COMP304: Knowledge Representation and Reasoning

Lecture 17: Probability Theory (1)

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Today

- Introduction to Probability Theory
- Some basic definitions
  - Sample spaces
  - Events
  - Combining events and computing their probability
  - Conditional probability

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Probability Theory - Introduction

- So far we have looked at logical aspects of KR&R; there are many different varieties of logic that contribute towards the adequate and efficient representation of particular forms of knowledge for computer manipulation.
- We now we turn our attention to a different aspect of KR&R, based on Probability and Decision Theory, to help represent and reason about certain things that logic cannot deal with.
- The purpose of this is to enable us to design and build agents that will operate in a wide range of complex environments.
- Suppose that we want a robot agent to perform an action, say picking up an object.
  - What happens if the environment is stochastic, so picking up the item is not reliable?
  - What happens if the environment is partially observable?

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Probability Theory - Introduction

- We have already considered approaches that distinguish between and represent 'possible' and 'necessary' truths.
- Another approach is to use probability theory.
- Thus, we can say things like: 90% of the time a *Pickup* will leave the item in the robot's hand, 10% of the time it will leave the item on the table.
- We can then perform computations to find out how likely a plan is to achieve a given goal.
- But this is only part of what we need to do: an agent also needs to figure out what to do.

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Probability Theory - Introduction

- We want to build agents which "do the right thing", which clearly this needs a notion of "rightness".
- One way to deal with "rightness" is to define it in terms of what is in an agent's best interests.
- Given a set of actions that can be performed, we can look at the situations which result from the actions and choose the one which is best.
- This requires a way of quantifying "best" from the perspective the agent.
- In addition, since the effects of actions are *uncertain*, we need to factor in a way of reasoning about how likely the situations which result from actions are.

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Some Basic Definitions

- **Definition 1:** *A sample space of an experiment is a set S of elements $E_1, E_2, \dots, E_k$ such that any outcome of the experiment corresponds to exactly one element in the set.*
- **Definition 2:** *Given an experiment with a sample space S, the elements $E_1, E_2, \dots, E_k$ of S are called sample points.*
- **Definition 3:** *An event is a subset of the sample space S.*
  We say that the event *E* has occurred if the outcome of the experiment corresponds to an element in the set *E*.
- **Definition 4**: *The sample space $S = \{E_1, E_2, \dots, E_k\}$ and the probabilities $\Pr(E_i)$, $i = 1, \dots, k$ determine the probability model for a random experiment.*

  Definition 5: *The probability of an event E is the sum of the probabilities of the simple events which constitute E.*

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Example - Sample Spaces

- Consider tossing a coin - there are two possible outcomes, a head (H) and a tail (T).
- The sample space for the experiment is $S = \{H, T\}$
- In a more complex experiment, such as tossing a dime and a nickel, there are a number of ways of defining the sample space.
- We could count the total number of heads giving: $S = \{0, 1, 2\}$
- Alternatively we could distinguish exactly what the outcome was, for instance using: $S = \{H_n H_d, H_n T_d, T_n H_d, T_n T_d\}$

  where H_n indicates that the nickel came up heads.
- The second sample space (but not the first) allows us, for instance, to identify whether a particular coin came up heads.

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Events

- But what if we want to talk about situations in which "at least one head appears"?
- Considering the nickel and dime example, there are particular outcomes which might be interesting.
- For instance:
  - U: Exactly one head appears.
  - V: Exactly two heads appear.
  - W: At least one head appears.
  - X: A head appears on the dime.
  - Y: A head appears on the nickel.
  - Z: No head appears.
- These outcomes are "events".

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Events

- Each event corresponds to a set of sample points:
  - $U = \{H_n T_d, T_n H_d\}$
  - $V = \{H_n H_d\}$
  - $W = \{H_n H_d, H_n T_d, T_n H_d\}$
  - $X = \{H_n H_d, T_n H_d\}$
  - $Y = \{H_n H_d, H_n T_d\}$
  - $Z = \{T_n T_d\}$
- It is clear that V and Z are different from the rest since they contain a single sample point; we call events which contain a single sample point *simple events*.
- Events made up of more than one sample point will be called *composite events*.
  - Clearly composite events can always be decomposed into simple events.
- Because experiments are random, listing the outcomes does not fully describe the experiment - we need probabilities as well.

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Probabilities

- Given a sample space: $S = \{E_1, E_2, \ldots, E_k\}$ we want to determine: $\Pr(E_i)$ for $i = 1, \ldots, k$
- There are three ways in which this might be done:

  (1) Assume that every outcome is equally likely: $\Pr(E_i) = \frac{1}{k}$ for $i = 1, \ldots, k$. This can be justified when nothing is known about the likelihood of the various outcomes.

  (2) Use observed relative frequencies obtained from a series of experiments: $\Pr(E_i)$ = *relative frequency of the event* E_i

  (3) Assign the probabilities based on our belief about the likelihood of the events.
- In all cases we assign probabilities so that:

$$0 \le \Pr(E_i) \le 1$$

$$\sum_{i=1}^{k} \Pr(E_i) = 1$$

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Tautologies and Contradictions

- Since all possible outcomes are enumerated in the sample space

$$\Pr(S) = 1$$

  This might be translated as "some event in S must occur".
- The event S is sometimes known as the *tautology* - the event which is always true - and is written as T:

$$\Pr(\mathrm{T}) = 1$$

- If an event is not a possible outcome of the experiment, then there are no corresponding sample points in S. The probability of such an event is zero.
- This event is sometimes known as the *contradiction*, and is written as $\perp$. It may also be written as $\emptyset$.

$$\Pr(\perp) = \Pr(\emptyset) = 0$$

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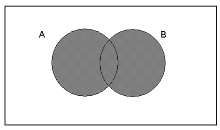
Combining Events

- As we have defined them, events are sets of outcomes. We thus combine events set-theoretically.
- In the following, we will consider events A and B in some sample space S.
- The event "A or B" is the union of A and B. Thus:

$$A \text{ or } B = A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

Depicted as a Venn Diagram:



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Combining Events

- The event "A and B" is the intersection of A and B. Thus:
  A and $B = A \cap B$
  $= \{x \mid x \in A \text{ and } x \in B\}$
- We can also write A,B or AB in place of $A \cap B$.
- The event "not A", also written as $\neg A$ or $\bar{A}$, is the set of all events in S which are not in A:
  $\neg A = \{x \in S \mid x \notin A\}$
- The difference of A and B, also called the relative complement of B with respect to A, is the set of points in A which belong to A and not B.
  $A - B = A \cap \neg B$
  $= \{x \mid x \in A \text{ and } x \notin B\}$

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Partitions

- Clearly the events A and $\neg A$ cannot occur at the same time; we say such sets of events are *mutually exclusive*.
- If A and B are mutually exclusive, they have no points in common, so:
  $A \cap B = \varnothing$
- The set of events $A_1, \ldots, A_n$ are mutually exclusive if:
  $A_i \cap A_j = \varnothing$

  for all i, j such that $i \neq j$.
- The set of events $A_1, \ldots, A_n$ is said to be exhaustive if:
  $A_1 \cup \ldots \cup A_n = S$
- A set of mutually exclusive and exhaustive events forms a *partition* of S.
- The simple events which describe a sample space are always a partition.

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Partitions

- For any event A, A and $\neg A$ form a partition of S since they are mutually exclusive and:

  $$A \cup \neg A = S$$
- If $A_1, \ldots, A_n$ form a partition of S, then any event F can be written as:

  $$F = (A_1 \cap F) \cup (A_2 \cap F) \cup \ldots \cup (A_n \cap F)$$
- This allows us to decompose F into the union of a set of mutually exclusive events.
- We can now consider how to establish the probabilities of combined events.

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Probabilities of Combined Events

- In our nickel and dime example we could establish the probabilities of the sample events, and so we could calculate the probability of any composite event.
- This is not always possible, so we will develop methods for dealing with composite events without having to know the probabilities of simple events.
- We start from the following axioms:

  $$0 \leq \Pr(A) \leq 1 \quad (1)$$
  $$\Pr(S) = 1 \quad (2)$$
  $$\Pr(A_1 \cup A_2) = \Pr(A_1) + \Pr(A_2) \quad (3)$$

  where A_1 and A_2 are mutually exclusive.

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Addition Law

- We can now show the following (called the addition law):
  Theorem 1: *If A and B are any two events in S, then the probability of the event "A or B" is given by:*

  $$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$
- We subtract the intersection of A and B to ensure that events in the intersection are not counted twice, when A and B are not mutually exclusive.
- In the nickel and dime example, consider the event of a head on the nickel or the dime, that is $X \cup Y$.
- Applying the addition law we find the probability of this event is:

  $$\Pr(X \cup Y) = \Pr(X) + \Pr(Y) - \Pr(X \cap Y)$$
  $$= \Pr(X) + \Pr(Y) - \Pr(V)$$

  where the probabilities of X, Y and V can be established, using axiom (3), from the probabilities of the original sample points.

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Probabilities of Combined Events

- Often we need to deal with more than two events, and for such situations we have:
- **Theorem 2:** *If $A_1, \ldots, A_n$ are a set of mutually exclusive events in S, the probability of their union is:*

  $$\Pr(A_1 \cup \ldots \cup A_n) = \sum_{i=1}^{n} \Pr(A_i)$$
- Of course, if $A_1, \ldots, A_n$ forms a partition of S, then:

  $$A_1 \cup \ldots \cup A_n = S$$

  and

  $$\Pr(A_1 \cup \ldots \cup A_n) = 1$$
- We also have:
  Theorem 3: *If $\neg A$ is the complement of the event A in S, then*

  $$\Pr(\neg A) = 1 - \Pr(A)$$

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Probabilities of Combined Events

- Another useful result is the extension of the addition law to more than two events which are not mutually exclusive.
- In this case we have:
  Theorem 4: *If A, B and C are any three events in S, then the probability of their union is:*

  $\Pr(A \cup B \cup C) = \Pr(A) + \Pr(B) + \Pr(C)$
  $- \Pr(A \cap B)$
  $- \Pr(A \cap C)$
  $- \Pr(B \cap C)$
  $+ \Pr(A \cap B \cap C)$

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Conditional Probability

- Frequently we are interested in just part of the sample space.
  - e.g. given a sample space of "English children" we may be interested in the sub-population with blue eyes.
- Conditional probability gives us a means of handling this kind of problem.
- Consider a family is chosen at random from a set of families having two children (but not having twins).
- What is the probability that both children are boys?
- A suitable sample space is:

  $S = \{(B, B), (G, B), (B, G), (G, G)\}$

- It is reasonable to assume that each of the sample points is equally likely, so that:

  Pr(*Both children are boys*) = ¼

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Conditional Probability

- Now you learn that the families were selected from those who have one child at a boys' school, how does this affect the probability?
- The new sample space (denoted by S^*) is:

  $S^* = \{(B, B), (G, B), (B, G)\}$

  and we are now looking for:

  Pr(*Two boys* | *At least one boy*)

  where the vertical line is read "given that".
- How do we assign probabilities to the events in S^*?
- The answer comes from considering how S^* relates to S.

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Normalisation

- S^* is a subset of S, so every sample point in S^* is a sample point in S and therefore has a probability we can determine.
- However, if we sum these probabilities they will sum to less than 1 (because the sum of the probabilities of all the sample points in S is 1) in violation of axiom (2) given earlier.
- We therefore *normalise* by dividing the probability of the sample point, calculated from S, by the sum of the probabilities of all the sample points in S^* (which is the probability of the event S^* in the sample space S):

  $$\Pr(\textit{Two boys} \mid \textit{At least one boy}) = \frac{1/4}{3/4}$$
  $$= 1/3$$

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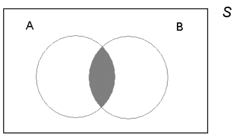
Conditioning

- We can generalise this. Consider a sample space S with two events A and B with a non-empty intersection $A \cap B$.
- We can picture this as:



- Now, consider we want to *condition on* event B, so that we are interested in discovering $\Pr(A \mid B)$, the probability that A will occur given that B is known to occur.

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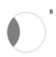
Conditioning

- The situation in which we are interested is thus the situation in which event B has occurred, a situation we can picture as:



- By comparing the two diagrams, it is clear that the sample points in A after the conditioning on B are exactly those which were in $\Pr(A \cap B)$ before the conditioning.
- Once again though we have to normalise, dividing by the probability of all the sample points in the new sample space:

  $$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

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Conditional Probability

- **Definition 6:** *The* conditional probability *of A given B, is denoted by* Pr $(A \mid B)$ *and is defined by:*

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

for $\Pr(B) \neq 0$.

- We can re-write this as:

$$\Pr(A \cap B) = \Pr(A \mid B) \Pr(B)$$

- Note: we can also write this the other way round: $\Pr(A \cap B) = \Pr(B \mid A) \Pr(A)$
- This is known as the *multiplication rule* or *product law*, and is useful is establishing joint probabilities.
- The rule can easily be extended for more events, thus for three events *A*, *B* and *C* we have:

$$\Pr(A \cap B \cap C) = \Pr(C \mid A \cap B) \Pr(B \mid A) \Pr(A)$$

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Multiplication Rule

- Consider choosing one family from a set of families with just one pair of twins (and thus no other children).
- What is the probability that both twins are boys?
- We can think of the choice of twins as occurring in two stages.
  - First we select whether the twins are identical, *I*, or fraternal, *Fr*. (Note that about one third of human twins are identical.)
  - Then we select the sex of the twins (*MM*, *MF*, *FM*, *FF*).
- The probabilities of the sexes of the fraternal twins will be the same as for any other two-child family. For the identical twins, the outcomes *MF* and *FM* are no longer possible.
- Essentially we are conditioning on the event "same sex" in families with two children.

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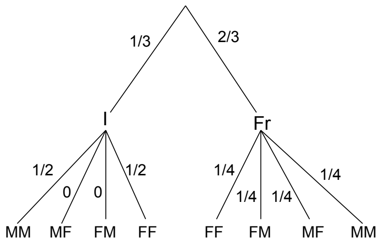
Tree Representation

- We can represent this two-stage experiment as a tree:



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Joint Probabilities

- Since the probabilities for the second stage are conditional, we can write:
- $\Pr(\textit{Twin boys}) = \Pr(I \cap MM) + \Pr(Fr \cap MM)$

$$= \Pr(MM \mid I) \Pr(I) + \Pr(MM \mid Fr) \Pr(Fr)$$

$$= 1/2 \cdot 1/3 + 1/4 \cdot 2/3$$

$$= 1/3$$

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Conditioning

- This example also illustrates the fact that it is often easier to obtain information in the form of conditional probabilities such as:

$$\Pr(MM \mid I)$$

than joint probabilities such as:

$$\Pr(MM \cap I)$$

- There is no real explanation for why this should be the case, but it is one of the reasons behind the use of Bayesian networks (which we will see later).

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Summary

- Motivation for using Probability Theory in KR&R
- Sample spaces
- Events
- Combining events
- Probabilities of combined events
- Addition law
- Conditional probabilities
- Product law

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