Metric Dimension: Results and open problems.

J. Díaz

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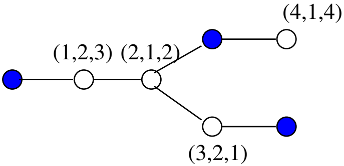
The Metric Dimension problem

Given $G(V, E)$ its metric dimension, $\beta(G)$ is the cardinality of the smallest $L \subset V$ s.t. $\forall x, y \in V$, $\exists z \in L$ with $d_G(x, z) \neq d_G(y, z)$. The set L is called a resolving set.

Harary, Melter, (1976), Slater, (1974)



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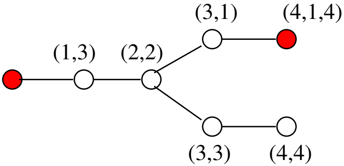
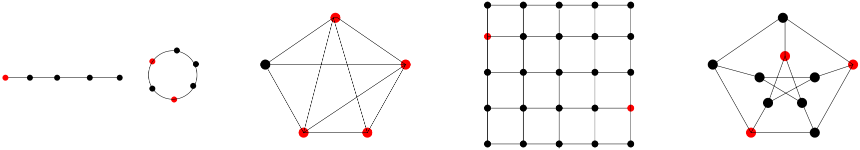


Examples MD



Complexity of Metric Dimension

- NPC for general graphs, Garey,Johnson (1979)
- P for trees, Khuller,Raghavachari,Rosenfeld (1996)
- There is a $2\log n$-approximation for general graphs, Khuller*
- If P$\neq$ NP, there is not a $o(2\log n)$-approximation, Beliova,Eberhard,Erlebach,Hall,Hoffmann,Mihálak,Ram (2006)
- $\forall\epsilon > 0$, There is no $(1-\epsilon)\log n$ for general graphs, unless NP$\subseteq$ DTIME $(n^{\log\log n})$, Hauptmann,Scmhied,Viehmann (2012)
- If P$\neq$ NP, not $o(\log n)$-approximation for general graphs with maximum degree 3, Hartung,Nichterlein (2013)

Characterizations of MD for particular graphs

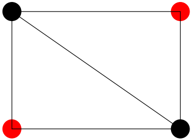
- $\beta(G) = 1$ iff G is a path.
- If $\beta(G) = 2 \Rightarrow G$ does not contain $K_{3,3}$ or K_5
  Khuller* (96).
- $\beta(G) = n - 1$ iff G is a $K_{s,t}$ or a split graph
  Chartrand, Eroh, Ollerman (2000)
- $\beta(G) = n - 1$ iff G is a n-clique.
- If G ha diameter D, $n \leq D^{\beta(G)-1} + \beta(G)$.

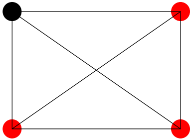
Complexity of Metric Dimension-2

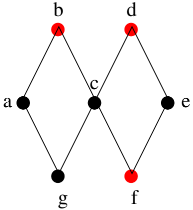
- NPC for bounded degree planar graphs
  Díaz, Pottonen, Serna, Van Leeuwen, (2012)
- NPC for Gabriel graphs Hoffman, Wanke (2012)
- NPC for weighted MD for a variety of graphs
  Epstein, Levin, Woeginger (2012)
- W[2]-complete for general graphs, Hartung, Nichterlein (2013)
- There is a poly-time algorithm for MD on outer-planar graphs
  Díaz*

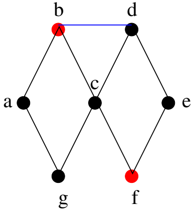
Why MD is difficult? 1

- *Strongly non-local.* A vertex in L can resolve vertices very far away.
- *Non-closed under vertex addition, subtraction, or subdivision.*



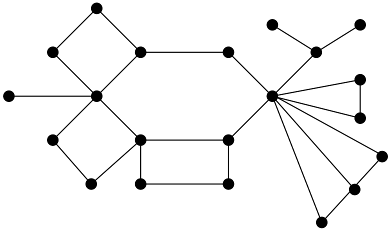






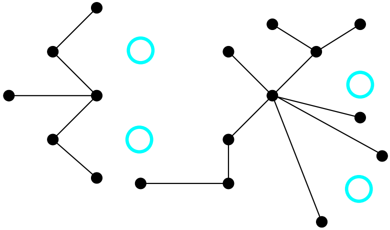
Baker's Technique

Outerplanar: planar with all vertices in the outer face



Baker's Technique (1944): to obtain FPTAS for NP planar problems: dual bounded tree width decomposition +DP.
The technique aims to produce FPTAS for problems that are known to be NPH on planar graphs. They decompose the planar realization into k-outerplanar, get an exact solution for each k-outerplanar slice and combine them.
Among the problems approximated: *vertex cover, maximum independent set, minimum dominating set,* minimum feedback vertex sets.

Min. feedback vertex set



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Why MD is difficult?

3 • MD does not have the bidimensionality behavior.

A problem is bidimensional if it does not increase when performing certain operations as contraction of edges and the solution value for the problem on a $n \times n$-grid is $\Omega(n^2)$ Demaine, Fomin, Hajiaghayi, Thilikos (2005)

Bidimensionality has been used as a tool to obtain sub-exponential time parameterized algorithms for problems on H-minor free graphs and to find PTAS for hard bidimensional problems on planar graphs. Demaine, Hajiaghayi (2005).

Examples: feedback vertex set, vertex cover, minimum maximal matching, face cover, dominating set, or edge dominating set.

Algorithm for outerplanar G

Our approach extended dual (unbounded) tree +DP + New Data structures.

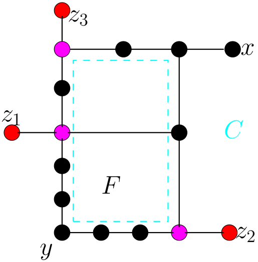
1. Charcterize the resolving sets by giving iff requirements for a v to be in L
2. Define a T where the vertices are the vertex and cut faces of G and the edges in T correspond to inner edges and bridges (separators) of G. Notice as size of an inner face could be arbitrarily large, the width of T could be arbitrary.
   Explor T in bottom-up fashion using two data structures:
   2.1 Boundary conditions
   2.2 Configurations

Characterization of resolving sets

Given G and L, let
$g(v, L) = \{u \in \mathcal{N}(()\, v) \mid d(z, u) = d(z, v) + 1, \forall z \in L\}$
R1. If $\forall v \in V(G)$, $|g(v, L)| \leq 1 \Rightarrow L$ resolves G.

C is a cycle implied by $z_1, z_2 \in L$, $x, y \in V(G)$ if z_1, z_2 do not resolve x, y, and $(z_1 \rightsquigarrow x) \cap (z_2 \rightsquigarrow y) = \emptyset$

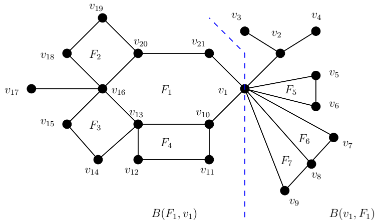
R2. If F is contained in an implied cycle C by z_1, z_2, x, y and L has two representatives in F there is a third representative in C s.t. resolves x, y.

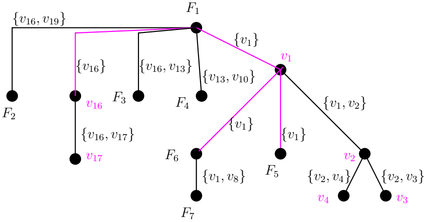


Theorem. *G connected outer planar with $|V| > 2$, then $L \subset V$ is resolving iff L satisfies R1 and R2*

Algorithm for outerplanar

Define the *extended dual tree* for the G outer-planar.
The extended dual tree contains the regular dual tree + cut vertices + trees in original graph.





Algorithm for outerplanar

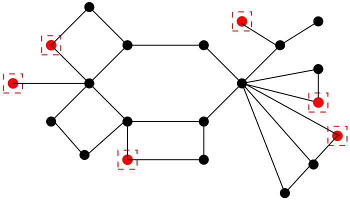
3.- Define use bottom up DP The algorithm will build a small set of members in L satisfying the previous requirements

When traversing the tree we need to combine the information of the children of the vertex and send the information to the parent. It also may expect certain landmarks present in the umprocesed part of the tree. We also need to combine properly the information of the verte and to satisfy R1 and R2.
Given a vertex $v \in V(T)$, this can be done using special DS:
boundary conditions to track the effects of the already placed resolving vertices, foresee the placement of new resolving sets in the unexplored part.
configurations control the process of combining the boundary conditions on the edges between v and its sons, into boundary conditions in the edge between v and its father. It dependa of what v represents (face, vertex cut,end vertex)

Algorithm for outerplanar

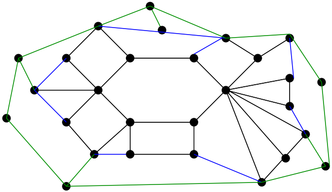
Even the number of vertices in G represented by $v \in V(T)$ could be unbounded, the total number of configurations is polynomial.

The algorithm works in $O(n^8)$ (plenty of room for possible improvement)



Open problems on the complexity of MD

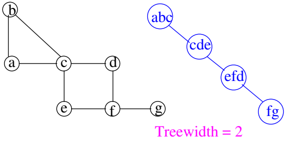
- Find if MD for K-outerplanar graphs is in P or in NPC.



- If there are polynomial time algorithms for K-outerplanar it could yield better approximations for MD on planar graphs. Prove of disprove that MD is APX-hard for planar graphs.

Background on parametrized complexity

The Treewidth of G



Parametrized complexity: Classify the problems according to their difficulty with respect an input parameter of the problem. Downey, Fellows (1999)

$$P \subseteq FPT \subseteq W[1] \subseteq W[2] \subseteq \cdots \subseteq XP$$

FPT is the class of problems solvable in time $f(k)\text{poly}(n)$

- k-vertex cover Given (G, k), does G have a VC $\leq k$?
  Time of k-VC $= (kn + 1.2^k)$. $\therefore$ k-VC $\in$ FPT.
- MD is W[2]-complete for general graphs. Hartung*(13)

Open problems for MD

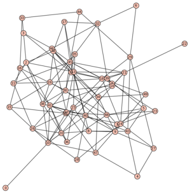
Courcell's Theorem *Any problem definable by* Monadic Second Order Logic *is* FPT *when parametrized by tree width and the length of the formula.*

So far, it seems to be difficult to formulate MD as an MSOL-formula $\Rightarrow$ Courcel's Theorem does not apply.

- Prove mathematically that MD can not be expressed as an MSOL formula.
- Show if MD $\in$ P (or not) for bounded treewidth graphs.
- Study the parametrized complexity of MD on planar graphs.

Binomial Graphs $G(n, p)$

$G \in G(n,p)$ if given n vertices $V(G)$, each possible edge e is included independently with probability $p = p(n)$.

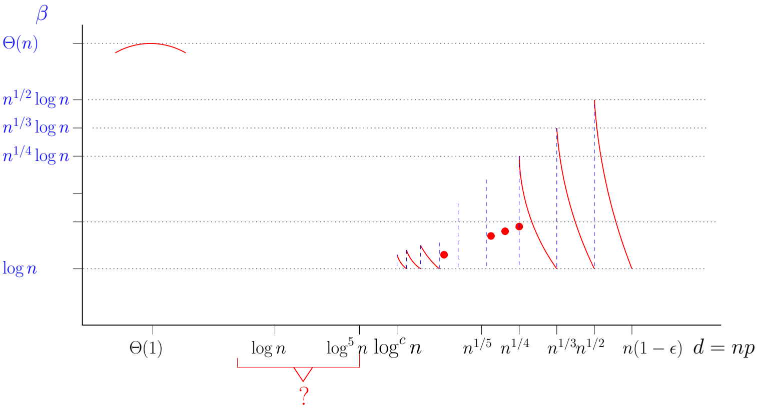


Whp $|E(G)| = p\binom{n}{2}$ and the expected degree of a vertex: $d = np$.
Giant component threshold: $p_t = (1+\epsilon)\frac{1}{n}$.
Connectivity threshold: $p_c = (1+\epsilon)\frac{\log n}{n}$.

Expected $\beta(G)$ in $G(n, p)$

Bollobas, Mitsche, Pralat (2013)

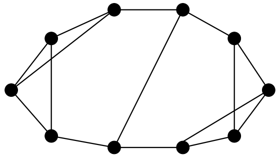
Given $G \in G(n, p)$, choose randomly the resolving set $S \subseteq V$ and bound $\mathbf{Pr}[\exists u, v$ not separated by $S]$.



Find $\mathbf{E}[\beta(G)]$ for $\Theta(1) < \beta(G) < \log^5 n$

Random t-regular Graphs $\mathcal{G}(n, t)$

$G \in \mathcal{G}(n, t)$ if it is uniformly sampled from the set of all graphs with n vertices and degree t. Assume $t = \Theta(1)$.



Let $G \in \mathcal{G}(n, t)$:

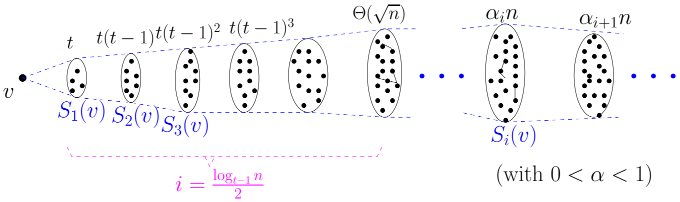
- For $t \geq 3$ aas G is *strongly connected* Cooper (93).
- For $t \geq 3$ aas G is *Hamiltonian* Robinson,Wormald (92,93), Cooper, Frieze (94).
- For $t \geq 3$ aas the diameter of $G = \log_{t-1} + o(\log n)$ Bollobas, Fernandez de la Vega (81)
- For $t \geq 3$, G is an *expander*, i.e. $\exists c > 1$ s.t. $\forall S \subset V(G)$ with $1 \leq |S| \leq \frac{n}{2}$, $\mathcal{N}(S) \geq c|S|$.

Expected $\beta(G)$ for $\mathcal{G}(n,t)$

Díaz, Mitsche, Pérez

Given $G \in \mathcal{G}(n,t)$, $|V| = n$ and $2 < t = \Theta(1)$, then whp $\mathbf{E}[\beta(G)] = \Theta(\log n)$.

Given $G \in \mathcal{G}(n,t), v \in V(G)$, let $S_i = \{u \in V(G) \mid d_G(v,u) = i\}$



Given $v \in V(G)$, for any pair $(u,w) \in V^2$:
v does not separate u and w if $u, w \in S_i$, and
v separates u and w if $u \in S_i$ & $w \in S_{i+1}$ (or vice versa).

Expected $\beta(G)$ for $\mathcal{G}(n, t)$

Therefore, $\mathbf{Pr}[v \text{ separates } u\&w] \geq 2\alpha_i\alpha_{i+1}$, and
$\mathbf{Pr}[v \text{ does not separate } u\&v] \geq \alpha_i^2 + \alpha_{i+1}^2$,
where α_i and α_{i+1} are constants between 0 and 1.

$$\underbrace{(1-\alpha_i\alpha_{i+1})}_{\alpha} \geq \mathbf{Pr}[v \text{ separates } u\&w] \geq \underbrace{2\alpha_i\alpha_{i+1}}_{\alpha'}$$

Upper Bound

Randomly choose a resolving $L \subset V(G)$ with $|L| = C \log n$, for large constant $C > 0$.

Then for a particular pair of vertices u, w
$\mathbf{Pr}[L \text{ does not separate } u\&w] < \alpha^{C \log n} \sim o(\frac{1}{n^2})$ (union bound)
Let $X_C =$ be the number of pairs not separated by L,

$$\mathbf{E}[X_C] < n^2\alpha^{C \log n} \to 0 \Rightarrow \mathbf{Pr}[X_C > 0] \to 0$$

Expected $\beta(G)$ for $\mathcal{G}(n,t)$: Lower Bound

Randomly choose a resolving set $L \subset V(G)$ with $|L| = c \log n$, for small constant $c > 0$.

$\mathbf{Pr}\left[L \text{ does not separate } u\&w\right] \geq \alpha'^{c \log n} \sim \omega(\frac{1}{n^2})$

$\Rightarrow$ If X_c = number pairs not separated by L, then

$$\mathbf{E}\left[X_c\right] > n^2 \alpha'^{c \log n} \to \infty \Rightarrow \mathbf{Pr}\left[X_c > 0\right] = 1 - o(1)$$

Therefore, $\boxed{\beta(G) = \Theta(\log n)}$.

Ongoing work: the expected value of α and C.

Thank you.