COMP210: Artificial Intelligence

Lecture 21. Predicate logic

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Pros and cons of propositional logic

- Propositional logic is *declarative*: pieces of syntax correspond to facts
- Propositional logic allows partial/disjunctive/negated information (unlike most data structures and databases)
- Propositional logic is *compositional*: meaning of $B_{1,1} \wedge P_{1,2}$ is derived from meaning of $B_{1,1}$ and of $P_{1,2}$
- Meaning in propositional logic is *context-independent* (unlike natural language, where meaning depends on context)
- Propositional logic has very limited expressive power (unlike natural language)

Example

- Consider

$$\frac{\frac{\text{Katy is a cat}}{\text{cats are mammals}}}{\text{Katy is a mammal}}$$

- In propositional logic this would be represented as

$$\frac{c, m}{k}$$

  - This derivation is not valid in propositional logic. If it were then from any c and m could derive any k. We need to capture the connection between c and m.
- We will use *first-order* or *predicate logic*.

First-order Logic

Whereas propositional logic assumes world contains *facts*, first-order logic (like natural language) assumes the world contains

- Objects: people, houses, numbers, theories, Ronald McDonald, colours, baseball games, wars, centuries ...
- Relations: red, round, bogus, prime, multistoried ..., brother of, bigger than, inside, part of, has colour, occurred after, owns, comes between, ...
- Functions: father of, best friend, third inning of, one more than, end of ...

Examples

$$course\_lecturer(Trevor, COMP210)$$
$$male(Trevor)$$
$$<(3, 4)$$
$$<(4, plustwo(1))$$
$$mammal(Katy)$$

- $Trevor$, $Katy$, $COMP210$, 3, 4 and 1 are *constants*.
- $course\_lecturer$, $male$, $mammal$, and $<$ are *predicates*.
- $male$, $mammal$ have *arity* one and the other predicates have arity two.
- $plustwo$ and are functions (that refer to other objects). For example $plustwo(1)$ refers to the constant 3

More Examples

- The mathematical relations $<, >, \leq, \geq, =, \neq$ are predicates.
- Note we would normally write the above as $3 < 4$.

Interpretations

- We need a domain to which we are referring. $course\_lecturer(Trevor, COMP210)$
- The name $Trevor$ is mapped to the object in the domain we are referring to (me).
- The name $COMP210$ is mapped to the object in the domain we are referring to (the course COMP210).
- The predicate name $course\_lecturer$ will be mapped to a set of pairs of objects where the first in the pair is the (real) person who teaches the second in the pair.
- Hence the above evaluates to true.

Interpretations

- Assuming $<$, 3, 4, 10 are mapped to what we would expect then $<(3, 4)$ evaluates to true whereas $<(10, 4)$ evaluates to false.
- Similarly with functions.

Terms, Sentences, Connectives

- *Terms* are expressions referring to objects. *Constants, variables* and *functions* are all terms.
- Thus $plustwo(1)$, $Trevor$ and 3 are all terms.

Atomic Sentences

- *Atomic sentences* are similar to propositions in propositional logic.
- *Atomic sentences* are predicates applied to a list of terms (in brackets).
  Thus
  - $male(father\_of(Trevor))$,
  - $cat(Katy)$,
  - $shares\_office(Paul, Tim)$
  - and $< (3, 4)$ are all atomic sentences.

Complex Sentences

- As in propositional logics we can combine atomic sentences using the connectives of propositional logic ($\wedge$, $\vee$, $\Rightarrow$, $\neg$, $\iff$).
  - $male(Trevor) \Rightarrow \neg female(Trevor)$
  - $male(Trevor) \wedge course\_lecturer(Trevor, COMP210)$

Quantifiers

- Quantifiers allow us to express properties about collections of objects.
- The quantifiers are

$\forall$	universal quantifier	*'For all . . . '*
$\exists$	existential quantifier	*'There exists . . . '*

- If $P(x)$ is a predicate then we can write
  - $\forall x \cdot P(x)$; and
  - $\exists x \cdot P(x)$;

  where x is a *variable* which can stand for any object in the domain.

Universal Quantification

- Note that universal quantification is similar to conjunction.
- Suppose the domain is the numbers $\{2, 4, 6\}$. Then

$$\forall n \cdot Even(n)$$

is the same as

$$Even(2) \wedge Even(4) \wedge Even(6).$$

Universal Quantification

- For all, x if x is a cat then x is a mammal. (or all cats are mammals).

$$\forall x \cdot cat(x) \Rightarrow mammal(x)$$

- For all x, if x is female then x is not male. (or all females are not males).

$$\forall x \cdot female(x) \Rightarrow \neg male(x)$$

- All blue cars are fast.

$$\forall x \cdot (blue(x) \wedge car(x)) \Rightarrow fast(x)$$

Universal Quantification and $\Rightarrow$

Typically, $\Rightarrow$ is the main connective with $\forall$

- If we have domain $\{Katy, Horace\}$ where $Katy$ is a cat (and a mammal) and $Horace$ is a lizard (not a mammal). $\forall x \cdot cat(x) \Rightarrow mammal(x)$ really means

$$(cat(Katy) \Rightarrow mammal(Katy)) \wedge$$
$$(cat(Horace) \Rightarrow mammal(Horace))$$

- This evaluates to true as both $cat(Katy)$ and $mammal(Katy)$ are true, also $cat(Horace)$ is false so the second implication is true.

A Common Mistake to Avoid

Common mistake: using $\wedge$ as the main connective with $\forall$:

$$\forall x \cdot At(x, Berkeley) \wedge Smart(x)$$

means "Everyone is at Berkeley and everyone is smart"

Existential Quantification

- Existential quantification is the same as *disjunction*. Thus with the same domain,

$$\exists n \cdot Even(n)$$

is the same as

$$Even(2) \vee Even(4) \vee Even(6).$$

Existential Quantification

Existential quantification allows us to make a statement about *some* object without naming it.

- There exists an x such that x is a man and x is a father (some men are fathers).

$$\exists x \cdot man(x) \wedge father(x)$$

- Some cats are white and have three legs.

$$\exists y \cdot cat(y) \wedge white(y) \wedge three\_legs(y)$$

Existential Quantification and $\wedge$

Typically, $\wedge$ is the main connective with $\exists$.

- If we have domain $\{Sam, Jim, Suzy\}$ where Sam and Jim are both men, Sam is a father but Jim is not and $Suzy$ is neither a man or a father. $\exists x \cdot man(x) \wedge father(x)$ really means

$$(man(Sam) \wedge father(Sam)) \vee$$
$$(man(Jim) \wedge father(Jim)) \vee$$
$$(man(Suzy) \wedge father(Suzy))$$

- This evaluates to true as both $man(Sam)$ and $father(Sam)$ are true. Using implication would be too weak.

Another Common Mistake to Avoid

Common mistake: using $\Rightarrow$ as the main connective with $\exists$:

$$\exists x \cdot At(x, Stanford) \Rightarrow Smart(x)$$

is true if there is anyone who is not at Stanford!

Examples

- $\forall x \cdot Man(x) \Rightarrow Mortal(x)$
  'For all x, if x is a man, then x is mortal.'
  (i.e. all men are mortal)
- $\forall x \cdot Man(x) \Rightarrow \exists y \cdot Woman(y) \wedge MotherOf(x, y)$
  'For all x, if x is a man, then there exists a y such that y is a woman and the mother of x is y.'
  (i.e., every man has a mother).
- $\exists m \cdot Monitor(m) \wedge MonitorState(m, ready)$
  'There exists a monitor that is in a ready state.'
- $\forall r \cdot Reactor(r) \Rightarrow \exists t \cdot (100 \leq t \leq 1000) \wedge temp(r) = t$
  'Every reactor will have a temperature in the range 100 to 1000.'

More Examples

- $\exists n \cdot posInt(n) \wedge n = (n * n)$
  'Some positive integer is equal to its own square.'
- $\exists c \cdot ECCountry(c) \wedge Borders(c, Albania)$
  'Some EC country borders Albania.'
- $\forall m, n \cdot Person(m) \wedge Person(n) \Rightarrow \neg Superior(m, n)$
  'No person is superior to another.'
- $\forall m \cdot Person(m) \Rightarrow \neg \exists n \cdot Person(n) \wedge Superior(m, n)$
  Ditto.

More Than One Quantifier

- For all x and y if x is the parent of y then y is the child of x.

$$\forall x\, \forall y \cdot parent(x, y) \Rightarrow child(y, x) \text{ or}$$
$$\forall x, y \cdot parent(x, y) \Rightarrow child(y, x)$$

The Order of Quantifiers is Important!

- Everyone ate something.

$$\forall x\, \exists y \cdot ate(x, y)$$

- There is something that was eaten by everyone.

$$\exists y\, \forall x \cdot ate(x, y)$$

- Everything was eaten by someone.

$$\forall y\, \exists x \cdot ate(x, y)$$

- Someone ate everything.

$$\exists x\, \forall y \cdot ate(x, y)$$

Syntax of Predicate Logic

The formulae of predicate logic are constructed from the following symbols.

- a set PRED of predicate symbols with arity;
- a set FUNC of function symbols with arity;
- a set CONS of constant symbols;
- a set VAR of variable symbols;
- the quantifiers $\forall$ and $\exists$;
- **true**, **false** and the connectives $\wedge$, $\vee$, $\Rightarrow$, $\neg$ $\iff$.

Note propositions can be viewed as predicates with arity 0.

Terms

The set of terms, TERM, is constructed by the following rules

- any constant is in TERM;
- any variable is in TERM;
- if $t_1, \ldots t_n$ are in TERM and f is a function symbol of arity n then $f(t_1, \ldots t_n)$ is a term.
  - `f(x,y)`
  - `add(2,4)`
  - `father_of(Trevor)`

Well-formed Formulae

The set of well-formed formulae of predicate logic are

- **true**, **false** and propositional formulae are in WFF.
- if $t_1, \ldots t_n$ are in TERM and p is a predicate symbol of arity n then $p(t_1, \ldots t_n)$ is in WFF.
- If A and B are in WFF so are $\neg A$, $A \wedge B$, $A \vee B$, $A \Rightarrow B$, $A \iff B$.
- If A is in WFF and x is a variable then $\forall x \cdot A$ and $\exists x \cdot A$ are in WFF.
- If A is in WFF so is (A).

Well-Formed Formulae Example

- The following are well-formed formulae of predicate logic.

$$\forall x \cdot p(x) \wedge \exists y \cdot r(y) \qquad \exists x \cdot q$$

- whereas the following are not

$$\forall \exists \cdot p(x) \qquad \exists x \cdot p(\neg x)$$

Binding

- A variable in the scope of a quantifier is said to be *bound*.
- A variable not in the scope of a quantifier is said to be *free*
  - Example: which are the free and bound variables in the following?

$$p(x) \wedge q(x)$$
$$\forall x \cdot (p(x) \Rightarrow q(x))$$
$$p(x) \wedge \exists x \cdot q(x)$$

Equivalences Between Quantifiers I

The universal and existential quantifiers are in fact *duals* of each other:

- *Saying that everything has some property is the same as saying that there is nothing that does not have the property.*

$$\forall x \cdot P(x) \equiv \neg \exists x \cdot \neg P(x)$$

  - *everyone doesn't like sprouts*

$$\forall x \cdot \neg likes\_sprouts(x)$$

  is the same as saying *it's not the case that someone likes sprouts.*

$$\neg \exists x \cdot likes\_sprouts(x)$$

Equivalences Between Quantifiers II

- *Saying that there is something that has the property is the same as saying that its not the case that everything doesn't have the property.*

$$\exists x \cdot P(x) \equiv \neg \forall x \cdot \neg P(x)$$

- Also

$$\forall x \cdot \neg P(x) \equiv \neg \exists x \cdot P(x)$$
$$\exists x \cdot \neg P(x) \equiv \neg \forall x \cdot P(x)$$

Domains & Interpretations

- Suppose we have a formula $\forall x \cdot P(x)$. What does x *range over*? Physical objects, numbers, people, times, ...?
- Depends on the *domain* that we intend. Often, we *name* a domain to make our intended interpretation clear.
  - Suppose our intended interpretation is the positive integers. Suppose $>, +, \times, \ldots$ have the usual mathematical interpretation.
  - Is this formula *satisfiable* under this interpretation? $\exists n \cdot n = (n \times n)$
  - Now suppose that our domain is negative integers (where $\times$ has the usual mathematical interpretation).
  - Is the formula satisfiable under this interpretation?

Semantics of Predicate Logic

- We haven't given the formal semantics of predicate logic.
- You should look at the formal semantics in a logic or AI book.
- Informally we've seen we need a domain of interest.
- Constants, predicates, functions have mappings into this domain.
- To evaluate quantifiers we must check whether all objects in the domain satisfy the formula ($\forall$) or some object does ($\exists$).
- Given the above, connectives behave in a similar way to propositional logic.

Decidability of Propositional Logic

- In propositional logic, we saw that some formulae were tautologies—they had the property of being true under all interpretations.
- We also saw that there was a procedure which could be used to tell whether any formula was a tautology — this procedure was the truth-table method.
  - A formula of FOL that is true under all interpretations is said to be *valid*.
  - So we could try to check for validity by writing down all the possible interpretations and looking to see whether the formula is true or not.

First-Order Example

- Unfortunately in general we can't use this method.
- Consider the formula:

$$\forall n \cdot Even(n) \Rightarrow \neg Odd(n)$$

  and the domain Natural Numbers, i.e. $\{1, 2, 3, 4, \ldots\}$
- There are an infinite number of interpretations.
- Is there any other procedure that we can use, that will be guaranteed to tell us, in a finite amount of time, whether a FOL formula is, or is not, valid?

Is FO Logic Decidable?

- The answer is *no*.
- FOL is for this reason said to be *undecidable*. FOL is often called *semi-decidable*, as given a formula that is not valid the procedure may not terminate.

Proof in FOL

- Proof in FOL is similar to that in propositional logic; we just need an extra set of rules, to deal with the quantifiers.
- FOL *inherits* all the rules of propositional logic.
- The most obvious rule, $\forall$-elimination.
  Tells us that if everything in the domain has some property, then we can infer that any *particular* individual has the property,

$$\frac{\forall x \cdot \phi(x)}{\phi(a)}$$

  for any a in the domain.
  Going from *general* to *specific*.

Example

Let's use $\forall$-Elimination to consider the cat/mammal example.

$$cat(Katy), \forall x \cdot cat(x) \Rightarrow mammal(x) \vdash mammal(Katy)$$

1.	$cat(Katy)$	[Given]
2.	$\forall x \cdot cat(x) \Rightarrow mammal(x)$	[Given]
3.	$cat(Katy) \Rightarrow mammal(Katy)$	[2, $\forall$-Elimination]
4.	$mammal(Katy)$	[1, 3, MP]

Other Rules

- There are other proof rules related to quantifiers.
- Consult a good logic or AI book to see what they are.

Summary

- We've given the formal syntax of predicate logic and informally considered the semantics.
- We've considered decidability for predicate logic.
- We've given an example of a proof rule for predicate logic.