

## Game Theory

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## Literature

- Fun and Games
  - Ken Binmore
- A Course in Game Theory
  - Martin Osborne en Ariel Rubinstein

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## What is it all about?

- game = interaction
  - traffic
  - supermarket
  - employee, employer, board, union
  - student / teacher
  - judge and lawyers
  - George en Osama
  - marriage and career

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## What is it all about?

- terminology of chess and bridge
- logic and systematics of interaction
- analysis takes you from irrational issues
  - strategic interaction is difficult
  - because reasoning is circular

If J and M play a game, J's strategy will typically depend on his prediction of M's strategy, which, on its turn depends on M's expectation of J's....

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## Surprise and Paradox

- does it make sense
  - to vote for a candidate you fancy least? **YES!!**
  - for a general, to toss a coin? **YES!!**
  - in poker, place a maximal bid with the worst cards? **YES!!**
  - to throw some goods away before starting to negotiate about them **YES!!**
  - to sell your house to the second best bidder? **YES!!**

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## Strategic voting

- Boris, Horace and Maurice determine who can be a member of the Dead Poet Society
  - proposal: allow Alice
  - amendment: allow Bob, rather than Alice
  - first vote over amendment, then over proposal

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## Strategic voting

Borice	Horace	Maurice
Alice	Nobody	Bob
Nobody	Alice	Alice
Bob	Bob	Nobody

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## Strategic voting

- first between A, B
  - winner Alice
- then between A, N
  - winner Alice
- strategic voting H:
  - first vote for Bob!
  - result ... B, N
- M anticipates: vote for A

Horace	Maurice
Nobody	Bob
Alice	Alice
Bob	Nobody

Borice
Alice
Nobody
Bob

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## History

- Von Neumann and Morgenstern
  - The Theory of Games and Economic Behaviour* (1944)
- ideas from economics and mathematics
- initially very optimistic, then draw-back
- revival since 1970's
  - Nash, Aumann, Shapley, Selten, Harsanyi

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## Game Theory

- theory of decision makers
  - are rational:
    - aware of alternatives
    - form expectations
    - have preferences
    - optimize after deliberation
  - set A of actions;
  - set C consequences;
  - $g: A \rightarrow C$ 
    - consequence function
  - preference relation  $\geq$  on A
    - or: utility function
    - $u: C \rightarrow \mathbb{R}$

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## Abstracts from 'emotions'

- suppose you're offered £ 1.000
- you make a deal with the first person you encounter:  $(1.000-x, x)$   $x = 1, 2 \dots$ 
  - if he accepts: (you, person) get  $(1.000-x, x)$
  - else  $(0, 0)$
- only money counts, and that is known
- both are rational: prefer  $y+1$  over  $y$
- what will you offer?

ONE £!!!!!!!

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## Game Theory

- theory of decision makers
  - are rational
  - reason strategically
- players anticipate on knowledge and expectations about behaviour of other decision makers

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## Strategic Games

- Definition  $G = \langle N, (A_i), (\succeq_i) \rangle$ 
  - finite set  $N$  (**players**)
  - set  $A_i$  (**actions**) for every player  $i$
  - preference relation**  $\succeq_i$  for every player  $i$
  - $u_i$  is **utility function**:  $A \rightarrow \mathbb{R}$  with
    - $a \succeq_i b \Leftrightarrow u_i(a) \geq u_i(b)$
    - also called **payoff-function**
    - (although not the same)

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## Representation strategic games

- $N = \{1,2\}$
- $A_1 = \{T,B\}$
- $A_2 = \{L,R\}$
- $u_1(T) = w_1$ , etc

	L	R
T	$w_1, w_2$	$x_1, x_2$
B	$y_1, y_2$	$z_1, z_2$

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## Representation strategic games

- Interpretation
  - one-shot
  - simultaneous
  - independent
  - utilities are known
  - not the choice of others

	L	R
T	$w_1, w_2$	$x_1, x_2$
B	$y_1, y_2$	$z_1, z_2$

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## Example: BoS

- $N = \{1,2\}$
- $A_1 = \{B,S\}$
- $A_2 = \{B,S\}$
- $u_1, u_2$  see figure
  - B: Bach
  - S: Strawinsky
- Battle of the Sexes

	B	S
B	2,1	0,0
S	0,0	1,2

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## Profiles

- $A_1, A_2, \dots, A_n$  are the action sets
- $(a_1, a_2, \dots, a_n) \in A_1 \times A_2 \times \dots \times A_n$  is a **profile**
- notation:  $(x)$ , or  $a^*$
- $x_i \in A_1 \times A_2 \times \dots \times A_{i-1} \times A_{i+1} \times \dots \times A_n$
- $(x_{-i}, x_i) = (x)$
- focus on  $i$ , given the profile of the others

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## Profiles: example

- $A_1, A_2, \dots, A_7$  are bids ( $\in \mathbb{R}$ )
- $(a_1, a_2, \dots, a_7)$  is a concrete bid
- notation:  $(x) = (25, 22, 20, 12, 0, 27, 22) = a^*$
- $x_{-6} \in (25, 22, 20, 12, 0, 22)$
- $(x_{-6}, x_6) = ((25, 22, 20, 12, 0, 22), 27)$ 
  - $(x_{-6}, x'_6) = ((25, 22, 20, 12, 0, 22), 26)$  would have been better for player 6, given the profiles of the others

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## Nash equilibrium

- John Nash
- equilibrium ("solution")
  - every player is rational
  - every player plays optimally
  - no use to deviate individually
  - not an algorithmic approach

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## Nash equilibrium (definitive)

- Given  $G = \langle N, (A_i), (\geq_i) \rangle$
- $a^* \in A = A_1 \times A_2 \times \dots \times A_n$  is Nash equilibrium iff
  - $\forall i \in N \forall a_i \in A_i (a^*_{-i}, a_i) \geq_i (a^*_{-i}, a_i)$ 
    - no player  $i$  can improve in  $a^*$ , if the other players still play  $a^*_{-i}$

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## Nash equilibrium (alternative)

- define for every  $a_{-i} \in A_{-i}$  the **best response for  $i$** ,  $B_i(a_{-i})$
- $B_i(a_{-i}) = \{a_i \in A_i \mid \forall a'_i \in A_i (a_{-i}, a_i) \geq_i (a_{-i}, a'_i)\}$
- $a^*$  is N.eq iff  $\forall i \in N a^*_{-i} \in B_i(a^*_{-i})$

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## Example: BoS (N.eq)

- $N = \{1, 2\}$
- $A_1 = \{B, S\}$
- $A_2 = \{B, S\}$
- $u_1, u_2$  see figure
  - B: Bach
  - S: Stravinsky
- two equilibria:
  - (bach, bach) and
  - (strawinsky, strawinsky)

	B	S
B	2,1	0,0
S	0,0	1,2

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## Example: coordination game

- Mozart or Mahler?
- same preferences
- two equilibria:
  - (Mozart, Mozart) and
  - (Mahler, Mahler)
- N.eq right concept?

	Mo	Ma
Mo	2,2	0,0
Ma	0,0	1,1

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## Pareto Efficiency

- (Mozart, Mozart) and
- (Mahler, Mahler)
- N.eq right concept?
- (2,2) is (strongly) Pareto efficient:
  - $\neg \exists x \neg \exists y (x, y) > (2, 2)$

	Mo	Ma
Mo	2,2	0,0
Ma	0,0	1,1

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## Pareto Optimality (definition)

- Given  $G = \langle N, (A_i), (\succeq_i) \rangle$
- $a^* = (a_1, a_2, \dots, a_n) \in A_1 \times A_2 \times \dots \times A_n$  is Pareto optimal iff
- $\forall i \in N \forall b^* \in A_1 \times A_2 \times \dots \times A_n (a^*) \succeq_i (b^*)$

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## Example: prisoner's dilemma

- C: cooperate with the other, keep silent
- D: justify against the other

	C	D
C	0,0	-2,3
D	3,-2	-1,-1

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## Example: prisoner's dilemma

- C: cooperate with the other, keep silent
- D: justify against the other
- Although cooperate would be better, every player has a preference for defeat

	C	D
C	0,0	-2,3
D	3,-2	-1,-1

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## Example: hawk-dove

- preference:
  - hawkish if other is dovish
  - dovish if other is hawkish
- N.eq: (Dove, Hawk)
- and (Hawk, Dove)

	D	H
D	3,3	1,4
H	4,1	0,0

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## Example: Matching Pennies

- Head and Tail
- if different, 1 pays a Pound to 2 if the same, 2 pays a Pound to 1
- no equilibrium!
- game is strictly competitief

	H	T
H	1,-1	-1,1
T	-1,1	1,-1

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## SCSG

- Strictly Competitive Strategic Game
- if  $G = \langle \{1,2\}, (A_i), (\succeq_i) \rangle$ ,
- and  $\forall a, b \in A: a \succeq_1 b \Leftrightarrow b \succeq_2 a$
- also called zero-sum game:
  - with  $u_1$  and  $u_2$  we have
  - $u_1 + u_2 = 0$

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### maximizer

- let  $G = \langle \{1,2\}, (A_i), (\geq_i) \rangle$  an SCSG
- action  $x^* \in A_1$  is maximizer for 1:
  - $\forall x \in A_1 \min_{y \in A_2} u_1(x^*, y) \geq_1 \min_{y \in A_2} u_1(x, y)$
- action  $y^* \in A_2$  is maximizer for 2:
  - $\forall y \in A_2 \min_{x \in A_1} u_2(x, y^*) \geq_2 \min_{x \in A_1} u_2(x, y)$

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### maximizer

- action  $x^* \in A_1$  is maximizer for 1:
  - $\forall x \in A_1 \min_{y \in A_2} u_1(x^*, y) \geq_1 \min_{y \in A_2} u_1(x, y)$
- solves for 1  $\max_x \min_y u_1(x, y)$
- solves for 2  $\max_y \min_x u_2(x, y)$
- 'maximises the minimum that i can guarantee'
- $x^*$  is a *security strategy* for 1

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### Equilibria and maximizers

- $(x^*, y^*)$  is N.eq for G, iff:
  - $x^*$  is a maximizer for 1;
  - $y^*$  is a maximizer for 2
  - $\max_x \min_y u_1(x, y) = \min_y \max_x u_1(x, y) = u_1(x^*, y^*)$

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### maximizers

- solves for 1
- $\max_x \min_y u_1(x, y) = \max\{ \min\{u_1(x, y) | y \in A_2\} | x \in A_1 \}$

...	$y_1$	$y_2$	$y_3$	$y_4$	$y_5$
$x_1$	2,-2	2,-2	3,-3	1,-1	1,-1
$x_2$	3,-3	5,-5	4,-4	6,-6	4,-4
$x_3$	5,-5	2,-2	4,-4	3,-3	3,-3
$x_4$	6,-6	8,-8	5,-5	7,-7	5,-5
$x_5$	3,-3	5,-5	4,-4	2,-2	3,-3
$x_6$	4,-4	3,-3	6,-6	5,-5	4,-4

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### maximizers

- solves for 1
- $\max_x \min_y u_1(x, y) = \max\{ \min\{u_1(x, y) | y \in A_2\} | x \in A_1 \}$
- $x_1: \min_y u_1(x_1, y) = 1$

...	$y_1$	$y_2$	$y_3$	$y_4$	$y_5$
$x_1$	2,-2	2,-2	3,-3	1,-1	1,-1
$x_2$	3,-3	5,-5	4,-4	6,-6	4,-4
$x_3$	5,-5	2,-2	4,-4	3,-3	3,-3
$x_4$	6,-6	8,-8	5,-5	7,-7	5,-5
$x_5$	3,-3	5,-5	4,-4	2,-2	3,-3
$x_6$	4,-4	3,-3	6,-6	5,-5	4,-4

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### maximizers

- solves for 1
- $\max_x \min_y u_1(x, y) = \max\{ \min\{u_1(x, y) | y \in A_2\} | x \in A_1 \}$
- $x_1: \min_y u_1(x_1, y) = 1$
- $x_2: \min_y u_1(x_2, y) = 3$
- ... .. = ..
- $x_4: \min_y u_1(x_4, y) = 3$

...	$y_1$	$y_2$	$y_3$	$y_4$	$y_5$
$x_1$	2,-2	2,-2	3,-3	1,-1	1,-1
$x_2$	3,-3	5,-5	4,-4	6,-6	4,-4
$x_3$	5,-5	2,-2	4,-4	3,-3	3,-3
$x_4$	6,-6	8,-8	5,-5	7,-7	5,-5
$x_5$	3,-3	5,-5	4,-4	2,-2	3,-3
$x_6$	4,-4	3,-3	6,-6	5,-5	4,-4

$\max = 5$  for  $x^* = x_4$

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### maxminimizers

- solves for 1
- $\max_x \min_y u_1(x,y) =$ 
  - $\max\{$
  - $\min\{u_1(x,y) | y \in A_2\}$
  - $| x \in A_1\} = 5$
- solves for 2
- $\max_x \min_y u_2(x,y) =$ 
  - $\max\{$
  - $\min\{u_2(x,y) | x \in A_1\}$
  - $| y \in A_2\} =$

...	$y_1$	$y_2$	$y_3$	$y_4$	$y_5$
$x_1$	2,-2	2,-2	3,-3	1,-1	1,-1
$x_2$	3,-3	5,-5	4,-4	6,-6	4,-4
$x_3$	5,-5	2,-2	4,-4	3,-3	3,-3
$x_4$	6,-6	8,-8	5,-5	7,-7	5,-5
$x_5$	3,-3	5,-5	4,-4	2,-2	3,-3
$x_6$	4,-4	3,-3	6,-6	5,-5	4,-4

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### maxminimizers

- solves for 1
- $\max_x \min_y u_1(x,y) =$ 
  - $\max\{$
  - $\min\{u_1(x,y) | y \in A_2\}$
  - $| x \in A_1\} = 5$
- solves for 2
- $\max_x \min_y u_2(x,y) =$ 
  - $\max\{$
  - $\min\{u_2(x,y) | x \in A_1\}$
  - $| y \in A_2\} = -5!$
- Equilibrium (5,-5)

...	$y_1$	$y_2$	$y_3$	$y_4$	$y_5$
$x_1$	2,-2	2,-2	3,-3	1,-1	1,-1
$x_2$	3,-3	5,-5	4,-4	6,-6	4,-4
$x_3$	5,-5	2,-2	4,-4	3,-3	3,-3
$x_4$	6,-6	8,-8	5,-5	7,-7	5,-5
$x_5$	3,-3	5,-5	4,-4	2,-2	3,-3
$x_6$	4,-4	3,-3	6,-6	5,-5	4,-4

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### security level vs equilibria

- consider cooperative game G
- (2,2) looks like 'the optimal' solution
- security strategy of 1 is r, gives 1!
- Nash equilibria?

	L	R
l	2,2	0,0
r	1,1	1,1

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### security level vs equilibria

- consider game G
- (2,2) looks like 'the optimal' solution
- security strategy of 1 is r, gives 1!
- Nash equilibria?

	L	R
l	2,2	0,0
r	1,1	1,1

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### bimatrix games

- m x n matrix
- 1 has strategies  $s_1$  and  $s_2$ , 2 has  $t_1$ ,  $t_2$  and  $t_3$
- payoff  $\pi_i(s_i, t_j) = ij$ 
  - $\pi_2(s_i, t_j) = (i-2)(j-2)$

	$t_1$	$t_2$	$t_3$
$s_1$	1,1	2,0	3,-1
$s_2$	2,0	4,0	6,0

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### bimatrix games

- m x n matrix
- 1 has strategies  $s_1$  and  $s_2$ , 2 has  $t_1$ ,  $t_2$  and  $t_3$
- Nash equilibrium  $(\sigma, \tau)$ :
  - $\forall s, t \pi_1(\sigma, \tau) \geq \pi_1(s, \tau)$
  - $\forall s, t \pi_2(\sigma, \tau) \geq \pi_2(\sigma, t)$

	$t_1$	$t_2$	$t_3$
$s_1$	1,1	2,0	3,-1
$s_2$	2,0	4,0	6,0

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## domination

- strategy  $s_d$  of 1 dominates  $s_i$  strongly:
  - $\forall t \pi_1(s_d, t) > \pi_1(s_i, t)$
- and weakly if:
  - $\forall t \pi_1(s_d, t) \geq \pi_1(s_i, t)$
  - $\exists t \pi_1(s_d, t) > \pi_1(s_i, t)$
- $t_1$  dominates  $t_2$  weakly

	$t_1$	$t_2$	$t_3$
$s_1$	1,1	2,0	3,-1
$s_2$	2,0	4,0	6,0

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## Iterated elimination

- $s_2$  of 1 strongly dominates  $s_1$
- No further (weak) domination: all is left are Nash Equilibria
- this is not generally so

	$t_1$	$t_2$	$t_3$
$s_1$	1,1	2,0	3,-1
$s_2$	2,0	4,0	6,0

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## Order of elimination

	C	D		C	D	
AE	2,0	1,1		2,0	1,1	lost equilibrium!
AF	0,2	1,1		0,2	1,1	
BE	3,3	3,3		3,3	3,3	
BF	3,3	3,3		3,3	3,3	

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## elimination: conclusions

- strict strategies: no problem
- with weakly dominated strategies:
  - some equilibria can get lost
  - order of elimination is important

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## Example: BoS

- $N = \{1,2\}$
- $A_1 = \{B,S\}$
- $A_2 = \{B,S\}$
- $u_1, u_2$  see figure
  - B: Bach
  - S: Strawinsky
- Battle of the Sexes

	B	S
B	2,1	0,0
S	0,0	1,2

no dominant strategies

still (two) Nash equilibria

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## Ex: coordination game

- Mozart of Mahler?
- Same preference

	Mo	Ma
Mo	2,2	0,0
Ma	0,0	1,1

No dominant strategy

still two Nash equilibria

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## Ex: prisoner's dilemma

- C: cooperate, and be silent
- D: justify against the other
- D dominates C
- D dominates C
- gives Nash equilibrium (-1,-1)

	C	D
C	0,0	-2,3
D	3,-2	-1,-1

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## Ex: Matching Pennies

- Head and Tail
- if different, 1 pays a Pound to 2; if the same, 2 pays a Pound to 1

	H	T
H	1,-1	-1,1
T	-1,1	1,-1

no dominant strategy  
No pure Nash equilibrium

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## mixed strategies

- don't always bid in the same way with poker
- being unpredictable can be an advantage
- sometimes a strategy is not dominated by another pure strategy, but by a mixed one

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## security level and strategy

- maximin for 1: 2, via  $s_2$
- note:  $(s_2, t_2)$  is a saddlepoint
- then 2 is also security level of player 1

	$t_1$	$t_2$	$t_3$
$s_1$	0,1	1,2	7,3
$s_2$	4,6	2,0	3,2
$s_3$	9,0	0,3	0,4

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## security level and strategy

- maximin for 1: 2 via  $s_3$
- $(s_3, t_2)$  is not a saddlepoint
- security level of 1 is indeed not 2, but  $2^{2/3}$
- How?

	$t_1$	$t_2$	$t_3$
$s_1$	1,0	6,4	0,9
$s_2$	2,1	0,2	3,0
$s_3$	3,7	2,3	4,0

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## mixed strategies

- remove strictly dominated strategy  $s_2$
- 2 has no pure dominating strategy, but, .....

	$t_1$	$t_2$	$t_3$
$s_1$	1,0	6,4	0,9
$s_2$	2,1	0,2	3,0
$s_3$	3,7	2,3	4,0

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### mixed strategies

- 2 has no pure dominating strategy, but, .....
- $q = (1/2, 0, 1/2)$  dominates  $t_2$  strongly!
- $\pi_2(s_1, q) = (1/2) \cdot 0 + (1/2) \cdot 9 = 4.5 > 4$
- $\pi_2(s_3, q) = (1/2) \cdot 7 + (1/2) \cdot 0 = 3.5 > 3$

	$t_1$	$t_2$	$t_3$
$s_1$	1,0	6,4	0,9
$s_2$	2,1	0,2	3,0
$s_3$	3,7	2,3	4,0

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### mixed strategies

- 2 has no pure dominating strategy, but, .....
- $q = (1/2, 0, 1/2)$  dominates  $t_2$  strongly!
- $\pi_2(s_1, q) = (1/2) \cdot 0 + (1/2) \cdot 9 = 4.5 > 4$
- $\pi_2(s_3, q) = (1/2) \cdot 7 + (1/2) \cdot 0 = 3.5 > 3$

	$t_1$	$t_2$	$t_3$
$s_1$	1,0	6,4	0,9
$s_2$	2,1	0,2	3,0
$s_3$	3,7	2,3	4,0

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### mixed strategies

- after iterated elimination
- what is security level of 1?
- suppose 1 plays mixed strategy  $(1-r, 0, r)$

$s$	$t_1$	$t_2$	$t_3$
$s_1$	1,0	6,4	0,9
$s_2$	2,1	0,2	3,0
$s_3$	3,7	2,3	4,0

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### mixed strategies

- suppose 1 plays mixed strategy  $(1-r, 0, r)$
- let  $E_k(r)$  be payoff of 1 if 2 plays  $t_k$ :
  - $E_1(r) = 1(1-r) + 3r = 1 + 2r$
  - $E_2(r) = 6(1-r) + 2r = 6 - 4r$
  - $E_3(r) = 0(1-r) + 4r = 4r$

$s$	$t_1$	$t_2$	$t_3$
$s_1$	1,0	6,4	0,9
$s_2$	2,1	0,2	3,0
$s_3$	3,7	2,3	4,0

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### mixed strategies

- $E_1(r) = 1(1-r) + 3r = 1 + 2r$
- $E_2(r) = 6(1-r) + 2r = 6 - 4r$
- $E_3(r) = 0(1-r) + 4r = 4r$
- $m(r) = \min\{E_1, E_2, E_3\}$
- max for  $r = 5/6$   
payoff is  $E_2(r) = 2/3$

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### Ex: Matching Pennies

- Head and Tail
- if different, 1 pays a Pound to 2; if the same, 2 pays a Pound to 1

	H	T
H	1,-1	-1,1
T	-1,1	1,-1

no dominant strategy  
No Nash equilibrium

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### Example: tictactoe

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### Strategies

- A *pure strategy* for player  $p$  specifies for every decision node of  $p$  what he will do there
- If all players choose such a strategy, the outcome of the game is determined there

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### Strategies

- A *pure strategy* for player  $p$  specifies for every decision node of  $p$  what he will do there
- Strategies for I:

ll lr rl rr

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### Strategies

- A *pure strategy* for player  $p$  specifies for every decision node of  $p$  what he will do there
- Strategies for II:

LLL, LLR, LML, LMR, LRL, LRR  
RLL, RLR, RML, RMR, RRL, RRR

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### Strategy profile

Example:

[lr, RMR]

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### extensive to strategic

AE	a	c
AF	b	c
BE	d	d
BF	d	d

reduced strategic form

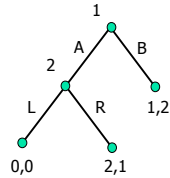
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## Equilibria: example

- Nash equilibria?
- via strategic form:

	L	R
A	0,0	2,1
B	1,2	1,2



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## Strategy profile

	L				R							
	LL	LR	ML	MR	RL	RR	LL	LR	ML	MR	RL	RR
ll	w	w	w	w	w	w	1	1	1	1	1	1
lr	w	w	w	w	w	w	1	1	1	1	1	1
rl	1	w	w	w	w	w	1	w	w	w	w	w
rr	w	w	w	w	w	w	w	w	w	w	w	w

a strategy profile: [rr, RLL]

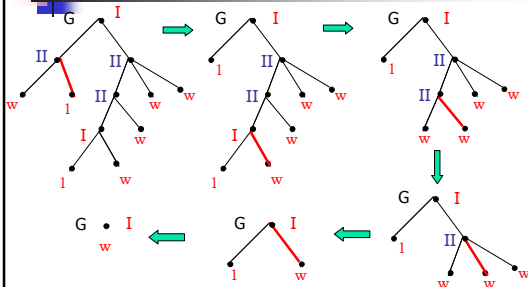
strategic form of G

extensive form of G

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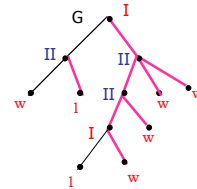
## Backward Induction



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## backward induction



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## I can win

	LLL	LLR	LML	LMR	LRL	LRR	RLL	RLR	RML	RMR	RRL	RRR
ll	w	w	w	w	w	w	1	1	1	1	1	1
lr	w	w	w	w	w	w	1	1	1	1	1	1
rl	1	w	w	w	w	w	1	w	w	w	w	w
rr	w	w	w	w	w	w	w	w	w	w	w	w

again: rr is winning strategy,  
since that row only contains a w

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## extensive games: definitions

- extensive games:  $G = \langle N, H, P, (\succeq_i) \rangle$ 
  - $N$ : set of players
  - $H$  histories:  $\emptyset, (a^k)_{k=1..K}$  (may be infinite)
    - closed under prefixes
    - terminals  $Z$ : no successor or infinite
  - $P$ :  $H \setminus Z \rightarrow N$  player who is to move
  - $\succeq_i$ : preference relation on  $Z$

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## extensive games: definitions

- $N$ : set of players
- $H$  histories:  $\emptyset, (a^k)_{k=1..K}$  (may be infinite)
  - closed under prefixes
  - terminals  $Z$ : no successor or infinite
  - $h \in H, a$  action  $\Rightarrow (h,a) \in H$
- $H$  is finite  $\Rightarrow G$  is **finite**
- $H$  only contains finite  $h \Rightarrow G$  has **finite horizon**

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## Subgame perfect solutions

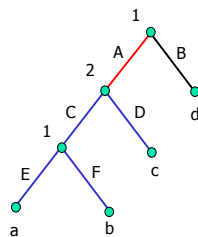
- extensive games:  $\Gamma = \langle N, H, P, (\succeq_i) \rangle$ 
  - $N$ : set of player
  - $H$  histories:  $\emptyset, (a^k)_{k=1..K}$  (may be infinite)
  - $P: H \setminus Z \rightarrow N$  player to play
  - $\succeq_i$ : preference relation on  $Z$
- subgames:  $\Gamma(h) = \langle N, H|_h, P|_h, (\succeq_i|_h) \rangle$ 
  - all continuations of  $h$

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## Subgames

- history  $h$
- subgame  $\Gamma(h)$



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## subgame perfect N.-eq

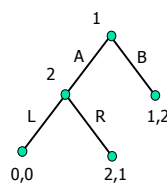
- let  $\Gamma = \langle N, H, P, (\succeq_i) \rangle$  extensive
  - $s^*$  is N.-eq if  $\forall i \forall s_i O(s_{-i}^*, s_i^*) \succeq_i O(s_{-i}^*, s_i)$
- $s^*$  is subgame perfect N.-eq if
  - $\forall i \forall h \in H \setminus Z (P(h)=i \Rightarrow$ 
    - $O_h(s_{-i}^*|_h, s_i^*|_h) \succeq_{i,h} O_h(s_{-i}^*|_h, s_i)$
    - for all strategies  $s_i$  for  $i$  in  $\Gamma(h)$
- $s^*|_h$  is N.-eq for all  $\Gamma(h)$

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## Equilibria: Example

- Nash equilibria?

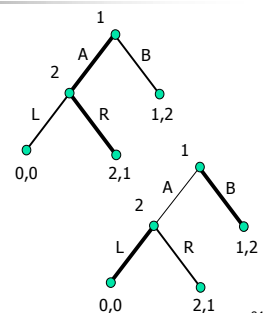


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## equilibria (ctd)

- so: (A,R) and (B,L)
- interpretation (B,L):
  - given that 2 plays L after A, 1 better choose B
- intuitive?
- what is optimal for 1?

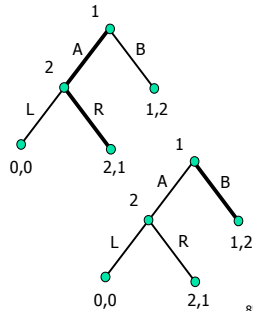


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## equilibria (ctd)

- so: (A,R) and (B,L)
- interpretation (B,L):
- given that 2 plays L after A, 1 better choose B
- AR is the only subgame perfect equilibrium

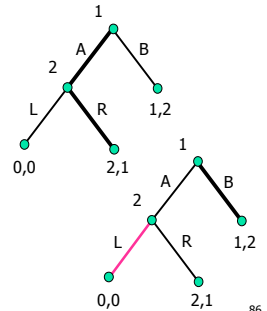


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## equilibria (ctd)

- so: (A,R) and (B,L)
- interpretation (B,L):
- given that 2 plays L after A, 1 better choose B
- AR is the only subgame perfect equilibrium
- not BL!



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## shop-chain game

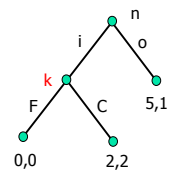
- chain  $k$  and  $n$  competitors
- every competitor can either enter challenge  $k$  ( $i$ ), or not ( $o$ )
- if so,  $k$  chooses between cooperate ( $c$ ) and fight ( $f$ )

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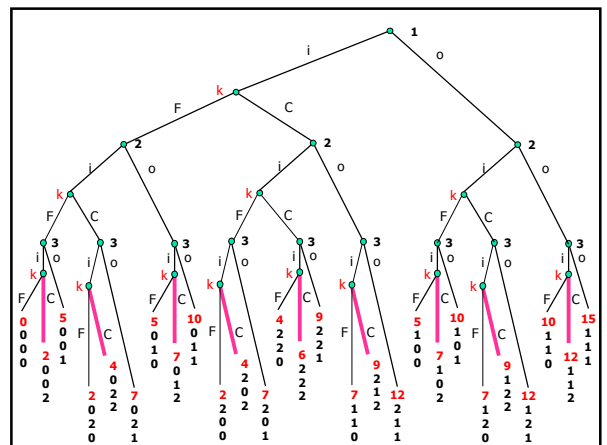
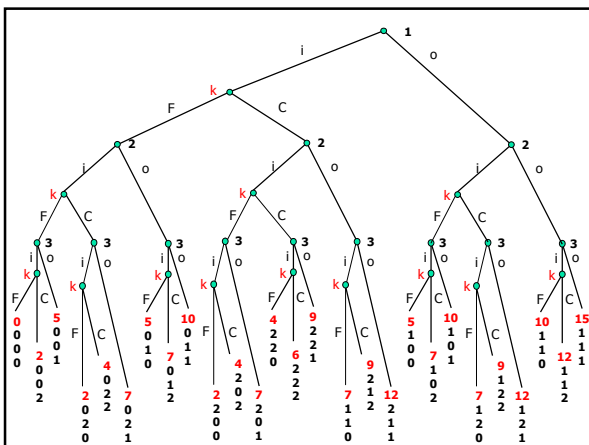
## shop-chain game

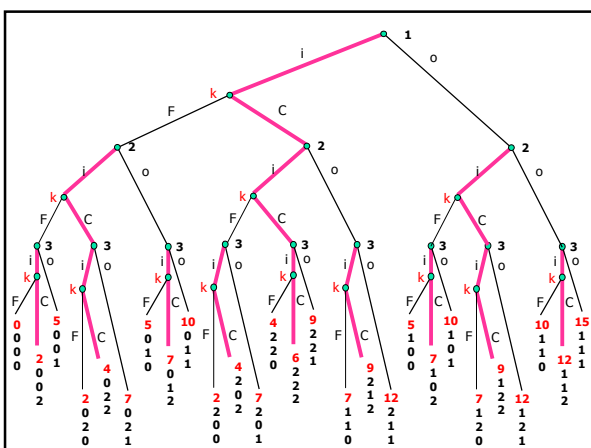
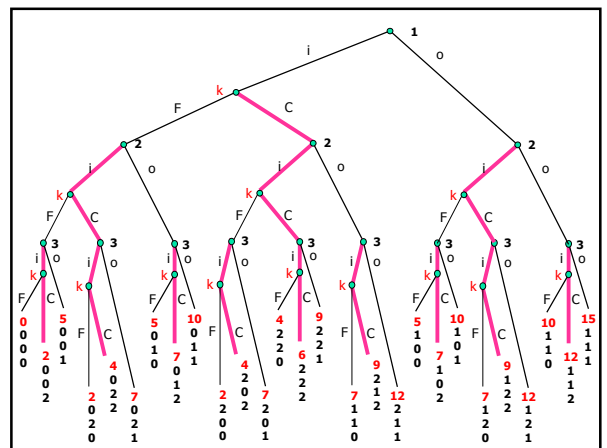
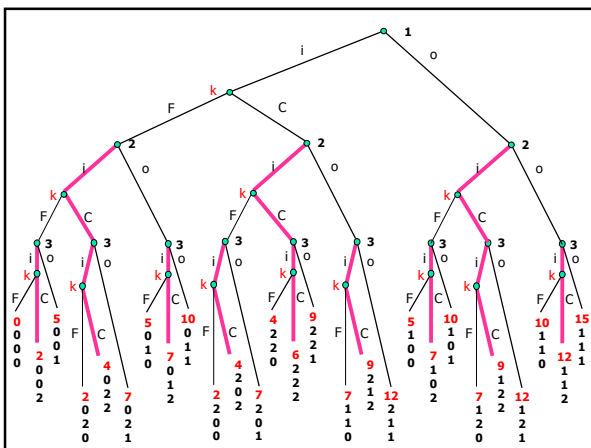
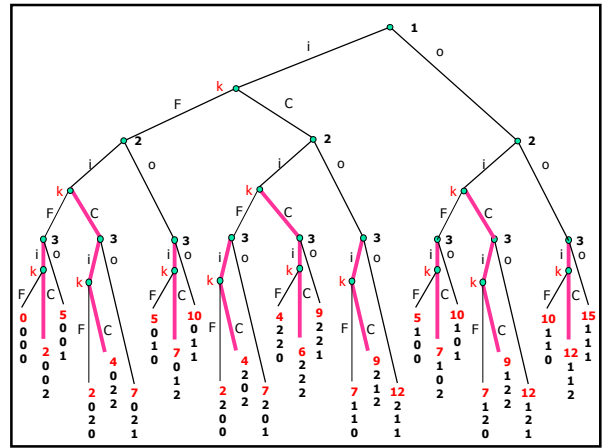
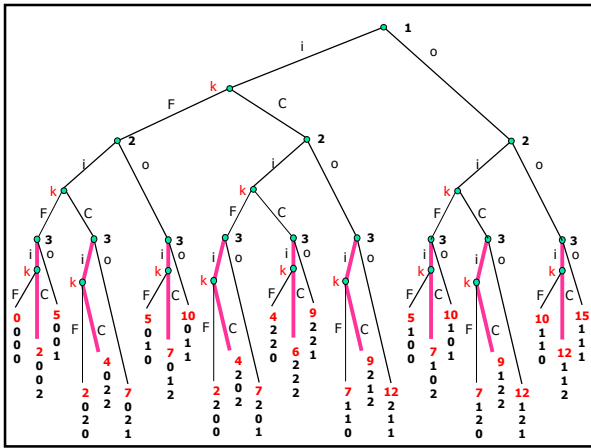
- chain  $k$  and  $n$  competitors
- every competitor can either enter challenge  $k$  ( $i$ ), or not ( $o$ )
- if so,  $k$  chooses between cooperate ( $c$ ) and fight ( $f$ )



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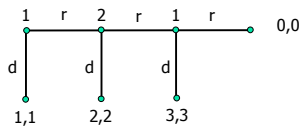


**shop-chain game**

- subgame perfect equilibrium:
  - all shops play i, chain k play c
- not realistic, if many more shops to fight
- solution: shops should be uncertain about the motives of k

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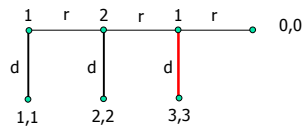
## Backward Induction



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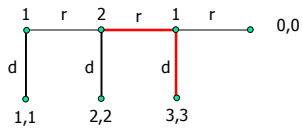
## Backward Induction



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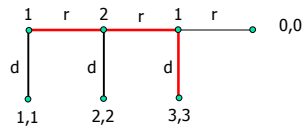
## Backward Induction



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## Backward Induction

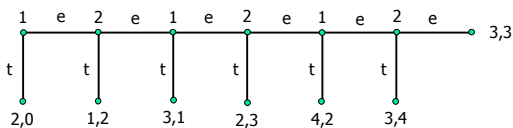


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## Centipede

- 1 and 2 divide  $n$  marbles; they choose in turn, if somebody picks two, the game is over

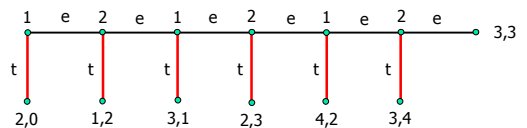


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## Centipede

- Intuitively correct?



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### Strategic voting

- Boris, Horace and Maurice determine who can be a member of the Dead Poet Society
  - proposal: allow Alice
  - counterprop: allow Bob, rather than Alice
  - first vote over counterprop, then over proposal

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### Strategic voting

- first between A, B
  - winner Alice
- then between A, N
  - winner Alice
- strategic voting H:
  - first vote Bob!
  - solution... B, N

Maurice		
Bob		
Alice		
Nobody		
Horace		
Nobody		
Alice		
Bob		
Borice		
Alice		
Nobody		
Bob		

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### Strategic voting

- first between A, B
  - winner Alice
- then between A, N
  - winner Alice
- strategic voting H:
  - first vote Bob!
  - solution... B, N
- M anticipates: vote for A

Maurice		
Bob		
Alice		
Nobody		
Horace		
Nobody		
Alice		
Bob		
Borice		
Alice		
Nobody		
Bob		

utility	B	H	M
a	3	2	2
b	1	1	3
n	2	3	1

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### Strategic voting

- first between A, B
  - winner Alice
- then between A, N
  - winnner Alice

Maurice		
Bob		
Alice		
Nobody		
Horace		
Nobody		
Alice		
Bob		
Borice		
Alice		
Nobody		
Bob		

utility	B	H	M
a	3	2	2
b	1	1	3
n	2	3	1

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### Strategic voting: extensive

u	B	H	M
a	3	2	2
b	1	1	3
n	2	3	1

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### Strategic voting: extensive

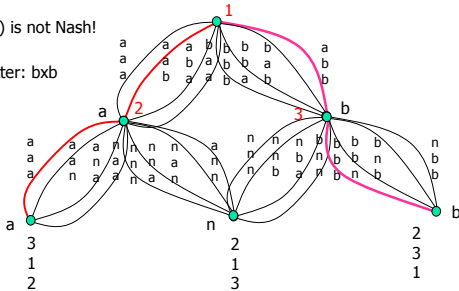
(aaa,aaa,xyz) is Nash

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## Strategic voting: extensive

(aab, aab, nnb) is not Nash!

H can do better: bxb



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## Pirates on an island

- Five pirates  $p_1, \dots, p_5$  are on an island
- There is also a bag of 100 diamonds
- And hence, a need to distribute them

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## Five Pirates: procedure

player  $i$  proposes a division  $D_i$  over  $p_1, \dots, p_5$

- with a majority for  $D_i$ : so be it done
- no majority for  $D_i$ :  $p_i$  gets shot, we move on to  $p_{i+1}$



Now you are  $p_1$ . What will  $D_1$  be?

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## Pirates on an island

- Assumptions: Any pirate**
  - values his life higher than 100 diamonds
  - values 1 diamond higher than another's life
  - votes in favour of a proposal iff others are worse

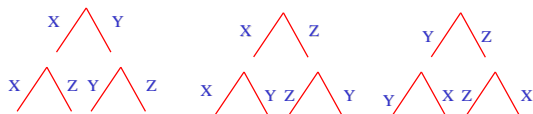
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## Voting agenda paradox

- 1:  $x > z > y$ ; 2:  $y > x > z$ ; 3:  $z > y > x$
- 40% type 1, 30% type 2, 30% type 3

majority rule: x wins  
binary protocol: chair decides!



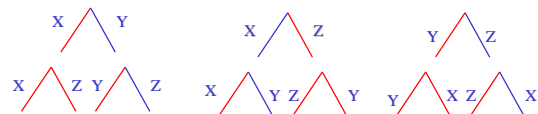
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## Voting agenda paradox

- 1:  $x > z > y$ ; 2:  $y > x > z$ ; 3:  $z > y > x$
- 40% type 1, 30% type 2, 30% type 3

majority rule: x wins  
binary protocol: chair decides!



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### Pareto dominated paradox

- 1:  $x > y > b > a$
- 2:  $a > x > y > b$
- 3:  $b > a > x > y$

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### Pareto dominated paradox

- 1:  $x > y > b > a$
- 2:  $a > x > y > b$
- 3:  $b > a > x > y$

but for all,  $x > y$  !!

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### Borda protocol

- allocate points: 4, 3, 2, 1.
  - 1:  $x > c > b > a$   $\Sigma$ :  $x: 22, a: 17, b: 16, c: 15$
  - 2:  $a > x > c > b$
  - 3:  $b > a > x > c$  If  $x$  withdraws:
  - 4:  $x > c > b > a$   $c: 15, b: 14, a: 13$  !!!
  - 5:  $a > x > c > b$
  - 6:  $b > a > x > c$
  - 7:  $x > c > b > a$

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### Arrow's theorem

- $m$  agents, each with preference  $\leq_i$  over  $D$
- Wanted:
  - $G(\leq_1, \dots, \leq_m, D) = \leq$

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### Arrow's theorem

- completeness:  $x \leq y$  or  $y \leq x$
- transitivity: if  $x \leq y \leq z$ , then  $x \leq z$
- unrestricted domain: all  $\leq$  satisfy 1 and 2
- Pareto: if  $\forall i, x \leq_i y$ , then  $x \leq y$
- independence of irrelevant choices  
if  $\leq_i$  is as  $\leq_i'$  regarding  $x$  and  $y$ ,  
then  $\leq = \leq'$  regarding  $x$  and  $y$
- no dictator: no  $i$  completely determines  $\leq$   
It is impossible to generate such a  $\leq$ !

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