathcal{A})$ the argument defined from this via Defn 2 and $T(V,E)$ a clausal tableaux refutation of $\Phi(Z_n)$.

We show how a TPI-dispute may progress using the clauses $C(v)$ labelling internal vertices of T and the literals labelling edges in such a way that after at most $(2n+1)\tau(T)$ moves the Defender has no available move to play. First observe that since $\rho(T)$ is closed, there is a (non-empty) subset W of V with the property: each $w \in W$ is an internal vertex all of whose children are leaf vertices. The TPI-dispute strategy for simulating $T(V,E)$ proceeds by simulating the vertex-edge labelling on a path from $\rho(v)$ to some $w \in W$ with the Challenger selecting clauses according to the labelling of vertices and the Defender selecting counter-attacks on these using the literals labelling edges. When the Challenger attacks Φ with the clause $C(w)$ the only available move to the Defender will be to RETRACT the literal subset forming the current defence. The dispute rules then indicate that in any continuation the label on the vertex w may be viewed as $\perp$ rather than $C(w)$: a defender cannot choose the literal leading from the parent of w to w in T since then a subset that had already been retracted would be instantiated.

We now expand upon this outline. It will be useful to think of the game proceeding via a sequence of trees $\langle T_1, T_2, \ldots, T_r\rangle$ in which T_1 is the clausal tableau refutation $T(V,E)$ and T_{j+1} for $j \geq 1$ is defined from T_j. The final tree T_r will contain a single internal vertex – the original root of $T(V,E)$ – so that when the Challenger attacks Φ with $C(\rho)$ the Defender can make no response. Thus, we say that $\langle T_1, T_2, \ldots, T_j, T_{j+1}, \ldots, T_r\rangle$ is a *reduction sequence* for the clausal tableaux refutation $T(V,E)$ if it satisfies the following conditions: $T_1 = T(V,E)$ the clausal tableau refutation for Φ; for $1 \leq j < r$, T_{j+1} is formed from T_j by choosing an internal vertex, w, all of whose children $\{w_1, w_2, \ldots, w_k\}$ are leaves, deleting these leaf vertices together with the edges $\langle w, w_j\rangle$ from T_j, and relabelling the (new) leaf vertex w in T_{j+1} with $\perp$ instead of $C(w)$ the clause that labelled it in T_j. Finally T_r consists of a single internal vertex all of whose children are leaf vertices (labelled $\perp$), and labelled with the clause $C(\rho)$ that was associated with the root vertex of T_1.

We first note that the length of a reduction sequence (r) is $\tau(T)$ since at each stage exactly one internal vertex (and its children) is replaced with a new leaf vertex. Given a reduction sequence $\langle T_1, T_2, \ldots, T_r\rangle$ of $T(V,E)$ let w_j be the internal vertex of T_j that becomes a leaf in T_{j+1} and let $\text{RUN}_j = \langle C^j_1, y^j_1, C^j_2, y^j_2, \ldots, C^j_k\rangle$ be the sequence of clauses and literals labelling the unique path from the root of T_j to w_j.

With A, C, and R denoting the moves ASSERT, COUNTER, and RETRACT, the upper bound is an easy consequence of the following claim.

Define M_1 to be the (non-terminated) TPI-dispute,

$$M_1 = \langle \text{A}(\Phi), \text{C}(C^1_1), \text{C}(y^1_1), \text{C}(C^1_2), \text{C}(y^1_2), \ldots, \text{C}(C^1_k)\rangle$$

and M_j (for $2 \leq j \leq r$) by

$$M_j = \langle M_{j-1}, \text{R}, \text{C}(C^j_1), \text{C}(y^j_1), \text{C}(C^j_2), \text{C}(y^j_2), \ldots, \text{C}(C^j_k)\rangle$$

We claim for all j with $1 \leq j \leq r$: M_j is a possible continuation of $\langle \text{A}(\Phi)\rangle$ and M_r is terminated.

We prove the first part by induction on j. For the inductive base, $j = 1$, we need to show that M_1 is a possible continuation of $\langle \text{A}(\Phi)\rangle$. To see this it suffices to observe that the sequence of clauses selected by the Challenger and literals chosen by the Defender are in accordance with the criteria defined for moves in Table 1 since they match exactly the labelling on vertices and edges in the path from ρ to w_1 in T_1: at most one vertex on this path is labelled with any given clause C_i of Φ; from the assumption following Fact 9 at most one edge is labelled with any given literal y_j (thus not only does the Defender avoid repeating literals, but the Challenger cannot select clauses to which a defence has already been chosen); from the definition of clausal tableau each y^1_j is a literal in the clause C^1_j; the Defender never adds $\neg y_i$ to its current defence set P since in the corresponding path of T_1 this would reach

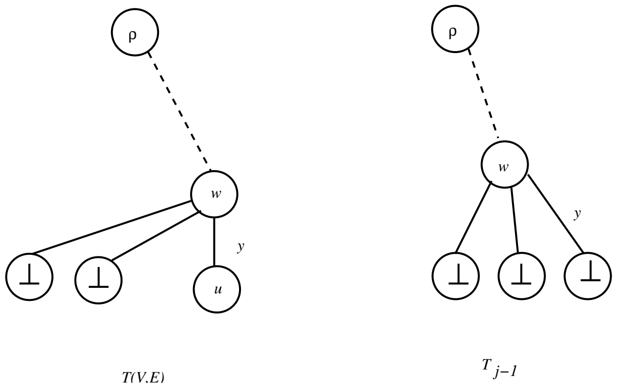

Fig. 1. Inductive Step: w_{j-1} had non-leaf children in T_1

a leaf vertex. These observations establish the Inductive Base: M_1 is a possible continuation of $\langle \mathrm{A}(\Phi)\rangle$.

Now assume that M_{j-1} is a possible continuation of $\langle \mathrm{A}(\Phi)\rangle$ for some $j \geq 2$, we complete the inductive argument by showing that M_j is a possible continuation of M_{j-1}. Consider the state $\sigma = \langle P, Q, R\rangle$ after the last move made in M_{j-1}. This has the current defence $P = \{y_1^{j-1}, \ldots, y_{k-1}^{j-1}\}$ and current attack $Q = \{C_1^{j-1}, \ldots, C_k^{j-1}\}$. In addition, from the Inductive Hypothesis, the set of retracted defences, R, is $\cup_{i=1}^{j-2}\{\{y_1^i, \ldots, y_{k-1}^i\}\}$. We need to show that if $P \neq \emptyset$ then R is the only available move with which D can continue M_{j-1}. To see this consider the path (in T_{j-1}) marked out by RUN_{j-1}. The (internal) vertex w_{j-1} that ends this path has only leaves as children. If the corresponding vertex in $T(V, E)$ has this property, then for each literal y labelling an edge *from* w_{j-1}, the literal $\neg y$ occurs on the path to w_{j-1} from the root of $T(V, E)$, and thus $\neg y \in P$. Thus in this case, the rules governing play by the Defender force a retraction, i.e. the move R must be played. The remaining case is when w_{j-1} had children that were not leaves in $T(V, E)$, cf. Fig.1.

Let y be any literal in $C(w_{j-1})$ such that the edge labelled y from w_{j-1} in $T(V, E)$ leads to an internal vertex u of T_1. Since u is a leaf in T_{j-1} there must be some point T_p in the reduction sequence at which u is replaced by a single leaf vertex in progressing from T_{p-1}. It is certainly the case that $p \leq j-1$, and therefore, from the Inductive Hypothesis, since M_p is a possible continuation of M_{p-1} the first continuation move of M_p must be a $\mathrm{RETRACT}$ move by D. The set of literals that are added to the retracted defences R is $\{y_1^{j-1}, \ldots, y_k^{j-1}, y\}$ (since these delineate the unique path to u in T_{p-1}. We deduce, therefore, that for each of the literals y in $C(w_{j-1})$ either $\neg y \in P$ or $\{P\} \cup \{y\}$ has been retracted prior to the end of M_{j-1}. It follows that the only move available to D to continue M_{j-1} is to retract $P = \{y_1^{j-1}, \ldots, y_k^{j-1}\}$. We have thus shown that $\langle M_{j-1}, \mathrm{R}\rangle$ is a possible continuation of M_{j-1} (for $j-1 < r$). That

$$M_j = \langle M_{j-1}, \mathrm{R}, \mathrm{C}(C_1^j), \mathrm{C}(y_1^j), \mathrm{C}(C_2^j), \mathrm{C}(y_2^j), \ldots, \mathrm{C}(C_k^j)\rangle$$

is a possible continuation of $\langle M_{j-1}, \text{R}\rangle$ now follows from a similar argument to that used in the Inductive Base, so completing the proof that M_j is a possible continuation of $\langle \text{A}(\Phi)\rangle$ for each $1 \le j \le r$. It remains to observe that M_r is terminated, since $M_r = \langle M_{r-1}, \text{R}, \text{C}(C_1^r)\rangle \;=\; \langle M_{r-1}, \text{R}, \text{C}(C(\rho))\rangle$ so that $P = \emptyset$ and in response to the Challenger's attack using $C(\rho)$ the Defender has no available move.

We now obtain the upper bound claimed by noting that the total number of moves in M_r is at most r (the length of the reduction sequence) multiplied by the total number of internal vertices and edges on any path from the root to a vertex with only leaves as children. The latter quantity is at most $2n+1$: no more than n literals can occur over all the edges on such a path; each internal vertex except for the final one has exactly one edge leaving it, thus there are at most $n+1$ vertices and n edges involved. We have already seen that r is equal to $\tau(T)$, thus

$$\delta(\mathcal{H}_\Phi, \Phi) \;\le\; (2n+1)\tau(T) \;\le\; (2n+1)\tau(\Phi)$$

as required. $\Box$ $\Box$

4 Conclusion

In this article we have dealt with an open problem raised in [5] concerning the efficiency of the TPI-dispute game of [12] as a mechanism for validating propositional tautologies of the form $\neg\Phi$ where Φ is a CNF-formula. Building on the simulation by CUT-free Gentzen Systems over connective $\{\vee, \neg\}$ presented in [5], the result of the present article characterises TPI-disputes as polynomially equivalent to this system and to Clausal Tableaux Refutations, informally a validation of $\neg\Phi$ in any one of these three systems can be translated into any other the other two without incurring a 'significant' increase in the proof length.

We conclude by, briefly, outlining some open issues. There are two immediate questions that may bear further investigation. One advantage of both Tableau approaches and Gentzen Systems is that both can handle validation of formulae Ψ when these are presented as arbitrary propositional forms, i.e. not restricted to CNF. In [4] a translation of arbitrary propositional formulae over the basis $\{\wedge, \vee, \neg\}$ into argument systems, $\mathcal{R}_\Phi$ similar to that of Defn. 2 is given. An analysis of TPI-disputes on these systems remains to be performed: in principle, it seems plausible that Tableaux refutation methods translate into TPI-dispute proofs of unsatisfiability when instantiated on instances $\langle \mathcal{R}_\Phi, \Phi\rangle$.

A related, and potentially more fruitful area, concerns the increasing use of Tableau methods with non-classical Logics. These in some ways are a more natural arena for the employment of argument systems as addressed in the current paper: formulation of appropriate dialogue protocols and comparison with existing proof calculi for non-classical Logics may reveal features of interest from both argumentation and proof-theoretic perspectives.

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