On Natural non-dcpo Domains

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Dedicated to my teacher, *Boris Abramovich Trakhtenbrot*,
in his 87th year whose influence on me and help
cannot be overstated

Abstract. As Dag Normann has recently shown, the fully abstract model for **PCF** of hereditarily sequential functionals is not ω-complete and therefore not continuous in the traditional terminology (in contrast to the old fully abstract continuous dcpo model of Milner). This is also applicable to a wider class of models such as the recently constructed by the author fully abstract (universal) model for $\mathbf{PCF}^+ = \mathbf{PCF} +$ parallel **if**. Here we will present an outline of a general approach to this kind of "natural" domains which, although being non-dcpos, allow considering "naturally" continuous functions (with respect to existing directed "pointwise", or "natural" least upper bounds) and also have appropriate version of "naturally" algebraic and "naturally" bounded complete "natural" domains. This is the non-dcpo analogue of the well-known concept of Scott domains, or equivalently, the complete f-spaces of Ershov. In fact, the latter version of natural domains, if considered under "natural" Scott topology, exactly corresponds to the class of f-spaces, not necessarily complete.

1 Introduction

The goal of this paper is to present a first brief outline of the so-called "natural" version of domain theory in the general setting, where domains are not necessary directed complete partial orders (dcpos). As Dag Normann has recently shown [6], the fully abstract model of hereditarily-sequential finite type functionals for **PCF** [1, 3, 5, 10][1] is not ω-complete (hence non-dcpo) and therefore not continuous in the traditional terminology. This is also applicable to a potentially wider class of models such as the fully abstract model of (hereditarily) wittingly consistent functionals for $\mathbf{PCF}^+$ (i.e. $\mathbf{PCF} +$ parallel **if**) [10]. Note that until the above mentioned negative result in [6] and further positive results in [10] the domain theoretical structure of such models was essentially unknown. The point of using the term "natural" for these kinds of domains is

[1] As to the language **PCF** for sequential finite type functionals see [4, 7, 9, 11]. Note also that the technical part of [10] — the source of considerations of the present paper — is heavily based on [8, 9].

that in the case of non-dcpos, the ordinary definitions of continuity and finite (algebraic) elements via arbitrary directed least upper bounds (lubs) prove to be inappropriate. A new, restricted concept of "natural" lub is necessary, and it leads to a generalized theory applicable also to non-dcpos. More informally, if some directed least upper bounds do not exist in a partial ordered set D then this can serve as an indication that even some existing least upper bounds can be considered as "unnatural" in a sense. Although "natural" lubs for functional domains can also be characterised technically as "pointwise" (in the well-known sense), using the latter term for the concepts of continuous functions or finite elements as defined in terms of pointwise lubs is, in fact, somewhat misleading. The term "pointwise continuous" is in this sense awkward and of course not intended to be considered as "continuous for each argument value", but rather as "continuous with respect to the pointwise lubs" which is lengthy. Thus, the more neutral and not so technical term "natural" is used instead of "pointwise". Moreover, for general non-functional non-dcpo domains the term "pointwise" does not seem to have the straightforward sense. However we should also note the terminological peculiarity of the term "natural". For example, the existence of "naturally finite but not finite" elements in such "natural" domains is quite possible (see Hypotheses 2.8 in [10] concerning sequential functionals). Although the main idea of the current approach has already appeared in [10], it was applied there only in a special situation of typed non-dcpo models with "natural" understood as (hereditarily) "pointwise". Here our goal is to make the first steps towards a general non-dcpo domain theory of this kind.

2 Natural Domains

A non-empty partially ordered set (poset) $\langle I, \leq \rangle$ is called *directed* if for all $i, j \in I$ there is a $k \in I$ such that $i, j \leq k$. By saying that a (non-empty) family of elements x_i in a poset $\langle D, \sqsubseteq \rangle$ is *directed*, we mean that I, the range of i, is a directed poset, and, moreover, the map $\lambda i.x_i : I \to D$ is *monotonic* in i, that is, $i \leq j \Rightarrow x_i \sqsubseteq x_j$. However in general, if it is not said explicitly or does not follow from the context, x_i may denote a not necessarily directed family. Moreover, we will usually omit mentioning the range I of i, relying on the context. Different subscript parameters i and j may range, in general, over different index sets I and J. As usual $\bigsqcup X$ denotes the ordinary least upper bound (lub) of a subset $X \subseteq D$ in a poset D which may exist or not. That is, this is a partial map $\bigsqcup : 2^D \dot{\to} D$ with 2^D denoting the powerset of D. If D has a least element, it is denoted as $\bot_D$ or $\bot$ and called *undefined*.

Definition 1.

(a) Any poset $\langle D, \sqsubseteq^D \rangle$ (not necessarily a dcpo) is also called a *domain*.
(b) Recall that a *directly complete partial order* (or dcpo domain) is required to be closed under taking directed least upper bounds $\bigsqcup x_i$.[2] (We omit the usual requirement that a dcpo should contain a least element $\bot$.)

[2] In general, by $\bigsqcup_i z_i$ we mean $\bigsqcup \{z_i \mid i \in I\}$, and analogously for $\biguplus$ below.

(c) A *natural pre-domain* is a domain D (in general non-dcpo) with a partially defined operator of *natural lub* $\biguplus : 2^D \dot{\rightarrow} D$ satisfying the first of the following *four* conditions. It is called a *natural domain* if all these conditions hold:

($\biguplus$1) $\biguplus \subseteq \bigsqcup$. That is, for all sets $X \subseteq D$, if $\biguplus X$ exists (i.e. X is in the domain of $\biguplus$) then $\bigsqcup X$ exists too and $\biguplus X = \bigsqcup X$.

($\biguplus$2) If $X \subseteq Y \subseteq D$, $\biguplus X$ exists, and Y is upper bounded by $\biguplus X$ then $\biguplus Y$ exists too (and is equal to $\biguplus X$).

($\biguplus$3) $\biguplus\{x\}$ exists (and is equal to x).

($\biguplus$4) Let $\{y_{ij}\}_{i\in I, j\in J}$ be an arbitrary non-empty family of elements in D indexed by I and J. Then

$$\biguplus_i \biguplus_j y_{ij} = (\biguplus_j \biguplus_i y_{ij} =) \biguplus_{ij} y_{ij} = \biguplus_i y_{ii}$$

provided that:

1. Assuming all the required internal natural lubs $\biguplus_j y_{ij}$ in $\biguplus_i \biguplus_j y_{ij}$ and one of the external natural lubs $\biguplus_i \biguplus_j y_{ij}$ or $\biguplus_{ij} y_{ij}$ exist, then both exist and the corresponding equality above holds. (The case of $\biguplus_j \biguplus_i y_{ij}$ is symmetrical.[3])
2. For the last equality to hold, the family y_{ij} is additionally required to be directed (and monotonic) in each parameter i and j ranging over the same I, and the existence of any natural lub in this equality implies the existence of the other.

The second part of ($\biguplus$4) (directed case) evidently follows also from ($\biguplus$1), ($\biguplus$2), and the following *optional clause* which might be postulated as well.

($\biguplus$5) If $X \subseteq Y \subseteq D$, $\biguplus Y$ exists, and X is cofinal with Y (i.e. $\forall y \in Y \exists x \in X.\ y \sqsubseteq x$) then $\biguplus X$ exists too (and $= \biguplus Y$).

But we will really use only ($\biguplus$1)–($\biguplus$4). Evidently, any pre-domain with unrestricted $\biguplus \rightleftharpoons \bigsqcup$ is a natural domain. As an extreme case any *discrete* D with $\sqsubseteq$ coinciding with $=$ and $\biguplus \rightleftharpoons \bigsqcup$ is a natural domain. But, as in the case of [10], it may happen that only under a restricted $\biguplus \sqsubseteq \bigsqcup$ a natural domain has some additional nice properties such as "natural" algebraicity properties discussed below in Sect. 3. Note that a natural domain is actually a second-order structure $\langle D, \sqsubseteq^D, \biguplus^D\rangle$ in contrast to the ordinary dcpo domains represented as a first-order poset $\langle D, \sqsubseteq^D\rangle$ structure.

Definition 2. *Direct product* of natural (pre-) domains $D \times E$ (or more generally, $\prod_{k\in K} D_k$) is defined by letting $\langle x, y\rangle \sqsubseteq^{D\times E} \langle x', y'\rangle$ iff $x \sqsubseteq^D x'$ & $y \sqsubseteq^E y'$, and additionally $\biguplus_i \langle x_i, y_i\rangle \rightleftharpoons \langle \biguplus_i x_i, \biguplus_i y_i\rangle$ for any family $\langle x_i, y_i\rangle$ of elements in $D \times E$ whenever each natural lub $\biguplus_i x_i$ and $\biguplus_i y_i$ exists.

[3] It follows that for the equality $\biguplus_i \biguplus_j y_{ij} = \biguplus_j \biguplus_i y_{ij}$ to hold it suffices to require that all the internal and either one of the external natural lubs or the mixed lub $\biguplus_{ij} y_{ij}$ exist.

Proposition 1. *The direct product of natural (pre-) domains is a natural (pre-) domain as well.* □

The poset of all monotonic maps $D \to E$ between any domains ordered pointwise ($f \sqsubseteq^{(D\to E)} f' \rightleftharpoons fx \sqsubseteq^E f'x$ for all $x \in D$) is denoted as $(D \to E)$. We will usually omit the superscripts to $\sqsubseteq$.

Definition 3.

(a) A monotonic map $f : D \to E$ between natural pre-domains is called *naturally continuous*[4] if $f(\biguplus_i x_i) = \biguplus_i f(x_i)$ for any directed natural lub $\biguplus_i x_i$, assuming it exists (that is, if $\biguplus_i x_i$ exists then $\biguplus_i f(x_i)$ is required to exist and satisfy this equality). The set of all (monotonic and) naturally continuous maps $D \to E$ is denoted as $[D \to E]$.

(b) Given an arbitrary family $f_i : D \to E$ of monotonic maps between natural pre-domains, define a natural lub $f = \biguplus_i f_i : D \to E$ pointwise, as

$$fx \rightleftharpoons \biguplus_i (f_i x),$$

assuming the latter natural lub exists for all x; otherwise $\biguplus_i f_i$ is undefined.

Proposition 2. *For the case of naturally continuous f_i the resulting f in (b) above is a naturally continuous map as well, assuming E is a natural domain.*

Proof. Use the first part of ($\biguplus$4): $f \biguplus_j x_j \rightleftharpoons \biguplus_i (f_i \biguplus_j x_j) = \biguplus_i \biguplus_j (f_i x_j) = \biguplus_j \biguplus_i (f_i x_j) \rightleftharpoons \biguplus_j f x_j$, for x_j directed and having a natural lub (with all other natural lubs evidently existing). □

Moreover, for any non-empty set F of monotonic functions $D \to E$ and a family $f_i \in F$, if the natural lub $\biguplus_i f_i$ exists and is also an element of F then it is denoted as $\biguplus_i^F f_i$; otherwise, $\biguplus_i^F f_i$, is considered as undefined. When defined, $\biguplus_i^F f_i = \bigsqcup_i^F f_i = \bigsqcup_i^{(D\to E)} f_i$. Here $\bigsqcup^F$ denotes the lub relativized to the poset F with the pointwise partial order $\sqsubseteq^{(D\to E)}$ restricted to F. Evidently, $F \subseteq F' \Longrightarrow \bigsqcup_i^{F'} f_i \sqsubseteq \bigsqcup_i^F f_i$ when both lubs exist. In contrast with $\bigsqcup^F$, the natural lub $\biguplus_i^F f_i = \biguplus_i f_i$ is essentially independent on F, except it is required to be in F. We will omit the superscript F when it is evident from the context. Further, it is easy to show (by pointwise considerations) that

[4] Using the adjective 'natural' here and in other definitions below is, in fact, rather annoying. We would be happy to avoid it at all, but we need to distinguish all these 'natural' non-dcpo versions of the ordinary definitions for dcpos relativized to the natural lub $\biguplus$ from similar definitions relativized to the ordinary lub $\bigsqcup$. In principle, if the context is clear, we could omit 'natural', and use this term as well as 'non-natural' only when necessary. Another way is to write '$\biguplus$-continuous' vs. '$\bigsqcup$-continuous', etc. to make the necessary distinctions.

Proposition 3. *For D and E natural pre-domains, any $F \subseteq (D \to E)$ is (trivially) a natural pre-domain under $\biguplus^F$ defined above. It is also a natural domain if E is, and, in particular, $(D \to E)$ and $[D \to E]$ are natural domains in this case with $[D \to E]$ closed under (existing, not necessarily directed) natural lubs in $(D \to E)$.*

Proof.

($\uplus$1) is trivial.

($\uplus$2) For a family of monotonic functions $\{f_j \in F\}_{j \in J}$ and $I \subseteq J$, assume that $\biguplus_{i \in I} f_i \in F$ and $f_j \sqsubseteq \biguplus_{i \in I} f_i$ for all $j \in J$. It follows that for all $j \in J$ and $x \in D$, $f_j x \sqsubseteq \biguplus_{i \in I}(f_i x)$. Therefore, by using ($\uplus$2) for E, $\biguplus_{j \in J}(f_j x)$ exists for all x in the natural domain E, and hence $\biguplus_{j \in J} f_j$ does exist too in $(D \to E)$ and therefore coincides with $\biguplus_{i \in I} f_i \in F$, as required.

($\uplus$3) For any f, $(\biguplus\{f\})x = \biguplus\{fx\} = fx$. Thus, $\biguplus\{f\} = f$, as required.

($\uplus$4) For arbitrary family of functions $f_{ij} \in F$ ($\uplus$4) reduces to the same in E for $y_{ij} = f_{ij}x$ with arbitrary $x \in D$.

1. Indeed, assume all the required internal natural lubs $\biguplus_j f_{ij}$ and one of the external natural lubs $\biguplus_i \biguplus_j f_{ij}$ or $\biguplus_{ij} f_{ij}$ exist and belong to F. Then for all $x \in D$ the corresponding asertion holds for $\biguplus_j f_{ij}x$ and $\biguplus_i \biguplus_j f_{ij}x$ or $\biguplus_{ij} f_{ij}x$, and therefore $\biguplus_i \biguplus_j f_{ij}x = \biguplus_{ij} f_{ij}x$ in E. This pointwise identity implies both existence of the required natural lubs in F and equality between them $\biguplus_i \biguplus_j f_{ij} = \biguplus_{ij} f_{ij}$.
2. For directed f_{ij}, $i, j \in I$, and one of the natural lubs $\biguplus_j f_{ij}$ or $\biguplus_j f_{ii}$ existing, we evidently have for all $x \in D$ that $f_{ij}x$ is directed in each parameter i and j, and $\biguplus_j f_{ij}x = \biguplus_j f_{ii}x$ holds in E, and therefore both the required lubs exist in F and the equality $\biguplus_j f_{ij} = \biguplus_j f_{ii}$ holds.

□

If natural domains D and E are dcpos with $\biguplus = \bigsqcup$ then the same holds both for $(D \to E)$ and $[D \to E]$, and the latter domain coincides with that of all (usual) continuous functions with respect to arbitrary directed lubs. This way natural domain theory generalizes that of dcpo domains, and we will see that other important concepts of domain theory over dcpos have natural counterparts in natural domains with all the ordinary considerations extending quite smoothly to the 'natural' non-dcpo case.

These considerations allow us to construct inductively some natural domains of finite type functionals by taking, for each type $\sigma = \alpha \to \beta$, an arbitrary subset $F_{\alpha\to\beta}$ of monotonic (or only naturally continuous) mappings $F_\alpha \to F_\beta$. Of course, we can additionally require that these F_σ are sufficiently closed (say, under λ-definability or sequential computability). This way, for example, the λ-model of hereditarily-sequential finite type functionals can be obtained. E.g. in [10] this was done inductively over level of types with an appropriate definition of sequentially computable functionals in $\mathbb{Q}_{\alpha_1,\ldots,\alpha_n\to\text{Basic-Type}} \subseteq (\mathbb{Q}_{\alpha_1},\ldots,\mathbb{Q}_{\alpha_n} \to \mathbb{Q}_{\text{Basic-Type}})$ (over the basic 'flat' domain $\mathbb{Q}_{\text{Basic-Type}} = \mathbf{N}_\perp$). It was proved only a posteriori and quite non-trivially that all sequential functionals are naturally continuous by embeddings: $\mathbb{Q}_{\alpha_1,\ldots,\alpha_n\to\text{Basic-Type}} \subseteq$

$[\mathbb{Q}_{\alpha_1}, \ldots, \mathbb{Q}_{\alpha_n} \to \mathbb{Q}_{\text{Basic-Type}}]$ and $\mathbb{Q}_{\alpha\to\beta} \hookrightarrow [\mathbb{Q}_\alpha \to \mathbb{Q}_\beta]$, and satisfy further "natural" algebraicity properties discussed in Sect. 3. It was while determining the domain theoretical nature of $\mathbb{Q}_\alpha$ that the idea of natural domains emerged; and, although it proved to be quite simple, it was unclear at that moment whether anything reasonable could be obtained. What is new here is a general, abstract presentation of natural domains that does not rely, as in [10], on a type structure like that of $\{\mathbb{Q}_\alpha\}$. Unfortunately, it would take too much space to consider here the construction of the λ-model $\{\mathbb{Q}_\alpha\}$ — the source of general considerations of this paper. (See also [1, 3, 5] where the same model was defined in a different way and where its domain theoretical structure was not described; it was even unknown whether it is different from the older dcpo model of Milner [4] which was shown later by Normann [6].)

Proposition 4. *Let D, E be natural pre-domains and F a natural domain. A two place monotonic function $f : D \times E \to F$ is naturally continuous iff it is so in each argument.*

Proof. "Only if" is trivial (and uses ($\biguplus$3) for F). Conversely, for arbitrary directed families x_i and y_i having natural lubs we have

$$f(\biguplus_i \langle x_i, y_i\rangle) \rightleftharpoons f(\langle \biguplus_i x_i, \biguplus_i y_i\rangle) = \biguplus_i \biguplus_j f(\langle x_i, y_j\rangle) = \biguplus_{ij} f(\langle x_i, y_j\rangle)$$
$$= \biguplus_i f(\langle x_i, y_i\rangle) \ ,$$

as required, by applying the natural continuity of f in each argument and using ($\biguplus$4) for F. □

Proposition 5. *There are the natural (in the sense of category theory) order isomorphisms over natural domains preserving additionally in both directions all the existing natural lubs, not necessarily directed*[5]*,*

$$(D \times E \to F) \cong (D \to (E \to F)) \ , \tag{1}$$
$$[D \times E \to F] \cong [D \to [E \to F]]. \tag{2}$$

This makes the class of natural domains with monotonic, resp., naturally continuous morphisms a Cartesian closed category (ccc) in two ways. Moreover, each side of the second isomorphism is a subset of the corresponding side of the first, with embedding making the square diagram commutative.

Proof. Indeed, the isomorphism (1) and its inverse are defined for any $f \in (D \times E \to F)$ and $g \in (D \to (E \to F))$, as usual, by

$$f^* \rightleftharpoons \lambda x.\lambda y.f(x, y) \in (D \to (E \to F)) \ ,$$
$$\hat{g} \rightleftharpoons \lambda(x, y).gxy \in (D \times E \to F) \ .$$

[5] and, of course, preserving the ordinary lubs

Then $\lambda f.f^*$ preserves (in both directions) all the existing natural lubs $(\biguplus_i f_i)^* = \biguplus_i f_i^*$:

$$(\biguplus_i f_i)^* xy \rightleftharpoons (\biguplus_i f_i)(x,y) \rightleftharpoons \biguplus_i f_i(x,y) \rightleftharpoons \biguplus_i ((f_i^* x)y) \rightleftharpoons (\biguplus_i (f_i^* x))y$$
$$\rightleftharpoons ((\biguplus_i f_i^*)x)y \rightleftharpoons (\biguplus_i f_i^*)xy$$

holds for all $x \in D$ and $y \in E$ where if the first natural lub exists then all the others exist too, and conversely. Here we used only the definitions of * and $\biguplus$ for functions. The second isomorphism (2) is just the restriction of the first. For its correctness we should check that f^* (resp. $\hat{g}$) is naturally continuous if f (resp. g) is:

$$f^* \biguplus_i x_i \rightleftharpoons \lambda y.f(\biguplus_i x_i, y) = \lambda y. \biguplus_i f(x_i, y) \rightleftharpoons \biguplus_i \lambda y.f(x_i, y) \rightleftharpoons \biguplus_i f^* x_i$$

by using additionally Proposition 4 in the second equality. Similarly,

$$\hat{g}(\biguplus_i x_i, \biguplus_i y_i) \rightleftharpoons g(\biguplus_i x_i)(\biguplus_i y_i) = \biguplus_i g x_i (\biguplus_i y_i) = \biguplus_i \biguplus_j g x_i y_j$$
$$= \biguplus_i g x_i y_i \rightleftharpoons \biguplus_i \hat{g}(x_i, y_i)$$

by using ($\biguplus$4) for F. □

Definition 4. An upward closed set U in a natural pre-domain D is called *naturally Scott open* if for all directed families x_i having the natural lub

$$\biguplus_i x_i \in U \Longrightarrow x_i \in U \text{ for some } i.$$

Such subsets constitute the *natural Scott topology* on D.

This is a straightforward generalization of the ordinary *Scott topology* on any poset defined in terms of the usual lub $\bigsqcup$ of directed families. Evidently, each Scott open set (in the standard sense) is naturally Scott open, and therefore the latter sets constitute a T_0-topology.

Proposition 6.

(a) Any natural pre-domain $\langle D, \sqsubseteq^D, \biguplus^D \rangle$ is a T_0-space under its natural Scott topology whose standardly generated partial ordering coincides with the original ordering $\sqsubseteq^D$ on D.
(b) Continuous functions between pre-domains defined as preserving the existing natural lubs are also continuous relative to the natural Scott topologies in the domain and co-domain.

(c) *But the converse holds only in the weakened form: continuity of a map f in the sense of natural Scott topologies implies $f(\biguplus_i x_i) = \bigsqcup_i f(x_i)$ for any directed family x_i with existing $\biguplus_i x_i$.*[6]

Proof.

(a) If $x \sqsubseteq y$ and $x \in U$ for any naturally Scott open $U \subseteq D$ then $y \in U$ because U is upward closed. Conversely, assume $x \not\sqsubseteq y$, and define $U_y \rightleftharpoons \{z \in D \mid z \not\sqsubseteq y\}$. This set is evidently upward closed. Let $\biguplus_i x_i \in U_y$ for a directed family. Then it is impossible that all $x_i \notin U_y$, i.e. $x_i \sqsubseteq y$, because then we should have $\biguplus_i x_i \sqsubseteq y$ — a contradiction. Therefore U_y is a naturally Scott open set (in fact, even Scott open in the standard sense) such that $x \not\sqsubseteq y$, $x \in U_y$ but $y \notin U_y$, as required.

(b) Assume monotonic $f : D \to E$ preserves natural directed lubs and $U \subseteq E$ is naturally Scott open in E. Then $f^{-1}(U)$ is evidently upward closed in D as U is such in E. Further, let $\biguplus_i x_i \in f^{-1}(U)$, i.e. $f(\biguplus_i x_i) = \biguplus_i f(x_i) \in U$ and hence $f(x_i) \in U$ and $x_i \in f^{-1}(U)$ for some i. Therefore $f^{-1}(U)$ is naturally Scott open. That is, f is continuous in the sense of natural Scott topologies in D and E.

(c) Conversely, assume $f : D \to E$ is continuous in the sense of natural Scott topologies in D and E, and $\biguplus_i x_i$ exists in D for a directed family. Let us show that $f(\biguplus_i x_i) = \bigsqcup_i f(x_i)$. The inequality $f(\biguplus_i x_i) \sqsupseteq f(x_i)$ follows by monotonicity of f. Assume y is an upper bound of all $f(x_i)$ in E but $f(\biguplus_i x_i) \not\sqsubseteq y$. Define like above the Scott open set $V_y \rightleftharpoons \{z \in E \mid z \not\sqsubseteq y\}$. Then $f^{-1}(V_y)$ is naturally Scott open containing $\biguplus_i x_i$ and therefore some x_i, implying $f(x_i) \in V_y$, i.e. $f(x_i) \not\sqsubseteq y$ — a contradiction. This means that $f(\biguplus_i x_i) = \bigsqcup_i f(x_i)$. □

3 Naturally Finite Elements

Definition 5. A *naturally finite* element d in a natural pre-domain D is such that for any directed natural lub (assuming it exists) if $d \sqsubseteq \biguplus X$ then $d \sqsubseteq x$ for some x in X. If arbitrary directed lubs $\bigsqcup X$ are considered then d is called just *finite*.

The last part of the definition is most reasonable in the case of dcpos. Otherwise (assuming $\biguplus \neq \bigsqcup$), 'finite' means rather 'non-natural finite'.

Definition 6. A natural pre-domain D is called *naturally (ω-) algebraic* if (it has only countably many naturally finite elements and) each element in D is a natural lub of a (non-empty) directed set of naturally finite elements.

[6] In the special case of $\biguplus \rightleftharpoons \bigsqcup$ and standard Scott topologies we have, as usual, the full equivalence of the two notions of continuity of maps with $f(\biguplus_i x_i) = \biguplus_i f(x_i)$. We will see below that the full equivalence of these two notion of continuity holds also for naturally algebraic and naturally bounded complete natural pre-domains.

If D is dcpo with $\uplus = \bigsqcup$ then the above reduces to the traditional concept of *(ω-) algebraic dcpo*. It follows, assuming additionally ($\uplus$2), that

$$x = \biguplus \hat{x} \tag{3}$$

where $\hat{x} \rightleftharpoons \{d \sqsubseteq x \mid d \text{ is naturally finite}\}$ for any $x \in D$.

Definition 7. If any two upper bounded elements c, d have least upper bound $c \sqcup d$ in D then D is called *bounded complete*, and it is called *finitely bounded complete* if, in the above, only finite c, d (and therefore $c \sqcup d$) are considered.

This is the traditional definition adapted to the case of an arbitrary poset D. If D is an algebraic dcpo then it is bounded complete iff it is finitely bounded complete. In fact, for dcpos bounded completeness means existence of a lub for any bounded set, not necessarily finite. Algebraic and bounded complete dcpos with least element $\bot$ are also known as *Scott domains* or as the *complete f_0-spaces* of Ershov [2] (or just *Scott-Ershov domains*). For the 'natural', non-dcpo version of these domains we need

Definition 8. A natural pre-domain D is called *naturally bounded complete* if any two naturally finite elements upper bounded in D have a lub (not necessarily natural lub, but evidently naturally finite element).

In such domains any set of the form $\hat{x}$ is evidently directed, if non-empty. (It is indeed non-empty in naturally algebraic pre-domains.)

Proposition 7. *For a naturally algebraic natural domain D the natural lub of an arbitrary family x_i can be represented as*

$$\biguplus_i x_i = \biguplus \bigcup_i \hat{x}_i \tag{4}$$

where both natural lubs either exist or not simultaneously.

Proof. Indeed, let $x_i^0 \sqsubseteq x_i$ denote an arbitrarily chosen naturally finite approximation of x_i, and let j range over naturally finite elements of D. Define $x_{ij} \rightleftharpoons j$ if $j \sqsubseteq x_i$, and $\rightleftharpoons x_i^0$ otherwise. Then $\biguplus_i x_i = \biguplus_i \biguplus \hat{x}_i = \biguplus_i \biguplus_j x_{ij} = \biguplus_{ij} x_{ij} = \biguplus \bigcup_i \hat{x}_i$ by (3) and the first part of ($\uplus$4). $\square$

Therefore, any naturally algebraic natural domain D is, in fact, defined by the quadruple $\langle D, D^{[\omega]}, \sqsubseteq^D, \mathcal{L}\rangle$ where $D^{[\omega]} \subseteq D$ consists of naturally finite elements in D and $\mathcal{L}$ is the set of all sets of naturally finite elements having a natural lub. Indeed, we can recover $\biguplus_i x_i \rightleftharpoons \bigsqcup \bigcup_i \hat{x}_i$ by (4) whenever $\bigcup_i \hat{x}_i \in \mathcal{L}$. Moreover, in the case of naturally algebraic and naturally bounded complete natural domains D their elements x can be identified, up to the evident order isomorphism, with the *ideals* $\hat{x} \in \mathcal{L}$ (non-empty directed downward closed sets of naturally finite elements ordered under set inclusion and having a natural lub). In particular,

$$x \sqsubseteq y \iff \hat{x} \subseteq \hat{y}\ . \tag{5}$$

Note 1. The above definition via naturally finite elements and ideals does not always work in practice. Thus, in the real application of this theory to the λ-model of hereditarily sequential finite type functionals $\{\mathbb{Q}_\alpha\}$ [10] we do not have naturally finite elements $\mathbb{Q}_\alpha^{[\omega]} \subseteq \mathbb{Q}_\alpha$ as given. We only have a priori that $\mathbb{Q}_\alpha$ are partial ordered sets with $\bot_\alpha$ and with monotonic application operators $\mathrm{App}_{\alpha,\beta} : \mathbb{Q}_{\alpha\to\beta} \times \mathbb{Q}_\alpha \to \mathbb{Q}_\beta$. That they are, in fact, naturally ω-algebraic, naturally bounded complete natural domains with $\mathrm{App}_{\alpha,\beta}$ naturally continuous requires quite complicated considerations (using appropriate theory of sequential computational strategies) for its proof. Even the fact that the natural (in fact, quite simply defined as pointwise) lub $\biguplus^\alpha$ on $\mathbb{Q}_\alpha$ is fruitful notion to use here was not self-evident at all.

Generalizing the case of dcpos we can improve an appropriate part in Proposition 6 (see also footnote 6):

Proposition 8.

(a) For D and E naturally algebraic and naturally bounded complete natural predomains, a monotonic map $f : D \to E$ is naturally continuous (in the sense of preserving directed natural lubs) iff for all $x \in D$ and naturally finite $b \sqsubseteq fx$ there exists naturally finite $a \sqsubseteq x$ such that $b \sqsubseteq fa$. This means that natural continuity of functions between such domains is equivalent to topological continuity with respect the natural Scott topology because

(b) Naturally Scott open sets in such domains are exactly arbitrary unions of the upper cones $\check{a} \rightleftharpoons \{x \mid a \sqsubseteq x\}$ for a naturally finite.

Proof. (a) Indeed, for f naturally continuous, $fx = \biguplus f(\hat{x})$, so $b \sqsubseteq fx$ implies $b \sqsubseteq fa$ for some $a \sqsubseteq x$ for naturally finite a, b.

Conversely, assume f satisfies the above b-a-continuity property and $x = \biguplus_i x_i$ be a natural directed lub in D. Let us show that $fx = \biguplus_i fx_i$. The inclusions $fx_i \sqsubseteq fx$ hold by monotonicity of f and imply $\bigcup_i \widehat{fx_i} \subseteq \widehat{fx}$. Now, it suffices to show, by (4) and (5) applied to E, the inverse inclusion $\widehat{fx} \subseteq \bigcup_i \widehat{fx_i}$. Thus, assume $b \sqsubseteq fx$ for a naturally finite b and hence $b \sqsubseteq fa$ for some naturally finite $a \sqsubseteq x = \biguplus_i x_i$ and, therefore, $a \sqsubseteq x_i$ for some i. Then $b \sqsubseteq fa \sqsubseteq fx_i$, as required.

(b) This follows straightforwardly from the definitions of naturally finite elements, naturally Scott open sets, and from the identity $x = \biguplus \hat{x}$ (with $\hat{x}$ directed). □

Note 2. In fact, it can be shown that naturally algebraic and naturally bounded complete natural (pre-) domains, if considered as topological spaces under the natural Scott topology, are exactly f-spaces of Ershov [2] (i.e. all, not necessary complete f-spaces). But here again we could apply the comments of Note 1. Indeed, $\mathbb{Q}_\alpha$ do not originally appear as f-spaces (represented as in [2] either topologically or order theoretically with finite (or f-) elements as given). This becomes clear only a posteriori, after complicated considerations based, in particular, on the general concept of natural domains (and on a lot of other things). That is why this concept is important in itself.

Further generalizing the traditional dcpo case and working in line with the theory of f-spaces [2], we can show

Proposition 9. *If natural domains D and E are naturally (ω-)algebraic and naturally bounded complete then so are $D \times E$ and $[D \to E]$, assuming additionally in the case of $[D \to E]$ that E contains the least element $\bot_E$. Then such a restricted class of domains with $\bot$ and with naturally continuous morphisms constitute a ccc.*

Proof. For $D \times E$ this is evident. Let us show this for $[D \to E]$. Indeed, let $a_0, \ldots, a_{n-1} \in D$ and $b_0, \ldots, b_{n-1} \in E$ be two arbitrary lists of naturally finite elements satisfying the

Consistency condition: for any $x \in D$ the set $\{b_i \mid a_i \sqsubseteq x, i < n\}$ is upper bounded in E, and hence its lub exists and is naturally finite.

(In general, assume that $a, b, c, d, \ldots$, possibly with subscripts, range over naturally finite elements.) Then define a *tabular* function $\left[\begin{smallmatrix} b_0,\ldots,b_{n-1} \\ a_0,\ldots,a_{n-1} \end{smallmatrix}\right] \in [D \to E]$ by taking for any $x \in D$

$$\left[\begin{smallmatrix} b_0,\ldots,b_{n-1} \\ a_0,\ldots,a_{n-1} \end{smallmatrix}\right] x \rightleftharpoons \bigsqcup\{b_i \mid a_i \sqsubseteq x, i < n\} \tag{6}$$

because this lub does always exist. (Here we use the fact that E contains the least element $\bot_E$ needed to get the lub defined if the set on the right is empty.) In particular, $\left[\begin{smallmatrix} b_0,\ldots,b_{n-1} \\ a_0,\ldots,a_{n-1} \end{smallmatrix}\right]$ is the least monotonic function $f : D \to E$ for which $b_i \sqsubseteq f a_i$ for all $i < n$, that is,

$$\left[\begin{smallmatrix} b_0,\ldots,b_{n-1} \\ a_0,\ldots,a_{n-1} \end{smallmatrix}\right] \sqsubseteq f \iff b_i \sqsubseteq f a_i \;\text{ for all } i < n. \tag{7}$$

Moreover, this is also a naturally continuous function. Indeed, for any directed family $\{x_k\}_{k \in K}$ in D with the natural lub existing

$$\left[\begin{smallmatrix} b_0,\ldots,b_{n-1} \\ a_0,\ldots,a_{n-1} \end{smallmatrix}\right] \biguplus_k x_k = \bigsqcup\{b_i \mid a_i \sqsubseteq \biguplus_k x_k\} = \left[\begin{smallmatrix} b_0,\ldots,b_{n-1} \\ a_0,\ldots,a_{n-1} \end{smallmatrix}\right] x_{k_0}$$

for some $k_0 \in K$ (due directedness of $\{x_k\}_{k \in K}$) so that, in fact, $\left[\begin{smallmatrix} b_0,\ldots,b_{n-1} \\ a_0,\ldots,a_{n-1} \end{smallmatrix}\right] x_k \sqsubseteq \left[\begin{smallmatrix} b_0,\ldots,b_{n-1} \\ a_0,\ldots,a_{n-1} \end{smallmatrix}\right] x_{k_0}$ for all $k \in K$ and hence, by ($\uplus$2) and ($\uplus$3) for E,

$$\left[\begin{smallmatrix} b_0,\ldots,b_{n-1} \\ a_0,\ldots,a_{n-1} \end{smallmatrix}\right] \biguplus_k x_k = \biguplus_k \left[\begin{smallmatrix} b_0,\ldots,b_{n-1} \\ a_0,\ldots,a_{n-1} \end{smallmatrix}\right] x_k \;.$$

It is also follows from (7) that $\left[\begin{smallmatrix} b_0,\ldots,b_{n-1} \\ a_0,\ldots,a_{n-1} \end{smallmatrix}\right] = \bigsqcup_{i<n} \left[\begin{smallmatrix} b_i \\ a_i \end{smallmatrix}\right]$. Moreover, this is a naturally finite element in $[D \to E]$. Thus, in the simplest case of $\left[\begin{smallmatrix} b \\ a \end{smallmatrix}\right]$

$$\left[\begin{smallmatrix} b \\ a \end{smallmatrix}\right] \sqsubseteq \biguplus_j f_j \overset{(7)}{\iff} b \sqsubseteq \biguplus_j f_j a \iff \exists j. b \sqsubseteq f_j a \iff \exists j. \left[\begin{smallmatrix} b \\ a \end{smallmatrix}\right] \sqsubseteq f_j$$

for any directed family of naturally continuous functions f_j with $\biguplus_j f_j$ and therefore $\biguplus_j f_j a$ existing. If $\left[\begin{smallmatrix} b_0,\dots,b_{n-1} \\ a_0,\dots,a_{n-1} \end{smallmatrix}\right] \sqsubseteq f$ and $\left[\begin{smallmatrix} d_0,\dots,d_{m-1} \\ c_0,\dots,c_{m-1} \end{smallmatrix}\right] \sqsubseteq f$ then evidently $\left[\begin{smallmatrix} b_0,\dots,b_{n-1} \\ a_0,\dots,a_{n-1} \end{smallmatrix}\right] \bigsqcup \left[\begin{smallmatrix} d_0,\dots,d_{m-1} \\ c_0,\dots,c_{m-1} \end{smallmatrix}\right] = \left[\begin{smallmatrix} b_0,\dots,b_{n-1},d_0,\dots,d_{m-1} \\ a_0,\dots,a_{n-1},c_0,\dots,c_{m-1} \end{smallmatrix}\right] \sqsubseteq f$. Thus, the set $\hat{f}$ of tabular approximations to any monotonic function f is directed. Moreover, any naturally continuous f is, in fact, the natural lub of this set:

$$f = \biguplus \hat{f} = \biguplus \{\varphi \mid \varphi \sqsubseteq f \ \&\ \varphi \text{ tabular}\} \tag{8}$$

because

$$\begin{aligned} fx &= \biguplus \widehat{fx} = \biguplus \{b \mid b \sqsubseteq fx\} = \biguplus \{b \mid \exists \text{ naturally finite } a \sqsubseteq x\ (b \sqsubseteq fa)\} \\ &= \biguplus \{\left[\begin{smallmatrix} b \\ a \end{smallmatrix}\right] x \mid \left[\begin{smallmatrix} b \\ a \end{smallmatrix}\right] \sqsubseteq f\} = \biguplus \{\varphi x \mid \varphi \sqsubseteq f \ \&\ \varphi \text{ tabular}\}. \end{aligned}$$

The last equality holds because, for tabular functions, $\varphi x = \left[\begin{smallmatrix} b \\ a \end{smallmatrix}\right] x$ for some $\left[\begin{smallmatrix} b \\ a \end{smallmatrix}\right] \sqsubseteq \varphi$ (where, in accordance with (6), $\left[\begin{smallmatrix} b \\ a \end{smallmatrix}\right]$ does not necessary is one of the columns of the tabular representation of φ). It also follows from (8) that tabular elements of $[D \to E]$ are exactly the naturally finite ones. Moreover, this domain is naturally (ω-)algebraic and naturally bounded complete. □

Note 3. For any finite list of tabular elements $\varphi_1, \dots, \varphi_k$ in $[D \to E]$, they are upper bounded in $[D \to E]$ iff the union of tables representing φ_i is consistent in the above sense. This reduces, essentially algorithmically, the problem of upperboundedness for naturally finite elements in $[D \to E]$ to those in D and E. But if we would consider a subset of $F \subseteq [D \to E]$ (say, of sequential or other kind of restricted function(al)s as in [10]) then no such algorithmic reduction for F is possible a priori, even if it is naturally algebraic and naturally bounded complete and its naturally finite elements are represented in the tabular way as above.

4 Semi-Formal Considerations on the More General Case of $F \subseteq [D \to E]$ Induced by [4][7]

Here most our of assertions will have a conditional character with intuitively appealing assumptions. Let $F \subseteq (D \to E)$ be an arbitrary natural domain of monotonic functions (for appropriate natural domains D and E). (See Proposition 3. For example, F could consist of naturally continuous sequential function(al)s only.) While postulating the additional requirement of natural continuity and ω-algebraicity property of a function domain F looks quite reasonable from the computational perspective, *the requirement of (natural) bounded completeness might seem questionable in general.* Why should the lub of two (naturally finite) sequential functionals exist at all and be sequentially computable, even if they

[7] Note that [4] was devoted only to the case of dcpos.

have a joint upper bound? However the following intuitive, semi-formal and sufficiently general argumentation in favour of natural bounded completeness can be given (and easily formalised for the case of finite type functionals).

The simplest, 'basic' domains D like flat ones may be reasonably postulated to be naturally bounded complete. Also, the greatest lower bound (glb) $x \sqcap y$ of any two elements can be considered computable/natural continuous. (Say, for flat domains we need only conditional **if** and equality $=$ to define $\sqcap$.) Then, assuming that F has the most basic computational closure properties, we can conclude that F is also closed under the naturally continuous operation glb $f \sqcap g = (\lambda x \in D. fx \sqcap gx) \in F$.

Moreover, it seems quite reasonable to assume that the set of naturally finite elements in any 'basic' D is a directed union, $D^{[\omega]} = \bigcup_k D^{[k]}$, of some finite sets $D^{[k]}$ of naturally finite objects which are suitably *finitely restricted* for each k where k (say, $0, 1, 2, \ldots$) may serve as a measure of restriction. For each $D^{[k]} \subseteq D$ we could expect that each $x \in D$ has a best naturally finite lower approximation $x^{[k]} = \Psi^{[k]}x \sqsubseteq x$ from $D^{[k]}$, assuming also $\Psi^{[k]}(\Psi^{[k]}x) = \Psi^{[k]}x$. Thus, $\Psi^{[k]}_D : D \to D$ is just a monotonic projection onto its finite range $D^{[k]}$. It easily follows that the family $\{x^{[k]}\}_k$ is directed for any $x \in D$. Also it is a reasonable assumption that such $\Psi^{[k]}_D$, for the basic domains, are computable and therefore naturally continuous.

Then the fact that each finitely restricted element $x^{[k]}$ is naturally finite can even be deduced as follows: $x^{[k]} \sqsubseteq \biguplus Z$ for a directed set Z implies $x^{[k]} \sqsubseteq \biguplus\{z^{[k]} \mid z \in Z\} = z^{[k]} \sqsubseteq z$ for some z by natural continuity of $\Psi^{[k]}$ and because $D^{[k]}$ is finite.

Further, we could additionally assume that $x = \biguplus_k x^{[k]}$ holds for all x. This implies formally (from our assumptions) that naturally finite and finitely restricted (i.e., of the form $x^{[k]}$) elements in D are the same.

It follows that any two upper bounded finitely restricted elements $d, e \in D^{[k]}$ must have a (not necessarily natural) lub $d \sqcup e$ in D which is also finitely restricted. Indeed, it can be obtained as the greatest lower bound in D of a finite nonempty set:

$$d \sqcup e = \sqcap\{x^{[k]} \mid x \sqsupseteq d, e\}. \tag{9}$$

By induction, given any (not necessary 'basic') naturally ω-algebraic and naturally bounded complete domains D and E with such projections, we should conclude that the composition $\Psi^{[k]}_E \circ f \circ \Psi^{[k]}_D$, denoted as $\Psi^{[k]}_F f$ or $f^{[k]}$ ($f^{[k]}x \rightleftharpoons (fx^{[k]})^{[k]}$), is computable/naturally continuous, assuming $f \in F \subseteq [D \to E]$ is such. Assuming that F has minimal reasonable closure properties, we can conclude that this composition should belong to F as well. But, once all $D^{[k]}$ and $E^{[k]}$ are finite sets consisting only of naturally finite elements, $\Psi^{[k]}_F f$ is just a naturally finite tabular function, which can be reasonably postulated as k-restricted in F, and $\Psi^{[k]}_F : F \to F$ is the corresponding directed family of projections having finite ranges $F^{[k]}$ consisting of some tabular k-restricted functions.

These projections are naturally continuous and, moreover, preserve all existing natural lubs (not necessarily directed) assuming $\Psi_D^{[k]}$ and $\Psi_E^{[k]}$ do:

$$(\Psi_E^{[k]} \circ (\biguplus_i f_i) \circ \Psi_D^{[k]})x = \Psi_E^{[k]}((\biguplus_i f_i)(\Psi_D^{[k]}x)) = \Psi_E^{[k]}(\biguplus_i (f_i(\Psi_D^{[k]}x)))$$
$$= \biguplus_i \Psi_E^{[k]}(f_i(\Psi_D^{[k]}x)) = \biguplus_i ((\Psi_E^{[k]} \circ f_i \circ \Psi_D^{[k]})x) = (\biguplus_i (\Psi_E^{[k]} \circ f_i \circ \Psi_D^{[k]}))x \ .$$

Moreover, having that F consists of only naturally continuous functions, $f = \biguplus_k f^{[k]}$ should hold for all f. Indeed, this follows from the same property in D and E: $fx = f(\biguplus_k x^{[k]}) = \biguplus_k (fx^{[k]}) = \biguplus_k \biguplus_m (fx^{[k]})^{[m]} = \biguplus_k (fx^{[k]})^{[k]} = \biguplus_k (f^{[k]}x)$. Then we can conclude that the tabular functions (of the form $f^{[k]}$ for any $f \in F$) are exactly the naturally finite elements of the natural domain F, and F is naturally ω-algebraic. Finally, having projections $\Psi_F^{[k]}$ and naturally continuous finite glb $\sqcap$ in F (definable by induction like above and therefore existing in F by the natural closure properties), natural bounded completeness of F follows exactly as above in (9) for the case of 'basic' domains.

To define a naturally ω-algebraic and naturally bounded complete natural domain $F \subseteq [D \to E]$, we can fix any (simply) bounded complete set $F^{[\omega]}$ of tabular elements in $[D \to E]$ containing $\perp_{[D \to E]}$, and take F to be any extension of $F^{[\omega]}$ by some (if exists in $[D \to E]$) directed natural lubs of these tabular elements. Then $F^{[\omega]}$ is exactly the set of all naturally finite elements in F. Two extreme versions of F are $F^{[\omega]}$, and the set of all existing directed natural lubs from $F^{[\omega]}$. Besides the fact that this construction looks quite natural in itself, it follows from the above considerations that naturally finite elements in F cannot be anything other than tabular elements, provided there are, as above, directed families of naturally continuous projections $\Psi_D^{[k]}$ and $\Psi_E^{[k]}$ to finite elements such that $x = \biguplus_k x^{[k]}$ and $y = \biguplus_k y^{[k]}$ hold for any $x \in D$ and $y \in E$, and that F is closed under projections $\Psi_F^{[k]}$ defined from $\Psi_D^{[k]}$ and $\Psi_E^{[k]}$.

5 Conclusion

Our presentation is that of the current state of affairs and has the peculiarity that really interesting concrete examples of non-dcpo domains (such as those of hereditarily sequential and wittingly consistent higher type functionals [10]) from which this theory has in fact arisen require too much space to be presented here. The theory is general, but the non-artificial and instructive non-dcpo examples on which it is actually based are rather complicated and in a sense exceptional (dcpo case being more typical and habitual). However we can hope that there will be many more examples where this theory can be used, similarly to the case of dcpos.

One important topic particularly important for applications which was not considered here in depth and which requires further special attention is the possibility of the effective version of naturally algebraic, naturally bounded complete natural domains. Unlike the ordinary dcpo version (Ershov-Scott domains), not everything goes so smoothly here as is noted in connection with the model of hereditarily sequential functionals in Sect. 2.4 of [10]; see also Note 3 above.

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