A Bounded Set Theory with Anti-Foundation Axiom and Inductive Definability*

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1 Introduction. Kripke-Platek Set Theory with Anti-Foundation Axiom

Let KP_0 be Kripke-Platek Set Theory [Bar75] with *omitted* Foundation Axiom. Denote by $\mathcal{V}$ any universe satisfying theory KP_0. Anti-Foundation Axiom, AFA, of P.Aczel [Acz88] says that there exists the unique set-theoretic *decoration* operation $\mathsf{D}(G,x)$ over graphs $G \in \mathcal{V}$ (as any sets of pairs $\langle x,y\rangle \rightleftharpoons \{\{x\},\{x,y\}\}$) satisfying the L.Gordeev's identity[2]

$$\mathsf{D}(G,y) = \{\mathsf{D}(G,x) : \langle x,y\rangle \in G\}, \quad \text{or } = \{\mathsf{D}(G,x) : x \to_G y\} \ .$$

Following originally to P.Aczel [Acz88] and to T.Fernando [Fer] in some details we define in terms of KP_0, a universe $\mathcal{G} \subseteq \mathcal{V}$, just a Δ_0-class in KP_0 of all *pointed graphs (pg's)* $g = \langle G,a\rangle$ (with G a graph and a a *distinguished vertex* or *point*) together with two Σ-relations over this class: $\in_0$ and a *congruence* $=_0$ with respect to $\in_0$ (and to many other predicates and operations over pg's). We say that a class $R \subseteq \mathcal{V}$ is *bisimulation relation* between graphs G and G', shortly $Bis(R,G,G')$, if R is a class of pairs and always bRb' implies $(\forall x \to_G b \exists y \to_{G'} b'.\ xRy)$ and $(\forall y \to_{G'} b' \exists x \to_G b.\ xRy)$. Let $\sim_{G,G'}$ be the *largest* such R. Then define

$$\langle G,a\rangle =_0 \langle G',a'\rangle \rightleftharpoons a \sim_{G,G'} a' \ (\text{iff } \exists R \in \mathcal{V}.(Bis(R,G,G') \ \&\ aRa')) \quad \text{and}$$
$$\langle G,a\rangle \in_0 \langle G',a'\rangle \rightleftharpoons \exists b' \to_{G'} a'.\langle G,a\rangle =_0 \langle G',b'\rangle \ .$$

Theorem 1 [Fer]. $\mathsf{KP}_0 + \mathsf{Power} \vdash$ "$\langle \mathcal{G}, =_0, \in_0\rangle \models \mathsf{KP}_0 + \mathsf{Power} + \mathsf{AFA}$".

It was essentially used in the proof of this theorem that Σ-relations $=_0$ and $\in_0$ become Δ in the presence of Powerset which seems too strong axiom from

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[2] In a Bounded Set Theory [Saz85, Saz87, Saz93] (exactly corresponding to PTIME-computability) it was used, instead of the decoration operation, analogous notion of A.Mostowski's *collapsing* (cf. [Bar75], Ch.I.9.13), however, only for the case of *well-founded graphs* where both these notions are equivalent. Also, our direction of edges in graphs is taken as in [Bar75] and in our previous papers, just converse to that in [Acz88].

various points of view. For example, Powerset is non-tractable operation giving rise to *Kalmar elementary* rather than to *polynomial-time* computability.

Our aim is to show in Sect. 2 with the details in the Appendix that analogous effect for BSTA, a *Bounded Set Theory* [Saz85, Saz87, Saz93] *with Anti-Foundation Axiom* (which extends KP_0 and also a *Basic Set Theory* [Gan74]) may be reached by using, instead of Power, a Recursive Δ-Separation axiom (added to KP_0 in op.cit.).

Another aim is to compare in Sect. 3 proof-theoretic and expressive power of BSTA with a *Logic of Inductive Definitions*, LID, whose relation to Generalized Computability was investigated in [Mos74], whose extension by arithmetical axioms is known rather as the system(s) ID_i [Fef70, Acz77] investigated also in [Jäg86] in connection with admissible sets and whose interpretation in finite *linear-ordered* models was shown to exactly correspond to PTIME in [Imm82, Liv82, Var82]; cf. also related works [Saz80, Gur83, Gur84, GS86]. Section 3 is written much more sketchy, in contrast to Sect. 2.

In the case of a previously considered in [Saz85, Saz87, Saz93] version of Bounded Set Theory with Bounded Foundation Axiom (or with Bounded $\in$-Induction Axiom) interpreted in the ordinary universe HF of *hereditarily-finite sets* it was possible to define in the language of BST a natural *linear order* on HF. Therefore that version was *exactly* corresponding to PTIME-computability (with respect to *acyclic,* or *well-founded* finite pointed graph representation of HF-sets) unlike the present version of BSTA for which it remains an *open question.* Only PTIME-computability of provably-total Σ-definable operations of BSTA can be shown.

We postpone some additional related aims to a future research. Also we will not touch here possible applications to Concurrent Transition Systems, Situation Logic and "Nested" Data Bases. (For the connection of the latter subject with a BST cf. [Saz93].)

2 Interpretation of set theory BSTA in KPR_0

Let us consider a set theory KPR_0 based on the following Δ_R-*language*

$$\begin{aligned} \Delta_R\text{-formulas} &::= a \in b \mid a = b \mid \varphi \& \psi \mid \varphi \vee \psi \mid \neg\varphi \mid \forall x \in a.\varphi \mid \exists x \in a.\varphi \\ \Delta_R\text{-terms} &::= \text{set-variables } p, q, x, y, \ldots \mid \{a, b\} \mid \cup a \mid \\ &\qquad \{t(x) : x \in a \& \varphi(x)\} \mid \mathsf{the\text{-}least}\, p.(p = \{x \in a : \varphi(x, p)\}) \end{aligned}$$

where a, b, t are Δ_R-terms and φ, ψ are Δ_R-formulas, set-variables x, p are not free in a and all occurrences of set variable p in Δ_R-formula φ are only *of the form* '- $\in p$', *positive* and *not inside* of any complex subterm of φ. The condition on p guarantees that φ is monotonic under the set inclusion on this set variable and therefore (at least in the framework of the classical set theory) the least such set p must exist. In fact, we will consider also the closure of Δ_R-formulas under $\&, \vee, \neg$ and unbounded $\forall$ and $\exists$ with the ordinary definition of the bounded quantifiers of Δ_R via unbounded ones.

Non-logical axioms of KPR_0 (which also explain a *meaning* of the above Δ_R-constructs) are the following ones:

Extensionality: $a \subseteq a' \ \& \ a' \subseteq a \Rightarrow a = a'$,
Pair: $t \in \{a, b\} \Leftrightarrow t = a \vee t = b$,
Union: $t \in \cup a \Leftrightarrow \exists x \in a.t \in x$,
Δ_R-Image: $s \in \{t(x) : x \in a \ \& \ \varphi(x)\} \Leftrightarrow \exists x \in a.(\varphi(x) \ \& \ s = t(x))$,
Recursive Δ_R-Separation: If $p_0 = \mathsf{the\text{-}least}\ p.(p = \{x \in a : \varphi(x, p)\})$ then

$$p_0 = \{x \in a : \varphi(x, p_0)\} \ \& \ (\{x \in a : \varphi(x, p)\} \subseteq p \Rightarrow p_0 \subseteq p) \ ,$$

Δ_R-Collection: $\forall x \in a \exists y.\varphi(x, y) \Rightarrow \exists z \forall x \in a \exists y \in z.\varphi(x, y)$.

where a, b, t, s are any Δ_R-terms and φ is any Δ_R-formula and $a \subseteq b \rightleftharpoons \forall x \in a.(x \in b)$. Note, that Δ_R-Collection is the only axiom of KPR_0 which itself is not a Δ_R-formula.

Let us call by Δ_{D} the extension of Δ_R-language by the *decoration* and *transitive closure* operations $\mathsf{D}(g, p)$ and $\mathsf{TC}(s)$ for any sets (or Δ_{D}-terms) g, p, s. Then the corresponding theory BSTA, called *Bounded Set Theory with Anti-Foundation Axiom*, consists of all axioms of KPR_0 reformulated for the language Δ_{D} (with Recursive Δ_{D}-Separation and Δ_{D}-Collection, etc.) plus axioms AFA and TC *formulated to be* Δ_{D}*-formulae.*

So, axioms for TC are as follows (where $\mathsf{Tran}(y) \rightleftharpoons \forall u \in y(u \subseteq y)$)

$$x \subseteq \mathsf{TC}(x), \ \mathsf{Tran}(\mathsf{TC}(x)) \ ,$$
$$(x \subseteq y \Rightarrow \mathsf{TC}(x) \subseteq \mathsf{TC}(y)) \quad \text{and} \quad \mathsf{Tran}(y) \Rightarrow \mathsf{TC}(y) \subseteq y \ .$$

We split AFA into two axioms AFA_1, *existence* or *correctness of the decoration,*

$$\forall z.(z \in \mathsf{D}(g, p) \Leftrightarrow \exists q \in \mathsf{field}(g).(\langle q, p\rangle \in g \& z = \mathsf{D}(g, q))) \ ,$$

(where $\mathsf{field}(R) \rightleftharpoons \cup\cup R = \{x, y : \langle x, y\rangle \in R\}$ for any set of pairs R) and AFA_2, *the uniqueness of decoration*, which is equivalent, as in [Acz88], to *strong extensionality property* for any set of pairs $R \in \mathcal{V}$

$$Bis_{\in}(R) \ \& \ xRy \Rightarrow x = y, \quad \text{where}$$
$$Bis_{\in}(R) \rightleftharpoons \forall x, y \in \mathsf{field}(R).(xRy \Rightarrow (\forall x' \in x \exists y' \in y.x'Ry') \ \& \ (\forall y' \in y \exists x' \in x.x'Ry')) \ .$$

In this paper let Δ-formulas and Δ-terms be those defined as Δ_R with the construct the-least omitted. (They define predicates and operations also known as *basic* [Gan74] or *rudimentary* [Jen72] ones.) Note, that provably-total Σ-definable operations in KP_0 coincide with those definable by Δ-terms [Saz85, Saz85a, Saz87]. Therefore, we may use the name KP_0 also for the sub-theory of KPR_0, based on this Δ*-language*, which does not involve both the term construct the-least and the corresponding axiom. Analogously, $\Delta_R(\Delta_{\mathsf{D}})$-terms and $\Delta_R(\Delta_{\mathsf{D}})$-formulas give all provably total Σ-definable operations of KPR_0

(BSTA) and provably-Δ-predicates (i.e. predicates over $\mathcal{V}$ represented simultaneously both by a Σ and a Π-formula which are provably equivalent). It follows that Δ-, Δ_R- and Δ_{D}-languages are *complete* in this sense for theories KP_0, KPR_0 and BSTA, respectively.

Another result from op.cit. is conservativeness relative to Δ-formulas of our version of KP_0 over KP_0 *minus* Δ-Collection axiom, i.e. over $\mathsf{KP}_0|_\Delta$, and, moreover [Saz87], over $(\mathsf{Extensionality\ axiom})|_{\Delta_0}$ in the A.Levy's language of Δ_0-formulas (i.e. Δ-formulas with variables as terms). Also KPR_0 is conservative over $\mathsf{KPR}_0|_{\Delta_R}$ and the same for BSTA and $\mathsf{BSTA}|_{\Delta_{\mathsf{D}}}$.

It is easy to represent in $\Delta_R(\Delta_{\mathsf{D}})$-language a construct dual to the-least,

$$\text{the-largest } p.(p = \{x \in a : \varphi(x,p)\}) \rightleftharpoons \\ a \setminus \text{the-least } q.(q = \{x \in a : \neg\varphi(x, a \setminus q)\}) \ ,$$

where $a \setminus b \rightleftharpoons \{x \in a : x \notin b\}$ and $t \in a \setminus q$ must be replaced everywhere in φ by $t \in a \ \& \ \neg t \in q$. So, the formula $\neg\varphi(x, a \setminus q)$ will involve q satisfying the same (positivity) requirement as p in $\varphi(x,p)$. Then the *largest bisimulation relation* $\sim_{GG'} \in \mathcal{V}$ between any two graphs (sets of pairs) $G, G' \in \mathcal{V}$ is expressible as a Δ_R-*term* of two variables G, G'. Therefore $=_0$ and $\in_0$ defined with the help of $\sim_{GG'} \in \mathcal{V}$ are also in Δ_R. Evidently, $=_0$ is an equivalence relation which is *congruent* with respect to $\in_0$. (We will need even more; cf. the Congruence Lemma 3 below). This allows to prove by a lengthy induction argument

Theorem 2. *$\mathsf{KPR}_0 \vdash$ "$\langle \mathcal{G}, =_0, \in_0 \rangle \models \mathsf{BSTA}$".*

Here we take formally "$\langle \mathcal{G}, =_0, \in_0 \rangle \models \varphi(\bar{x})$" $\rightleftharpoons \forall \bar{x} \in \mathcal{G}.[\![\varphi]\!](\bar{x})$ where $[\![\varphi]\!]$ is defined below a (Δ_R-)translation of any (Δ_{D}-)formula φ which will give the intended meaning of φ in the universe $\langle \mathcal{G}, =_0, \in_0 \rangle$. *Simultaneously,* we must properly define (by a structural induction) for any Δ_{D}-term $t = t(\bar{x})$ corresponding Δ_R-term $[\![t]\!]$. In particular, we will have $\mathsf{KPR}_0 \vdash \forall \bar{x} \in \mathcal{G}.([\![t]\!](\bar{x}) \in \mathcal{G})$. Some denotations used in the following inductive definition of $[\![\text{-}]\!]$ will be explained in Appendix 4.1. However, they have suitable mnemonics. Let us define here only the point-changing operation for any pg $\langle G, x \rangle$ by $\langle G, x \rangle * y \rightleftharpoons \langle G, y \rangle$ and also $\mathsf{elements}\langle G, x \rangle \rightleftharpoons \{y \in \mathsf{field}(G) : y \rightarrow_G x\}$. Then let

$[\![a \in b]\!] \rightleftharpoons [\![a]\!] \in_0 [\![b]\!]$; $[\![a = b]\!] \rightleftharpoons [\![a]\!] =_0 [\![b]\!]$;

$[\![\varphi \ \& \ \psi]\!] \rightleftharpoons [\![\varphi]\!] \ \& \ [\![\psi]\!]$; $[\![\varphi \vee \psi]\!] \rightleftharpoons [\![\varphi]\!] \vee [\![\psi]\!]$; $[\![\neg\varphi]\!] \rightleftharpoons \neg[\![\varphi]\!]$;

$[\![\forall x \in a.\varphi(x)]\!] = [\![\forall x.(x \in a \Rightarrow \varphi(x)]\!] \rightleftharpoons \forall x \in \mathsf{elements}[\![a]\!].[\![\varphi]\!]([\![a]\!] * x)$;

$[\![\exists x \in a.\varphi(x)]\!] = [\![\exists x.(x \in a \& \varphi(x)]\!] \rightleftharpoons \exists x \in \mathsf{elements}[\![a]\!].[\![\varphi]\!]([\![a]\!] * x)$;

$[\![\forall x.\varphi(x)]\!] \rightleftharpoons \forall x \in \mathcal{G}.[\![\varphi]\!](x)$,   if $\varphi(x)$ has not the form $x \in a \Rightarrow \psi(x)$;

$[\![\exists x.\varphi(x)]\!] \rightleftharpoons \exists x \in \mathcal{G}.[\![\varphi]\!](x)$,   if $\varphi(x)$ has not the form $x \in a \& \psi(x)$;

$[\![\text{set-variable}]\!] \rightleftharpoons$ the same variable;

$[\![\{a, b\}]\!] \rightleftharpoons \sum \{[\![a]\!], [\![b]\!]\}$ (cf. Appendix 4.1);

$[\![\cup a]\!] \rightleftharpoons (\mathsf{collect\ elements\text{-}of\text{-}elements}[\![a]\!] \ \mathsf{in\ graph}[\![a]\!])$ (cf. Appendix 4.1);

$$[\![\{t(x) : x \in a \;\&\; \varphi(x)\}]\!] \rightleftharpoons$$
$$\sum \{[\![t]\!]([\![a]\!] * x) : x \in \mathsf{elements}[\![a]\!] \;\&\; [\![\varphi]\!]([\![a]\!] * x)\};$$
$$[\![\mathsf{the\text{-}least}\ p.(p = \{x \in a : \varphi(x,p)\})]\!] \rightleftharpoons (\mathsf{collect}\ q_0\ \mathsf{in}\ \mathsf{graph}[\![a]\!]) \quad \text{where}$$
$$q_0 \rightleftharpoons \mathsf{the\text{-}least}\ q.(q = \{x \in \mathsf{elements}[\![a]\!] : [\![\varphi]\!]([\![a]\!] * x, (\mathsf{collect}\ q\ \mathsf{in}\ \mathsf{graph}[\![a]\!]))\}).$$

To be correct, in the last use of the construct $\mathsf{the\text{-}least}$ we should analyze all occurrences in $[\![\varphi]\!]([\![a]\!]*x, (\mathsf{collect}\ q\ \mathsf{in}\ \mathsf{graph}[\![a]\!]))$ of the term $(\mathsf{collect}\ q\ \mathsf{in}\ \mathsf{graph}[\![a]\!])$, or shortly $(\mathsf{collect}\ q\ \mathsf{in}\ G)$. Each such occurrence has the form $t \in_0 (\mathsf{collect}\ q\ \mathsf{in}\ G)$ or $v \in \mathsf{elements}(\mathsf{collect}\ q\ \mathsf{in}\ G)$ (the last case arise from $[\![\text{-}]\!]$-translation of a bounded quantifier $Qv \in q$ in $\varphi(x,q)$, if any, with v a quantified variable) and is evidently positive. Moreover, these formulas immediately prove to be equivalent respectively to $\exists v \in q.t =_0 \langle G, v\rangle$ and $v \in q$ (by using the definition of $\mathsf{collect}$ construct in Appendix 4.1) so that variable q remains positive, as required.

Finally, the translations of the operations TC, and D in terms of the original Δ_R-language are as follows (cf. also Appendix 4.1).

$$[\![\mathsf{TC}]\!](g) \rightleftharpoons (\mathsf{collect}\ \mathsf{elements}^*(g)\ \mathsf{in}\ \mathsf{graph}(g)) \ ,$$

$$[\![\mathsf{D}]\!](g,p) \rightleftharpoons \langle \mathsf{hgraph}(\tilde{g}), \mathsf{the\text{-}unique}\ x \in \mathsf{hvertices}(\tilde{g}).\tilde{g} * x =_0 p\rangle \ .$$

Here g, p are considered as any pg's in $\mathcal{G}$ and $\tilde{g}$ denotes a pointed *factor-graph* of $g = \langle G, \pi\rangle$ under the largest bisimulation (equivalence) relation $\sim_G$ on $G \rightleftharpoons \mathsf{graph}(g)$ with any $\langle x,y\rangle \in G$ defining an edge between corresponding two equivalence classes $[x], [y]$, and with the point π of g defining the point $[\pi]$ of $\tilde{g}$. Then it is clear, that any such $\tilde{g}$ is isomorphic to some pointed "transitive set" of the universe $\langle \mathcal{G}, =_0, \in_0\rangle/_{=_0}$.

We can prove in KPR_0 the *congruence property* of the equivalence relation $=_0$ with respect to the above definition of translation semantics $[\![\text{-}]\!]$ by using the structural induction on any Δ_{D}-formula $\varphi(g)$ and Δ_{D}-term $t(g)$.

Congruence Lemma 3.

$$g =_0 g' \Rightarrow ([\![\varphi]\!](g) \Leftrightarrow [\![\varphi]\!](g')) \;\&\; ([\![t]\!](g) =_0 [\![t]\!](g')) \ .$$

Note, that in the proof of this lemma given in the Appendix 4.2 it is simultaneously shown that also for bounded quantifiers the equivalence

$$[\![Qx \in a(g).\psi(x,g)]\!] \Leftrightarrow QX \in_0 [\![a]\!](g).[\![\psi]\!](X,g)$$

holds, the right-hand-side being in general not a Δ_R-formula even if ψ is. So, this is not just a definition. (Unfortunately, it is unclear how to prove this equivalence more directly; otherwise the validity in $\langle \mathcal{G}, =_0, \in_0\rangle$ of equality axioms considered after this paragraph could be deduced simply from their partial cases for $\in$ and $=$ in place of φ and for a variable v in place of a, without using the previous lemma.) This together with the definition of $[\![\varphi]\!]$ means that for all logical connectives and quantifiers $[\![\varphi]\!]$ behaves as required. Also all logical axioms remain true under

$[\![\text{-}]\!]$, i.e. they hold in $\langle \mathcal{G}, =_0, \in_0 \rangle$, provably in KPR_0. For example, translations of the (logical) equality axioms (cf. [Men64]),

$$[\![t = t]\!] \quad \text{and} \quad [\![t = s \Rightarrow (\varphi(t) \Leftrightarrow \varphi(s)) \ \& \ (a(t) = a(s))]\!]$$

for any formula $\varphi(v)$ and terms $a(v)$, t and s, are deciphered, respectively, as

$$[\![t]\!] =_0 [\![t]\!] \quad \text{and} \quad [\![t]\!] =_0 [\![s]\!] \Rightarrow ([\![\varphi(t)]\!] \Leftrightarrow [\![\varphi(s))]\!]) \ \& \ ([\![a(t)]\!] =_0 [\![a(s)]\!])$$

and we also should use, besides the above congruence lemma, the following theorems of KPR_0 proved by induction on φ and a with any t:

$$[\![\varphi(t)]\!] \Leftrightarrow [\![\varphi]\!]([\![t]\!]) \quad \text{and} \quad [\![a(t)]\!] = [\![a]\!]([\![t]\!]) \ .$$

This also allows to prove translation of the quantifier axiom $\forall v.\varphi(v) \Rightarrow \varphi(t)$:

$$\forall v \in \mathcal{G}.[\![\varphi]\!](v) \Rightarrow [\![\varphi]\!]([\![t]\!]) \ .$$

Another one $\forall v.(\psi \Rightarrow \varphi(v)) \Rightarrow (\psi \Rightarrow \forall v.\varphi(v))$ (with v non-free in ψ) is translated to its own partial case

$$\forall v \in \mathcal{G}.([\![\psi]\!] \Rightarrow [\![\varphi]\!](v)) \Rightarrow ([\![\psi]\!] \Rightarrow \forall v \in \mathcal{G}.[\![\varphi]\!](v))$$

as well as all propositional axioms. We finish the logical part by noting that the *logical inference rules* from [Men64], just *Modus ponens*, $\varphi, \varphi \Rightarrow \psi / \psi$, and *Generalization*, $\varphi(x)/\forall x \varphi(x)$, also preserve validity in $\langle \mathcal{G}, =_0, \in_0 \rangle$. All of these means that our semantics is *logically correct.*

Finally, and most important, we can prove in KPR_0 translations of all *non-logical axioms* of BSTA (cf. Appendix 4.3).

3 Interpretation of $\mathsf{BSTA}|_{\boldsymbol{\Delta}_\mathsf{D}}$ in LID

Let us consider a *Logic of Inductive Definitions,* LID. The corresponding Language of Inductive Definitions goes back to Y.Moschovakis; cf. [Mos74] and also [Imm82, Var82, Gur84, GS86]. It is called sometimes $\mathsf{FO{+}LFP}$ and defined here as *many-sorted* First-Order Logic with predicate variables and with the (positive) Least Fixed Point construct

$$[\mathsf{the\text{-}least}\ P \forall \bar{x}.(P(\bar{x}) \Leftrightarrow \varphi(\bar{x}, \bar{y}, P, \bar{Q}))]$$

abbreviated below as P_φ $(= P_\varphi(\text{-}) = P_\varphi(\text{-}, \bar{y}, \bar{Q}))$. Here, $\bar{x}, \bar{y}$ and $P, \bar{Q}$ are, respectively, all free individual and predicate variables of φ and the predicate variable P is required to have only *positive* occurrences in φ. The last condition guarantees that φ is monotonic on the predicate variable P and therefore (at least in the framework of the classical set theory) *the least* required value of P must exist. This new predicate construct may participate in other LID-formulas with any nesting except that its predicate variable P *is not allowed to appear in the corresponding body* $\varphi(\bar{x}, P)$ *inside any other such construct which is a part of* *this body* (essentially the same as in Δ_R-language[3]). In general, LID-formulas are constructed from atomic formulas of the kind $R(\bar{v})$, with R a predicate (a variable or a constant or a binary equality relation $=$ or, inductively, the above the-least construct) and $\bar{v}$ individuals (variables or constants), by using the ordinary first order connectives $\&, \vee, \neg, \Rightarrow, \Leftrightarrow, \forall, \exists$ with the restrictions on the shape of the-least construct mentioned. No second order quantification is allowed in LID-formulas, however, n-ary predicates are considered as comprising a *second order sort*.

LID as a *logic* extends the ordinary Firs-Order Logic with equality also by the following axiom schemes (for φ and P_φ as above):

$$\forall \bar{x}.(\varphi(\bar{x}, P_\varphi) \Rightarrow P_\varphi(\bar{x})) \quad \text{and} \quad \forall \bar{x}.(\varphi(\bar{x}, P) \Rightarrow P(\bar{x})) \Rightarrow \forall \bar{x}.(P_\varphi(\bar{x}) \Rightarrow P(\bar{x})).$$

In particular, when φ does not depend on P these axioms essentially give rise just to the ordinary comprehension axiom for first-order formulas.

A semantics of this language and calculus in the framework of any set theory universe $\mathcal{V}$ is defined via fixing any second-order structure $\mathcal{M} = \langle M, \mathcal{P} \rangle$ consisting of a many-sorted first-order structure $M \in \mathcal{V}$ (e.g. with sorts being just (the sets of vertices of) some pg's $\bar{g} = g_1, \ldots, g_n$ in $\mathcal{G}$[4] in which case M is essentially a tuple or a join $\langle g_1, \ldots, g_n \rangle$ of one-sorted structures) together with some *class* $\mathcal{P} \subseteq \mathcal{V}$ of many-sorted predicates over M. It is required that always (the value of) $P_\varphi = P_\varphi(\text{-}, \bar{y}, \bar{Q})$ is in $\mathcal{P}$ when parameters $\bar{y}, \bar{Q}$ are in $\mathcal{M}$ and also that both the above axioms hold in $\mathcal{M}$. In the case $\mathcal{V}$ satisfies $\mathsf{KPR}_0|_{\Delta_R}$ it suffices to take $\mathcal{P}$ just *all* the subsets $\in \mathcal{V}$ of the corresponding direct products of fitst-order sorts and we also will write M *instead of* $\mathcal{M}$. Let us call such $\mathcal{M}$ induced by $M \in \mathcal{V}$ an *inner model* in $\mathcal{V}$ of LID. Bearing in mind this case, Recursive Δ_R-Separation axiom of KPR_0 allows to interpret the-least construct of LID by the-least construct of KPR_0. Evidently, "$M \models \varphi$" is represented by a Δ_R-formula of KPR_0 for any $\varphi \in$ LID. Then let $\models_{\mathcal{V}} \varphi$ abbreviate (non-Δ_R-formula) $\forall M \in \mathcal{V} \tilde{\forall}.(M \models \varphi)$ where $\tilde{\forall}$ denotes universal quantification over all free individual and predicate variables of φ (ranging eventually over $\mathcal{V}$). Evidently, this gives a natural embedding of LID in KPR_0 (and, therefore, also in BSTA):

$$\mathsf{LID} \vdash \varphi \quad \text{implies} \quad \mathsf{KPR}_0 \vdash \text{“} \models_{\mathcal{V}} \varphi \text{”} \quad (\text{with } \mathcal{V} = \{x : x = x\})$$

which is, in fact, *conservative* according to the Corollary 8 below. It is well known that (for variations of the language of LID)

Theorem 4. ([Imm82, Liv82, Var82]; cf. also related results [Saz80, Gur83, Gur84]) *If* LID *has in its signature a binary predicate for a linear order then the expressive power of this language in finite linear ordered structures exactly corresponds to* PTIME-*computability.*

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[3] Otherwise we could not embed below LID into KPR_0. This restriction on LID seems also essential for infinite models and sets which are not excluded by KPR_0 or BSTA.

[4] Any pg g may be considered as a one-sorted structure with one binary relation and one individual constant, the point of the graph

The translation $[\![t]\!]$ of Δ_{D}-terms defined in Sect. 2 gives for each such a term an operation over pointed graphs $(\lambda\bar{g} \in \mathcal{G}.[\![t]\!](\bar{g})) : \mathcal{G}^n \to \mathcal{G}$. This operation was defined by a Δ_R-term and its values do matter only up to an isomorphism of pg's. Analogously, translation $[\![\varphi]\!](\bar{g})$ of a Δ_{D}-formula gives a predicate over $\mathcal{G}$ which is also expressed by a Δ_R-formula.

As for the case of a well-founded universe [Saz91], we can *imitate* these translations by a more elementary LID-language (than Δ_R dealing with "nested" sets and predicates) with binary predicate constants and individual constants, each pair for each sort, corresponding to each input pointed graph. We may just redefine translations $[\![\varphi]\!](\bar{g})$ and $[\![t]\!](\bar{g})$, written now respectively as LID-formulas $\langle\!\langle\varphi\rangle\!\rangle(\bar{c})$ and $\langle\!\langle t\rangle\!\rangle(\bar{u},\bar{v},\bar{c})$, with no free individual variables in the first and with designated two lists of all free variables $\bar{u},\bar{v}$ in the second and one list $\bar{c}$ of constants in both (just a list of the points of the given input pg's $\bar{g}$ in any order and possibly with repetitions) all of the same nonzero length (may be different from $\bar{g}$). This may be done in a full correspondence with the old $[\![\text{-}]\!]$-translation:

1. $\bar{g} \models \langle\!\langle\varphi\rangle\!\rangle(\bar{c})$ (i.e. $\langle\!\langle\varphi\rangle\!\rangle(\bar{c})$ is true in a given many-sorted input structure $\bar{g} \in \mathcal{G}$) *iff* $[\![\varphi]\!](\bar{g})$ is true in $\mathcal{G}$ for the same input graphs $\bar{g}$, and
2. the graph defined by $\lambda\bar{u}\bar{v}.$"$\bar{g} \models \langle\!\langle t\rangle\!\rangle(\bar{u},\bar{v},\bar{c})$" is *isomorphic* to that defined by $[\![t]\!](\bar{g})$ with the tuple $\bar{c}$ in the role of the corresponding point,

all of these being provable in KPR_0 (with the isomorphism definable by a Δ_R-term).

Then it proves that $\langle\!\langle\text{-}\rangle\!\rangle$-translation of all Δ_{D}-axioms of BSTA (i.e. all, except Δ_{D}-$\mathsf{Collection}$[5]) including appropriate logical axioms for Δ_{D}-language are provable in LID. Also, logical inference rules for Δ_{D}-language will preserve validity in corresponding many-sorted structures under this translation. All this gives rise to

Theorem 5. $\mathsf{LID} \vdash \langle\!\langle \mathsf{BSTA}|_{\Delta_{\mathsf{D}}}$, *i.e. all* Δ_{D}*-theorems of* $\mathsf{BSTA}\rangle\!\rangle$. [6]

On the other hand, *any* model $\mathcal{M} = \langle M, \mathcal{P}\rangle$ of LID gives rise to a supply $\mathcal{G}^{\mathcal{M}}$ of pointed graphs, just graphs with tuples as vertices defined by $\mathcal{P}$-predicates $\lambda\bar{u}\bar{v}.P(\bar{u},\bar{v})$, each one endowed with corresponding tuple of elements $\bar{m}$ of M as a point. Naturally definable in LID relations $=_0^{\mathcal{M}}$ and $\in_0^{\mathcal{M}}$ also allow to consider the triple $\langle \mathcal{G}^{\mathcal{M}}, =_0^{\mathcal{M}}, \in_0^{\mathcal{M}}\rangle$ almost in the *same* way as $\langle \mathcal{G}, =_0, \in_0\rangle$ in Theorem 2 (just by an imitation of the whole argument).

Theorem 6. $\langle \mathcal{G}^{\mathcal{M}}, =_0^{\mathcal{M}}, \in_0^{\mathcal{M}}\rangle \models \mathsf{BSTA}|_{\Delta_{\mathsf{D}}}$.

[5] Remember, that the full version of BSTA with Δ_{D}-$\mathsf{Collection}$ is a conservative extension of $\mathsf{BSTA}|_{\Delta_{\mathsf{D}}}$, cf. Sect.2 and [Saz85, Saz85a, Saz87].

[6] Note, that Δ_{D}-formulas and terms are formally defined so that each always involves a non-empty list of free variables $\bar{x}$ corresponding as above to a many sorted structure $\bar{g}$ consisting of pg's with length$(\bar{x})$ = length$(\bar{g})$. Otherwise we would consider here also 0-sorted structures, i.e. structures with *no* sorts at all.

Theorem 7 (on a strong completeness of LID). *If any theory Th based on the logic* LID *is formally consistent then it has such a model $\mathcal{M}$ for which in the corresponding universe $\langle \mathcal{G}^{\mathcal{M}}, =_0^{\mathcal{M}}, \in_0^{\mathcal{M}} \rangle$ there exists an inner model which is isomorphic to $\mathcal{M}$.*

Therefore by taking here $Th = \neg\varphi$ and by using also Theorem 6, the conservativity of BSTA over $\mathsf{BSTA}|_{\Delta_{\mathsf{D}}}$ and the abovementioned embedding of LID in KPR_0 and BSTA we obtain

Corollary 8. LID *is conservatively embedded in* BSTA*, i.e. for all* LID*-formulas*

$$\mathsf{LID} \vdash \varphi \quad \text{iff} \quad \mathsf{BSTA} \vdash \text{“} \models_{\mathcal{V}} \varphi \text{”} .$$

Also we can characterize in terms of LID the operations and predicates in the universe $\langle \mathcal{G}, =_0, \in_0 \rangle$ (of Theorem 2) definable in Δ_{D}-language (up to $=_0$).

Theorem 9. (i) *Δ_{D}-definable (or, equivalently, provably-total Σ-definable in* BSTA*) operations and predicares over $\langle \mathcal{G}, =_0, \in_0 \rangle$ coincide (up to $=_0$) with those operations and predicates definable in the language* LID*, for which $=_0$ is a congruence relation.*

(ii) *For $\mathcal{G}$ based on $\mathcal{V} = HF$ each such an operation over graphs is polynomial-time computable (*cf. Theorem 4 *) with respect to the number of vertices of the graphs in $\mathcal{G}$.*

One direction of (i) is based just on $\langle\!\langle\text{-}\rangle\!\rangle$-translation of Δ_{D}-language. Conversely, any LID-definable operation $g' = F(g)$ in $\langle \mathcal{G}, =_0, \in_0 \rangle$ over pg's (as sorts for LID) which preserves $=_0$ may be expressed in Δ_{D} as a composition of the corresponding Δ_{D}-term $\lambda g \in \mathcal{G}.t_F(g)$ (as the second factor of the composition) directly imitating the given LID-definition of F (up to an isomorphism of the output graph g') with $\lambda x \in \mathcal{V}.\langle \in |_{\mathsf{TC}(\{x\})}, x \rangle$ (as the first factor) and with D (as the third factor).

The converse to (ii) (or, more precisely, if any PTIME-computable transformation over finite pg's preserving $=_0$ is Δ_{D}-definable) is an *open question.* Unfortunately, it is also unknown to the author how to Δ_{D}-define, if possible at all, any (natural or not) linear order on the whole (or even on each set of the) universe $\langle \mathcal{G}, =_0, \in_0 \rangle /_{=_0}$ based on $\mathcal{V} = HF$. (Note, that in the universe HF itself a natural linear order *is* definable in the language $\Delta_R + \mathsf{TC}$ [Saz85, Saz87, Saz93].) Otherwise it would be possible to prove the converse of the clause (ii) of this theorem for such universe $\langle \mathcal{G}, =_0, \in_0 \rangle /_{=_0}$ and therefore identify PTIME-computability over such $\langle \mathcal{G}, =_0, \in_0 \rangle /_{=_0}$ with Δ_{D}-expressibility (as it was done for the universe HF and $\Delta_{\mathsf{C}} \rightleftharpoons \Delta_R + \mathsf{TC} + \mathsf{C}$ in op. cit. where C is the *collapsing operation* the same as D except to be nontrivially defined only on *well-founded* graphs).

4 Appendix

4.1 Some technical definitions needed for $[\![\text{-}]\!]$

Let us define for every pg $\langle G, a \rangle$

$$\mathsf{point}\langle G, a \rangle \rightleftharpoons a, \quad \mathsf{graph}\langle G, a \rangle \rightleftharpoons G, \quad \mathsf{vertices}\langle G, a \rangle \rightleftharpoons \mathsf{field}(G) \cup \{a\} \ ,$$

$$\mathsf{elements\text{-}of\text{-}elements}\langle G,a\rangle \rightleftharpoons \{x \in \mathsf{vertices}\langle G,a\rangle : \exists y \in \mathsf{vertices}\langle G,a\rangle.(x \to_G y \to_G a)\} \ ,$$

$$\mathsf{elements}^*\langle G,a\rangle \rightleftharpoons \{x \in \mathsf{vertices}\langle G,a\rangle : (x \to_G \cdots \to_G a)\} \ ,$$

$$\mathsf{new}(v) \rightleftharpoons \{x \in v : x \notin x\}, \quad \mathsf{new\text{-}vertex}(G,A) \rightleftharpoons \mathsf{new}(\mathsf{vertices}\langle G,a\rangle) \ .$$

Here, in the definition of $\mathsf{elements}^*$ $\mathsf{the\text{-}least}$ construct must be actually involved. Evidently, by using the idea of B.Russell's paradox, $\mathsf{new}(v) \notin v$ for any set v. For any pg $x = \langle G,a\rangle$ we will need to consider its *isomorphic copy* $\mathsf{copy}(x) = \mathsf{copy}\langle G,a\rangle$ so that for different x and y the sets of vertices of $\mathsf{copy}(x)$ and $\mathsf{copy}(x)$ do not intersect. This can be done by labelling each vertex of the pg x just by x itself. Given any graph G with a set of its distinguished vertices A, define corresponding pg with A as the set of its $\mathsf{elements}$

$$(\mathsf{collect}\ A\ \mathsf{in}\ G) \rightleftharpoons \langle G \cup \{\langle x, \mathsf{new\text{-}vertex}(G,A)\rangle : x \in A\}, \mathsf{new\text{-}vertex}(G,A)\rangle.$$

Also, given any set of pg's X we can define its *sum*

$$\sum X \rightleftharpoons (\mathsf{collect}\ A\ \mathsf{in}\ G_X) \ ,$$

where $G_X \rightleftharpoons \cup_{x\in X} \mathsf{graph}(\mathsf{copy}(x))$ and $A \rightleftharpoons \{\mathsf{point}(\mathsf{copy}(x)) : x \in X\}$. We will need also the abbreviation

$$\mathsf{the\text{-}unique}\ x \in t.\varphi(x) \rightleftharpoons \cup\{x \in t : \varphi(x)\} \ .$$

Then for $x_0 = \mathsf{the\text{-}unique}\ x \in t.\varphi(x)$

$$\exists! x \in t.\varphi(x) \Rightarrow x_0 \in t \ \&\ \varphi(x_0) \quad \text{and} \quad \neg\exists x \in t.\varphi(x) \Rightarrow x_0 = \emptyset \ .$$

For example, for $\mathsf{the\text{-}unique}\ q \in \mathsf{hvertices}(\tilde{g}).(\tilde{g} * q =_0 p)$ in the definition of $[\![D]\!](g,p)$ one of these two alternatives holds. In the case that required q does not exist, $\emptyset = \mathsf{the\text{-}unique}\ldots$ will be an (improper) isolated point of pg $[\![\mathsf{D}]\!](g,p)$.

The so called "horizontal" notions hpair, etc. are defined as follows:

$$\begin{aligned}
&\mathsf{hpair}(\langle G,\pi\rangle, \langle x,y\rangle) \rightleftharpoons \exists u,v \in \mathsf{field}(G).\\
&\quad (x \to_G u \to_G \pi \ \&\ \ x \to_G v \to_G \pi \ \&\ \ y \to_G v \ \&\\
&\quad\ \forall z \in \mathsf{field}(G).((z \to_G u \Rightarrow z = x) \ \&\\
&\qquad\qquad (z \to_G v \Rightarrow z = x \vee z = y) \ \&\\
&\qquad\qquad (z \to_G \pi \Rightarrow z = u \vee z = v))) \ ,\\
&\mathsf{hgraph}(g) \rightleftharpoons \{\langle x,y\rangle \in (\mathsf{vertices}(g))^2 : \exists \pi \in \mathsf{elements}(g).(\mathsf{hpair}(g*\pi), \langle x,y\rangle))\} \ ,\\
&\mathsf{hvertices}(g) \rightleftharpoons \{x \in \mathsf{vertices}(g) : \exists y \in \mathsf{vertices}(g) \exists \pi \in \mathsf{elements}(g).\\
&\qquad\qquad (\mathsf{hpair}(g*\pi, \langle x,y\rangle) \vee \mathsf{hpair}(g*\pi, \langle y,x\rangle))\} \ .
\end{aligned}$$

All the above constructs are easily definable by Δ_R-terms. However, only the cases of $=_0, \in_0$, $\mathsf{elements}^*$ and $\tilde{g}$ require $\mathsf{Recursive}\ \Delta_R\text{-}\mathsf{Separation}$.

4.2 Proof of the Congruence Lemma

Consider the following cases of the induction step (according to the form of φ or t).

- Let $\varphi(g)$ be $\forall x \in a(g).\psi(x,g)$. Then

$$[\![\varphi(g)]\!] \rightleftharpoons \forall x \in \mathsf{elements}[\![a]\!](g).[\![\psi]\!]([\![a]\!](g) * x, g) \Leftrightarrow \forall X \in_0 [\![a]\!](g).[\![\psi]\!](X,g)$$

by induction hypothesis for $[\![\psi]\!](X,g)$ and because $[\![a]\!](g) * x \in_0 [\![a]\!](g)$ for all $x \in \mathsf{elements}[\![a]\!](g)$. By using induction hypothesis $[\![a]\!](g) =_0 [\![a]\!](g')$ for all $g =_0 g'$ and easily provable the congruence property of $=_0$ with respect to $\in_0$ we also obtain, as required, that

$$[\![\varphi(g)]\!] \Leftrightarrow \forall X \in_0 [\![a]\!](g).[\![\psi]\!](X,g) \Leftrightarrow \forall X \in_0 [\![a]\!](g').[\![\psi]\!](X,g') \Leftrightarrow [\![\varphi(g')]\!] \ .$$

- Let $t(g)$ be $\{a(g), b(g)\}$. Then $[\![t(g)]\!]$ is $\sum\{[\![a]\!](g), [\![b]\!](g)\}$. By induction hypothesis for a and b we have, as required,

$$\sum\{[\![a]\!](g), [\![b]\!](g)\} =_0 \sum\{[\![a]\!](g'), [\![b]\!](g')\} \quad \text{for all } g =_0 g'$$

by constructing a bisimulation from those between $[\![a]\!](g)$ and $[\![a]\!](g')$ and, respectively, between $[\![b]\!](g)$ and $[\![b]\!](g')$.

- Let $t(g)$ be $\cup a(g)$. Then $[\![t(g)]\!]$ is

$$(\mathsf{collect\ elements\text{-}of\text{-}elements}[\![a]\!](g) \mathsf{\ in\ graph}[\![a]\!](g)) =_0$$
$$(\mathsf{collect\ elements\text{-}of\text{-}elements}[\![a]\!](g') \mathsf{\ in\ graph}[\![a]\!](g')) \quad \text{for all } g =_0 g'$$

by constructing a bisimulation from that one between $[\![a]\!](g)$ and $[\![a]\!](g')$.

- Let $t(g)$ be $\{s(x,g) : x \in a(g)\ \&\ \varphi(x,g)\}$. Then $[\![t(g)]\!]$ is

$$\sum\{[\![s]\!]([\![a]\!](g) * x, g) : x \in \mathsf{elements}[\![a]\!](g)\ \&\ [\![\varphi]\!]([\![a]\!](g) * x, g)\} =_0$$
$$\sum\{[\![s]\!]([\![a]\!](g') * x', g') : x' \in \mathsf{elements}[\![a]\!](g')\ \&\ [\![\varphi]\!]([\![a]\!](g') * x', g')\} \ ,$$

as required, for all $g =_0 g'$ by constructing an appropriate bisimulation relation from the given bisimulations respectively between $[\![a]\!](g)$ and $[\![a]\!](g')$, between $[\![s]\!]([\![a]\!](g) * x, g)$ and $[\![s]\!]([\![a]\!](g') * x', g')$, and the equivalence between statements $[\![\varphi]\!]([\![a]\!](g) * x, g)$ and $[\![\varphi]\!]([\![a]\!](g') * x', g')$ with corresponding x and x', via the given bisimulation between $[\![a]\!](g)$ and $[\![a]\!](g')$.

- Let $t(g)$ be $\mathsf{the\text{-}least}\ p.(p = \{x \in a(g) : \varphi(x,p,g)\})$. Then $[\![t(g)]\!] =_0 [\![t(g')]\!]$ is just $(\mathsf{collect}\ q_0(g) \mathsf{\ in\ graph}[\![a]\!](g)) =_0 (\mathsf{collect}\ q_0(g') \mathsf{\ in\ graph}[\![a]\!](g'))$ where

$$q_0(g) \rightleftharpoons \mathsf{the\text{-}least}\ q.(q = \{x \in \mathsf{elements}[\![a]\!](g) :$$
$$[\![\varphi]\!]([\![a]\!](g) * x, (\mathsf{collect}\ q \mathsf{\ in\ graph}[\![a]\!](g)), g)\}) \quad \text{and}$$
$$q_0(g') = \mathsf{the\text{-}least}\ q'.(q' = \{x' \in \mathsf{elements}[\![a]\!](g') :$$
$$[\![\varphi]\!]([\![a]\!](g') * x', (\mathsf{collect}\ q' \mathsf{\ in\ graph}[\![a]\!](g')), g')\}).$$

Suppose $g =_0 g'$. Then by induction hypothesis for a and φ we have a bisimulation relation R witnessing $[\![a]\!](g) =_0 [\![a]\!](g')$, and an equivalence between statements

$$[\![\varphi]\!]([\![a]\!](g) * x, (\mathsf{collect}\ q\ \mathsf{in}\ \mathsf{graph}[\![a]\!](g)), g)) \quad \text{and}$$

$$[\![\varphi]\!]([\![a]\!](g') * x', (\mathsf{collect}\ q'\ \mathsf{in}\ \mathsf{graph}[\![a]\!](g')), g'))$$

with corresponding x and x' and q and q' via the given bisimulation R. Here we say that sets of vertices q and q' *correspond* by R if for each $v \in q$ there exists a $v' \in q'$ such that vRv' and vice versa. We need also use the fact that for each set $q \subseteq \mathsf{elements}[\![a]\!](g)$ satisfying the inequality

$$\{x \in \mathsf{elements}[\![a]\!](g) : [\![\varphi]\!]([\![a]\!](g) * x, (\mathsf{collect}\, q\, \mathsf{in}\, \mathsf{graph}[\![a]\!](g)), g)\} \subseteq q$$

the set $R(q) \rightleftharpoons \{v' \in \mathsf{elements}[\![a]\!](g') : \exists v \in q. vRv'\}$ corresponds to q via R and satisfies the corresponding inequality for q'

$$\{x' \in \mathsf{elements}[\![a]\!](g') : [\![\varphi]\!]([\![a]\!](g') * x', (\mathsf{collect}\, q'\, \mathsf{in}\, \mathsf{graph}[\![a]\!](g')), g')\} \subseteq q' \ .$$

Also if q_0 is the least solution of the first (in)equality then $R(q_0)$ is the least solution of the second. Indeed, if q' is any solution of the second inequality then (as above) $R^{-1}(q')$ is corresponding solution for q of the first and therefore $q_0 \subseteq R^{-1}(q')$. Any $x' \in R(q_0)$ is related by bisimulation R with some $x \in q_0$. It follows that x is an element of the right hand side of the first inequality (which proves to be just equality) with q_0 in place of q and therefore, by monotonicity, with $R^{-1}(q')$ in place of q. Then appropriate bisimulation gives rise to the analogous membership x' to the right-hand-side of the second inequation for q'. Since this right-hand-side is included in q' we have $x' \in q'$ and therefore $R(q_0) \subseteq q'$ what means that $R(q_0)$ is the least solution $q_0(g')$, as required. It follows also that $q_0 = q_0(g)$ is related by the required bisimulation relation with $R(q_0) = q_0(g')$. This finally gives $(\mathsf{collect}\ q_0(g)\ \mathsf{in}\ \mathsf{graph}[\![a]\!](g)) =_0 (\mathsf{collect}\ q_0(g')\ \mathsf{in}\ \mathsf{graph}[\![a]\!](g'))$.

- We leave to the reader the non-considered cases of $\&, \vee, \neg, \exists, \mathsf{TC}$ and D.

4.3 Provability in KPR_0 of $[\![$non-logical axioms of $\mathsf{BSTA}]\!]$

- The case of axioms Pair, Union and Image is straightforward.

- Extensionality follows from the following easy consideration. For any two pg's $\langle G, a\rangle$ and $\langle G', a'\rangle$ if for any $x \to_G a$ there exists $x' \to_{G'} a'$ and a bisimulation relation R such that xRx', and vice versa, then the same holds for the unique, the largest bisimulation relation $\sim_{G,G'}$ between G and G' (extending all these R) which must also connect a and a'. It follows that $\langle G, a\rangle =_0 \langle G', a'\rangle$.

Analogously, if $[\![\langle G, a\rangle \subseteq \langle G', a'\rangle]\!]$ then for some set $A \subseteq \mathsf{elements}\langle G', a'\rangle$ we have $\langle G, a\rangle =_0 (\mathsf{collect}\ A\ \mathsf{in}\ G')$.

- $\mathsf{Recursive}\ \Delta_{\mathsf{D}}\text{-}(\mathsf{and}\ \Delta_R\text{-})\mathsf{Separation}$ axiom may be equivalently rewritten for $p_0 \rightleftharpoons \mathsf{the\text{-}least}\ p.(p = \{x \in a : \varphi(x,p)\})$ as

$$\forall x \in a.(\varphi(x,p_0) \Rightarrow x \in p_0)\ \&\ (\forall x \in a.(\varphi(x,p) \Rightarrow x \in p) \& p \subseteq a \Rightarrow p_0 \subseteq p)\ \ .$$

Its $[\![\text{-}]\!]$-translation looks as follows. If q_0 is Δ_R-defined as

$$q_0 \rightleftharpoons \mathsf{the\text{-}least}\ q.(q = \{x \in \mathsf{elements}[\![a]\!] : [\![\varphi]\!]([\![a]\!] * x, (\mathsf{collect}\ q\ \mathsf{in}\ \mathsf{graph}[\![a]\!]))\})$$

with $[\![p_0]\!] = [\![\mathsf{the\text{-}least}\ p.(p = \{x \in a : \varphi(x,p)\})]\!] \rightleftharpoons (\mathsf{collect}\ q_0\ \mathsf{in}\ \mathsf{graph}[\![a]\!])$ (cf. the definition of $[\![\text{-}]\!]$) then

$$\begin{aligned}&\forall x \in \mathsf{elements}[\![a]\!].([\![\varphi]\!]([\![a]\!] * x, [\![p_0]\!]) \Rightarrow [\![a]\!] * x \in_0 [\![p_0]\!])\ \&\\ &(\forall x \in \mathsf{elements}[\![a]\!].([\![\varphi]\!]([\![a]\!] * x, p) \Rightarrow [\![a]\!] * x \in_0 p)\ \&\\ &\quad \&\ [\![p \subseteq a]\!] \Rightarrow [\![[\![p_0]\!] \subseteq p]\!])\end{aligned}$$

where all free variables such as p are supposed to run over $\mathcal{G}$. To show this let us present the above form of $\mathsf{Recursive}\ \Delta_R\text{-}\mathsf{Separation}$ axiom for q_0:

$$\begin{aligned}&\forall x \in \mathsf{elements}[\![a]\!].([\![\varphi]\!]([\![a]\!] * x, (\mathsf{collect}\ q_0\ \mathsf{in}\ \mathsf{graph}[\![a]\!])) \Rightarrow x \in q_0)\ \&\\ &(\forall x \in \mathsf{elements}[\![a]\!].([\![\varphi]\!]([\![a]\!] * x, (\mathsf{collect}\ q\ \mathsf{in}\ \mathsf{graph}[\![a]\!])) \Rightarrow x \in q)\ \&\\ &\quad \&\ q \subseteq \mathsf{elements}[\![a]\!] \Rightarrow q_0 \subseteq q]\!])\end{aligned}$$

where all free variables such as q run over arbitrary sets (of the original universe $\mathcal{V}$ of KPR_0). Then to prove the above $[\![\text{-}]\!]$-translation of $\mathsf{Recursive}\ \Delta_{\mathsf{D}}\text{-}\mathsf{Separation}$ it is sufficient to note, that

1. $x \in q_0 \Rightarrow [\![a]\!] * x \in_0 (\mathsf{collect}\ q_0\ \mathsf{in}\ \mathsf{graph}[\![a]\!]) = [\![p_0]\!]$,
2. $[\![p \subseteq a]\!]$ is equivalent to existence of $q \subseteq \mathsf{elements}[\![a]\!]$ such that $p =_0 (\mathsf{collect}\ q$-$\mathsf{in}\ \mathsf{graph}[\![a]\!])$ and, moreover, $x \in q \Leftrightarrow [\![a]\!] * x \in_0 (\mathsf{collect}\ q\ \mathsf{in}\ \mathsf{graph}[\![a]\!]) = [\![p_0]\!]$ for all x and for this q, and
3. $q_0 \subseteq q \Rightarrow [\![(\mathsf{collect}\ q_0\ \mathsf{in}\ \mathsf{graph}[\![a]\!]) \subseteq (\mathsf{collect}\ q\ \mathsf{in}\ \mathsf{graph}[\![a]\!])]\!] \Leftrightarrow [\![[\![p_0]\!] \subseteq p]\!]$.

- $\Delta_{\mathsf{D}}\text{-}\mathsf{Collection}$ axiom is translated to

$$\begin{aligned}&\forall x \in \mathsf{elements}[\![a]\!] \exists y \in \mathcal{G}.[\![\varphi]\!]([\![a]\!] * x, y) \Rightarrow\\ &\ \exists z \in \mathcal{G} \forall x \in \mathsf{elements}[\![a]\!] \exists v \in \mathsf{elements}(z).[\![\varphi]\!]([\![a]\!] * x, z * v)\ \ .\end{aligned}$$

According to the original $\Delta_R\text{-}\mathsf{Collection}$ the antecedent of this formula implies

$$\exists Z \forall x \in \mathsf{elements}[\![a]\!] \exists y \in Z.[\![\varphi]\!]([\![a]\!] * x, y)\ \ .$$

It remains to take $z \rightleftharpoons \sum Z$.

- Consider $[\![\text{-}]\!]$-translations of AFA axioms. First note, that AFA_1 may be rewritten also as conjunction of the following two formulas:

$$\forall z \in \mathsf{D}(g,p) \exists \langle q,p \rangle \in g.(z = \mathsf{D}(g,q)) \quad \text{and} \quad \forall \langle q,p \rangle \in g.(\mathsf{D}(g,q) \in \mathsf{D}(g,p))$$

or, more detailed, for the first formula

$$\forall z \in \mathsf{D}(g,p) \exists w \in g \exists u,v \in w \exists q \in u.$$
$$(p,q \in v \;\&\; \forall x \in u.(x=q) \;\&$$
$$\forall x \in v.(x=q \vee x=p) \;\&\; \forall x \in w.(x=u \vee x=v) \;\&\; z=\mathsf{D}(g,q)) \;\;,$$

and analogously for the second formula. Their translations

$$\forall z \in \mathsf{elements}([\![\mathsf{D}]\!](g,p))$$
$$\exists w \in \mathsf{elements}(g) \exists u,v \in \mathsf{elements}(g*w) \exists q \in \mathsf{elements}(g*u).$$
$$(p, g*q \in_0 g*v \;\&$$
$$\forall x \in \mathsf{elements}(g*u).(g*x =_0 g*q) \;\&$$
$$\forall x \in \mathsf{elements}(g*v).(g*x =_0 g*q \vee g*x =_0 p) \;\&$$
$$\forall x \in \mathsf{elements}(g*w).(g*x =_0 g*u \vee g*x =_0 g*v) \;\&$$
$$[\![\mathsf{D}]\!](g,p)*z =_0 [\![\mathsf{D}]\!](g,g*q)) \;\;,$$

and analogously for the second formula, are provable in KPR_0 straightforwardly (just take for the first formula w, u, v, q, s.t. $q \in z$, i.e. $z = [q]$).

$[\![\mathsf{AFA}_2]\!]$ is $[\![Bis_\in(R)]\!] \& [\![xRy]\!] \Rightarrow x =_0 y$ where $[\![Bis_\in(R)]\!]$ is equivalent to

$$\forall x,y \in \mathcal{G}.([\![xRy]\!]) \Rightarrow$$
$$(\forall x' \in \mathsf{elements}(x) \exists y' \in \mathsf{elements}(y).[\![x*x'Ry*y']\!]) \;\&$$
$$(\forall y' \in \mathsf{elements}(y) \exists x' \in \mathsf{elements}(x).[\![x*x'Ry*y']\!]))$$

and implies that for any $x, y \in \mathcal{G}$ the relation $\lambda x'y'.[\![x*x'Ry*y']\!]$ is a bisimulation relation between $\mathsf{graph}(x)$ and $\mathsf{graph}(y)$. It follows that also $\mathsf{KPR}_0 \vdash [\![\mathsf{AFA}_2]\!]$.

- Finally, the transitive closure axioms are translated as

$$\forall v \in \mathsf{elements}(x).(x*v \in_0 [\![\mathsf{TC}(x)]\!]) \;\;,$$
$$\forall u \in \mathsf{elements}[\![\mathsf{TC}(x)]\!] \forall v \in \mathsf{elements}([\![\mathsf{TC}(x)]\!]*u).([\![\mathsf{TC}(x)]\!]*v \in_0 [\![\mathsf{TC}(x)]\!]) \;\;,$$
$$(\forall v \in \mathsf{elements}(x).(x*v \in_0 y) \Rightarrow$$
$$(\forall u \in \mathsf{elements}([\![\mathsf{TC}(x)]\!]).([\![\mathsf{TC}(x)]\!]*u \in_0 [\![\mathsf{TC}(y)]\!]) \quad \text{and}$$
$$\forall u \in \mathsf{elements}(x) \forall v \in \mathsf{elements}(x*u).(x*v \in_0 x) \Rightarrow [\![\mathsf{TC}(x) \subseteq x]\!] \;\;.$$

The first two formulas are evident. For the third formula note, that the set $\{u \in \mathsf{elements}^*(x) : \exists w \in \mathsf{elements}^*(y).(x*u =_0 y*w)\}$ contains $\mathsf{elements}^*(x)$ because it satisfies corresponding equation for $\mathsf{elements}^*(x)$. Analogously, antecedent of the last formula implies that the set $\{w \in \mathsf{vertices}(x) : \exists w' \in \mathsf{elements}(x).(x*w =_0 x*w')\}$ includes $\mathsf{elements}^*(x)$ what is essentially we need.

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