Representation and Complexity in Boolean Games

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Abstract. Boolean games are a class of two-player games which may be defined via a Boolean form over a set of atomic actions. A particular game on some form is instantiated by partitioning these actions between the players – player 0 and player 1 – each of whom has the object of employing its available actions in such a way that the game's outcome is that sought by the player concerned, i.e. player i tries to bring about the outcome i. In this paper our aim is to consider a number of issues concerning how such forms are represented within an algorithmic setting. We introduce a concept of *concise form* representation and compare its properties in relation to the more frequently used "extensive form" descriptions. Among other results we present a "normal form" theorem that gives a characterisation of winning strategies for each player. Our main interest, however, lies in classifying the computational complexity of various decision problems when the game instance is presented as a concise form. Among the problems we consider are: deciding existence of a winning strategy given control of a particular set of actions; determining whether two games are "equivalent".

1 Introduction

The combination of logic and games has a significant history: for instance, a game theoretic interpretation of quantification goes back at least to the 19th century with C.S. Peirce ([9]), but it took until the second half of the previous century, when Henkin suggested a way of using games to give a semantics for infinitary languages ([5]), that logicians used more and more game-theoretic techniques in their analysis, and that game theoretic versions for all essential logical notions (truth in a model, validity, model comparison) were subsequently developed. In addition, connections between game theory and modal logic have recently been developed.

Where the above research paradigm can be summarised as 'game theory in logic', toward the end of the last decade, an area that could be coined 'logic in game theory' received a lot of attention (we only mention [7]; the reader can find more references there); especially when this field is conceived as formalising the dynamics and the powers of coalitions in multi-agent systems, this boost in attention can be explained.

Boolean Games, as introduced by Harrenstein *et. al* ([4]) fit in this research paradigm by studying two-player games in which each player has control over a set of atomic propositions, is able to force certain states of affairs to be true, but is dependent on the other player concerning the realisation of many other situations. This seems to model

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a rather general set of multi-agent scenarios, where it is quite common that each agent has its own goals, but, at the same time, only limited control over the resources (the particular propositional variables); at the same time, all two-player games in strategic form can be modelled as such.

The issues treated in [3,4], are primarily driven by semantic concerns, thus: presenting inductive definitions of two-player fully competitive games; introducing viable ideas of what it means for two games to be "strategically equivalent"; constructing sound and complete calculi to capture winning strategies; and exploring the logical and algebraic relationship between the game-theoretic formalism and propositional logic functions.

Our concerns in the present paper are two-fold: firstly, we revisit the correspondence between propositional logic functions and Boolean forms presented in [4] under which a given form g involving atomic actions A maps to a propositional function $\lceil g \rceil$ of the propositional arguments A. We establish a strong connection between the concept of winning strategy in a game and classical representation theorems for propositional functions. Among the consequences following from this connection we may,

a. Characterise all winning strategies for either player in a form g in terms of particular representations of $\lceil g \rceil$.
b. Obtain various "normal form" theorems for Boolean forms.
c. Characterise the "minimal winning control allocations" for a player,
d. Characterise precisely those forms, g, for which neither player has a winning strategy unless they have control over every atomic action.

Our second and principal interest, however, is to examine the computational complexity of a number of natural decision problems arising in the domain of Boolean game forms. In particular,

e. Given a description of a form g and an allocation of atomic actions, σ_i, to player i, does σ_i give a winning strategy for i?
f. Given descriptions of two forms g and h are these *strategically equivalent*?
g. Given a description of some form g, is there *any* subset of actions with which player i has a winning strategy in g?
h. Given g and h together with instantiations of these to games via σ for g and τ for h, are the resulting *games* – (g, σ) and (h, τ) – "equivalent"?

The computational complexity of these decision problems turns on how much freedom is permitted in representing the forms within an instance of the problem. If we insist on an "extensive form" description then all may be solved by deterministic polynomial-time methods: this bound, however, is polynomial in the size of the instance, which for extensive forms will typically be exponential in the number of atomic actions. If, on the other hand, we allow so-called "concise forms", each of the problems above is, under the standard assumptions, computationally intractable, with complexities ranging from NP–complete to completeness for a class believed to lie strictly between the second and third levels of the polynomial hierarchy.

As one point of interest we observe that the classification results for the problems described by (e)–(g) are fairly direct consequences of the characterisations given by (a) and (b). The constructions required in our analysis of the "game equivalence" problem (h) are, however, rather more intricate and we omit the details of this.

In total these results suggest a trade-off between extensive, computationally tractable descriptions that, in general, will be unrealistically large and concise descriptions for which decision methods for those questions of interest are unlikely to be viable. It may, however, be the case that this apparent dichotomy is not as clearly defined as our results imply: we show that there are natural questions which even in terms of extensive forms are computationally intractable.

The remainder of this paper is organised as follows. In the next section we review the definitions of Boolean form, game and "winning strategy" originating in [3,4]. In Section 3 we examine the relationship between forms and propositional functions from [4] and show that this can be used to characterise winning strategies as well as yielding various "normal form" representations for Boolean forms. We then introduce the ideas that form the central concern of this paper, namely the notion of concise form descriptions, and discuss the properties of these in relation to extensive forms. We continue this analytic comparison in Section 5, wherein various decision problems are defined and classified with respect to their computational complexity. The final section outlines some directions for further research.

2 Boolean Forms and Games

Definition 1. *(Harrenstein* et al. *[4]) Let $A = \{a_1, \ldots, a_n\}$ be a (finite) set of* atomic actions *and $\mathcal{G}_A$ the smallest set that satisfies both:*

a. $\{\mathbf{0}, \mathbf{1}\} \subset \mathcal{G}_A$.
b. If $g_0,\ g_1 \in \mathcal{G}_A$ and $\alpha \in A$ then $\alpha(g_0, g_1) \in \mathcal{G}_A$.

Thus $\mathcal{G}_A$ defines the set of Boolean forms with atomic strategies A.

A Boolean form $g \in \mathcal{G}_A$ is instantiated as a Boolean game*, $\langle g, \sigma\rangle$, by specifying for each player $i \in \{0, 1\}$ which subset of the actions the player has complete control over. Thus $\sigma : A \to \{0, 1\}$ describes a partition of A as $\langle \sigma_0, \sigma_1\rangle$ with σ_i the subset of A mapped to i by σ. We refer to σ_i as the* control allocation *for player i.*

Given some Boolean form $g \in \mathcal{G}_A$, we may think of g depicted as a *binary tree* so that: if $g \in \{\mathbf{0}, \mathbf{1}\}$, then this tree is a single vertex labelled with the relevant outcome **0** or **1**; if $g = \alpha(g_0, g_1)$ this tree comprises a root labelled with the action α one of whose out-going edges is directed into the root of the tree for g_0, with the other out-going edge directed into the root of the tree for g_1. An outcome of $\alpha(g_0, g_1)$ is determined by whether the action α is invoked – in which event play continues with g_0 – or abstained from – resulting in play continuing from g_1. Thus given some subset s of A, and the form g, the outcome $\mathbf{sf}(g)(s)$ (the "strategic form" of [4]) resulting from this process is inductively defined via

$$\mathbf{sf}(g)(s) \ := \ \begin{cases} 0 & \text{if } \ g = \mathbf{0} \\ 1 & \text{if } \ g = \mathbf{1} \\ \mathbf{sf}(g_0)(s) & \text{if } \ g = \alpha(g_0, g_1) \text{ and } \alpha \in s \\ \mathbf{sf}(g_1)(s) & \text{if } \ g = \alpha(g_0, g_1) \text{ and } \alpha \notin s \end{cases}$$

The forms g and $h \in \mathcal{G}_A$ are regarded as *strategically equivalent* ($g \equiv_{\mathbf{sf}} h$) if for every choice $s \subseteq A$ it holds that $\mathbf{sf}(g)(s) = \mathbf{sf}(h)(s)$.

We note one important feature of this model is its requirement that having committed to playing or refraining from playing an action α the player allocated control of α does not subsequently reverse this choice. In consequence it may be assumed in terms of the binary tree presentation of forms that a path from the root to an outcome contains at most one occurrence of any atomic action.

Given an instantiation of g to a game $\langle g, \sigma \rangle$ the aim of player i is to choose for each action, α, within its control, i.e. in the set σ_i, whether or not to play α. Thus a *strategy* for i in $\langle g, \sigma \rangle$ can be identified as the *subset* of σ_i that i plays. This gives rise to the following definition of *winning strategy* in [4].

Definition 2. *A player $i \in \{0, 1\}$ has a* winning strategy *in the game $\langle g, \sigma \rangle$ if there is a subset ν of σ_i such that regardless of which instantiation τ of σ_{1-i} is chosen, $\mathbf{sf}(g)(\nu \cup \tau) = i$.*

A control allocation, σ_i, is minimal *for i in g if there is a winning strategy, ν for player i in $\langle g, \sigma \rangle$* and *changing the allocation of any $\alpha \in \sigma_i$ to σ_{1-i}, results in a game where i no longer has a winning strategy, i.e. in the control allocation, σ' for which $\sigma(\alpha) = i$ becomes $\sigma'(\alpha) = 1 - i$, i does not have a winning strategy in the game $\langle g, \sigma' \rangle$.*

The basic definition of Boolean form is extended by introducing a number of operations on these.

Definition 3. *Let $g_0,\ g_1,\ h \in \mathcal{G}_A$, and $\alpha \in A$. The operations $+$, $\cdot$, $\otimes$ and $^-$ are,*

1. $\mathbf{0} + h := h$; $\quad \mathbf{1} + h := \mathbf{1}$; $\quad \alpha(g_0, g_1) + h := \alpha(g_0 + h, g_1 + h)$
2. $\mathbf{0} \cdot h := \mathbf{0}$; $\quad \mathbf{1} \cdot h := h$; $\quad \alpha(g_0, g_1) \cdot h := \alpha(g_0 \cdot h, g_1 \cdot h)$
3. $\overline{\mathbf{0}} := \mathbf{1}$; $\quad \overline{\mathbf{1}} := \mathbf{0}$; $\quad \overline{\alpha(g_0, g_1)} := \alpha(\overline{g_0}, \overline{g_1})$
4. $\otimes(\mathbf{0}, g, h) := h$; $\quad \otimes(\mathbf{1}, g, h) := g$; $\quad \otimes(\alpha(g_0, g_1), h_0, h_1) :=$
$\alpha(\otimes(g_0, h_0, h_1), \otimes(g_1, h_0, h_1))$

Informally we may regard the operations of Definition 3 in terms of transformations effected on the associated binary tree form describing g. Thus, $g + h$ is the form obtained by replacing each 0 outcome in g with a copy of h; $g \cdot h$ that resulting by replacing each 1 outcome in g with a copy of h; and $\overline{g}$ the form in which 0 outcomes are changed to 1 and *vice-versa*. The operation $\otimes$ is rather more involved, however, the form resulting from $\otimes(g, h_0, h_1)$ is obtained by *simultaneously* replacing 0 outcomes in g with h_1 and 1 outcomes with h_0.

Finally we recall the following properties,

Fact 1. *The set of Boolean forms $\mathcal{G}_A$ is such that,*

1. *The relationship $\equiv_{\mathbf{sf}}$ is an equivalence relation.*
2. *Let $[g]$ denote $\{h : g \equiv_{\mathbf{sf}} h\}$ and $\mathcal{G}_{\equiv}$ be $\{[g] : g \in \mathcal{G}_A\}$. The operations of Definition 3 may be raised to corresponding operations on equivalence classes of forms so that,*

$$[g] + [h] = [g + h] \ ; \ [g] \cdot [h] = [g \cdot h]$$

$$\overline{[g]} = [\,\overline{g}\,] \ ; \ \otimes([g], [h_0], [h_1]) = [\otimes(g, h_0, h_1)]$$

3. *Each form g may be associated with a propositional function,* $\lceil g \rceil$ *with arguments A so that* $\lceil g \rceil[s] = 0$ *if and only if* $\mathbf{sf}(g)(s) = 0$ *(where for a propositional function* $f(X)$ *and* $Q \subseteq X$*,* $f[Q]$ *denotes the value of the function f under the instantiation* $x := 1$ *if* $x \in Q$*,* $x := 0$ *if* $x \notin Q$*). For g and h forms in* $\mathcal{G}_A$

$$g \equiv_{\mathbf{sf}} h \;\;\Leftrightarrow\;\; \lceil g \rceil \equiv \lceil h \rceil$$

3 Characterising Winning Strategies

We recall the following standard definitions concerning propositional logic functions.

Definition 4. *Let* $X_n = \{x_1, \ldots, x_n\}$ *be a set of n propositional variables and* $f(X_n)$ *some propositional logic function defined over these. A* literal*, y over* X_n*, is a term x or* $\neg x$ *for some* $x \in X_n$*. A set of literals,* $S = \{y_1, \ldots, y_r\}$*, is well-formed if for each* x_k *at most one of* x_k*,* $\neg x_k$ *occurs in S. A well-formed set S defines an* implicate *of* $f(X_n)$ *if the (partial) instantiation,* α_S*, of* X_n *by* $x_k := 1$ *if* $\neg x_k \in S$*,* $x_k := 0$ *if* $x_k \in S$ *is such that* $f(\alpha) = 0$ *for all* α *that agree with* α_S*. A well-formed set S is an* implicant *of* $f(X_n)$ *if the (partial) instantiation,* β_S*, of* X_n *by* $x_k := 0$ *if* $\neg x_k \in S$*,* $x_k := 1$ *if* $x_k \in S$ *is such that* $f(\alpha) = 1$ *for all* α *that agree with* β_S*. An implicate (resp. implicant) is a* prime *implicate (resp. implicant) of* $f(X_n)$ *if no* strict *subset* S' *of S is an implicate (implicant) of* $f(X_n)$*.*

We require some further notation in order to describe the main results of this section.

Definition 5. *For a form* $g \in \mathcal{G}_A$*,*

$$W_i(g) \mathrel{\hat{=}} \{(\sigma_i, \nu) \;:\; \nu \subseteq \sigma_i \text{ and } \forall\, \tau \subseteq \sigma_{1-i} \;:\; \mathbf{sf}(g)(\nu \cup \tau) = i\}$$

$$\begin{aligned} M_i(g) \mathrel{\hat{=}} \{(\sigma_i, \nu) \;:\;& (\sigma_i, \nu) \in W_i(g) \text{ and} \\ & \forall \alpha \in \nu\; \exists \tau \subseteq \sigma_{1-i} \;:\; \mathbf{sf}(g)(\nu \setminus \{\alpha\} \cup \tau) = 1 - i \text{ and} \\ & \forall \alpha \in \sigma_i \setminus \nu\; \exists \tau \subseteq \sigma_{1-i} \;:\; \mathbf{sf}(g)(\nu \cup \{\alpha\} \cup \tau) = 1 - i\} \end{aligned}$$

Thus, $W_i(g)$ is the set of allocations (σ_i) paired with winning strategies for *i* in these (ν), while $M_i(g)$ is the subset of these in which σ_i is a minimal control allocation for *i*.

Now consider the set of *play choices*, $\mathcal{P}(A)$, given by

$$\mathcal{P}(A) \;=\; \{(\nu, \pi) \;:\; \nu \subseteq A,\; \pi \subseteq A, \;\text{ and } \nu \cap \pi = \emptyset\}$$

and define a (bijective) mapping, *Lits*, between these and well-formed literal sets over the propositional variables $\{a_1, \ldots, a_n\}$ by

$$Lits(\;(\nu, \pi)\;) \;\;:=\;\; \bigcup_{a_i \in \nu} \{a_i\} \;\cup\; \bigcup_{a_i \in \pi} \{\neg a_i\}$$

We now obtain,

Theorem 1. *Let* $\langle g, \sigma \rangle$ *be a Boolean game with* $g \in \mathcal{G}_A$*.*

a. *Player* 0 *has a winning strategy in* $\langle g, \sigma \rangle$ *if and only if there exists* $\nu \subseteq \sigma_0$ *for which* $Lits((\nu, \sigma_0 \setminus \nu))$ *is an implicate of* $\lceil g \rceil(A)$*.*

b. Player 1 *has a winning strategy in* $\langle g, \sigma \rangle$ *if and only if there exists* $\nu \subseteq \sigma_1$ *for which* $Lits((\nu, \sigma_1 \setminus \nu))$ *is an implicant of* $\lceil g \rceil(A)$.

c. The control allocation σ_0 *is minimal for player* 0 *in* g *if and only if there exists* $\nu \subseteq \sigma_0$ *for which* $Lits((\nu, \sigma_0 \setminus \nu))$ *is a prime implicate of* $\lceil g \rceil(A)$.

d. The control allocation σ_1 *is minimal for player* 1 *in* g *if and only if there exists* $\nu \subseteq \sigma_1$ *for which* $Lits((\nu, \sigma_1 \setminus \nu))$ *is a prime implicant of* $\lceil g \rceil(A)$.

Corollary 1. *Let* $\sum$ *and* $\prod$ *denote the raised versions of the operations* $+$ *and* $\cdot$ *when applied over a collection of forms. For* $(\nu, \mu) \in \mathcal{P}(A)$ *the forms* $c_{(\nu,\mu)}$ *and* $d_{(\nu,\mu)}$ *are*

$$c_{(\nu,\mu)} \hat{=} \sum_{\alpha \in \nu} \alpha(0,1) + \sum_{\alpha \in \mu} \alpha(1,0)$$

$$d_{(\nu,\mu)} \hat{=} \prod_{\alpha \in \nu} \alpha(1,0) \cdot \prod_{\alpha \in \mu} \alpha(0,1)$$

a. For all $g \in \mathcal{G}_A$,

$$g \equiv_{\mathbf{sf}} \sum_{(\sigma_1,\nu) \in W_1(g)} d_{(\nu,\sigma_1 \setminus \nu)} \equiv_{\mathbf{sf}} \prod_{(\sigma_0,\nu) \in W_0(g)} c_{(\nu,\sigma_0 \setminus \nu)}$$

b. For all $g \in \mathcal{G}_A$,

$$g \equiv_{\mathbf{sf}} \sum_{(\sigma_1,\nu) \in M_1(g)} d_{(\nu,\sigma_1 \setminus \nu)} \equiv_{\mathbf{sf}} \prod_{(\sigma_0,\nu) \in M_0(g)} c_{(\nu,\sigma_0 \setminus \nu)}$$

Proof. Both are immediate from Theorem 1 and the fact that every propositional function $f(X_n)$ is equivalent to formulae described by the disjunction of product terms matching (prime) implicants and the conjunction of clause terms matching (prime) implicates. □

Corollary 2. *For any set* $A^{(n)} = \{a_1, \ldots, a_n\}$ *of* n *atomic actions, there are (modulo strategic equivalence)* exactly two $g \in \mathcal{G}_{A^{(n)}}$ *with the property that neither player has a winning strategy in the game* $\langle g, \sigma \rangle$ *unless* $\sigma_i = A^{(n)}$.

Thus, with these two exceptions, in any g at least one *of the players can force a win without having control over the entire action set.*

Proof. We first show that there are at least two such cases.

Consider the form, $Par_n \in \mathcal{G}_{A^{(n)}}$, inductively defined as

$$Par_n := \begin{cases} a_1(0,1) & \text{if } n = 1 \\ \otimes(Par_{n-1}, a_n(0,1), a_n(1,0)) & \text{if } n \geq 2 \end{cases}$$

Then, it is easy to show that $\lceil Par_n \rceil \equiv a_1 \oplus a_2 \oplus \ldots \oplus a_n$, where $\oplus$ is binary exclusive or. This function, however, cannot have its value determined by setting any *strict* subset of its arguments, i.e. i has a winning strategy only if i has control over the entire action set $A^{(n)}$. The second example class is obtained simply by considering $\overline{Par_n}$.

To see that there at most two cases, it suffices to note that if $f(A^{(n)}) \not\equiv \lceil Par_n \rceil$ and $f(A^{(n)}) \not\equiv \neg \lceil Par_n \rceil$ then there is either a prime implicate or prime implicant of f that depends on at most $n-1$ arguments. Theorem 1(c–d) presents an appropriate winning strategy in either case. □

4 Representation Issues – Concise Versus Extensive

Representational issues are one aspect that must be considered in assessing the computational complexity of particular decision problems on Boolean forms/games: thus one has a choice of using *extensive forms* whereby $g \in \mathcal{G}_A$ is described in terms of Definition 1(a,b) or, alternatively, in some *"concise" form*, e.g. $g \in \mathcal{G}_A$ is described by its construction in terms of the operations in Definition 3. Formally,

Definition 6. *A game $g \in \mathcal{G}_A$, is described in* extensive form *if it is presented as a labelled binary tree $T(V, E)$ each internal vertex of which is labelled with some $a_i \in A$ and whose leaves are labelled with an outcome from the set $\{0, 1\}$. The* size *of an extensive form representation T of g, denoted $S^{ext}(g, T)$ is the total number of* internal *vertices in T, i.e. the number of occurrences of atomic strategy labels.*

A game $g \in \mathcal{G}_A$ is described in concise form *if given as a well-formed expression in the language $\mathcal{L}_A$ defined as follows:*

- *a.* $\{0, 1\} \subset \mathcal{L}_A$.
- *b. For each $\alpha \in A$ and F_0, $F_1 \in \mathcal{L}_A$, $\alpha(F_0, F_1) \in \mathcal{L}_A$.*
- *c. For each $\theta \in \{+, \cdot\}$, and F, $G \in \mathcal{L}_A$, $F\ \theta\ G \in \mathcal{L}_A$.*
- *d. For each F, G, $H \in \mathcal{L}_A$, $\otimes(F, G, H) \in \mathcal{L}_A$.*
- *e. For each $F \in \mathcal{L}_A$, $\overline{F} \in \mathcal{L}_A$*
- *f. For each $F \in \mathcal{L}_A$, $(F) \in \mathcal{L}_A$ (bracketing).*
- *g. $F \in \mathcal{L}_A$ if and only if F is formed by a finite number of applications of the rules (a)–(f).*

The size *of a concise representation, F of g, denoted $S^{con}(F, g)$ is the total number of applications of rule (b) used to form F.*

It will be useful, subsequently, to consider the following measures in capturing ideas of representational complexity.

Definition 7. *For $g \in \mathcal{G}_A$, the* extensive complexity *of (the class) $[g]$ is,*

$$S^{ext}([g]) \ :=\ \min\ \{S^{ext}(h, T)\ :\ T \text{ is an extensive representation of } h \text{ with } g \equiv_{\mathbf{sf}} h\}$$

Similarly, the concise complexity *of (the class) $[g]$, is*

$$S^{con}([g]) \ :=\ \min\ \{S^{con}(h, F)\ :\ F \text{ is a concise representation of } h \text{ with } g \equiv_{\mathbf{sf}} h\}$$

A significant difference between extensive and concise descriptions of $g \in \mathcal{G}_A$ is that the latter may be exponentially shorter (in terms of $n = |A|$) than the former, even for quite "basic" game forms. Thus we have the following result concerning these measures,

Proposition 1.

- *a. For all $g \in \mathcal{G}_A$, $S^{con}([g])\ \leq\ S^{ext}([g])\ \leq\ 2^{|A|} - 1$.*
- *b. For each $n \geq 1$, let $A^{(n)}$ denote the set of atomic strategies $\{a_1, a_2, \ldots, a_n\}$. There is a sequence $\{g^{(n)}\}$ of forms with $g^{(n)} \in \mathcal{G}_{A^{(n)}}$ for which:*

$$S^{ext}([g^{(n)}])\ =\ 2^n - 1 \quad ; \quad S^{con}([g^{(n)}])\ =\ O(n)$$

Proof. (*outline*)

a. The upper bound $S^{con}([g]) \le S^{ext}([g])$ is obvious. For $S^{ext}([g]) \le 2^{|A|} - 1$ it suffices to observe that in any extensive representation of h for which $g \equiv_{\mathbf{sf}} h$ no atomic strategy $\alpha \in A$ need occur more than once on any path leading from the root to a leaf. Thus each path contains at most $|A|$ distinct labels. Since the representation is a binary tree the upper bound is immediate.
b. Use the form Par_n of Corollary 2 whose concise form complexity is $2n - 1$[1]. It may be shown that a minimal extensive form representation of Par_n must be a complete binary tree of depth n. □

We can, in fact, obtain a rather stronger version of Proposition 1(a), by considering another idea from the study of propositional logic functions, namely:

Definition 8. *A set $\mathcal{C}$ of implicates (implicants) is a* covering set *for $f(X_n)$ if for any instantiation α under which $f(\alpha)$ takes the value 0 (1) there is at least one $S \in \mathcal{C}$ that evaluates to 0 (1) under α. For a propositional function $f(X_n)$ we denote by $R_i(f)$ the sizes of the smallest covering set of implicates (when $i = 0$) and implicants ($i = 1$).*

From this,

Proposition 2. *For all $g \in \mathcal{G}_A$,*

a. $S^{ext}([g]) \ge R_0(\lceil g \rceil) + R_1(\lceil g \rceil) - 1.$
b. $S^{con}([g]) \le |A| \min\{R_0(\lceil g \rceil), R_1(\lceil g \rceil)\}.$

Proof. (*outline*)

a. Suppose T is a minimal size extensive representation of $g \in \mathcal{G}_A$. It must be the case that for $i \in \{0,1\}$, T has at least $R_i(\lceil g \rceil)$ leaves labelled with the outcome i. For if this failed to be the case for $i = 0$ say, then from Theorem 1(a) we could construct a covering set of implicates for $\lceil g \rceil$ via the choice of actions on each path from the root of T to an outcome 0: this covering set, however, would contain fewer than $R_0(\lceil g \rceil)$ implicates. We deduce that T has at least $R_0(\lceil g \rceil) + R_1(\lceil g \rceil)$ *leaves* and hence, $R_0(\lceil g \rceil) + R_1(\lceil g \rceil) - 1$ internal vertices.
b. Consequence of Corollary 1. □

We note that Proposition 2(a) implies the first part of Propostion 1(b) since $R_0(\lceil Par_n \rceil) = R_1(\lceil Par_n \rceil) = 2^{n-1}$.

5 Decision Problems for Boolean Games

Given a description of some Boolean game (g, σ), probably the most basic question one would be concerned with is whether one or other of the players can force a win with their allotted set of actions, σ_i. Similarly, if one is just presented with the form $g \in \mathcal{G}_A$,

[1] A similar result can be shown even if the concise form representation may only use the operations $\{+, \cdot, -\}$. The gap between $S^{ext}(g^{(n)})$ and $S^{con}_{\{+,\cdot,-\}}(g^{(n)})$, however, is not fully exponential in this case: $S^{con}_{\{+,\cdot,-\}}(g^{(n)}) = \Theta(n^2)$.

a player may wish to discover which subset of A it would be beneficial to gain control of, since it would admit a winning strategy in g. We note that our notion of "winning strategy" for i using a set of actions σ_i has one consequence: if σ_i admits a winning strategy for i in g, then any superset of σ_i also admits a winning strategy for i. From this observation it follows that in order to decide if i has *any* winning strategy in g it suffices to determine if setting $\sigma_i = A$, that is giving i control of *every* atomic action, admits such.

As well as these questions concerning two different aspects of whether one player has the capability to win a game, another set of questions arises in determining various concepts of games being "equivalent". At one level, we have the idea of *strategic* equivalence: thus deciding of given forms g and h in $\mathcal{G}_A$ whether these are strategically equivalent. The concept of strategic equivalence, however, is in some ways rather too precise: there may be forms g and h which, while not strategically equivalent, we may wish to regard as "equivalent" under certain conditions, for example if it were the case that particular *instantiations* of g and h as *games* were "equivalent" in some sense. Thus, Harrenstein (personal communication), has suggested a concept of two Boolean games – $\langle g, \sigma\rangle$ and $\langle h, \tau\rangle$ – being *equivalent* based on the players being able to bring about "similar" outcomes: the formal definition being presented in terms of a "forcing relationship" similar in spirit to those introduced in [2,8]. We may very informally describe this notion as follows,

Definition 9. *Given a game $\langle g, \sigma\rangle$, a subset s of the action set A and a* possible outcome*, X, from the set $\{\{0\}, \{1\}, \{0,1\}\}$, the relationship $s\ \rho^i_{\langle g,\sigma\rangle}\ X$ holds whenever: there is a choice (s') for player i that is consistent with the profile s and such that no matter what choice of actions (s'') is made by the other player, the outcome of the g will be in X. The notation $\rho^i_{\langle g,\sigma\rangle}\ X$ indicates that $s\ \rho^i_{\langle g,\sigma\rangle}\ X$ for every choice of $s \subseteq A$.*

We write $\langle g, \sigma\rangle \equiv_{\mathbf{H}} \langle h, \tau\rangle$ if and only if: for each player i and possible outcome X, $\rho^i_{\langle g,\sigma\rangle}\ X \ \Leftrightarrow\ \rho_{\langle h,\tau\rangle}\ X$.

We have given a rather informally phrased definition of the "game equivalence" concept engendered via $\equiv_{\mathbf{H}}$, since the definition we employ is given in,

Definition 10. *Let $\langle g, \sigma\rangle$ and $\langle h, \tau\rangle$ be Boolean games with $g \in \mathcal{G}_A$ and $h \in \mathcal{G}_B$, noting that we do* not *require $A = B$ or even $A \cap B \neq \emptyset$. We say that $\langle g, \sigma\rangle$ and $\langle h, \tau\rangle$ are* equivalent games*, written $\langle g, \sigma\rangle \equiv_{\mathbf{g}} \langle h, \tau\rangle$, if the following condition is satisfied:*

For each $i \in \{0,1\}$, player i has a winning strategy in $\langle g, \sigma\rangle$ if and only if player i has a winning strategy in $\langle h, \tau\rangle$.

Thus, we interpret $\langle g, \sigma\rangle \equiv_{\mathbf{g}} \langle h, \tau\rangle$ as indicating that i's capabilities in $\langle g, \sigma\rangle$ *are identical to* its capabilities in $\langle h, \tau\rangle$: if i can force a win in one game then it can force a win in the other; if *neither* player can force a win in one game then neither player can force a win in the other. The following properties of $\equiv_{\mathbf{g}}$ with respect to $\equiv_{\mathbf{H}}$ may be shown

Lemma 1. *For a control allocation mapping, σ, let $\bar{\sigma}$ denote the control allocation in which $\bar{\sigma}_0 := \sigma_1$ and $\bar{\sigma}_1 := \sigma_0$, i.e. $\bar{\sigma}$ swops the control allocations around. For games $\langle g, \sigma\rangle$ and $\langle h, \tau\rangle$, it holds: $\langle g, \sigma\rangle \ \equiv_{\mathbf{H}}\ \langle h, \tau\rangle$ if and only if* both $\langle g, \sigma\rangle \equiv_{\mathbf{g}} \langle h, \tau\rangle$ and $\langle g, \bar{\sigma}\rangle \equiv_{\mathbf{g}} \langle h, \bar{\tau}\rangle$.

The relationship between $\equiv_{\mathbf{H}}$ and $\equiv_{\mathbf{g}}$ described in Lemma 1 merits some comment. Certainly it is clear that $\equiv_{\mathbf{g}}$ does not capture the entire class of equivalent games within the sense of being able to "enforce similar outcomes" defined in $\equiv_{\mathbf{H}}$. There is, however, a simple mechanism by which those missed can be recovered, i.e. by requiring the additional condition $\langle g, \bar{\sigma}\rangle \equiv_{\mathbf{g}} \langle h, \bar{\tau}\rangle$ to hold. We have interpreted $\langle g, \sigma\rangle \equiv_{\mathbf{g}} \langle h, \tau\rangle$ as "player i can force a win in $\langle g, \sigma\rangle$ if and only if player i can force a win in $\langle h, \tau\rangle$". In this light, $\langle g, \bar{\sigma}\rangle \equiv_{\mathbf{g}} \langle h, \bar{\tau}\rangle$ indicates "player i can force *their opponent* to win in $\langle g, \sigma\rangle$ if and only if player i can force their opponent to win in $\langle h, \tau\rangle$". In passing we observe that while "forcing one's opponent to win" may seem rather eccentric, examples of problems involving such strategies are well-known, e.g. "self-mate" puzzles in Chess. In total the view of game equivalence engendered by the "traditional" notion of state-to-outcome-set relationship, is identical to an arguably rather more intuitive notion of equivalence, namely: for each possible choice of i and j, player i can force a win for player j in $\langle g, \sigma\rangle$ if and only if player i can force a win for player j in $\langle h, \tau\rangle$.

The formal definition of the decision problems examined in the remainder of this section is given below.

- Winning Strategy (WS)
  Instance: a game $\langle g, \sigma\rangle$; set A of atomic strategies, a player $i \in \{0, 1\}$.
  Question: Does i have a winning strategy in the game (g, σ)?
- Equivalence (EQUIV)
  Instance: forms $g, h \in \mathcal{G}_A$.
  Question: Is it the case that $g \equiv_{\mathbf{sf}} h$, i.e. are g and h *strategically equivalent*?
- Win Existence (WE)
  Instance: a form $g \in \mathcal{G}_A$ and a player $i \in \{0, 1\}$.
  Question: Is there *some* set of actions with which i has a winning strategy?
- Game Equivalence (GE)
  Instance: two games $\langle g, \sigma\rangle$, $\langle h, \tau\rangle$ with $g \in \mathcal{G}_A$, $h \in \mathcal{G}_B$.
  Question: Is it the case that $\langle g, \sigma\rangle \equiv_{\mathbf{g}} \langle h, \tau\rangle$, i.e. are $\langle g, \sigma\rangle$ and $\langle h, \tau\rangle$ equivalent *games*?

We note that our definitions have not specified any restrictions on how a given form is described within an instance of these decision problems. We have, as noted in our discussion opening Section 4 above, (at least) two choices: allow $g \in \mathcal{G}_A$ to be described using an equivalent *concise form*, F_g; or insist that g is presented as an *extensive form*, T_g. One difference that we have already highlighted is that the space required for some descriptions F_g may be significantly less than that required for the minimum extensive form descriptions of g. While this fact appears to present a powerful argument in favour of employing concise form descriptions, our first set of results indicate some drawbacks.

If Q is a decision problem part of the instance for which is a Boolean form, we shall use the notation Q^c to describe the case which permits *concise* representations and Q^e for that which insists on *extensive* form descriptions.

Theorem 2.

a. WS^c *is* Σ_2^p*–complete.*
b. EQUIV^c *is co-*NP*–complete.*

Proof. (*Outline*)For (a), that $\text{WS}^c \in \Sigma_2^p$ follows directly from Definition 2 and the fact that $\mathbf{sf}(g)(\pi) = i$ can be tested in time polynomial in $S^{con}(F_g, g)$. The proof of

Σ_2^p–hardness uses a reduction from QSAT_2^{Σ}, instances of which comprise a propositional formula $\varphi(X_n, Y_n)$, these being accepted if there is some instantiation, α_X of X_n, under which every instantiation β_Y of Y_n renders $\varphi(\alpha_X, \beta_Y)$ false. An instance $\varphi(X_n, Y_n)$ can be translated into a Boolean form F_φ with atomic actions $A = X_n \cup Y_n$. Player 0 has a winning strategy with $\sigma_0 = X_n$ in F_φ if and only if $\varphi(X_n, Y_n)$ is a positive instance of QSAT_2^{Σ}.

For (b) given G and H, concise representations of forms $g,\ h \in \mathcal{G}_A$, we note that $g \ \equiv_{\mathbf{sf}} \ h$ if and only if $G{\cdot}\overline{H} + \overline{G} \cdot H \quad \equiv_{\mathbf{sf}} \quad \mathbf{0}$ Thus, a CO-NP algorithm to decide $g \ \equiv_{\mathbf{sf}} \ h$ simply tests $\forall\sigma\ \mathbf{sf}(g \cdot \overline{h} + \overline{g} \cdot h)(\sigma) = 0$ employing the forms G and H. That EQUIV^c is CO-NP–hard is an easy reduction from UNSAT: given $\varphi(X_n)$ an instance of UNSAT, construct the form F_φ (with action set X_n) by similar methods to those used in the proof of (a). For the instance $\langle F_\varphi, \mathbf{0}\rangle$ we have $F_\varphi \ \equiv_{\mathbf{sf}} \ \mathbf{0}$ if and only if $\varphi(X_n)$ is unsatisfiable. □

Corollary 3. WE^c *is* NP*–complete.*

Proof. Given $\langle F_g, i\rangle$ with F_g a concise representation of $g \in \mathcal{G}_A$ use an NP computation to test if $\sigma_i = A$ admits a winning strategy for i: since $\sigma_{1-i} = \emptyset$ this only requires finding a single $\nu \subseteq A$ for which $\mathbf{sf}(g)(\nu) = i$. The proof of NP–hardness follows from Theorem 2(b) and the observation that i has some choice yielding a winning strategy in g if and only if $g \not\equiv_{\mathbf{sf}} 1 - i$. □

While it may seem odd that deciding if there is *some* winning strategy for i in g is more tractable than deciding if a *specific* profile gives a winning strategy, the apparent discrepancy is easily resolved: i has *some* profile that admits a winning strategy if and only if giving i control of the set of *all* actions admits a winning strategy. Thus testing $\langle\langle g, \sigma\rangle, A, i\rangle$ as a positive instance of WS^c can be done in NP in those cases where $|\sigma_{1-i}| = O(\log |A|)$ since there are only polynomially many possible choices of $\tau \subseteq \sigma_{1-i}$ to test against.

Our final problem – Game Equivalence – turns out to be complete for a complexity class believed strictly to contain $\Sigma_2^p \cup \Pi_2^p$, namely the complement of the class D_2^p formed by those languages expressible as the intersection of a language $L_1 \in \Sigma_2^p$ with a language $L_2 \in \Pi_2^p$. We note that "natural" complete problems for this class are extremely rare.[2]

Theorem 3. GE^c *is* $co - \text{D}_2^p$*–complete.*

Proof. Omitted. □

In total Theorems 2, 3 and Corollary 3 indicate that the four decision problems considered are unlikely to admit efficient algorithmic solutions when a concise representation of forms is allowed. In contrast,

Theorem 4. *The decision problems* WS^e, EQUIV^e, WE^e, *and* GE^e *are all in* P.

Proof. (*Outline*) All follow easily from the fact that $\text{WS}^e \in \text{P}$: for EQUIV^e constructing an extensive form equivalent to $\overline{f} \cdot g + f \cdot \overline{g}$ can be done in time polynomial in the size

[2] i.e. other than those comprising pairs of the form $\langle x, y\rangle$ with x in some Σ_2^p-complete language L_1 and y in some Π_2^p-complete language L_2. The only exception we know of is the problem *Incomplete Game* of [10] from which our notation D_2^p for this class is taken.

of the extensive representations; after simplifying to ensure no path repeats any action, it suffices to verify that only the outcome 0 can be achieved. For WE^e we simply have to test that $\sigma_i := A$ admits a winning strategy for i. Similar arguments give $\text{GE}^e \in \text{P}$. □

In combination, Theorems 2, 3, and 4, might seem to offer a choice between efficient in length but computationally intractable concise forms and infeasible length though computationally malleable extensive forms. We can, however, identify a number of problems whose solution is hard even measured in terms of extensive representations. For example, we may interpret the problem of identifying a *minimal* size extensive form in terms of the following "high-level" algorithmic process given T_g: identify an action $\beta \in A$ and forms h_0, h_1 in $\mathcal{G}_{A\setminus\{\beta\}}$ for which $\beta(h_0, h_1) \equiv_{\mathbf{sf}} g$ and for which $1 + S^{ext}([h_0]) + S^{ext}([h_1])$ is minimised. One key feature of such an approach is its requirement to find a suitable choice of β. Thus, for extensive forms we have the following decision problem:

Optimal Branching Action OBA

Instance: T_g extensive form description of $g \in \mathcal{G}_A$.

Question: Is there an action β and extensive forms T_{h_0}, T_{h_1} for which $\beta(h_0, h_1) \equiv_{\mathbf{sf}} g$ and $1 + S^{ext}(T_{h_0}, h_0) + S^{ext}(T_{h_1}, h_1) < S^{ext}(T_g, g)$?

Informally, OBA asks of a given extensive form whether it is possible to construct a *smaller* equivalent form. An immediate consequence of [6] is that OBA is NP–hard.

6 Conclusions and Further Work

Our principal goal in this paper has been to examine the computational complexity of a number of natural decision problems arising in the context of the Boolean game formalism from [3,4]. Building from the correspondence between the underlying forms of such games and propositional functions we can characterise winning strategies and their structure for any game. Using the operational description developed in [4] allows a precise formulation for the concept of "concise form" to be defined in contrast to the usual idea of "extensive form". We have indicated both advantages and drawbacks with both approaches so that the apparent computational tractability of extensive forms is paid for in terms of potentially exponentially large (in the number of atomic actions) descriptions; on the other hand, while the concise forms may admit rather terser descriptions of a given game all four of the basic decision questions raised are unlikely to admit feasible algorithmic solutions.

We conclude by, briefly, summarising some potential developments of the ideas discussed above. We have already noted that our concept of "game equivalence" – $\equiv_{\mathbf{g}}$ is defined independently of ideas involving the "forcing relations" of [2,8] and have observed in Lemma 1 that it describes a proper subset of games equivalent under $\equiv_{\mathbf{H}}$. A natural open question is to determine the computational complexity of deciding if $\langle g, \sigma \rangle \equiv_{\mathbf{H}} \langle h, \tau \rangle$: noting the correspondence presented in Lemma 1, it is certainly the case that this is within the class $\text{P}^{\Sigma_2^p[4]}$ of languages decidable by (deterministic) polynomial-time algorithms that may make up to 4 calls on a Σ_2^p oracle. We are unaware of natural complete problems for $\text{P}^{\Sigma_2^p[4]}$ and thus, it would be of some interest to determine whether deciding $\langle g, \sigma \rangle \equiv_{\mathbf{H}} \langle h, \tau \rangle$ is $\text{P}^{\Sigma_2^p[4]}$–complete

One final issue concerns the algebraic relationship between forms and propositional functions. Our analysis in Section 3 could be viewed as relating winning strategies in g to terms occurring in a "standard" representation of $\lceil g \rceil$ over the complete logical basis $\{\wedge, \vee, \neg\}$. An alternative complete logical basis is given by $\{\wedge, \oplus, \top\}$ ($\top$ being the nullary function evaluating to true) within which a normal form theorem holds, i.e. the ringsum expansion of Zhegalkin [11], see, e.g. [1, p. 14]: it is unclear, but may be of some interest, to what extent terms within this descriptive form correspond to properties of winning strategies.

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