COMP521: Knowledge Representation (Part 1: Modal and Description Logics)

Wiebe van der Hoek

Department of Computer Science
University of Liverpool
Liverpool, UK

`http://www.csc.liv.ac.uk/~wiebe/Teaching/COMP521/`

Knowledge Representation Part 1: Modal and Description Logics

Wiebe van der Hoek

Lecture 7: Modal logic (3)
Model Checking

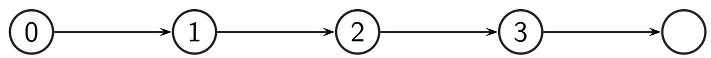
Truth Definition

A Kripke Model $M = \langle W, R, \pi \rangle$
From definitions and exercises we distill $M, w \models \varphi$:

$M, w \models \top$	and	$M, w \not\models \bot$
$M, w \models p$	iff	$\pi(w)(p) = \mathit{true}$
$M, w \models \varphi \wedge \psi$	iff	$M, w \models \varphi$ and $M, w \models \psi$
$M, w \models \neg\varphi$	iff	not: $M, w \models \varphi$
$M, w \models \varphi \vee \psi$	iff	$M, w \models \varphi$ or $M, w \models \psi$
$M, w \models \varphi \rightarrow \psi$	iff	if $M, w \models \varphi$ then $M, w \models \psi$
$M, w \models \varphi \leftrightarrow \psi$	iff	$M, w \models \varphi$ iff $M, w \models \psi$
$M, w \models \Box\varphi$	iff	for all $v : (Rwv \Rightarrow M, v \models \varphi)$
$M, w \models \Diamond\varphi$	iff	for some $v : (Rwv \ \& \ M, v \models \varphi)$

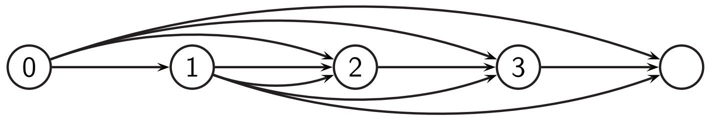
Example

$M_1 = \langle \mathbb{N}, <, \pi \rangle$



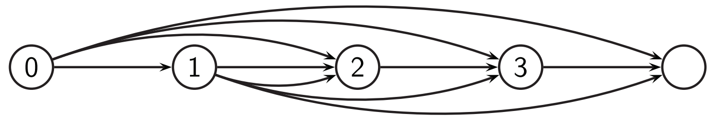
Example

$M_1 = \langle \mathbb{N}, <, \pi \rangle$



Exercise

$M_1 = \langle \mathbb{N}, <, \pi \rangle$



- $\pi(x)(p) = \textit{true}$ iff x is even;
- $\pi(x)(q) = \textit{true}$ iff x is prime;

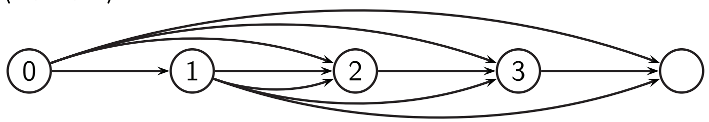
Exercise 7.1

Verify whether:

(1) $M_1, 0 \models \Diamond(p \wedge q) \wedge \Diamond\Box\neg(p \wedge q)$
(2) $M_1, 0 \models \Box\Diamond p$
(3) $M_1, 0 \models \Box(p \longrightarrow \Diamond q)$
(3) $M_1, 0 \models \Diamond\Box q \vee \Box\Diamond q$

Example

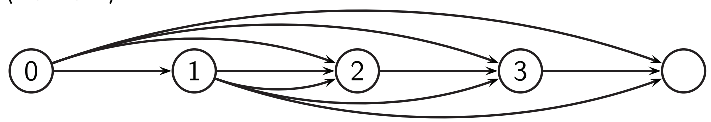
$M_1 = \langle \mathbb{N}, <, \pi \rangle$



- $\pi(x)(p) = \textit{true}$ iff x is even;
- $\pi(x)(q) = \textit{true}$ iff x is prime;
- $\pi(x)(r) = \textit{true}$ iff x is multiple of 5

Exercise

$M_1 = \langle \mathbb{N}, <, \pi \rangle$



Exercise 7.2

Show whether, for all π and φ,

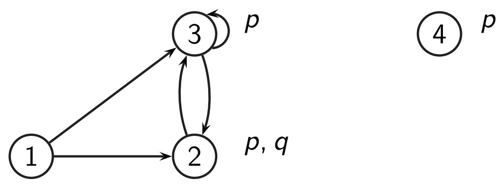
(1) $M_1, 0 \models \Box\varphi \longrightarrow \Box\Box\varphi$

(2) $M_1, 0 \models \Box\varphi$

(3) $M_1, 0 \models \Diamond\top$

Modal logics: Graph queries

M_1:

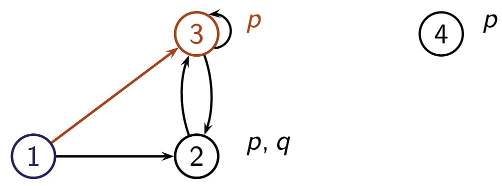


- Is there a node which we can reach in one step from node 1 that is not marked by p?

$$M_1, 1 \models \Diamond\neg p$$

Modal logics: Graph queries

M_1:

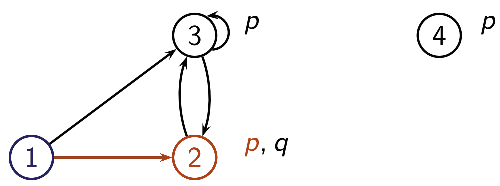


- Is there a node which we can reach in one step from node 1 that is not marked by p?

$$M_1, 1 \models \Diamond\neg p$$

Modal logics: Graph queries

M_1:

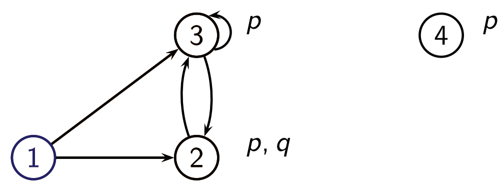


- Is there a node which we can reach in one step from node 1 that is not marked by p?

$$M_1, 1 \models \Diamond\neg p$$

Modal logics: Graph queries

M_1:

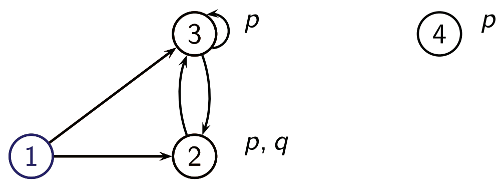


- Is there a node which we can reach in one step from node 1 that is not marked by p?

$$M_1, 1 \models \Diamond\neg p \quad \rightsquigarrow \quad \textit{no!}$$

Modal logics: Graph queries

M_1:



- Is there a node which we can reach in one step from node 1 that is not marked by p?

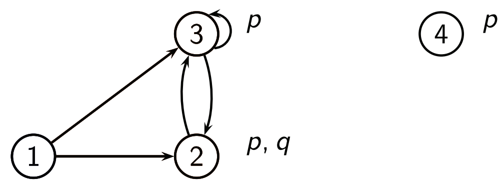
$$M_1, 1 \models \Diamond\neg p \quad \rightsquigarrow \quad no!$$

- Is it the case that all nodes that we can reach in two steps from node 1 are marked by p?

$$M_1, 1 \models \Box\Box p$$

Modal logics: Graph queries

M_1:



- Is there a node which we can reach in one step from node 1 that is not marked by p?

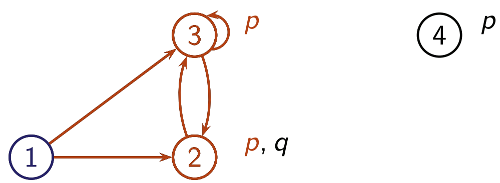
$$M_1, 1 \models \Diamond\neg p \quad \rightsquigarrow \quad no!$$

- Is it the case that all nodes that we can reach in two steps from node 1 are marked by p?

$$M_1, 1 \models \Box\Box p \quad \rightsquigarrow \quad yes!$$

Modal logics: Graph queries

M_1:



- Is there a node which we can reach in one step from node 1 that is not marked by p?

$$M_1, 1 \models \Diamond\neg p \quad \rightsquigarrow \quad no!$$

- Is it the case that all nodes that we can reach in two steps from node 1 are marked by p?

$$M_1, 1 \models \Box\Box p$$

Modal logics: Graph queries

Why are graph queries interesting?

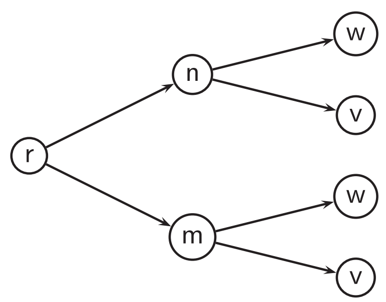
The set of all execution sequences of a program naturally form a graph:

- The states of a program execution form the nodes of the graph
- There is a link from node n_1 to node n_2 in the graph when state s_2 corresponding to n_2 is a possible successor state of state s_1 corresponding to n_1

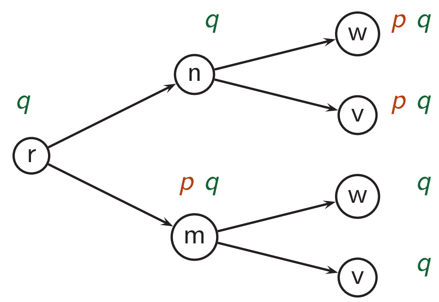
Then graph queries correspond to properties of program executions, e.g.

- we can ask whether the program can reach a state satisfying a particular property
- we can ask whether all states the program can ever reach satisfy a particular property

Trees as Computations

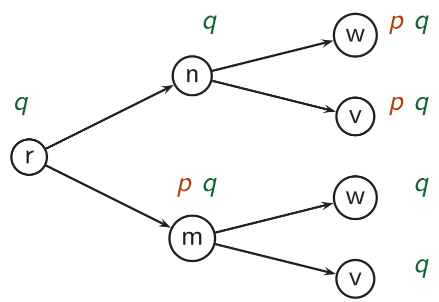


Trees as Computations



- on all paths, p is always eventually true (liveness)
- q is always true (safety)

Trees as Computations



Model checking

The problem of determining for a given Kripke model M, a world w, and a modal formula φ whether φ is true at w in M $(M, w \models \varphi)$ is called the model checking problem.

In the following we will define an algorithm which solves the model checking problem.

Notation

Definition 7.1

A Kripke Model $M = \langle W, R, \pi \rangle$ where

- W is a set of worlds
- $R \subseteq W \times W$ is a binary relation
- $\pi : W \longrightarrow \mathbf{P} \longrightarrow \{true, false\}$

Notation

Definition 7.1

A Kripke Model $M = \langle W, R, \pi \rangle$ where

- W is a set of worlds
- $R \subseteq W \times W$ is a binary relation
- $\pi : W \longrightarrow \mathbf{P} \longrightarrow \{true, false\}$

To emphasize that M is a graph with a valuation, we sometimes write $M = \langle (W, R), \pi \rangle$ or $M = \langle F, \pi \rangle$ where $F = (W, R)$ is called a frame.

A Kripke model is sometimes also denoted $M = \langle W, R, I \rangle$, where $I : \mathbf{P} \longrightarrow 2^W$

$\pi(w)(p) = true$ corresponds then to $w \in I(p)$.

Notation

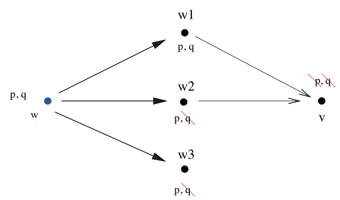
Definition 7.1

A Kripke Model $M = \langle W, R, \pi \rangle$ where

- W is a set of worlds
- $R \subseteq W \times W$ is a binary relation
- $\pi : W \longrightarrow \mathbf{P} \longrightarrow \{true, false\}$

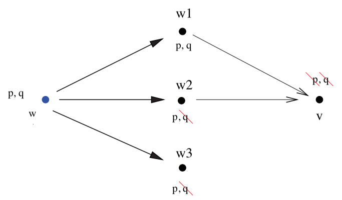
To emphasize that M is a graph with a valuation, we sometimes write $M = \langle (W, R), \pi \rangle$ or $M = \langle F, \pi \rangle$ where $F = (W, R)$ is called a frame.

Notation



W	$=$	$\{w, w1, w2, w3, v\}$
R	$=$	$\{(w, w1), (w, w2), (w, w3), (w1, v), (w2, v)\}$
π	$:$	$\pi(w)(p) = \pi(w)(q) = true$
		$\pi(v)(p) = \pi(v)(q) = false$
		$\pi(w2)(p) = \pi(w3)(p) = true$
		$\pi(w2)(q) = \pi(w3)(q) = false$
		$\pi(w1) = \pi(w)$

Notation



$$
\begin{array}{lcl}
W & = & \{w, w1, w2, w3, v\} \\
R & = & \{(w, w1), (w, w2), (w, w3), \\
 & & (w1, v), (w2, v)\} \\
\boldsymbol{\pi} & : & \boldsymbol{\pi}(w)(p) = \boldsymbol{\pi}(w)(q) = \mathit{true} \\
 & & \boldsymbol{\pi}(v)(p) = \boldsymbol{\pi}(v)(q) = \mathit{false} \\
 & & \boldsymbol{\pi}(w2)(p) = \boldsymbol{\pi}(w3)(p) = \mathit{true} \\
 & & \boldsymbol{\pi}(w2)(q) = \boldsymbol{\pi}(w3)(q) = \mathit{false} \\
 & & \boldsymbol{\pi}(w1) = \boldsymbol{\pi}(w)
\end{array}
$$

Alternatively, $M = \langle W, R, I \rangle$ where

- W is as above
- $R \subseteq W \times W$ is as above
- $I(p) = \{w, w1, w2, w3\}$, $I(q) = \{w, w1\}$
  or $I = \{(p, \{w, w1, w2, w3\}), (q, \{w\})\}$

Model Checking

- is an (automated) *verification* technique
- given a finite set of *states* S
  - an elevator, a railway (cross-road), a communication protocol, . . .
- and a *temporal property* φ
  - fairness: ‘bad things will never happen’
  - liveness: ‘something good will eventually be achieved’
- *exhaustively check* whether φ is true of S
  - if YES: that’s fine
  - if NO: read off trace of states
    - if bad: improve the system
    - if desired: use the trace as a plan

Symbolic model checking algorithm

Given a Kripke model $M = ((W, R), I)$,
a world $w \in W$,
and a modal formula φ
we check that $M, w \models \varphi$ holds by iteratively

(1) unfolding the semantic definition of $M, w \models \varphi$

(2) replacing occurrences of expressions of the form $(u, v) \in R$ and $v \in I(p)$ for arbitrary $u, v \in W$ and $p \in \mathbf{P}$ by true/false depending on R and I in M

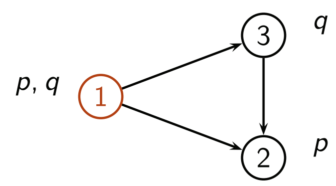
(3) simplifying the expression we obtain logically

(4) expanding quantifiers with respect to W

until the truth or falsehood of the expression is evident.
Steps (1)–(4) can be performed in an arbitrary order.

Unfolding the semantic definition: Example

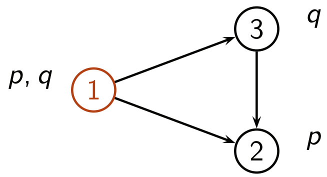
$M_2 = ((W_2, R_2), I_2)$:



$M_2, 1 \models p \longrightarrow \Box q$

Unfolding the semantic definition: Example

$M_2 = ((W_2, R_2), I_2)$:



$M_2, 1 \models p \longrightarrow \Box q$ iff (if $M_2, 1 \models p$ then $M_2, 1 \models \Box q$)

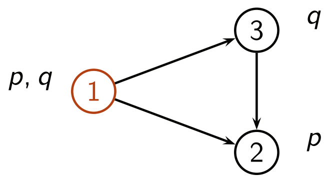
since

$M, w \models \varphi \longrightarrow \psi$ iff (if $M, w \models \varphi$ then $M, w \models \psi$)

for every Kripke model $M = ((W, R), I)$, world $w \in W$,
and modal formulae φ, ψ

Unfolding the semantic definition: Example

$M_2 = ((W_2, R_2), I_2)$:



$M_2, 1 \models p \longrightarrow \Box q$ iff (if $M_2, 1 \models p$ then $M_2, 1 \models \Box q$)
iff (if $1 \in I_2(p)$ then $M_2, 1 \models \Box q$)

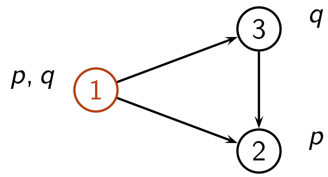
since

$M, w \models r$ iff $w \in I(r)$

for every Kripke model $M = ((W, R), I)$, world $w \in W$,
and propositional variable r

Replacing primitive expressions: Example

$M_2 = ((W_2, R_2), I_2)$:



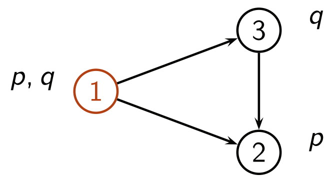
$M_2, 1 \models p \longrightarrow \Box q$ iff (if $M_2, 1 \models p$ then $M_2, 1 \models \Box q$)
iff (if $1 \in I_2(p)$ then $M_2, 1 \models \Box q$)
iff (if true then $M_2, 1 \models \Box q$)

since

$1 \in I_2(p)$ by definition of M_2: $W_2 = \{1, 2, 3\}$
$R_2 = \{(1, 2), (1, 3), (2, 3)\}$
$I_2 = \{(p, \{1, 2\}), (q, \{1, 3\})\}$

Simplifying expressions logically: Example

$M_2 = ((W_2, R_2), I_2)$:



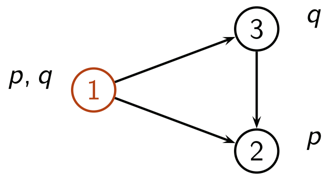
$M_2, 1 \models p \longrightarrow \Box q$ iff (if $M_2, 1 \models p$ then $M_2, 1 \models \Box q$)
iff (if $1 \in I_2(p)$ then $M_2, 1 \models \Box q$)
iff (if true then $M_2, 1 \models \Box q$)
iff $M_2, 1 \models \Box q$

since

(if true then E) is equivalent to E

Unfolding the semantic definition: Example

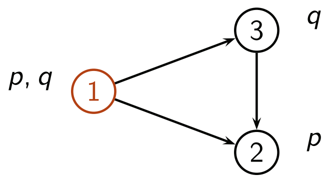
$M_2 = ((W_2, R_2), I_2)$:



$M_2, 1 \models p \longrightarrow \Box q$ iff $M_2, 1 \models \Box q$

Unfolding the semantic definition: Example

$M_2 = ((W_2, R_2), I_2)$:



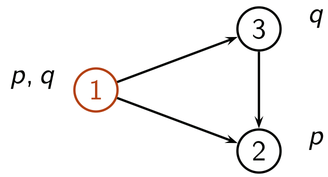
$M_2, 1 \models p \longrightarrow \Box q$ iff $M_2, 1 \models \Box q$
iff for every $v \in \{1, 2, 3\}$,
if $(1, v) \in R_2$ then $M_2, v \models q$

since

$M, w \models \Box\varphi$ iff for every $v \in W$, if $(w, v) \in R$ then $M, v \models \varphi$
for every Kripke model $M = ((W, R), I)$, world $w \in W$,
and modal formula φ

Unfolding the semantic definition: Example

$M_2 = ((W_2, R_2), I_2)$:



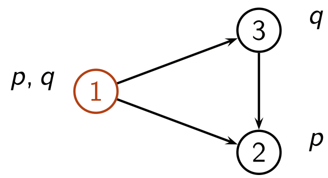
$M_2, 1 \models p \longrightarrow \Box q$ iff $M_2, 1 \models \Box q$
iff for every $v \in \{1, 2, 3\}$,
if $(1, v) \in R_2$ then $M_2, v \models q$
iff for every $v \in \{1, 2, 3\}$,
if $(1, v) \in R_2$ then $v \in I_2(q)$

since

$M, w \models r$ iff $w \in I(r)$
for every Kripke model $M = ((W, R), I)$, world $w \in W$,
and propositional variable r

Expanding quantifiers wrt. *W*: Example

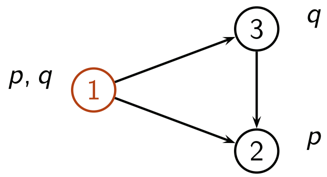
$M_2 = ((W_2, R_2), I_2)$:



$M_2, 1 \models p \longrightarrow \Box q$ iff for every $v \in \{1, 2, 3\}$,
if $(1, v) \in R_2$ then $v \in I_2(q)$

Expanding quantifiers wrt. W: Example

$M_2 = ((W_2, R_2), I_2)$:



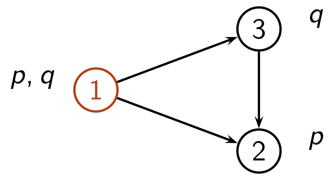
$M_2, 1 \models p \longrightarrow \Box q$ iff for every $v \in \{1, 2, 3\}$,
if $(1, v) \in R_2$ then $v \in I_2(q)$
iff (if $(1,1) \in R_2$ then $1 \in I_2(q)$)
and (if $(1,2) \in R_2$ then $2 \in I_2(q)$)
and (if $(1,3) \in R_2$ then $3 \in I_2(q)$)

since

(for every $x \in \{e_1, \ldots, e_n\}$ $\varphi(x)$) iff ($\varphi(e_1)$ and … and $\varphi(e_n)$)

Replacing primitive expressions: Example

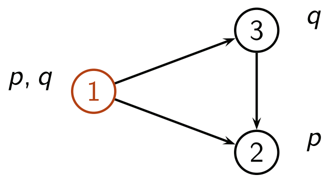
$M_2 = ((W_2, R_2), I_2)$:



$M_2, 1 \models p \longrightarrow \Box q$ iff (if $(1,1) \in R_2$ then $1 \in I_2(q)$)
and (if $(1,2) \in R_2$ then $2 \in I_2(q)$)
and (if $(1,3) \in R_2$ then $3 \in I_2(q)$)
iff (if false then true)
and (if true then false)
and (if true then true)
by definition of R_2 and I_2 in M_2.

Replacing primitive expressions: Example

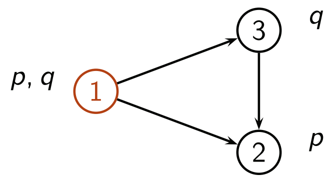
$M_2 = ((W_2, R_2), I_2)$:



$M_2, 1 \models p \longrightarrow \Box q$ iff (if $(1,1) \in R_2$ then $1 \in I_2(q)$)
and (if $(1,2) \in R_2$ then $2 \in I_2(q)$)
and (if $(1,3) \in R_2$ then $3 \in I_2(q)$)

Simplifying expressions logically: Example

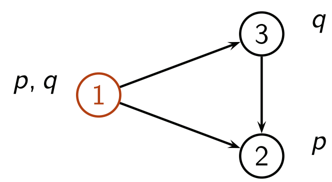
$M_2 = ((W_2, R_2), I_2)$:



$M_2, 1 \models p \longrightarrow \Box q$ iff (if false then true)
and (if true then false)
and (if true then true)

Simplifying expressions logically: Example

$M_2 = ((W_2, R_2), I_2)$:



$M_2, 1 \models p \longrightarrow \Box q$ iff (if false then true)
and (if true then false)
and (if true then true)
iff (true and false and true)

since

(if false then F) is equivalent to true, and

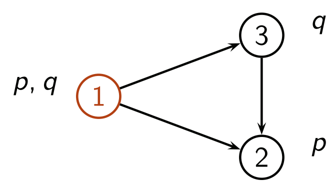
(if true then F) is equivalent to F

Summary of the example

$M_2, 1 \models p \longrightarrow \Box q$
iff (if $M_2, 1 \models p$ then $M_2, 1 \models \Box q$)
iff (if $1 \in I_2(p)$ then $M_2, 1 \models \Box q$)
iff (if true then $M_2, 1 \models \Box q$)
iff $M_2, 1 \models \Box q$
iff for every $v \in \{1, 2, 3\}$, if $(1, v) \in R_2$ then $M_2, v \models q$
iff for every $v \in \{1, 2, 3\}$, if $(1, v) \in R_2$ then $v \in I_2(q)$
iff (if $(1, 1) \in R_2$ then $1 \in I_2(q)$)
and (if $(1, 2) \in R_2$ then $2 \in I_2(q)$)
and (if $(1, 3) \in R_2$ then $3 \in I_2(q)$)
iff (if false then true) and (if true then false) and (if true then true)
iff (true and false and true)
iff false

Simplifying expressions logically: Example

$M_2 = ((W_2, R_2), I_2)$:



$M_2, 1 \models p \longrightarrow \Box q$ iff (if false then true)
and (if true then false)
and (if true then true)
iff (true and false and true)
iff false

since

(F_1 and ... and F_n) is equivalent to false given that one of the F_i is false

Exercise

Exercise 7.3

Let the set **P** of propositional variables be given by $\{p, q\}$.
Let the Kripke model $M = ((W, R), I)$ be given by

$$((\{1, 2, 3\}, \{(1, 2), (1, 3), (2, 2)\}), \{(p, \{2, 3\}), (q, \{3\})\}).$$

(1) Depict the labelled directed graph for M.

(2) Is $M, 1 \models \Diamond q$ true?

Checking the truth of a modal formula: Shortcuts

- (F_1 and ... and F_n) is equivalent to false iff one of the F_i is false
- (true and F) is equivalent to F
- (F_1 or ... or F_n) is equivalent to true iff one of the F_i is true
- (false or F) is equivalent to F
- (if false then F) is equivalent to true
- (if true then F) is equivalent to F
- $M, w \models \Box\varphi$ is true given that there is no possible world for w (equivalently, given that there is no outgoing edge from w)
- $M, w \models \Diamond\varphi$ is false given that there is no possible world for w (equivalently, given that there is no outgoing edge from w)
- $M, w \models \Box\top$ is true
- $M, w \models \Diamond\bot$ is false

Knowledge Representation
Part 1: Modal and Description Logics

Wiebe van der Hoek

Lecture 8: Modal logic (5)
Model Checking

Summary

- Symbolic model checking algorithm
  - Four rules: Unfolding, Replacing, Simplifying, Expanding
- Examples for each of the four rules
- Shortcuts for model checking

Last time ...

- Model checking examples
  - Propositional variables and negated propositional variables
  - Propositional formulae
  - Box formulae
  - Diamond formulae

Model checking and heuristics

- The last example we have looked at illustrates that the order in which we apply the rules (1)–(4) influences how quickly we can conclude whether $M, w \models \varphi$ holds or not
- For problems of realistic size, it is in general not possible to predict in advance which particular order will be the fastest Instead we will have to rely on heuristics which guide the application of the rules
- However, the order will not influence the final outcome

Model checking

- The problem of determining for a given Kripke model M, a world w, and a modal formula φ whether φ is true at w in M ($M, w \models \varphi$) is called the model checking problem.
- Variants of the model checking problem are the following two problems:
  (1) Given a Kripke model M and a modal formula φ check whether φ is true at every world in M
  (2) Given a Kripke model M and a modal formula φ determine the set of all worlds v in M such that $M, v \models \varphi$

Model checking: Theorem

We call a Kripke model $M = ((W, R), I)$ finite iff the set W of worlds is finite.

Theorem 8.1
Let A be an algorithm which is able to solve one of the three problems stated on the previous slide for any finite Kripke model M and any modal formula φ.
Then we can use A to solve the other two problems.

Model checking: Proof (1)

Proof
We only show that, if we have an algorithm A that can solve the model checking problem (as we have defined it), then we can solve the other two problems.
The other directions are left as an exercise.

(1) If we are given a Kripke model M and a modal formula φ and we want to check whether φ is true at every world in M, then we simply take algorithm A and check that $M, w \models \varphi$ holds for every world w in the set W. If this is the case, then φ is true for every world in M, otherwise it is not.
The procedure always terminates since W is finite. So, it solves problem (1).

Model checking: Proof (2)

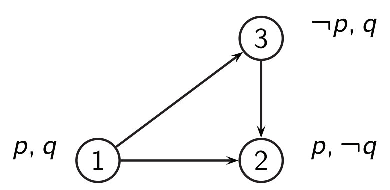
Proof

(2) If we are given a Kripke model M and a modal formula φ and we want to determine the set of all worlds v in M such that $M, v \models \varphi$ we proceed as follows:
Our procedure starts with an empty set W'. Using algorithm A, we check for each world w in W whether $M, w \models \varphi$ holds. If this is the case, then we add w to W', otherwise we leave W' unchanged. In the end the procedure returns W'.
The procedure always terminates since W is finite and the set W' it returns is the set we wanted to determine. So, the procedure solves problem (2).

Exercise

Exercise 8.4
Complete the proof of Theorem 8.1

Exercise 8.5



(1) Is $p \longrightarrow \Diamond q$ true at world 2? Justify!

(2) Is $p \vee q$ true at every world? Justify!

(3) At which worlds is $\Box p$ true? Justify!

Applications of model checking

- Model checking has been used in a number of industry projects for verification purposes
- Scenario: The behaviour of a technical artifact is described by a finite Kripke model. The desired properties of the artifact are described by modal logic formulae.
  Use model checking to show that the artifact has the desired properties.
- Examples are the verification of
  - circuit designs
  - cache coherence protocols for multiprocessors
  - authentication protocols
  - e-commerce protocols

Applications of model checking: Example (1)

We take mobile phones as an example.

- At each moment a mobile phone is in one of finitely many states, e.g. turned-off, idle, composing SMS, ringing, receiving call, etc.
- It changes from one state to another due to some event, e.g. the user presses a particular button, a call is received, etc.
- One of the desired properties of a mobile phone is that whatever events take place, you can always get back into the idle state.

Applications of model checking: Example (2)

- We model a mobile phone by a Kripke model as follows
  - With each state we associate a world and a propositional variable which is true only in this particular world, e.g. there is world w_{idle} associated with the idle state, and a propositional variable p_{idle} which is only true in w_{idle}
  - Worlds w_{s_1} and w_{s_2} are linked iff there is an event taking us from state s_1 to s_2.
- If we read '$\Box\varphi$' as 'in all worlds reachable by an arbitrary number of steps φ is true' and $\Diamond\varphi$ as 'in some world reachable by an arbitrary number of steps φ is true', then the desired property can be represented by the formula $\Box\Diamond p_{idle}$

Modal formulae: Satisfiability and validity

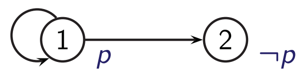
- A modal formula φ is satisfiable iff
  there exists a Kripke model $M = ((W, R), I)$ and
  there exists a world $w \in W$ such that
  φ is true at w in M
- A modal formula φ is valid iff
  for every a Kripke model $M = ((W, R), I)$ and
  for every a world $w \in W$ it holds that
  φ is true at w in M
- Two modal formulae φ and ψ are equivalent iff
  for every a Kripke model $M = ((W, R), I)$ and
  for every a world $w \in W$ it holds that
  φ is true at w in M if and only if ψ is true at w in M

Satisfiability: Examples

- Question: Is $\Diamond p \wedge \Diamond\neg p$ satisfiable?
- To show that $\Diamond p \wedge \Diamond\neg p$ is satisfiable we have to
  find a model M and a world w in M
  such that $M, w \models \Diamond p \wedge \Diamond\neg p$ holds

Satisfiability: Examples

- Question: Is $\Diamond p \wedge \Diamond\neg p$ satisfiable?
- To show that $\Diamond p \wedge \Diamond\neg p$ is satisfiable we have to
  find a model M and a world w in M
  such that $M, w \models \Diamond p \wedge \Diamond\neg p$ holds
- Let $M_2 = ((W_2, R_2), I_2)$ be the following Kripke model



Then $M_2, 1 \models \Diamond p$ since $(1, 1) \in R_2$ and $M_2, 1 \models p$

and $M_2, 1 \models \Diamond\neg p$ since $(1, 2) \in R_2$ and $M_2, 2 \models \neg p$

So, $M_2, 1 \models \Diamond p \wedge \Diamond\neg p$

Satisfiability: Examples

- Question: Is $\Diamond p \wedge \Box\neg p$ satisfiable?
- To show that $\Diamond p \wedge \Box\neg p$ is satisfiable we have to
  find a model M and a world w in M
  such that $M, w \models \Diamond p \wedge \Box\neg p$ holds

Satisfiability: Examples

- $\Diamond p \wedge \Box\neg p$ is satisfiable (if and) only if
  there exists a model $M = ((W, R), I)$ and a world $w \in W$
  such that $M, w \models \Diamond p \wedge \Box\neg p$

  iff

  $M, w \models \Diamond p$ and $M, w \models \Box\neg p$

  iff

  for some world $v \in W$, $(w, v) \in R$ and $M, v \models p$, and
  $M, w \models \Box\neg p$

  iff

  for some world $v \in W$, $(w, v) \in R$ and $M, v \models p$, and
  for every world $u \in W$, if $(w, u) \in R$ then $M, u \models \neg p$

Satisfiability: Examples

- Question: Is $\Diamond p \wedge \Box\neg p$ satisfiable?
- To show that $\Diamond p \wedge \Box\neg p$ is satisfiable we have to
  find a model M and a world w in M
  such that $M, w \models \Diamond p \wedge \Box\neg p$ holds
- However, this turns out to be impossible!

Satisfiability: Examples

- $\Diamond p \wedge \Box\neg p$ is satisfiable (if and) only if
  there exists a model $M = ((W, R), I)$ and a world $w \in W$
  such that
  for some world $v \in W$, $(w, v) \in R$ and $M, v \models p$, and
  for every world $u \in W$, if $(w, u) \in R$ then $M, u \models \neg p$

  iff

  for some world $v \in W$, $(w, v) \in R$ and $v \in I(p)$, and
  for every world $u \in W$, if $(w, u) \in R$ then $u \notin I(p)$

  iff

  for some world $v \in W$, $(w, v) \in R$, $v \in I(p)$, and
  if $(w, v) \in R$ then $v \notin I(p)$

Satisfiability: Examples

- $\Diamond p \wedge \Box\neg p$ is satisfiable (if and) only if
  there exists a model $M = ((W, R), I)$ and a world $w \in W$
  such that
  for some world $v \in W$, $(w, v) \in R$, $v \in I(p)$, and
  if $(w, v) \in R$ then $v \notin I(p)$
  iff
  for some world $v \in W$, $(w, v) \in R$, $v \in I(p)$, and
  $v \notin I(p)$
- Since there is no Kripke model $M = ((W, R), I)$ such that $v \in I(p)$ and at the same time $v \notin I(p)$ for any $p \in \mathbf{P}$ and $v \in W$, $\Diamond p \wedge \Box\neg p$ is unsatisfiable

Exercise: satisfiability

Exercise 8.6
Verify of the following formulas whether they are satisfiable:

(1) $\neg\Box(p \rightarrow q) \wedge (\Box p \rightarrow \Box q)$

(2) $\Box(p \rightarrow q) \wedge \neg(\Box p \rightarrow \Box q)$

(3) $\Diamond p \wedge \Diamond\neg p$

(4) $\Diamond p \wedge \neg\Diamond p$

(5) $\Box p \wedge \Box\neg p$

(6) $\Box p \wedge \neg\Box p$

(7) $\Diamond p \wedge \Diamond q \wedge \neg\Diamond(\wedge q)$

Summary

- Satisfiability and Validity
  - Example of a satisfiable formula (with justification)
  - Example of an unsatisfiable formula (with justification)

Knowledge Representation
Part 1: Modal and Description Logics

Wiebe van der Hoek

Lecture 9: Modal logic (5)
validities

Truth Definition

A Kripke Model $M = \langle W, R, I \rangle$

From definitions and exercises we distill $M, w \models \varphi$:

$M, w \models \top$	and	$M, w \not\models \bot$
$M, w \models p$	iff	$w \in I(p)$
$M, w \models \varphi \wedge \psi$	iff	$M, w \models \varphi$ and $M, w \models \psi$
$M, w \models \neg\varphi$	iff	not: $M, w \models \varphi$
$M, w \models \varphi \vee \psi$	iff	$M, w \models \varphi$ or $M, w \models \psi$
$M, w \models \varphi \rightarrow \psi$	iff	if $M, w \models \varphi$ then $M, w \models \psi$
$M, w \models \varphi \leftrightarrow \psi$	iff	$M, w \models \varphi$ iff $M, w \models \psi$
$M, w \models \Box\varphi$	iff	for all $v : (Rwv \Rightarrow M, v \models \varphi)$
$M, w \models \Diamond\varphi$	iff	for some $v : (Rwv \ \& \ M, v \models \varphi)$

Modal formulae: Satisfiability and validity

Definition 9.2

- A modal formula φ is satisfiable iff
  there exists a Kripke model $M = ((W, R), I)$ and
  there exists a world $w \in W$ such that
  $M, w \models \varphi$
- A modal formula φ is valid in model $M = ((W, R), I)$ iff
  for every a world $w \in W$ it holds that $M, w \models \varphi$
  We write $M \models \varphi$
- A modal formula φ is valid iff
  for every a Kripke model M it holds that $M \models \varphi$
  We write $\models \varphi$
- Two modal formulae φ and ψ are equivalent iff
  $(\varphi \leftrightarrow \psi)$ is valid: $\models (\varphi \leftrightarrow \psi))$

Modal formulae: Satisfiability and validity

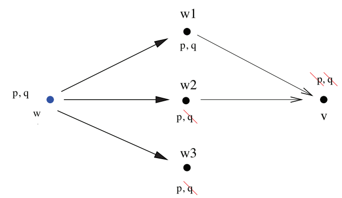
Definition 9.2

- A modal formula φ is satisfiable iff
  there exists a Kripke model $M = ((W, R), I)$ and
  there exists a world $w \in W$ such that
  $M, w \models \varphi$
- A modal formula φ is valid in model $M = ((W, R), I)$ iff
  for every a world $w \in W$ it holds that $M, w \models \varphi$
- A modal formula φ is valid iff
  for every a Kripke model M it holds that $M \models \varphi$
- Two modal formulae φ and ψ are equivalent iff
  $(\varphi \leftrightarrow \psi)$ is valid: $\models (\varphi \leftrightarrow \psi))$

Satisfiabilty: Examples

We know already some satisfiable formulas from Exercise 8.6.

(1) $\Diamond p \wedge \neg\Diamond p$ is not satisfiable
(2) $\Box(q \longrightarrow p)$ is valid in M below, but not valid sec.
(3) $\top$ and $\Box\top$ are valid
(4) $\Diamond(p \vee q)$ and $(\Diamond p \vee \Diamond q)$ are equivalent



Exercise 9.7

Prove the claims above

Modal logic: Duality of validity and satisfiability

Theorem 9.2

(a) A modal formula φ is valid if and only if $\neg\varphi$ not satisfiable

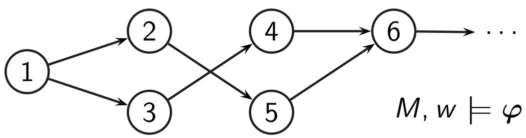
(b) A modal formula φ is satisfiable if and only if $\neg\varphi$ is not valid

(c) Two modal formulae φ and ψ are equivalent if and only if $\varphi \leftrightarrow \psi$ is valid

What is it good for?

Why is it interesting to know which modal formulae are valid in the class of all Kripke frames?

- Shortcuts for model checking



$M, w \models \varphi$ for all w?

Checking that φ is a valid modal formula can take less time than checking that φ is true at every world.

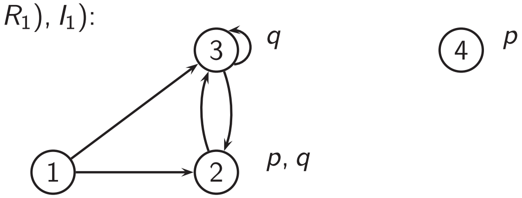
- Will be important for deductive systems for modal logics

Validity in the class of all Kripke frames

- We have just defined what it means for a modal formula to be valid in the class of all Kripke frames.
- However, that does not imply that there are any modal formulae which are valid in the class of all Kripke frames, that is, the set of formulae which is valid in the class of all Kripke frames could be empty.
- In the following we will see that this is not the case.
- We start by looking at the 'propositional part' of modal logic.

Propositional tautologies: Example

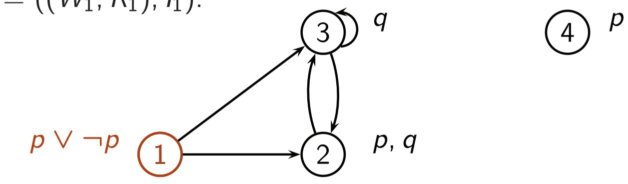
$M_1 = ((W_1, R_1), I_1)$:



Let us consider the propositional tautology $p \vee \neg p$.

Propositional tautologies: Example

$M_1 = ((W_1, R_1), I_1)$:

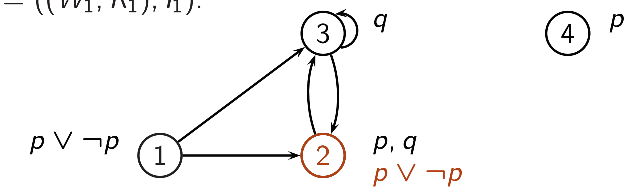


Let us consider the propositional tautology $p \vee \neg p$.

$M_1, 1 \models p \vee \neg p$ iff $M_1, 1 \models p$ or $M_1, 1 \models \neg p$
iff $1 \in I_1(p)$ or not $M_1, 1 \models p$
iff $1 \in I_1(p)$ or not $1 \in I_1(p)$
iff false or not false
iff false or true
iff true

Propositional tautologies: Example

$M_1 = ((W_1, R_1), I_1)$:

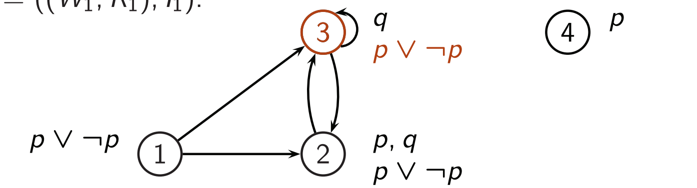


Let us consider the propositional tautology $p \vee \neg p$.

$M_1, 2 \models p \vee \neg p$ iff $M_1, 2 \models p$ or $M_1, 2 \models \neg p$
iff $2 \in I_1(p)$ or not $M_1, 2 \models p$
iff $2 \in I_1(p)$ or not $2 \in I_1(p)$
iff true or not true
iff true or false
iff true

Propositional tautologies: Example

$M_1 = ((W_1, R_1), I_1)$:

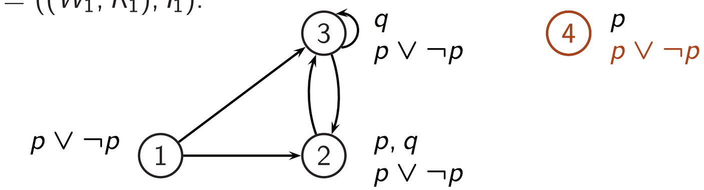


Let us consider the propositional tautology $p \vee \neg p$.

$M_1, 3 \models p \vee \neg p$ iff $M_1, 3 \models p$ or $M_1, 3 \models \neg p$
iff $3 \in I_1(p)$ or not $M_1, 3 \models p$
iff $3 \in I_1(p)$ or not $3 \in I_1(p)$
iff false or not false
iff false or true
iff true

Propositional tautologies: Example

$M_1 = ((W_1, R_1), I_1)$:

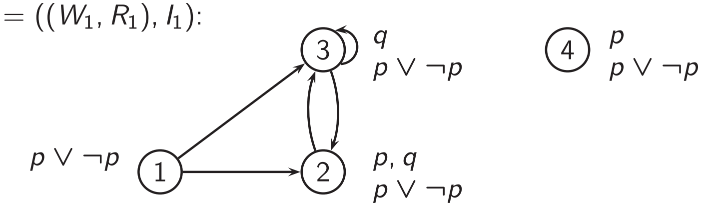


Let us consider the propositional tautology $p \vee \neg p$.

$M_1, 4 \models p \vee \neg p$ iff $M_1, 4 \models p$ or $M_1, 4 \models \neg p$
iff $4 \in I_1(p)$ or not $M_1, 4 \models p$
iff $4 \in I_1(p)$ or not $4 \in I_1(p)$
iff true or not true
iff true or false
iff true

Propositional tautologies: Example

$M_1 = ((W_1, R_1), I_1)$:



Let us consider the propositional tautology $p \vee \neg p$.

So it turns out that $p \vee \neg p$ is true at every world in M_1.
Thus, $p \vee \neg p$ is valid in M_1.

Also, we have not used any particular property of M_1.
Thus, $p \vee \neg p$ is also valid in any other Kripke model.

Tautologies: Theorem

We are now able to characterise the first class of modal formulae which is valid in the class of all Kripke frames, namely, the class of propositional tautologies:

Theorem 9.3
Let φ be a propositional tautology over the set **P** of propositional variables.
Then φ is valid in the class of all Kripke frames.

Corollary 9.1
Any propositional formula that is valid in propositional logic, is valid in the class of all Kripke frames.

Instances of tautologies

- In the case of propositional logic we know that any substitution instance of a propositional tautology is again a propositional tautology.
- So, any propositional substitution instance of a propositional tautology is valid in the class of all Kripke frames.
- However, what about the non-propositional substitution instances of a propositional tautology?

Exercise 6

Let the Kripke model $M = ((W, R), I)$ be given by

$$W = \{1, 2, 3\}$$
$$R = \{(1, 2), (2, 3), (3, 1)\}$$
$$I = \{(p, \{2\}), (q, \{3\})\}$$

(1) Depict the labelled directed graph corresponding to M.

(2) Give a formal derivation which determines whether the formula $\Diamond(p \longrightarrow q)$ is true at world 1 in the Kripke model M defined above.

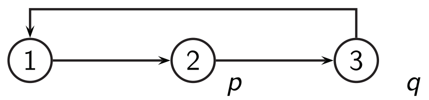
Exercise 6: Answer (1)

Let the Kripke model $M = ((W, R), I)$ be given by

$$W = \{1, 2, 3\}$$
$$R = \{(1,2), (2,3), (3,1)\}$$
$$I = \{(p, \{2\}), (q, \{3\})\}$$

(1) Depict the labelled directed graph corresponding to M.

Answer:



Exercise 6: Answer (2)

(2) Give a formal derivation which determines whether the formula $\Diamond(p \longrightarrow q)$ is true at world 1 in the Kripke model M defined above.

$M, 1 \models \Diamond(p \longrightarrow q)$
iff for some $v \in \{1,2,3\}$, $(1, v) \in R$ and $M, v \models p \longrightarrow q$
iff $((1,1) \in R$ and $M, 1 \models p \longrightarrow q)$
or $((1,2) \in R$ and $M, 2 \models p \longrightarrow q)$
or $((1,3) \in R$ and $M, 3 \models p \longrightarrow q)$
iff (false and $M, 1 \models p \longrightarrow q$)
or (true and $M, 2 \models p \longrightarrow q$)
or (false and $M, 3 \models p \longrightarrow q$)
iff false or $(M, 2 \models p \longrightarrow q)$ or false
iff $(M, 2 \models p \longrightarrow q)$

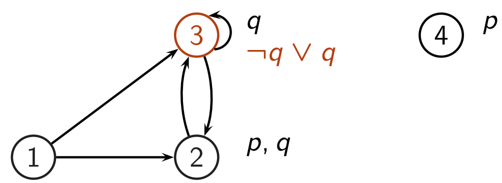
Exercise 6: Answer (2)

(2) Give a formal derivation which determines whether the formula $\Diamond(p \longrightarrow q)$ is true at world 1 in the Kripke model M defined above.

$M, 1 \models \Diamond(p \longrightarrow q)$
iff $(M, 2 \models p \longrightarrow q)$
iff (if $M, 2 \models p$ then $M, 2 \models q$)
iff (if $2 \in I(p)$ then $M, 2 \models q$)
iff (if true then $M, 2 \models q$)
iff $M, 2 \models q$
iff $2 \in I(q)$
iff false

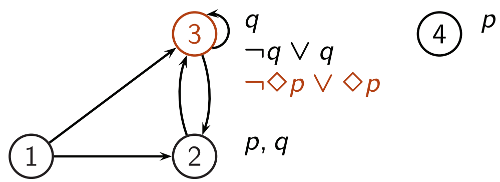
So, $\Diamond(p \longrightarrow q)$ is not true at 1 in M.

Instances of tautologies: Example



Focus on world 3. Since $\neg q \vee q$ is a tautology, it is true at this world.

Instances of tautologies: Example



Focus on world 3. Since $\neg q \vee q$ is a tautology, it is true at this world.

Now, if we apply the substitution $\{q/\Diamond p\}$ to $\neg q \vee q$, then we obtain $\neg\Diamond p \vee \Diamond p$.

It is straightforward to see that $\neg\Diamond p \vee \Diamond p$ is true at world 3.

Instances of tautologies

Corollary 9.2

Let φ be a propositional tautologies over the set **P** of propositional variables, and let σ be a substitution mapping propositional variables to arbitrary modal formulae.
Then $\varphi\sigma$ is valid in the class of all Kripke frames.

Variations of validity

- A formula φ is valid in a (Kripke) model $M = ((W, R), I)$ iff φ is true at every world $w \in W$ in M
- A formula φ is valid in a (Kripke) frame $\mathcal{F} = (W, R)$ iff φ is valid in every Kripke model based on $\mathcal{F}$
- A formula φ is valid in a class $\mathcal{C}$ of (Kripke) frames iff φ is valid in every frame $\mathcal{F}$ in $\mathcal{C}$.
- A formula φ is valid in the class of all (Kripke) frames iff φ is valid in every frame $\mathcal{F}$

Knowledge Representation
Part 1: Modal and Description Logics

Wiebe van der Hoek

Lecture 10: Modal logic (6)
Modal Validities

Modal formulae: Satisfiability and validity

Definition

- A modal formula φ is satisfiable iff
  there exists a Kripke model $M = ((W, R), I)$ and
  there exists a world $w \in W$ such that
  $M, w \models \varphi$
- A modal formula φ is valid in model $M = ((W, R), I)$ iff
  for every a world $w \in W$ it holds that $M, w \models \varphi$
  We write $M \models \varphi$
- A modal formula φ is valid iff
  for every a Kripke model M it holds that $M \models \varphi$
  We write $\models \varphi$
- Two modal formulae φ and ψ are equivalent iff
  $(\varphi \leftrightarrow \psi)$ is valid: $\models (\varphi \leftrightarrow \psi)$

Validity: some meta properties

Exercise 10.1

Verify whether:

(1) $\models (\varphi \wedge \psi)$ iff $\models \varphi$ and $\models \psi)$

(2) $\models (\varphi \vee \psi)$ iff $\models \varphi$ or $\models \psi)$

(3) $\models \neg\varphi$ iff *not* $\models \varphi$

(4) $\models (\varphi \longrightarrow \psi)$ iff if $\models \varphi$ then $\models \psi)$

Hint: only one item is true!

Give for the others the correct direction.

Modal Validity 1: Propositional Tautologies

Theorem 10.1

For any instance φ of a propositional tuautology, we have $\models \varphi$

Modal Validity 1: Propositional Tautologies

Theorem 10.1

For any instance φ of a propositional tuautology, we have $\models \varphi$

Example 10.1

(1) $(p \vee \neg p)$

(2) $(p \longrightarrow (q \longrightarrow p))$

(3) $(\Diamond p \vee \neg\Diamond p)$

(4) $((\Box r \wedge \Diamond\neg s) \longrightarrow (\neg\Box(p \longrightarrow \Diamond q) \longrightarrow (\Box r \wedge \Diamond\neg s)))$

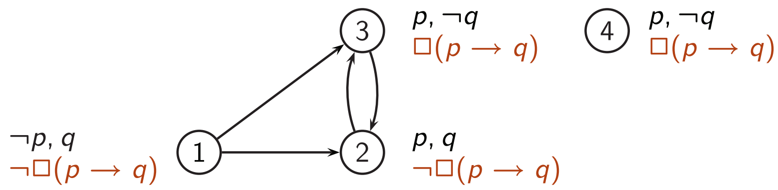
(5) $(\Diamond\neg p \longrightarrow \Box p) \longrightarrow (\Diamond\neg p \longrightarrow \Box p)$

Modal Validity 2: K

Theorem 10.2

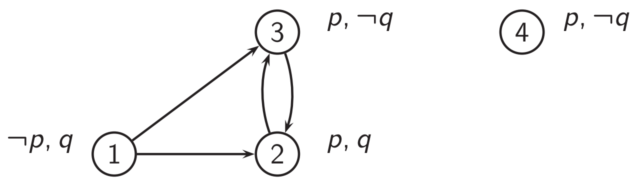
$\models \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$

The axiom (K): Example

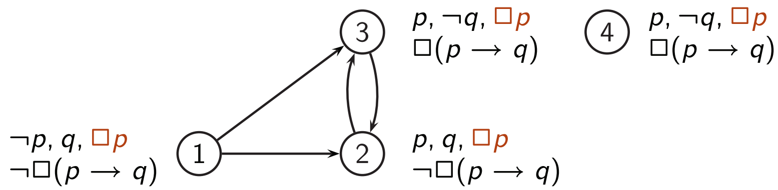


We observe that $\Box(p \to q)$ is true at $\{3, 4\}$ and false at $\{1, 2\}$.

The axiom (K): Example

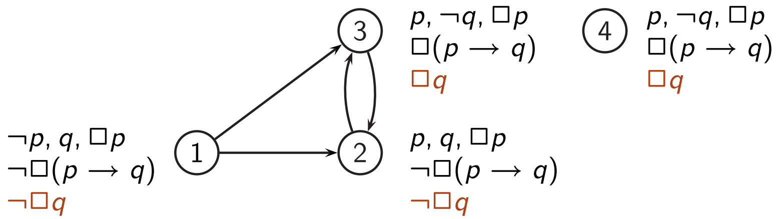


The axiom (K): Example



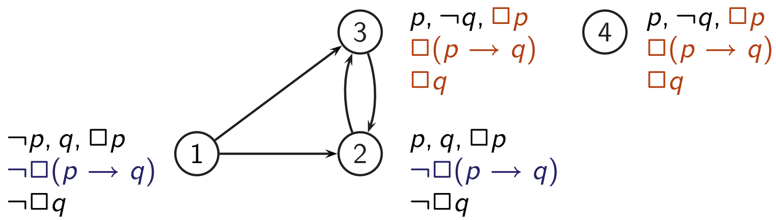
We observe that $\Box(p \to q)$ is true at $\{3, 4\}$ and false at $\{1, 2\}$.
We can also check that $\Box p$ is true at all worlds in this model.

The axiom (K): Example



We observe that $\Box(p \longrightarrow q)$ is true at $\{3, 4\}$ and false at $\{1, 2\}$.
We can also check that $\Box p$ is true at all worlds in this model.
Finally, $\Box q$ is true at $\{3, 4\}$ and false at $\{1, 2\}$.

The axiom (K): Example



We observe that $\Box(p \longrightarrow q)$ is true at $\{3, 4\}$ and false at $\{1, 2\}$.
We can also check that $\Box p$ is true at all worlds in this model.
Finally, $\Box q$ is true at $\{3, 4\}$ and false at $\{1, 2\}$.

Taking everything together, we see that
$\Box(p \longrightarrow q) \longrightarrow (\Box p \longrightarrow \Box q)$ is true at every world in this model.
This illustrates the validity of $\Box(\varphi \longrightarrow \psi) \longrightarrow (\Box\varphi \longrightarrow \Box\psi)$ in the class of all Kripke frames.

The axiom (K): Theorem

Theorem 10.2
Let φ and ψ be arbitrary modal formulae.
The formula

$$\Box(\varphi \longrightarrow \psi) \longrightarrow (\Box\varphi \longrightarrow \Box\psi) \qquad \text{(K)}$$

is valid in the class of all Kripke models.

The axiom (K): Proof (1)

Proof Let $M = ((W, R), I)$ be an arbitrary Kripke model.
We have to show that for every world w in W,

$$M, w \models \Box(\varphi \longrightarrow \psi) \longrightarrow (\Box\varphi \longrightarrow \Box\psi).$$

Assume that $M, w \models \Box(\varphi \longrightarrow \psi)$ and $M, w \models \Box\varphi$.
We have to show that $M, w \models \Box\psi$,
that is, for every world $v \in W$, if $(w, v) \in R$ then $M, v \models \psi$.

Assume for $v \in W$ that $(w, v) \in R$.

The axiom (K): Proof (2)

- Assume for $v \in W$ that $(w, v) \in R$.
  - Since by assumption, $M, w \models \Box\varphi$, for every world $u \in W$, if $(w, u) \in R$ then $M, u \models \varphi$. So, $M, v \models \varphi$.
  - Since by assumption, $M, w \models \Box(\varphi \rightarrow \psi)$, for every world $u \in W$, if $(w, u) \in R$ then
    $$M, u \models (\varphi \rightarrow \psi).$$
    So, $M, v \models (\varphi \rightarrow \psi)$, that is, if $M, v \models \varphi$ then $M, v \models \psi$.

The axiom (K): Proof (3)

- Since $M, v \models \varphi$ and
  if $M, v \models \varphi$ then $M, v \models \psi$,
  we conclude that $M, v \models \psi$.
- Since v has been an arbitrary world in W with $(w, v) \in R$, we conclude that for every world $v \in W$, if $(w, v) \in R$ then $M, v \models \psi$.
- Thus, $M, w \models \Box\psi$.

This concludes our proof.

Implications of K

Corollary 10.1

Let φ, ψ be arbitrary formulas. Then:

(1) $\models \Box(\varphi \wedge \psi) \rightarrow (\Box\varphi \wedge \Box\psi)$

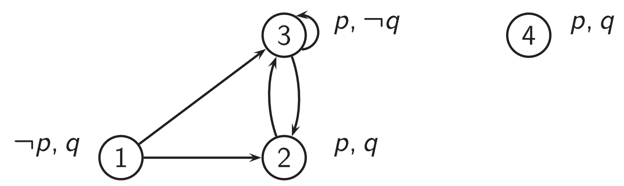
(2) $\models (\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$

(3) $\models (\Box(\varphi \rightarrow \psi) \wedge \Box\varphi) \rightarrow \Box\psi$

Exercise 10.2

Prove Corollary 10.1

Duality of $\Box$ and $\Diamond$: Example

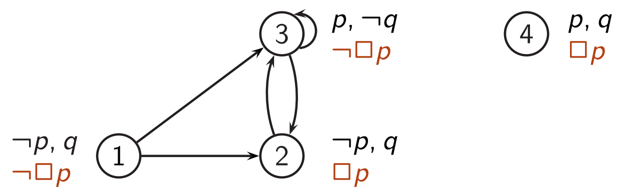


Recall that

$M, w \models \Box\varphi$ if for every $v \in W$, if $(w, v) \in R$ then $M, v \models \varphi$

$M, w \models \Diamond\varphi$ if for some $v \in W$, $(w, v) \in R$ and $M, v \models \varphi$

Duality of □ and ◇: Example



Recall that

$M, w \models \Box\varphi$ if for every $v \in W$, if $(w, v) \in R$ then $M, v \models \varphi$

$M, w \models \Diamond\varphi$ if for some $v \in W$, $(w, v) \in R$ and $M, v \models \varphi$

In our model, $\Box p$ is true at $\{2, 4\}$ and false at $\{1, 3\}$

Duality of □ and ◇: Theorem

Theorem 10.3

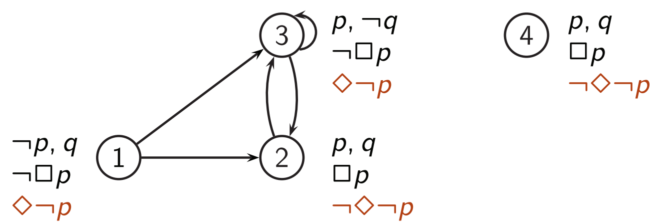
Let φ be an arbitrary modal formula.

$\Box\varphi$ and $\neg\Diamond\neg\varphi$ are equivalent, as are $\Diamond\varphi$ and $\neg\Box\neg\varphi$. Hence:

(1) $\models \Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$

(2) $\models \Diamond\varphi \leftrightarrow \neg\Box\neg\varphi$

Duality of □ and ◇: Example



Recall that

$M, w \models \Box\varphi$ if for every $v \in W$, if $(w, v) \in R$ then $M, v \models \varphi$

$M, w \models \Diamond\varphi$ if for some $v \in W$, $(w, v) \in R$ and $M, v \models \varphi$

In our model, $\Box p$ is true at $\{2, 4\}$ and false at $\{1, 3\}$
and $\Diamond\neg p$ is true at $\{1, 3\}$ and false at $\{2, 4\}$.

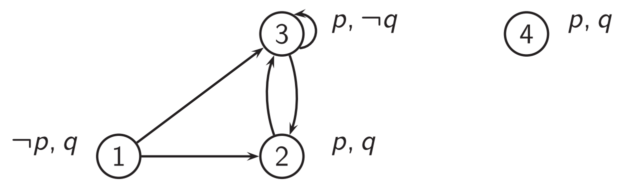
Thus, $\Box\varphi$ is true iff $\Diamond\neg\varphi$ is false in all worlds of a Kripke frame.

Conditionally valid modal formulae

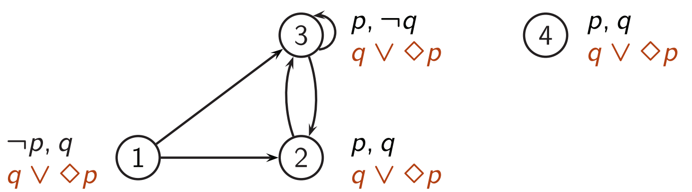
- Until now all the classes of modal formulae which are valid in the class of all Kripke frames are characterised by the syntactical form of the modal formulae.
- Could it be that there are modal formulae which are valid in the class of all Kripke frames, not because of their syntactical form, but because some other modal formulae are valid in the class of all Kripke frames?
- We show that this is the case by looking at
  - Necessitation
  - Modus Ponens

  (The reason behind these names will become clear later.)

Necessitation: Example

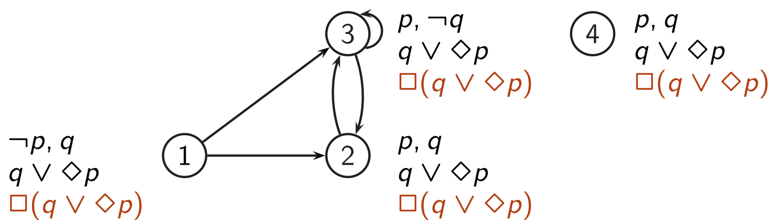


Necessitation: Example



It is straightforward to check that $q \vee \Diamond p$ is true at all worlds in this model

Necessitation: Example



It is straightforward to check that $q \vee \Diamond p$ is true at all worlds in this model

Is is also straightforward to check that also $\Box(q \vee \Diamond p)$ is true at all worlds in this model

This illustrates that if a modal formula φ is true at every world in a model, then also $\Box\varphi$ is true at every world of this model

Necessitation: Theorem

Theorem 10.4
Let φ be an arbitrary formula and M an arbitrary Kripke model.
If φ is valid in M then $\Box\varphi$ is valid in M.
Formally: for all φ $(M \models \varphi \Rightarrow M \models \Box\varphi)$

Necessitation: Theorem

Theorem 10.4

Let φ be an arbitrary formula and M an arbitrary Kripke model.
If φ is valid in M then $\Box\varphi$ is valid in M.
Formally: for all φ $(M \models \varphi \Rightarrow M \models \Box\varphi)$

Exercise 10.3

(1) Prove Theorem 10.4

(2) Show that $M \models \varphi \longrightarrow \Box\varphi$ is generally not true.

Corollary 10.2

For all M: $M \models \Box\top$

Necessitation

Corollary 10.3

If φ is valid in the class of all Kripke models, then $\Box\varphi$ is valid in the class of all Kripke models.
Formally: for all φ $(\models \varphi \Rightarrow \models \Box\varphi)$

Exercise 10.4

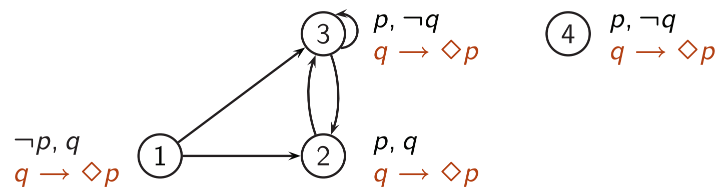
(1) Prove Corollary 10.3

(2) Show that $\not\models \varphi \longrightarrow \Box\varphi$.

Corollary 10.4

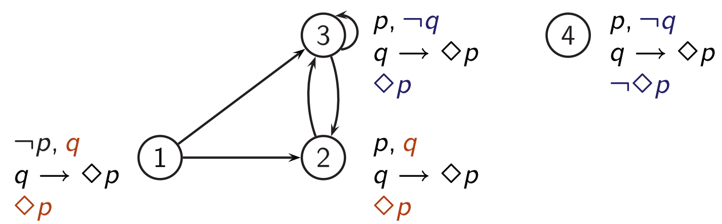
We have $\models \Box\top$

Modus Ponens: Example



We see that $q \longrightarrow \Diamond p$ is true at every world in this Kripke model.

Modus Ponens: Example



We see that $q \longrightarrow \Diamond p$ is true at every world in this Kripke model.
We also see that whenever q is true at a world w in this model, then also $\Diamond p$ is true at w.
(If q is false at w, then $\Diamond p$ may or may not be true.)

This illustrates that if $(\varphi \longrightarrow \psi)$ and φ are true at a world, so is ψ.

Modus Ponens: Theorem

Theorem 10.5
Let φ and ψ be modal formulae,
$M = ((W, R), I)$ be a Kripke model, and
w be a world in W.
If $M, w \models \varphi \rightarrow \psi$ and $M, w \models \varphi$ then $M, w \models \psi$.

Corollary 10.5
Let φ and ψ be arbitrary modal formulae.
If $\varphi \rightarrow \psi$ and φ are both valid in the class of all Kripke frames, then ψ is valid in the class of all Kripke frames.

Knowledge Representation
Part 1: Modal and Description Logics

Wiebe van der Hoek

Lecture 11: Modal logic (7)
Correspondence Theory

Validity in all Kripke models: Summary

- We have shown
  - $\models \varphi$ if φ is instance of propositional tautolgy
  - $\models \Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$
- Furthermore,
  - $\models \varphi \ \Rightarrow\ \models \Box\varphi$
  - $\models \varphi \ \& \models \varphi \rightarrow \psi \ \Rightarrow\ \models \psi$

Last time . . .

- We have shown that all modal formulae of the form
  - $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$, and

  as well as
  - all substitution instances of propositional tautologies

  are valid in the class of all Kripke frames.
- Furthermore,
  - if φ is valid in the class of all Kripke frames, then $\Box\varphi$ is valid in the class of all Kripke frames, and
  - if φ and $\varphi \rightarrow \psi$ are valid in the class of all Kripke frames, then ψ is valid in the class of all Kripke frames.

Logical omniscience (1)

- Suppose we wanted to model a logic-based agent A using modal logic, and we read '$\Box\varphi$' as 'agent A knows φ'.
- It follows from our theorems that
  - agent A knows all propositional tautologies, and
  - if agent A knows φ and agent A knows that φ implies ψ, then agent A also knows ψ
- Thus, agent A knows all the logical consequences of its knowledge
- This is called logical omniscience

Logical omniscience (2)

- Some researchers think that logical omniscience is unrealistic, because
  (1) real people are resource-bounded, that is, we do not have the time or mental capacity to work out all the logical consequences of our knowledge
  (2) real people are not rational, that is, they can claim to know things which actually contradict each other

Logical omniscience (3)

- But computers are resource-bounded too. So, if we model a logic-based agent A using modal logic but implement it by a program which runs on any kind of computer, then agent A will not actually be logical omniscient: There will be theorems of modal logic, which the program will not be able to derive given the memory limits and finite life time of any computer.
- The argument that real people are not rational has more weight. Here we have to concede that (basic) modal logic is inadequate. However, there are modal logics which allow to model various forms of irrationality.

Logical omniscience (4)

- Note that most applications of modal logics involve the modelling of technical artifacts, which can be assumed to behave in a rational and consistent way.
- The notion of Knowledge defined as above is oftern referred to as *Implicit Knowledge*; the agent A implicitly knows φ, if given sufficiently many resourses, he would be able to make it explicit!

Logical omniscience (5)

Example 11.1

Suppose that agent A knows the prime factorization of 2. Also, assume that if he knows the factorization of n, he also knows that of $n+1$. Then, using logical omnisience, we conclude that he is able to factorize any number!

Logical omniscience (6)

Exercise 11.2

Let us assume that agent A knows about today which day of the week it is, and that he knows the date. Moreover, assume that, if A knows of a specific date which day of the week it is, then he also knows about the next day the date, and which day of the week that is.

Logical omniscience (5)

Example 11.1

Suppose that agent A knows the prime factorization of 2. Also, assume that if he knows the factorization of n, he also knows that of $n+1$. Then, using logical omnisience, we conclude that he is able to factorize any number!

Exercise 11.1

Discuss what the effects of Example 11.1 are for security theory. Also, relate the example to the notion of *Implicit Knowledge*

Logical omniscience (6)

Exercise 11.2

Let us assume that agent A knows about today which day of the week it is, and that he knows the date. Moreover, assume that, if A knows of a specific date which day of the week it is, then he also knows about the next day the date, and which day of the week that is.

$$K_A(\text{date\_today} \ \wedge \ \text{weekday\_today}) \ \wedge$$
$$K_A(\text{date\_x} \ \wedge \ \text{weekday\_x}) \longrightarrow K_A(\text{date\_next(x)} \ \wedge \ \text{weekday\_next(x)})$$

Use logical omniscience to prove that A knows which day of the week it is on 27 July 2183.

Logical omniscience (7)

Exercise 11.3
Formalise and discuss the prerequisited needed to obtain the following conclusion:

> If agent A is logically omniscient, and if there exists a winning strategy for player white in chess, then agent A can win any game of chess, if he plays white

Correspondence theory

- So far we have looked at (classes of) modal formulae which are true in the class of all Kripke frames, that is, modal formulae that are true at every world in any Kripke model $M = ((W, R), I)$, whatever the particular set W of worlds, accessibility relation R, and interpretation function I is.
- Interestingly, there are (classes of) modal formulae which are true if and only if the accessibility relation R satisfies certain properties.
- The body of research concerned with the study of the relationship between modal formulae and properties of the accessibility relation is called correspondence theory.

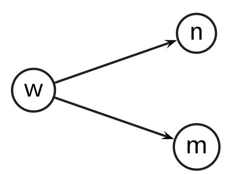
Frames

Definition 11.1
(1) A Kripke frame $\mathcal{F} = (W, R)$ is a set of worlds with a binary relation. Hence, $M = \langle F, I \rangle$ is a Kripke model.
(2) $\mathcal{F}, w \models \varphi$ iff for all I, $\langle F, I \rangle, w \models \varphi$
(3) $\mathcal{F} \models \varphi$ iff for $w \in W$, $\mathcal{F}, w \models \varphi$

Frames

Definition 11.1
(1) A Kripke frame $\mathcal{F} = (W, R)$ is a set of worlds with a binary relation. Hence, $M = \langle F, I \rangle$ is a Kripke model.
(2) $\mathcal{F}, w \models \varphi$ iff for all I, $\langle F, I \rangle, w \models \varphi$
(3) $\mathcal{F} \models \varphi$ iff for $w \in W$, $\mathcal{F}, w \models \varphi$

Example 11.2
Consider frame $\mathcal{F}_0$:



Frames

Definition 11.1

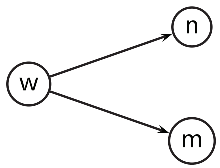
(1) A Kripke frame $\mathcal{F} = (W, R)$ is a set of worlds with a binary relation. Hence, $M = \langle F, I \rangle$ is a Kripke model.

(2) $\mathcal{F}, w \models \varphi$ iff for all I, $\langle F, I \rangle, w \models \varphi$

(3) $\mathcal{F} \models \varphi$ iff for $w \in W$, $\mathcal{F}, w \models \varphi$

Example 11.2

Consider frame $\mathcal{F}_0$:



$\mathcal{F}_0, w \models \Box\Box\bot$
$\mathcal{F}_0, w \models \Diamond(p \wedge q) \wedge \Diamond(\neg p \wedge q) \longrightarrow \Box q$
$\mathcal{F}_0 \models \Box\bot \vee \Box\Box\bot$
$\mathcal{F}_0 \models \Diamond(p \wedge q) \wedge \Diamond(\neg p \wedge q) \longrightarrow \Box q$

Properties of binary relations

Definition 11.2

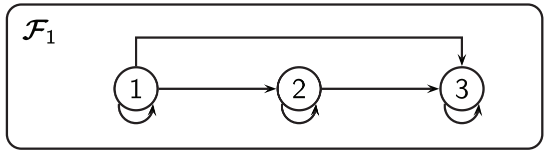
A binary relation R over a set W is

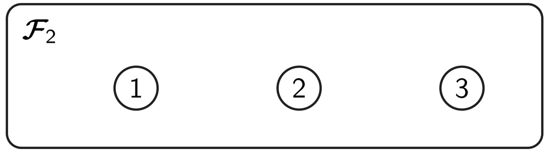
- reflexive iff for all $x \in W$, $(x, x) \in R$
- irreflexive iff for all $x \in W$, $(x, x) \notin R$
- serial iff for all $x \in W$ exists $y \in W$ such that $(x, y) \in R$
- symmetric iff for all $x, y \in W$, if $(x, y) \in R$ then $(y, x) \in R$
- transitive iff for all $x, y, z \in W$, if $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$
- euclidean iff for all $x, y, z \in W$, if $(x, y) \in R$ and $(x, z) \in R$ then $(y, z) \in R$

Exercise

Exercise 11.4

(1) Determine which of the properties mentioned in Definition 11.2 hold for the accessibility relation of the following two Kripke frames.





Frames and axiom schemata: Reflexivity (1)

Theorem 11.1

Let $\mathcal{F} = (W, R)$ be a frame. Let φ be an arbitrary modal formula and p be an arbitrary propositional variable.
The following statements are equivalent:

(1) R is reflexive

(2) $\Box\varphi \longrightarrow \varphi$ is valid in $\mathcal{F}$ (for every modal formula φ)

(3) $\Box p \longrightarrow p$ is valid in $\mathcal{F}$ (for every atom p)

(4) $\varphi \longrightarrow \Diamond\varphi$ is valid in $\mathcal{F}$

(5) $p \longrightarrow \Diamond p$ is valid in $\mathcal{F}$

Proof We show (1) implies (2) implies (3) implies (1). And equivalence of (2), (4), and (5).

Frames and axiom schemata: Reflexivity (2)

(1) $\Rightarrow$ (2): Suppose R is reflexive. Let I be an arbitrary interpretation function.
We need to show that for every world $v \in W$,

$$(\mathcal{F}, I), v \models \Box\varphi \rightarrow \varphi.$$

Let v be an arbitrary element of W. Then:
Suppose $(\mathcal{F}, I), v \models \Box\varphi$. Then by definition of the semantics of $\Box$, for every $u \in W$, if $(v, u) \in R$ then $(\mathcal{F}, I), u \models \varphi$. Since R is reflexive, $(v, v) \in R$ and therefore $(\mathcal{F}, I), v \models \varphi$. So, $(\mathcal{F}, I), v \models \Box\varphi \rightarrow \varphi$.

Frames and axiom schemata: Reflexivity (3)

(2) $\Rightarrow$ (3): Assume that $\Box\varphi \rightarrow \varphi$ is valid in $\mathcal{F}$ for every modal formula φ.
We need to show that $\Box p \rightarrow p$ is valid in $\mathcal{F}$ where p is an arbitrary propositional variable.

Straightforward: Let the modal formula φ in $\Box\varphi \rightarrow \varphi$ be the propositional variable p. Then we obtain that $\Box p \rightarrow p$ is valid in $\mathcal{F}$.

Frames and axiom schemata: Reflexivity (4)

(3) $\Rightarrow$ (1): Assume that $\Box p \rightarrow p$ is valid in $\mathcal{F} = (W, R)$ where p is an arbitrary propositional variable.
That is for any interpretation function I and any world $w \in W$ we have

$$\text{if } (\mathcal{F}, I), w \models \Box p \text{ then } (\mathcal{F}, I), w \models p.$$

We need to show that R is reflexive.

Assume R is not reflexive, that is, there is a world $w \in W$ such that $(w, w) \notin R$.
Let I be an interpretation function such that $I(p) = W \setminus \{w\}$, that is, w is the only world where p is not true.
Then $(\mathcal{F}, I), w \models \Box p$, but $(\mathcal{F}, I), w \not\models p$, a contradiction to our assumption.

Frames and axiom schemata: Reflexivity (5)

$\Box\varphi \rightarrow \varphi$ is valid in $\mathcal{F}$ for every modal formula φ (2) iff
$\varphi \rightarrow \Diamond\varphi$ is valid in $\mathcal{F}$ for every modal formula φ (4)

Proof

$\forall\varphi$: $\mathcal{F} \models \Box\varphi \rightarrow \varphi$	iff	(φ is just a variable)
$\forall\psi : \mathcal{F} \models \Box\psi \rightarrow \psi$	iff	(use contraposition) [a]
$\forall\psi : \mathcal{F} \models \neg\psi \rightarrow \neg\Box\psi$	iff	$(\neg\Box\psi \equiv \neg\Box\neg\neg\psi \equiv \Diamond\neg\psi)$
$\forall\psi : \mathcal{F} \models \neg\psi \rightarrow \Diamond\neg\psi$	iff	Schema Lemma (Lemma 11.1)
$\forall\varphi : \mathcal{F} \models \varphi \rightarrow \Diamond\varphi$		

Finally, (4) $\Rightarrow$ (5) is similar to (2) $\Rightarrow$ (3) and (5) $\Rightarrow$ (4) follows from Atoms Lemma (Lemma 11.2)

[a] which says $A \rightarrow B \equiv \neg B \rightarrow \neg A$

Schema Lemma

Lemma 11.1 (Schema Lemma)
Let $\alpha(\psi)$ be a formula with occurrences of ψ, and $\mathcal{F}$ an arbitrary frame. Then:

$$\forall\psi : \mathcal{F} \models \alpha(\neg\psi) \text{ iff } \forall\varphi : \mathcal{F} \models \alpha(\varphi)$$

Schema Lemma

Lemma 11.1 (Schema Lemma)
Let $\alpha(\psi)$ be a formula with occurrences of ψ, and $\mathcal{F}$ an arbitrary frame. Then:

$$\forall\psi : \mathcal{F} \models \alpha(\neg\psi) \text{ iff } \forall\varphi : \mathcal{F} \models \alpha(\varphi)$$

Proof:
$\Rightarrow$: Given $\forall\psi : \mathcal{F} \models \alpha(\neg\psi)$, the aim is to prove $\forall\varphi : \mathcal{F} \models \alpha(\varphi)$. So let φ be arbitrary. Choose $\psi = \neg\varphi$ to conclude $\mathcal{F} \models \alpha(\neg\neg\varphi)$, which gives $\mathcal{F} \models \alpha(\varphi)$.

Schema Lemma

Lemma 11.1 (Schema Lemma)
Let $\alpha(\psi)$ be a formula with occurrences of ψ, and $\mathcal{F}$ an arbitrary frame. Then:

$$\forall\psi : \mathcal{F} \models \alpha(\neg\psi) \text{ iff } \forall\varphi : \mathcal{F} \models \alpha(\varphi)$$

Proof:
$\Rightarrow$: Given $\forall\psi : \mathcal{F} \models \alpha(\neg\psi)$, the aim is to prove $\forall\varphi : \mathcal{F} \models \alpha(\varphi)$. So let φ be arbitrary. Choose $\psi = \neg\varphi$ to conclude $\mathcal{F} \models \alpha(\neg\neg\varphi)$, which gives $\mathcal{F} \models \alpha(\varphi)$.
$\Leftarrow$: Given $\forall\varphi : \mathcal{F} \models \alpha(\varphi)$, the aim is to prove $\Rightarrow$: Given $\forall\psi : \mathcal{F} \models \alpha(\neg\psi)$. Let ψ be arbitrary. Take $\varphi = \neg\psi$ to conclude $\mathcal{F} \models \alpha(\neg\psi)$

Atoms Lemma

Lemma 11.2 (Atoms Lemma)
Let $\alpha(\psi)$ be a formula with occurrences of ψ, and $\mathcal{F} = (W, R)$ an arbitrary frame. Then:

$$\forall p : \mathcal{F} \models \alpha(p) \text{ iff } \forall\varphi : \mathcal{F} \models \alpha(\varphi)$$

Atoms Lemma

Lemma 11.2 (Atoms Lemma)
Let $\alpha(\psi)$ be a formula with occurrences of ψ, and $\mathcal{F} = (W, R)$ an arbitrary frame. Then:

$$\forall p: \ \mathcal{F} \models \alpha(p) \ \text{ iff } \ \forall \varphi: \ \mathcal{F} \models \alpha(\varphi)$$

$\Leftarrow$: If $\alpha(\varphi)$ is valid for all formulas φ, then also for all atoms p.

Wrapping it up: Reflexivity (6)

We proved: For every frame $\mathcal{F} = (W, R)$ and every formula φ

$$\mathcal{F} \models \Box\varphi \rightarrow \varphi \text{ iff } R \text{ is reflexive}$$

We now say that

(1) $\Box\varphi \rightarrow \varphi$ corresponds to reflexivity, or

(2) that $\Box\varphi \rightarrow \varphi$ defines reflexivity on frames

Atoms Lemma

Lemma 11.2 (Atoms Lemma)
Let $\alpha(\psi)$ be a formula with occurrences of ψ, and $\mathcal{F} = (W, R)$ an arbitrary frame. Then:

$$\forall p: \ \mathcal{F} \models \alpha(p) \ \text{ iff } \ \forall \varphi: \ \mathcal{F} \models \alpha(\varphi)$$

Proof (Sketch):
$\Rightarrow$: Given $\forall p: \ \mathcal{F} \models \alpha(p)$, the aim is to prove $\forall \varphi: \ \mathcal{F} \models \alpha(\varphi)$.
So let φ be arbitrary and suppose $\mathcal{F} \not\models \alpha(\varphi)$. That is, for some w and I, $\langle W, R, I \rangle, w \models \neg\alpha(\varphi)$. Let $W' \subseteq W$ the set of worlds w' such that $\mathcal{F}, w' \models \varphi$. Let p be atom not in $\alpha(\varphi)$.
Define $I'(p) = W'$ and $I'(q) = I(q)$ if $q \neq p$.
Then $\langle W, R, I' \rangle, w \models \neg\alpha(p)$, a contradiction.
$\Leftarrow$: If $\alpha(\varphi)$ is valid for all formulas φ, then also for all atoms p.

Wrapping it up: Reflexivity (6)

We proved: For every frame $\mathcal{F} = (W, R)$ and every formula φ

$$\mathcal{F} \models \Box\varphi \rightarrow \varphi \text{ iff } R \text{ is reflexive}$$

We now say that

(1) $\Box\varphi \rightarrow \varphi$ corresponds to reflexivity, or

(2) that $\Box\varphi \rightarrow \varphi$ defines reflexivity on frames

This is a correspondence between modal and first order properties:

$$\mathcal{F} \models \Box\varphi \rightarrow \varphi \text{ iff } \mathcal{F} \models \forall x Rxx$$

Frames and axiom schemata: Seriality (1)

Theorem 11.2

Let $\mathcal{F} = (W, R)$ be a frame. Let φ be an arbitrary modal formula and p be an arbitrary propositional variable.
The following statements are equivalent:

(1) R is serial

(2) $\Box\varphi \longrightarrow \Diamond\varphi$ is valid in $\mathcal{F}$

(3) $\Box p \longrightarrow \Diamond p$ is valid in $\mathcal{F}$

(4) $\Diamond\top$ is valid in $\mathcal{F}$

Proof

We show (1) implies (2) and (3) implies (1). And that (2) and (4) are equivalent. The proof that (2) implies (3) follows the lines of the corresponding case in the proof of Theorem 11.1.

Frames and axiom schemata: Seriality (2)

(1) $\Rightarrow$ (2): Suppose R is serial. Let I be an arbitrary interpretation function. We need to show that for every world $v \in W$,

$$(\mathcal{F}, I), v \models \Box\varphi \longrightarrow \Diamond\varphi.$$

Let v be an arbitrary element of W and uppose $(\mathcal{F}, I), v \models \Box\varphi$.
Since R is serial, there exists $w \in W$ such that $(v, w) \in R$.
By definition of the semantics of $\Box$, for every $u \in W$, if $(v, u) \in R$ then $(\mathcal{F}, I), u \models \varphi$ and therefore $(\mathcal{F}, I), w \models \varphi$.
Since $(v, w) \in R$ and $(\mathcal{F}, I), w \models \varphi$, we conclude that $(\mathcal{F}, I), v \models \Diamond\varphi$.
So, assuming $(\mathcal{F}, I), v \models \Box\varphi$, we obtain $(\mathcal{F}, I), v \models \Diamond\varphi$.
Therefore, $(\mathcal{F}, I), v \models \Box\varphi \longrightarrow \Diamond\varphi$.

Frames and axiom schemata: Seriality (3)

(3)$\Rightarrow$ (1): Assume that $\Box p \longrightarrow \Diamond p$ is valid in $\mathcal{F} = (W, R)$ where p is an arbitrary propositional variable.
That is, for any I and any world $w \in W$ we have

$$\text{if } (\mathcal{F}, I), w \models \Box p \text{ then } (\mathcal{F}, I), w \models \Diamond p.$$

Let I be an arbitrary interpretation function.
We need to show that R is serial.

Frames and axiom schemata: Seriality (4)

Assume R is not serial, that is, there is a world $w \in W$ such that there is no world $v \in W$ such that $(w, v) \in R$.
In this case, by the definition of the semantics of $\Box$, $(\mathcal{F}, I), w \models \Box p$.
Since we also have $(\mathcal{F}, I), w \models \Box p \longrightarrow \Diamond p$, it follows that $(\mathcal{F}, I), w \models \Diamond p$.

However, by the definition of the semantics of $\Diamond$ $(\mathcal{F}, I), w \models \Diamond p$ implies that there exists a world $v \in W$ such that $(w, v) \in R$ (and $(\mathcal{F}, I), v \models p$), a contradiction to our assumption.

Frames and axiom schemata: Seriality (5)

We now prove:

(2) $\mathcal{F} \models \Box\varphi \rightarrow \Diamond\varphi$ iff (4) $\mathcal{F} \models \Diamond\top$, with $\mathcal{F} = (W, R)$

$\Rightarrow$:We know that $\mathcal{F} \models \Box\top$ (see Corollary 11.10.2).

Now, use Modus Ponens and (2) to conclude $\mathcal{F} \models \Diamond\top$

$\Leftarrow$:Suppose $\mathcal{F} \models \Diamond\top$.

Let w be an arbitary world in W. Then $\mathcal{F}, w \models \Diamond\top$.

Hence, there is a v such that $(w, v) \in R$ and $\mathcal{F}, v \models \top$

Hence, there is a v such that $(w, v) \in R$.

Sinc w was arbitrary, this means that R is serial

We already know that then $\mathcal{F} \models \Box\varphi \rightarrow \Diamond\varphi$.

Wrapping it up: Seriality (6)

We proved: For every frame $\mathcal{F} = (W, R)$ and every formula φ

$$\mathcal{F} \models \Box\varphi \rightarrow \Diamond\varphi \text{ iff } R \text{ is serial}$$

We now say that

(1) $\Box\varphi \rightarrow \Diamond\varphi$ corresponds to seriality, or

(2) that $\Box\varphi \rightarrow \Diamond\varphi$ defines seriality on frames

This is a correspondence between modal and first order properties:

$$\mathcal{F} \models \Box\varphi \rightarrow \Diamond\varphi \text{ iff } \mathcal{F} \models \forall x \exists y Rxy$$

Wrapping it up: Seriality (6)

We proved: For every frame $\mathcal{F} = (W, R)$ and every formula φ

$$\mathcal{F} \models \Box\varphi \rightarrow \Diamond\varphi \text{ iff } R \text{ is serial}$$

We now say that

(1) $\Box\varphi \rightarrow \Diamond\varphi$ corresponds to seriality, or

(2) that $\Box\varphi \rightarrow \Diamond\varphi$ defines seriality on frames

Wrapping it up: Seriality (6)

We proved: For every frame $\mathcal{F} = (W, R)$ and every formula φ

$$\mathcal{F} \models \Box\varphi \rightarrow \Diamond\varphi \text{ iff } R \text{ is serial}$$

We now say that

(1) $\Box\varphi \rightarrow \Diamond\varphi$ corresponds to seriality, or

(2) that $\Box\varphi \rightarrow \Diamond\varphi$ defines seriality on frames

This is a correspondence between modal and first order properties:

$$\mathcal{F} \models \Box\varphi \rightarrow \Diamond\varphi \text{ iff } \mathcal{F} \models \forall x \exists y Rxy$$

The same holds for the scheme $\Diamond\top$.

Frames and axiom schemata: Transitivity (1)

Theorem 11.3
Let $\mathcal{F} = (W, R)$ be a frame. Let φ be an arbitrary modal formula and p be an arbitrary propositional variable.
The following statements are equivalent:

(1) R is transitive

(2) $\Box\varphi \longrightarrow \Box\Box\varphi$ is valid in $\mathcal{F}$

(3) $\Box p \longrightarrow \Box\Box p$ is valid in $\mathcal{F}$

(4) $\Diamond\Diamond\varphi \longrightarrow \Diamond\varphi$ is valid in $\mathcal{F}$

(5) $\Diamond\Diamond p \longrightarrow \Diamond p$ is valid in $\mathcal{F}$

Proof We show (1) implies (2) and (3) implies (1). The proof of (2) implies (3) follows the lines of the corresponding case in the proof of Theorem 11.1. Rest is exercise.

Frames and axiom schemata: Transitivity (2)

(1) $\Rightarrow$ (2): Suppose R is transitive. Let I be an arbitrary interpretation function.
We need to show that for every world $v \in W$,

$$(\mathcal{F}, I), v \models \Box\varphi \longrightarrow \Box\Box\varphi.$$

Let v be an arbitrary element of W.
Suppose $(\mathcal{F}, I), v \models \Box\varphi$.
We have to show that $(\mathcal{F}, I), v \models \Box\Box\varphi$.

Frames and axiom schemata: Transitivity (3)

By the definition of the semantics of $\Box$,
$(\mathcal{F}, I), v \models \Box\Box\varphi$ iff for every $u, w \in W$ if $(v, u) \in R$ and $(u, w) \in R$ then $(\mathcal{F}, I), w \models \varphi$.

If there are no $u, w \in W$ with $(v, u) \in R$ and $(u, w) \in R$, then is trivially the case.

So, assume that there are $u, w \in W$ with $(v, u) \in R$ and $(u, w) \in R$. Since R is transitive, we also have $(v, w) \in R$.

By the assumption that $(\mathcal{F}, I), v \models \Box\varphi$ and the semantics of $\Box$, if $(v, w) \in R$ then $(\mathcal{F}, I), w \models \varphi$.
Since we have just established that $(v, w) \in R$, we conclude that $(\mathcal{F}, I), w \models \varphi$, which is what we needed to prove.

Frames and axiom schemata: Transitivity (4)

(3) $\Rightarrow$ (1): Assume that $\Box p \longrightarrow \Box\Box p$ is valid in $\mathcal{F} = (W, R)$ where p is an arbitrary propositional variable.
That is for any interpretation function I and any world $w \in W$ we have

$$\text{if } (\mathcal{F}, I), w \models \Box p \text{ then } (\mathcal{F}, I), w \models \Box\Box p.$$

We need to show that R is transitive, that is, let u, v, w be arbitrary elements of W with $(w, v) \in R$ and $(v, u) \in R$, then we need to show that $(w, u) \in R$.
If $w = v$ or $v = u$, then this already follows from $(w, v) \in R$ and $(v, u) \in R$. Thus, for the rest of the proof we assume that u, v, w are pairwise distinct.

Frames and axiom schemata: Transitivity (5)

Let I be an interpretation function with $I(p) = W \setminus \{u\}$ and assume that $(w, u) \notin R$. Note that $u \notin I(p)$.
Since for any world x either $(w, x) \notin R$ or $(w, x) \in R$ and $x \in I(p)$, we have $(\mathcal{F}, I), w \models \Box p$. Since we have assumed that if $(\mathcal{F}, I), w \models \Box p$ then $(\mathcal{F}, I), w \models \Box\Box p$, we also have $(\mathcal{F}, I), w \models \Box\Box p$.
However, by the semantics of $\Box$, $(\mathcal{F}, I), w \models \Box\Box p$ implies that if $(w, v) \in R$ and $(v, u) \in R$ then $(\mathcal{F}, I), u \models p$. Furthermore, $(\mathcal{F}, I), u \models p$ iff $u \in I(p)$.
By assumption $(w, v) \in R$ and $(v, u) \in R$, which means we have $u \in I(p)$. Thus, we have obtained a contradiction and the assumption $(w, u) \notin R$ has to be wrong.

Frames and axiom schemata: Transitivity (6)

Exercise 11.5

(a) Let $\mathcal{F} = (W, R)$ be a frame. Let φ be an arbitrary modal formula and p be an arbitrary propositional variable.
Prove that the following statements are equivalent:

(1) R is transitive

(4) $\Diamond\Diamond\varphi \rightarrow \Diamond\varphi$ is valid in $\mathcal{F}$

(5) $\Diamond\Diamond p \rightarrow \Diamond p$ is valid in $\mathcal{F}$

(b) Argue that this prove seems 'easier' than the equivalence of (1), (2) and (3) in Theorem 11.3.

Wrapping it up: Transitivity (7)

We proved: For every frame $\mathcal{F} = (W, R)$ and every formula φ

$$\mathcal{F} \models \Box\varphi \rightarrow \Box\Box\varphi \text{ iff } R \text{ is transitive}$$

We now say that

(1) $\Box\varphi \rightarrow \Box\Box\varphi$ corresponds to transitivity, or

(2) that $\Box\varphi \rightarrow \Box\Box\varphi$ defines transitivity on frames

Wrapping it up: Transitivity (7)

We proved: For every frame $\mathcal{F} = (W, R)$ and every formula φ

$$\mathcal{F} \models \Box\varphi \rightarrow \Box\Box\varphi \text{ iff } R \text{ is transitive}$$

We now say that

(1) $\Box\varphi \rightarrow \Box\Box\varphi$ corresponds to transitivity, or

(2) that $\Box\varphi \rightarrow \Box\Box\varphi$ defines transitivity on frames

This is a correspondence between modal and first order properties:

$$\mathcal{F} \models \Box\varphi \rightarrow \Box\Box\varphi \text{ iff } \mathcal{F} \models \forall x \forall y \forall z((Rxy \& Ryz) \Rightarrow Rxz)$$

Wrapping it up: Transitivity (7)

We proved: For every frame $\mathcal{F} = (W, R)$ and every formula φ

$$\mathcal{F} \models \Box\varphi \longrightarrow \Box\Box\varphi \text{ iff } R \text{ is transitive}$$

We now say that

(1) $\Box\varphi \longrightarrow \Box\Box\varphi$ corresponds to transitivity, or

(2) that $\Box\varphi \longrightarrow \Box\Box\varphi$ defines transitivity on frames

This is a correspondence between modal and first order properties:

$$\mathcal{F} \models \Box\varphi \longrightarrow \Box\Box\varphi \text{ iff } \mathcal{F} \models \forall x \forall y \forall z((Rxy \& Ryz) \Rightarrow Rxz)$$

The same holds for the scheme $\Diamond\Diamond\varphi \longrightarrow \Diamond\varphi$.

Frames and axiom schemata: Symmetry (1)

Theorem 11.4
Let $\mathcal{F} = (W, R)$ be a frame. Let φ be an arbitrary modal formula and p be an arbitrary propositional variable.
The following statements are equivalent:

(1) R is symmetric

(2) $\varphi \longrightarrow \Box\Diamond\varphi$ is valid in $\mathcal{F}$

(3) $p \longrightarrow \Box\Diamond p$ is valid in $\mathcal{F}$

Proof We show (1) implies (2) and (3) implies (1). The proof of (2) implies (3) follows the lines of the corresponding case in the proof of Theorem 11.1.

Frames and axiom schemata: Symmetry (2)

(1) $\Rightarrow$ (2): Suppose R is symmetric. Let I be an arbitrary interpretation function. We need to show that for every world $v \in W$,

$$(\mathcal{F}, I), v \models \varphi \longrightarrow \Box\Diamond\varphi.$$

Let v be an arbitrary element of W and suppose $(\mathcal{F}, I), v \models \varphi$.
We have to show that $(\mathcal{F}, I), v \models \Box\Diamond\varphi$.
Choose $u \in W$ with $(v, u) \in R$.
Since R is symmetric, we have $(u, v) \in R$ and $(\mathcal{F}, I), u \models \Diamond\varphi$.
Since u is an arbitrary world with $(v, u) \in R$, we have $(\mathcal{F}, I), v \models \Box\Diamond\varphi$ which is what we needed to prove.

Frames and axiom schemata: Symmetry (3)

(3) $\Rightarrow$ (1): Assume that $p \longrightarrow \Box\Diamond p$ is valid in $\mathcal{F} = (W, R)$ where p is an arbitrary propositional variable. That is, for any interpretation function I and any world $w \in W$ we have

$$\text{if } (\mathcal{F}, I), w \models p \text{ then } (\mathcal{F}, I), w \models \Box\Diamond p.$$

We need to show that R is symmetric, that is, let w and v be arbitrary elements of W with $(w, v) \in R$, then we need to show that $(v, w) \in R$.
If $w = v$ then this trivially the case. Thus, for the rest of the proof we assume that w and v are distinct.

Frames and axiom schemata: Symmetry (4)

Let I be an interpretation function with $I(p) = \{w\}$ and assume that $(v, w) \notin R$.
Since $I(p) = \{w\}$, $(\mathcal{F}, I), w \models p$. We also know that if $(\mathcal{F}, I), w \models p$ then $(\mathcal{F}, I), w \models \Box\Diamond p$. So, we obtain $(\mathcal{F}, I), w \models \Box\Diamond p$.
By the semantics of $\Box$, $(\mathcal{F}, I), w \models \Box\Diamond p$ implies that if $(w, v) \in R$ then there exists $u \in W$ such that $(v, u) \in R$ and $(\mathcal{F}, I), u \models p$.
Since $(w, v) \in R$ there has to exist $u \in W$ such that $(v, u) \in R$ and $(\mathcal{F}, I), u \models p$. However, only for w we have $(\mathcal{F}, I), w \models p$, but $(v, w) \notin R$.
Thus, we have obtained a contradiction and the assumption $(v, w) \notin R$ has to be wrong.

Wrapping it up: Symmetry (5)

We proved: For every frame $\mathcal{F} = (W, R)$ and every formula φ

$$\mathcal{F} \models \varphi \rightarrow \Box\Diamond\varphi \text{ iff } R \text{ is symmetric}$$

We now say that

(1) $\varphi \rightarrow \Box\Diamond\varphi$ corresponds to symmetry, or

(2) that $\varphi \rightarrow \Box\Diamond\varphi$ defines symmetry on frames

This is a correspondence between modal and first order properties:

$$\mathcal{F} \models \varphi \rightarrow \Box\Diamond\varphi \text{ iff } \mathcal{F} \models \forall x \forall y (Rxy \Rightarrow Ryx)$$

Wrapping it up: Symmetry (5)

We proved: For every frame $\mathcal{F} = (W, R)$ and every formula φ

$$\mathcal{F} \models \varphi \rightarrow \Box\Diamond\varphi \text{ iff } R \text{ is symmetric}$$

We now say that

(1) $\varphi \rightarrow \Box\Diamond\varphi$ corresponds to symmetry, or

(2) that $\varphi \rightarrow \Box\Diamond\varphi$ defines symmetry on frames

Frames and axiom schemata: Euclideanness (1)

Theorem 11.5
Let $\mathcal{F} = (W, R)$ be a frame. Let φ be an arbitrary modal formula and p be an arbitrary propositional variable.
The following statements are equivalent:

(1) R is euclidean

(2) $\Diamond\varphi \rightarrow \Box\Diamond\varphi$ is valid in $\mathcal{F}$

(3) $\Diamond p \rightarrow \Box\Diamond p$ is valid in $\mathcal{F}$

(4) $\neg\Box\varphi \rightarrow \Box\neg\Box\varphi$ is valid in $\mathcal{F}$

(5) $\neg\Box\varphi \rightarrow \Box\neg\Box\varphi$ is valid in $\mathcal{F}$

Proof We show (1) implies (2) and (3) implies (1) and equivalence of (2) and (4). The other cases are as in Theorem 11.1.

Frames and axiom schemata: Euclideanness (2)

(1)$\Rightarrow$ (2): Suppose R is euclidean. Let I be an arbitrary interpretation function.We need to show that for every world $v \in W$,

$$(\mathcal{F}, I), v \models \Diamond\varphi \rightarrow \Box\Diamond\varphi.$$

Let v be an arbitrary element of W and suppose $(\mathcal{F}, I), v \models \Diamond\varphi$.
We have to show that $(\mathcal{F}, I), v \models \Box\Diamond\varphi$.
Let u be an arbitrary element of W with $(v, u) \in R$.
We need to show that there exists $w \in W$ such that $(u, w) \in R$ and $(\mathcal{F}, I), w \models \varphi$.

Frames and axiom schemata: Euclideanness (4)

Since $(\mathcal{F}, I), v \models \Diamond\varphi$, there exists $u' \in W$ such that $(v, u') \in R$ and $(\mathcal{F}, I), u' \models \varphi$.
Since R is euclidean, it follows from $(v, u) \in R$ and $(v, u') \in R$ that also $(u, u') \in R$.
So, there is an element of W, namely u', such that $(u, u') \in R$ and $(\mathcal{F}, I), v \models \varphi$, which is what we needed to prove.
Thus, $(\mathcal{F}, I), v \models \Box\Diamond\varphi$.

Frames and axiom schemata: Euclideanness (5)

(3) $\Rightarrow$ (1): Assume that $\Diamond p \rightarrow \Box\Diamond p$ is valid in $\mathcal{F} = (W, R)$ where p is an arbitrary propositional variable.
That is for any interpretation function I and any world $w \in W$ we have

$$\text{if } (\mathcal{F}, I), w \models \Diamond p \text{ then } (\mathcal{F}, I), w \models \Box\Diamond p.$$

We need to show that R is euclidean, that is, let w, v, and u be arbitrary elements of W with $(w, v) \in R$ and $(w, u) \in R$, then we need to show that $(v, u) \in R$.
If $w = v$ or $w = u$ then this trivially the case. Thus, for the rest of the proof we assume that w and v as well as w and u are distinct.

Frames and axiom schemata: Euclideanness (6)

Let I be an interpretation function with $I(p) = \{u\}$ and assume that $(v, u) \notin R$.
Since $(w, u) \in R$ and $I(p) = \{u\}$, $(\mathcal{F}, I), w \models \Diamond p$. We also know that if $(\mathcal{F}, I), w \models \Diamond p$ then $(\mathcal{F}, I), w \models \Box\Diamond p$. So, we obtain $(\mathcal{F}, I), w \models \Box\Diamond p$.
By the semantics of $\Box$, $(\mathcal{F}, I), w \models \Box\Diamond p$ implies that if $(w, v) \in R$ then there exists $u' \in W$ such that $(v, u') \in R$ and $(\mathcal{F}, I), u' \models p$.
Since $(w, v) \in R$ there has to exist $u' \in W$ such that $(v, u') \in R$ and $(\mathcal{F}, I), u' \models p$. However, only for u we have $(\mathcal{F}, I), u \models p$, but $(v, u) \notin R$.
Thus, we have obtained a contradiction and the assumption $(v, u) \notin R$ has to be wrong.

Frames and axiom schemata: Euclideanness (5)

$\Diamond\varphi \rightarrow \Box\Diamond\varphi$ is valid in $\mathcal{F}$ for every modal formula φ (2) iff
$\neg\Box\varphi \rightarrow \Box\neg\Box\varphi$ is valid in $\mathcal{F}$ for every modal formula φ (4)

Proof

$\forall\varphi$: $\mathcal{F} \models \Diamond\varphi \rightarrow \Box\Diamond\varphi$	iff	(φ is just a variable)
$\forall\psi : \mathcal{F} \models \Diamond\psi \rightarrow \Box\Diamond\psi$	iff	(use $\Diamond\psi \equiv \neg\Box\neg\psi$)
$\forall\psi : \mathcal{F} \models \neg\Box\neg\psi \rightarrow \Box\neg\Box\neg\psi$	iff	(Schema Lemma (Lemma 1
$\forall\varphi : \mathcal{F} \models \neg\Box\varphi \rightarrow \Box\neg\Box\varphi$		

Finally, (4) $\Rightarrow$ (5) is similar to (2) $\Rightarrow$ (3) and (5) $\Rightarrow$ (4) follows from Atoms Lemma (Lemma 11.2)

Wrapping it up: Euclideanness (7)

We proved: For every frame $\mathcal{F} = (W, R)$ and every formula φ

$$\mathcal{F} \models \Diamond\varphi \rightarrow \Box\Diamond\varphi \text{ iff } R \text{ is euclidean}$$

We now say that

(1) $\Diamond\varphi \rightarrow \Box\Diamond\varphi$ corresponds to euclideanness, or

(2) that $\Diamond\varphi \rightarrow \Box\Diamond\varphi$ defines euclideanness on frames

This is a correspondence between modal and first order properties:

$$\mathcal{F} \models \Diamond\varphi \rightarrow \Box\Diamond\varphi \text{ iff } \mathcal{F} \models \forall x\forall y\forall z((Rxy \& Rxz) \Rightarrow Ryz)$$

Wrapping it up: Euclideanness (7)

We proved: For every frame $\mathcal{F} = (W, R)$ and every formula φ

$$\mathcal{F} \models \Diamond\varphi \rightarrow \Box\Diamond\varphi \text{ iff } R \text{ is euclidean}$$

We now say that

(1) $\Diamond\varphi \rightarrow \Box\Diamond\varphi$ corresponds to euclideanness, or

(2) that $\Diamond\varphi \rightarrow \Box\Diamond\varphi$ defines euclideanness on frames

Wrapping it up: Euclideanness (7)

We proved: For every frame $\mathcal{F} = (W, R)$ and every formula φ

$$\mathcal{F} \models \Diamond\varphi \rightarrow \Box\Diamond\varphi \text{ iff } R \text{ is euclidean}$$

We now say that

(1) $\Diamond\varphi \rightarrow \Box\Diamond\varphi$ corresponds to euclideanness, or

(2) that $\Diamond\varphi \rightarrow \Box\Diamond\varphi$ defines euclideanness on frames

This is a correspondence between modal and first order properties:

$$\mathcal{F} \models \Diamond\varphi \rightarrow \Box\Diamond\varphi \text{ iff } \mathcal{F} \models \forall x\forall y\forall z((Rxy \& Rxz) \Rightarrow Ryz)$$

The same holds for the scheme $\neg\Box\varphi \rightarrow \Box\neg\Box\varphi$.

Correspondence theory: Summary

We determine the following correspondences:

Name	Formula	Formula'	Property of the accessibility relation
(T)	$\Box p \longrightarrow p$	$p \longrightarrow \Diamond p$	reflexivity
(D)	$\Box p \longrightarrow \Diamond p$	$\Diamond \top$	seriality
(B)	$p \longrightarrow \Box \Diamond p$	$\Diamond \Box p \longrightarrow p$	symmetry
(4)	$\Box p \longrightarrow \Box \Box p$	$\Diamond \Diamond p \longrightarrow \Diamond p$	transitivity
(5)	$\Diamond p \longrightarrow \Box \Diamond p$	$\neg \Box p \longrightarrow \Box \neg \Box p$	euclideanness

Table 1: Summarising the correspondences

Knowledge Representation
Part 1: Modal and Description Logics

Wiebe van der Hoek

Lecture 12: Modal logic (8)
Deduction

Correspondence: Exercise

Exercise 11.6

Prove the remaining cases for reflexivity and symmetry in Table 1.

Last time ...

- Correspondence theory
  Formulae or formula schemata that enforce certain properties of the accessibility relation in a Kripke frame

Name	Formula	Property of the accessibility relation
(T)	$\Box p \longrightarrow p$	reflexivity
(D)	$\Box p \longrightarrow \Diamond p$	seriality
(B)	$p \longrightarrow \Box \Diamond p$	symmetry
(4)	$\Box p \longrightarrow \Box \Box p$	transitivity
(5)	$\Diamond p \longrightarrow \Box \Diamond p$	euclideanness

Modal logic: Proof theory

- Until now, whenever we wanted to establish that a modal formula φ is valid in a class $\mathcal{C}$ of Kripke frames, we have done this by showing in a more or less formal way that φ is true at every world in every Kripke model based on a frame in $\mathcal{C}$.
- Proof theory is concerned with the problem of finding a fully formal and syntactical way of establishing the validity of formulae for some given logic.

Deductive systems (1)

- We formalise this idea of a 'fully formal and syntactical way of establishing validity' by the notion of a deductive system.
- A deductive system D for a logic L consists of a (possibly empty) set of axioms and rules of inference which are used to define the notion of a deduction in D and the notion of a theorem of D.
- If we define a deductive system correctly, then every theorem of D is a valid formula of L (soundness of D) and every valid formula of L is a theorem of D (completeness of D).
- A deductive system D is terminating iff any deduction in D can be computed/performed in finite time.
- A sound, complete, and terminating deductive system D for a logic L is a decision procedure for L.

Deductive systems (2)

- In the following we look at various deductive systems for modal logic:
  - Hilbert axiomatisations or Hilbert systems
  - Translation to first-order logic combined with a deductive system for first-order logic
- Other deductive systems include
  - Tableaux calculi
  - Sequent calculi
  - Inverse method
  - Translation to (weak) monadic second-order logic combined with a deductive system for this logic

Hilbert axiomatisations

- A Hilbert axiomatisation or Hilbert system consists of axioms and rules of inference.
- The axioms are 'given truths'.
  In the case of modal logic, they should be modal formulae which are valid in a given class of Kripke frames.
- The rules of inference allow us to derive new truths from already established ones.
  In the case of modal logic, modal formulae derived from modal formulae which are valid in a given class of Kripke frames should again be valid in the same class.

The system K

Definition 12.1

The system **K** consists of the following:

(1) the axioms

*A*1 all instances of propositional tautologies

*A*2 $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$

(2) the rules of inference

*R*1 $\dfrac{\varphi \qquad \varphi \to \psi}{\psi}$

*R*2 $\dfrac{\varphi}{\Box\varphi}$

The axiom *A*2 is sometimes also called *K*, and the rules *R*1 and *R*2 as Modus Ponens and Necessitation, respectively.

Derivations

Definition 12.2

Let **X** a logical system. Define $\mathbf{X} \vdash \varphi$ iff there exists a sequence of formulas $\alpha_1, \ldots \alpha_n$ such that, for every $\alpha_i (i \leq k)$:

(1) α_i is an instance of one of the axioms

(2) α_i is obtained by applying a derivaition rule to previous derived formulas $\alpha_j (j \leq i)$

(3) $\alpha_n = \alpha$

We say that φ is a theorem in **X**.

Deduction in K

Let **K** be as before

- A deduction in a Hilbert axiomatisation is a sequence

$$\varphi_0, \varphi_1, \ldots$$

of formulae such that for each index i, $0 \leq i$ one of the following holds.

(ax) The formula φ_i is an axiom in $\mathcal{A}$,

(mp) The formula φ_i is obtained by *R*1 from two earlier formulae, that is, there are indices $j, l < i$ with $\varphi_j = (\varphi_l \to \varphi_i)$,

(nec) The formula φ_i is obtained by *R*2 from an earlier formula, that is, there is an index $j < i$ with $\varphi_i = \Box\varphi_j$.

Deduction: Example

- The following sequence of modal formulae is a deduction in the Hilbert axiomatisation K:

1 $p \vee \neg p$	(axiom of K)
2 $\Box(p \vee \neg p)$	(by (Nec) from 1)
3 $(p \vee \neg p) \to (q \to q \vee \Diamond q)$	(axiom of K)
4 $\Box((p \vee \neg p) \to (q \to q \vee \Diamond q))$	(by (Nec) from 3)
5 $\Box((p \vee \neg p) \to (q \to q \vee \Diamond q))$ $\to \Box(p \vee \neg p) \to \Box(q \to q \vee \Diamond q)$	(axiom of K)
6 $\Box(p \vee \neg p) \to \Box(q \to q \vee \Diamond q)$	(by (MP) from 4 and 5)
7 $\Box(q \to q \vee \Diamond q)$	(by (MP) from 2 and 6)

Derived Rule

Theorem 12.1
the following rule is derivable:

$\Box D \qquad \dfrac{\varphi \rightarrow \psi}{\Box\varphi \rightarrow \Box\psi}$

- Proof:

Derived Rule

Theorem 12.1
the following rule is derivable:

$\Box D \qquad \dfrac{\varphi \rightarrow \psi}{\Box\varphi \rightarrow \Box\psi}$

- Proof:

1 $\mathbf{K} \vdash \varphi \rightarrow \psi$ assumption

2 $\mathbf{K} \vdash \Box(\varphi \rightarrow \psi)$ *R2*; 1

Derived Rule

Theorem 12.1
the following rule is derivable:

$\Box D \qquad \dfrac{\varphi \rightarrow \psi}{\Box\varphi \rightarrow \Box\psi}$

- Proof:

1 $\mathbf{K} \vdash \varphi \rightarrow \psi$ assumption

Derived Rule

Theorem 12.1
the following rule is derivable:

$\Box D \qquad \dfrac{\varphi \rightarrow \psi}{\Box\varphi \rightarrow \Box\psi}$

- Proof:

1 $\mathbf{K} \vdash \varphi \rightarrow \psi$ assumption

2 $\mathbf{K} \vdash \Box(\varphi \rightarrow \psi)$ *R2*; 1

3 $\mathbf{K} \vdash \Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$ *A2*

Derived Rule

Theorem 12.1

the following rule is derivable:

$$\Box D \qquad \frac{\varphi \rightarrow \psi}{\Box\varphi \rightarrow \Box\psi}$$

- Proof:

1	$\mathbf{K} \vdash \varphi \rightarrow \psi$	assumption
2	$\mathbf{K} \vdash \Box(\varphi \rightarrow \psi)$	*R2*; 1
3	$\mathbf{K} \vdash \Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$	*A2*
4	$\mathbf{K} \vdash \Box\varphi \rightarrow \Box\psi$	*R1*, 2, 3

Derived Axiom

Theorem 12.2

$\mathbf{K} \vdash (\Box\varphi \vee \Box\psi) \rightarrow \Box(\varphi \vee \psi)$

1	$\mathbf{K} \vdash \varphi \rightarrow (\varphi \vee \psi)$	*A1*
2	$\mathbf{K} \vdash \Box\varphi \rightarrow \Box(\varphi \vee \psi)$	$\Box D$; 1

Derived Axiom

Theorem 12.2

$\mathbf{K} \vdash (\Box\varphi \vee \Box\psi) \rightarrow \Box(\varphi \vee \psi)$

1	$\mathbf{K} \vdash \varphi \rightarrow (\varphi \vee \psi)$	*A1*

Derived Axiom

Theorem 12.2

$\mathbf{K} \vdash (\Box\varphi \vee \Box\psi) \rightarrow \Box(\varphi \vee \psi)$

1	$\mathbf{K} \vdash \varphi \rightarrow (\varphi \vee \psi)$	*A1*
2	$\mathbf{K} \vdash \Box\varphi \rightarrow \Box(\varphi \vee \psi)$	$\Box D$; 1
3	$\mathbf{K} \vdash \psi \rightarrow (\varphi \vee \psi)$	*A1*

Derived Axiom

Theorem 12.2

$\mathbf{K} \vdash (\Box\varphi \lor \Box\psi) \longrightarrow \Box(\varphi \lor \psi)$

1 $\mathbf{K} \vdash \varphi \longrightarrow (\varphi \lor \psi)$ — *A1*
2 $\mathbf{K} \vdash \Box\varphi \longrightarrow \Box(\varphi \lor \psi)$ — $\Box D$; 1
3 $\mathbf{K} \vdash \psi \longrightarrow (\varphi \lor \psi)$ — *A1*
4 $\mathbf{K} \vdash \Box\psi \longrightarrow \Box(\varphi \lor \psi)$ — *R1*, 2, 3

Derived Axiom

Theorem 12.2

$\mathbf{K} \vdash (\Box\varphi \lor \Box\psi) \longrightarrow \Box(\varphi \lor \psi)$

1 $\mathbf{K} \vdash \varphi \longrightarrow (\varphi \lor \psi)$ — *A1*
2 $\mathbf{K} \vdash \Box\varphi \longrightarrow \Box(\varphi \lor \psi)$ — $\Box D$; 1
3 $\mathbf{K} \vdash \psi \longrightarrow (\varphi \lor \psi)$ — *A1*
4 $\mathbf{K} \vdash \Box\psi \longrightarrow \Box(\varphi \lor \psi)$ — *R1*, 2, 3
5 $\mathbf{K} \vdash (\Box\varphi \longrightarrow \Box(\varphi \lor \psi))$
$\longrightarrow ((\Box\psi \longrightarrow \Box(\varphi \lor \psi)) \longrightarrow ((\Box\varphi \lor \Box\psi) \longrightarrow \Box(\varphi \lor \psi)))$ — *A1*

Derived Axiom

Theorem 12.2

$\mathbf{K} \vdash (\Box\varphi \lor \Box\psi) \longrightarrow \Box(\varphi \lor \psi)$

1 $\mathbf{K} \vdash \varphi \longrightarrow (\varphi \lor \psi)$ — *A1*
2 $\mathbf{K} \vdash \Box\varphi \longrightarrow \Box(\varphi \lor \psi)$ — $\Box D$; 1
3 $\mathbf{K} \vdash \psi \longrightarrow (\varphi \lor \psi)$ — *A1*
4 $\mathbf{K} \vdash \Box\psi \longrightarrow \Box(\varphi \lor \psi)$ — *R1*, 2, 3
5 $\mathbf{K} \vdash (\Box\varphi \longrightarrow \Box(\varphi \lor \psi))$
$\longrightarrow ((\Box\psi \longrightarrow \Box(\varphi \lor \psi)) \longrightarrow ((\Box\varphi \lor \Box\psi) \longrightarrow \Box(\varphi \lor \psi)))$ — *A1*
6 $\mathbf{K} \vdash (\Box\psi \longrightarrow \Box(\varphi \lor \psi)) \longrightarrow ((\Box\varphi \lor \Box\psi) \longrightarrow \Box(\varphi \lor \psi))$
R1,2,5

Derived Axiom

Theorem 12.2

$\mathbf{K} \vdash (\Box\varphi \lor \Box\psi) \longrightarrow \Box(\varphi \lor \psi)$

1 $\mathbf{K} \vdash \varphi \longrightarrow (\varphi \lor \psi)$ — *A1*
2 $\mathbf{K} \vdash \Box\varphi \longrightarrow \Box(\varphi \lor \psi)$ — $\Box D$; 1
3 $\mathbf{K} \vdash \psi \longrightarrow (\varphi \lor \psi)$ — *A1*
4 $\mathbf{K} \vdash \Box\psi \longrightarrow \Box(\varphi \lor \psi)$ — *R1*, 2, 3
5 $\mathbf{K} \vdash (\Box\varphi \longrightarrow \Box(\varphi \lor \psi))$
$\longrightarrow ((\Box\psi \longrightarrow \Box(\varphi \lor \psi)) \longrightarrow ((\Box\varphi \lor \Box\psi) \longrightarrow \Box(\varphi \lor \psi)))$ — *A1*
6 $\mathbf{K} \vdash (\Box\psi \longrightarrow \Box(\varphi \lor \psi)) \longrightarrow ((\Box\varphi \lor \Box\psi) \longrightarrow \Box(\varphi \lor \psi))$
R1,2,5
7 $\mathbf{K} \vdash (\Box\varphi \lor \Box\psi) \longrightarrow \Box(\varphi \lor \psi)$ — *R1*,4,6

Derived Axiom

Theorem 12.2

$\mathbf{K} \vdash (\Box\varphi \vee \Box\psi) \rightarrow \Box(\varphi \vee \psi)$

1	$\mathbf{K} \vdash \varphi \rightarrow (\varphi \vee \psi)$	*A1*
2	$\mathbf{K} \vdash \Box\varphi \rightarrow \Box(\varphi \vee \psi)$	$\Box D$; 1
3	$\mathbf{K} \vdash \psi \rightarrow (\varphi \vee \psi)$	*A1*
4	$\mathbf{K} \vdash \Box\psi \rightarrow \Box(\varphi \vee \psi)$	*R1*, 2, 3
5	$\mathbf{K} \vdash (\Box\varphi \rightarrow \Box(\varphi \vee \psi)) \rightarrow ((\Box\psi \rightarrow \Box(\varphi \vee \psi)) \rightarrow ((\Box\varphi \vee \Box\psi) \rightarrow \Box(\varphi \vee \psi)))$	*A1*
6	$\mathbf{K} \vdash (\Box\psi \rightarrow \Box(\varphi \vee \psi)) \rightarrow ((\Box\varphi \vee \Box\psi) \rightarrow \Box(\varphi \vee \psi))$	*R1*,2,5
7	$\mathbf{K} \vdash (\Box\varphi \vee \Box\psi) \rightarrow \Box(\varphi \vee \psi)$	*R1*,4,6

Exercise

Exercise 12.1

Prove the following rule of inference:

$$HS \quad \frac{\varphi \rightarrow \varphi_1,\ \varphi_1 \rightarrow \varphi_2,\ \ldots \varphi_n \rightarrow \psi}{\varphi \rightarrow \psi}$$

Exercise 12.2

Prove the following theorems:

(1) $\mathbf{K} \vdash \Box(\varphi \wedge \psi) \leftrightarrow (\Box\varphi \wedge \Box\psi)$

(2) $\mathbf{K} \vdash \neg\Box\bot \leftrightarrow (\Box\varphi \rightarrow \neg\Box\neg\varphi)$

Soundness of K

Theorem 12.3

For all φ: $\mathbf{K} \vdash \varphi \Rightarrow \models \varphi$

Proof:

For *A1*, this follows from Theorem 10.1

For *A2*, this follows from Theorem 10.2

For *R1*, this follows from Theorem 10.5

For *R2*, this follows from Theorem 10.4

Hilbert axiomatisation of other modal logics

- Other modal logics are axiomatised by extending the set of axioms. Thus, for instance, let

  KD **KT** **KB** **K4** **K5**

  be the systems whose axioms comprise the smallest acceptable set of axioms containing the formulae

  D T B 4 5

  respectively.
- Similarly, let Σ be a sequence of symbols chosen from D, T, B, 4, 5. Then $\mathbf{K\Sigma}$ denotes the modal logic whose axioms comprise the smallest acceptable set of axioms containing the fomulae indicated by the symbols in Σ.

KD45: The modal logic of belief (1)

- Following the convention we have just defined, **KD45** is the modal logic whose axioms comprise the smallest acceptable set of axioms containing the formulae

  $$(D)\ \Box p \longrightarrow \Diamond p \quad (4)\ \Box p \longrightarrow \Box\Box p \quad (5)\ \Diamond p \longrightarrow \Box\Diamond p$$

  which is commonly used to describe the notion of belief.
- That is, to describe the belief of an agent A, we use the modal logic **KD45** and
  read $\Box\varphi$ as 'A believes φ' and
  $\Diamond\varphi$ as 'A does not believe $\neg\varphi$'
  (that is, $\Diamond\neg\varphi$ reads 'A does not believe φ'.)
- Remember that the set of axioms of **KD45** not only contains (D), (4), and (5), but also all their substitution instances.

KD45: The modal logic of belief (2)

- (D) means that if A believes φ then A does not believe $\neg\varphi$, that is, the beliefs of A are non-contradictory.
- (4) means that if A believes φ then A believes that it believes φ. This property is called positive introspection.
- (5) means that if A does not believe φ then A believes that it does not believe φ. This property is called negative introspection.
- Whether human beliefs are always non-contradictory and whether humans have full positive and negative introspection is, of course, debatable.

KT45: The modal logic of knowledge (1)

- Similarly, **KT45** (also called **S5**) is the modal logic whose axioms comprise the smallest acceptable set of axioms containing the formulae

  $$(T)\ \Box p \longrightarrow p \quad (4)\ \Box p \longrightarrow \Box\Box p \quad (5)\ \Diamond p \longrightarrow \Box\Diamond p$$

  which is commonly used to describe the notion of knowledge.
- That is, to describe the knowledge of an agent A, we use the modal logic **KT45** and
  read $\Box\varphi$ as 'A knows φ' and
  $\Diamond\varphi$ as 'A does not know $\neg\varphi$'
  (that is, $\Diamond\neg\varphi$ reads 'A does not know φ')
- Remember that the set of axioms of **KT45** not only contains (T), (4), and (5), but also all their substitution instances.

KT45: The modal logic of knowledge (2)

- (T) means that if A knows φ then φ is true, that is, the knowlege of A is true.
- (4) means that if A knows φ then A knows that it knows φ. Again, this property is called positive introspection.
- (5) means that if A does not know φ then A knows that it does not know φ. Again, this property is called negative introspection.
- For humans it is doubtful whether they possess full negative introspection, however for logic-based agents it is a useful property.

Soundness and Completeness (1)

- A Hilbert system H is sound with respect to some class $\mathcal{C}$ of Kripke frames iff every theorem of H is valid in the class $\mathcal{C}$.
- A Hilbert system H is complete with respect to some class $\mathcal{C}$ of Kripke frames iff every modal formula that is valid in the class $\mathcal{C}$ is a theorem of H.

Soundness and completeness (2)

Theorem 12.4
Let Σ be a sequence of symbols chosen from D, T, B, 4, 5.
Let $\mathcal{C}$ be the class of all those Kripke frames whose accessibility relation satisfies all the properties associated with the symbols in Σ.
Then the Hilbert system KΣ is sound and complete with respect to the class $\mathcal{C}$ of Kripke frames.

Soundness and completeness (3)

- Recall the Hilbert systems KD45 and KT45.
- What Theorem **??** says is the following:
  - KD45 is sound and complete with respect to the class of all those Kripke frames whose accessibility relation satisfies all the properties all the properties associated with D, 4, and 5, that is, seriality, transitivity and euclideanness.
    That is, the accessibility relation R in these Kripke frames has to be serial, transitive, and euclidean.
  - KT45 is sound and complete with respect to the class of all those Kripke frames whose accessibility relation satisfies all the properties all the properties associated with T, 4, and 5, that is, reflexivity, transitivity and euclideanness.

Hilbert systems: Conclusion

- Hilbert systems provide a concise way to describe the theorems of a modal logic.
- It is common to start from a Hilbert system H for a modal logic L, and to determine the appropriate semantics for L from H.
  In same cases this can be done automatically using systems like SCAN, for example.
- However, Hilbert systems alone provide no practical way to derive the theorems of a modal logic.
- We now turn our attention to a more practical approach, namely translation to first-order logic.

Modal logic and first-order logic (1)

- If you recall the definition of the semantics of basic modal logic including, for example,

$$M, w \models (\varphi \wedge \psi) \text{ iff } (M, w \models \varphi \text{ and } M, w \models \psi)$$
$$M, w \models \Box\varphi \quad \text{iff for every } v \in W, \text{ if } (w, v) \in R \text{ then } M, v \models \varphi$$

  then we observe that on the right-hand side of the 'iff' we find logical expressions using operators like 'and', 'or', and quantifiers like 'for every' and 'for some'.
- Operators and quantifiers like these are part of the language of first-order logic which gives a formal semantics and a proof theory for this language.

Modal logic and first-order logic (2)

- So, one may ask
  - whether we could fully formalise the semantics of the basic modal logic K and others like KT45 and KD45 in first-order logic and
  - whether we can use this formalisation as an alternative way to establish the validity of modal formulae.

Translation to first-order logic (1)

- Let **P** be a set of propositional variables. With every propositional variable p in **P** we associate a unary prediate symbol q_p of first-order logic.
- In addition, we use a binary predicate symbol r which will represent the accessibility relation R of a Kripke model.
- We now define a function π that takes a modal formula and a variable symbol of first-order logic as arguments and returns a first-order formula.

Translation to first-order logic (2)

$$\pi(p, x) = q_p(x) \quad \text{for any propositional variable } p$$
$$\pi(\top, x) = \top$$
$$\pi(\bot, x) = \bot$$
$$\pi(\neg\varphi, x) = \neg\pi(\varphi, x)$$
$$\pi(\varphi \wedge \psi, x) = \pi(\varphi, x) \wedge \pi(\psi, x)$$
$$\pi(\varphi \vee \psi, x) = \pi(\varphi, x) \vee \pi(\psi, x)$$
$$\pi(\varphi \rightarrow \psi, x) = \pi(\varphi, x) \rightarrow \pi(\psi, x)$$
$$\pi(\varphi \leftrightarrow \psi, x) = \pi(\varphi, x) \leftrightarrow \pi(\psi, x)$$
$$\pi(\Box\varphi, x) = \forall y \cdot r(x, y) \rightarrow \pi(\varphi, y) \quad \text{where } y \text{ is new}$$
$$\pi(\Diamond\varphi, x) = \exists y \cdot r(x, y) \wedge \pi(\varphi, y) \quad \text{first-order variable}$$

Translation to first-order logic: Example

- Consider the modal formula $\Box p \wedge \Diamond\neg r$. We can compute the translation of this formula as follows.

$$\begin{aligned}
&\pi(\Box p \wedge \Diamond\neg q, x)\\
&\quad = \pi(\Box p, x) \wedge \pi(\Diamond\neg q, x)\\
&\quad = (\forall y \cdot r(x,y) \longrightarrow \pi(p,y)) \wedge \pi(\Diamond\neg q, x)\\
&\quad = (\forall y \cdot r(x,y) \longrightarrow q_p(y)) \wedge \pi(\Diamond\neg q, x)\\
&\quad = (\forall y \cdot r(x,y) \longrightarrow q_p(y)) \wedge (\exists z \cdot r(x,z) \wedge \pi(\neg q, z))\\
&\quad = (\forall y \cdot r(x,y) \longrightarrow q_p(y)) \wedge (\exists z \cdot r(x,z) \wedge \neg\pi(q, z))\\
&\quad = (\forall y \cdot r(x,y) \longrightarrow q_p(y)) \wedge (\exists z \cdot r(x,z) \wedge \neg q_q(z))
\end{aligned}$$

Translation to first-order logic (3)

- It is straightforward to see that all the properties of the accessibility relation enforced by the formulae T, D, B, 4, and 5 are expressible by first-order formulae.
- For $s \in \{T, D, B, 4, 5\}$ let Ax_s be given by

	Property of R	Ax_s
(T)	reflexivity	$\forall x \cdot r(x,x)$
(D)	seriality	$\forall x \exists y \cdot r(x,y)$
(B)	symmetry	$\forall x, y \cdot r(x,y) \longrightarrow r(y,x)$
(4)	transitivity	$\forall x, y, z \cdot (r(x,y) \wedge r(y,z)) \longrightarrow r(x,z)$
(5)	euclideanness	$\forall x, y, z \cdot (r(x,y) \wedge r(x,z)) \longrightarrow r(y,z)$

Translation to first-order logic: Theorem

Theorem 12.5

Let Σ be a sequence of symbols chosen from D, T, B, 4, 5.
Let $\mathcal{C}$ be the class of all those Kripke frames whose accessibility relation satisfies all the properties associated with the symbols in Σ.
Let φ be an arbitrary modal formula. Then φ is satisfiable in the class $\mathcal{C}$ of Kripke frames iff $\pi(\varphi, x) \wedge \bigwedge_{s\in\Sigma} Ax_s$ is satisfiable in first-order logic.

Recall that φ is valid iff $\neg\varphi$ is unsatisfiable.

Corollary 12.1

Let φ be an arbitrary modal formula. Then φ is valid iff $\pi(\neg\varphi, x)$ is unsatisfiable in first-order logic.

Translation to first-order logic: Conclusion

- Once a modal formula has been translated to first-order logic, the satisfiability of the first-order formulae we obtain can be determined by a number of means.
- One possibility is the use of first-order resolution which is a sound and complete deductive system for first-order logic.
- In fact, the deductions of particular refinements of first-order resolution will always terminate on translated modal formulae. This provides us with a decision procedure for the modal logic K and its extensions by D, T, B, 4, and 5.
- There are a number of implementations of first-order resolution, for example Bliksem, SPASS, and Vampire.

Modal logic: Summary

- Syntax
- Semantics: Kripke frames, Kripke models, labelled directed graphs, truth of a modal formula
- Model checking: Simple model checking procedure, some shortcuts
- Satisfiability and validity of modal formulae
- Classes of modal formulae that are valid in the class of all Kripke frames, logical omniscience problem
- Correspondence theory
- Proof theory: Hilbert axiomatisations, the logic of belief KD45, the logic of knowledge KT45, translation to first-order logic