Phase-Transitions in Algorithmically Undecidable Languages*)

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Abstract

In this paper phase-transition phenomena occurring in a particular class of intractable decisions problems are considered: those which are *undecidable*. This is in contrast to earlier studies in which the (widely held to be) intractable class of *NP*-complete decision problems have formed the principal area of interest. We determine an asymptotic threshold function for an encoding of Post's Correspondence Problem that establishes conditions for a random instance 'almost certainly' to have (respectively, not to have) a solution. In addition, we prove a partial result concerning thresholds for the Matrix Mortality Problem. The aim of considering this particular intractable class is to attempt to glean further understanding of the apparent correspondence between a purely combinatorial property (that of almost all instances of a problem meeting certain constraints having/not having a solution) and the experimentally observed improved performance of search processes on such instances, when the general problem is intractable. While there can be no generally effective search process to solve undecidable problems, our proof of the combinatorial property trivially leads to an algorithm for the Correspondence Problem which is effective for 'almost all' instances 'outside' the threshold regions. The nature of this algorithm raises a number of possible analytic explanations for the apparent speed-up of search processes when dealing with analogous effects in *NP*-complete decision problems.

1. Introduction

Numerous studies, both analytic and empirical, have reported a property shared by many combinatorial search problems: that of having a 'phase-transition', e.g. (Cheeseman, Kanefsky, and Taylor, 1991), (Chvátal and Szemerédi, 1988), (Crawford and Auton, 1996), (Gent and Walsh, 1996), (Hogg and Williams, 1994), (Kirousis, Kranakis, and Krizanc, 1996), (Koršunov, 1976), (Mitchell, Selman, and Levesque, 1992), (Schrag and Crawford, 1996) etc. Informally, this is the phenomenon whereby typical instances of a given problem progress from 'almost certainly' having a solution to 'almost certainly' not having a solution as some parameter of the instances is varied. Furthermore, in many problems this shift in the likelihood of a solution existing is characterised by a sharp 'threshold' value of the parameter in terms of the instance size, e.g. for the decision problem 3-*SAT*, experimental studies indicate that random instances in which the ratio of number of clauses (m) to number of variables (n) exceeds 4.24 tend to be unsatisfiable, whereas those for which this ratio is less than 4.24 are usually satisfiable. It has been, almost invariably, the case that studies of such phase-transitions have examined these with respect to *NP*-complete decision problems (significant exceptions are analytic studies of tractable variants of the Satisfiability problem, e.g. 2-*SAT*: Goerdt (1992), (Chvátal and Reed, 1992); Horn-clause *SAT*: (Istrate and Ogihara, 1998); k-Horn-clause *SAT*: (Dunne and Bench-Capon, 1997). The motivation for concentrating on this particular class of intractable problems comes from experimental studies on the performance of search heuristics when dealing with typical instances lying outside the neighbourhood of the threshold. In the case of 3-*SAT*, such methods appear to be able to resolve typical n-variable, m-clause instances very quickly whenever $m/n < 4.24$ or $m/n > 4.24$. Thus, there appears to be a correlation between the combinatorial property of a decision problem exhibiting a phase-transition and the efficiency of search processes for solving the problem. While some researchers have attempted to study this connection in a general experimental framework, notably (Gent *et al.*, 1996) through the concept of 'constrainedness', a generally applicable analytic demonstration that search performance is enhanced outside the threshold regions has yet to be presented. Indeed, the results proved in (Dunne, Gibbons, and Zito, 1998) provide some evidence that even with the so-called 'easy' regions *NP*-complete problems remain fundamentally intractable, since search methods meeting

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particular performance guarantees for instances in the 'easy' regions could be used to construct methods with similar performance for general instances.

Our purpose in the present note, is to present some further examples of problems for which there can be no clear correlation between the presence of phase-transition effects (as regards the likelihood of an instance having a solution) and the typical performance of a search process for the problem. The significant difference between the decision problems addressed in this note and previous work, is that the examples considered correspond to *undecidable* decision problems: namely, Post's Correspondence Problem over a finite alphabet Σ - hereafter denoted $PCP(\Sigma)$; and, a version of the Matrix Mortality Problem. We first present a representation of the former problem as a monotone language, i.e. instances of PCP of 'size N' are subsets of an underlying N element 'basis' set, so that if W is a set for which $PCP(\Sigma)$ is 1, then any super-set of W also has $PCP(\Sigma)=1$. We then show that if $I_{N,m(N)}$ is the set of instances of $PCP(\Sigma)$, containing exactly $m(N)$ elements from the basis set of N elements, then there is a function $\theta:\mathbf{N}\rightarrow\mathbf{N}$ satisfying: for any strictly monotone increasing function $f:\mathbf{N}\rightarrow\mathbf{N}$ such that $\lim_{n\rightarrow\infty} f(n)/\theta(n)=0$

$$\lim_{N\rightarrow\infty}\frac{|\{\, x\in I_{N,\lceil f(N)\rceil} : PCP(\Sigma)(x)=1\,\}|}{|I_{N,\lceil f(N)\rceil}|}=0$$

For any strictly monotone increasing function, $f:\mathbf{N}\rightarrow\mathbf{N}$, such that $\lim_{n\rightarrow\infty}\theta(n)/f(n)=0$

$$\lim_{N\rightarrow\infty}\frac{|\{\, x\in I_{N,\lceil f(N)\rceil} : PCP(\Sigma)(x)=1\,\}|}{|I_{N,\lceil f(N)\rceil}|}=1$$

The function $\theta(n)$ is explicitly identified as $\sqrt{n}/\log n$. We observe that $\theta(n)$ is a threshold function in the sense of (Bollobás and Thomason, 1987). We also note that, since our representation is not a symmetric property, the existence of a phase-transition with *sharp* threshold is not guaranteed by the results of (Friedgut and Kalai, 1996). For the Matrix Mortality Problem, we prove a lower bound on the size of instances such that solutions 'almost certainly' exist.

2. Results

Definition 1: Let Σ be a finite alphabet containing at least two symbols with Σ^* denoting the set of all finite length words over Σ. The *Correspondence Problem* over Σ is the decision problem defined as follows: an instance of the correspondence problem is a finite set $W\subseteq\Sigma^*\times\Sigma^*$ of ordered pairs of words over Σ. An instance

$$W \;=\; \{\,(x_1,y_1)\,;(x_2,y_2)\,;\cdots;(x_m,y_m)\,\}$$

of the Correspondence Problem has a *solution*, i.e. the answer returned is 1, if there exists a finite sequence $<i_1,i_2,\ldots,i_r>$ of (not necessarily distinct) integers with $1\le i_j\le m$ such that

$$x_{i_1}\,x_{i_2}\cdots x_{i_r}=y_{i_1}\,y_{i_2}\cdots y_{i_r}$$ •

We denote by $L_{PCP(\Sigma)}$ the language comprising those sets of pairs W from $\Sigma^*\times\Sigma^*$ such that W has a solution. •

Fact 1: For any alphabet Σ comprising at least 2 symbols, the Correspondence Problem over Σ is undecidable.

Proof: Proofs may be found in most introductory texts on Computability Theory, e.g. Dunne (1992), (Hopcroft and Ullman, 1979). □

Consider the set of words

$$\Sigma_{\le n} =_{def} \bigcup_{r=0}^{n}\{\,\sigma_1,\ldots,\sigma_k\,\}^r$$

Then it is clear that

$$L_{PCP(\Sigma)} \;=\; \bigcup_{n=0}^{\infty} L^n_{PCP(\Sigma)}$$

where

$$L^n_{PCP(\Sigma)} \; = \; \{ \; W \subseteq \Sigma_{\leq n} \times \Sigma_{\leq n} \; : \; W \; has \; a \; solution \; \}$$

In other words, $L_{PCP(\Sigma)}$ can be split into sets of instances W in which every word in W has length at most n, for $n = 0, 1, 2, \ldots,$.

Definition 2: Let Σ be a finite alphabet of k symbols (with $k \geq 2$) as before. The *basis set* for n, denoted $B(n, k)$, is the set $B(n, k) = \Sigma_{\leq n} \times \Sigma_{\leq n}$. If $W \subseteq B(n, k)$ then W is an instance of $PCP(\Sigma)$ of *size* $N(n, k)$ where

$$N(n, k) \; =_{def} \; |B(n, k)| \; = \; |\Sigma_{\leq n}|^2$$

We denote by $PCP_{N(n,k)}(\Sigma)$ the restriction of the Correspondence Problem to instances $W \subseteq B(n, k)$. Finally we say that $W \subseteq B(n, k)$ has the *correspondence property* if W has a solution to the Correspondence Problem. •

We note that if there were an algorithm which for *all* n, k, and $W \subseteq B(n, k)$ determined whether W had the correspondence property, then this would contradict the undecidability of $PCP(\Sigma)$.

We subsequently drop the dependence on n and k in $N(n, k)$ and $B(n, k)$ where this will not create any ambiguity.

Fact 2: The Correspondence Property is monotone, i.e. if $V \subseteq W \subseteq B$ and V has the correspondence property then W has the correspondence property.

Proof: Obvious, since any solution for V will also be a solution for W. □

We can now establish the main result of this note, namely to show that the Correspondence Property has a threshold function and to determine this, i.e. we show for a specific function $\theta : \mathbf{N} \to \mathbf{N}$ that: for any function $f : \mathbf{N} \to \mathbf{N}$

a) If $f(N) = o(\theta(N))$ then 'almost all' instances of size N containing at most $f(N)$ elements of B do not have the Correspondence Property.

b) If $f(N) = \omega(\theta(N))$ then 'almost all' instances of size N containing at least $f(N)$ elements of B do have the the Correspondence Property.

Let W_p be the probability space comprising random subsets, W of B constructed as follows: for a probability p each pair $(x, y) \in B$ is included in W with probability p and excluded with probability $(1 - p)$ (all choices being made independently). In this model the expected number of pairs in $W \in W_p$ is pN. It is well-known that results proved in this model can be translated into the model in which W is chosen uniformly at random from the set of subsets containing pN elements. We denote the probability of $W \in W_p$ having some property X by $P[\, W_p \; has \; X \,]$.

Theorem 1: Let $\theta(n) = (\sqrt{n} \log n)$.

a) If $\lim_{N \to \infty} p(N)\,\theta(N) = 0$, i.e. $p(N) = 1/\omega(\theta(N))$ then

$$\lim_{N \to \infty} P[\, W_p \; has \; the \; Correspondence \; Property \,] \; = \; 0$$

b) If $\lim_{N \to \infty} (\, p(N)\theta(N)\,)^{-1} = 0$, i.e. $p(N) = 1/o(\theta(N))$ then

$$\lim_{N \to \infty} P[\, W_p \; has \; the \; Correspondence \; Property \,] \; = \; 1$$

Proof: We denote by $P[\, W \in PCP(\Sigma) \,]$ the probability of the random variable $W \in W_p$ having the Correspondence Property.

a): A necessary condition for $W \subseteq B$ to have the Correspondence Property is that W has a non-empty intersection with the set $P(n, k) \bigcup S(n, k)$ defined by

$$P(n, k) \; =_{def} \; \{ \, (\, x, xy \,) : x \in \Sigma_{\leq n} \; and \; xy \in \Sigma_{\leq n} \, \}$$
$$S(n, k) \; =_{def} \; \{ \, (\, xy, x \,) : x \in \Sigma_{\leq n} \; and \; xy \in \Sigma_{\leq n} \, \}$$

Without this condition being satisfied, it is not possible to choose a pair to start the correspondence solution sequence. It follows that

$$P[\,W \notin PCP(\Sigma)\,] \;\geq\; P[\,\forall\, x, y\ (x, xy) \notin W \ and\ (xy, x) \notin W\,]$$

$$\geq\; (1-p)^{|P(n,k)|+|S(n,k)|} \;\geq\; (1-p)^{2|S(n,k)|}$$

$$\geq\; 1 - 2\,p\,|S(n,k)| \tag{1}$$

To complete the proof of part (a), we determine the size of $S(n, k)$. We have

$$|\,S(n,k)\,| \;=\; |\,\{\,(xy, x) : x \in \Sigma_{\leq n} \ and\ xy \in \Sigma_{\leq n}\,\}\,|$$

$$=\; \sum_{r=0}^{n} k^r \sum_{s=0}^{n-r} k^s \;=\; \sum_{r=0}^{n} k^r \left(\frac{k^{n-r+1}-1}{k-1} \right) \geq\; \frac{(n+1)\,k^{n+1}}{(k-1)^2} \tag{2}$$

By definition,

$$N \;=\; (\sum_{r=0}^{n} k^r\,)^2 \;=\; \Theta(\,k^{2(n+1)}\,) \tag{3}$$

Combining (2) and (3) it follows that $|S(n,k)| = \Theta(\sqrt{N}\log N\,)$, thus if $p(N) = 1/\omega(\sqrt{N}\log N\,)$ then the probability in (1) approaches 1.

For part (b), a sufficient condition for W to have the Correspondence Property is that, for some x, y, z in $\Sigma_{\leq n}$ W contains both $(\,x, xy\,)$ and $(\,yz, z\,)$ (where xy and yz are in $\Sigma_{\leq n}$). Using this condition, gives

$$P[\,W \in PCP(\Sigma)\,] \;\geq\; P[\,\exists\, y \in \Sigma_{\leq n}\ \exists\, x, z \in \Sigma_{\leq n} : (\,x, xy\,) \in W \ and\ (\,yz, z\,) \in W\,]$$

$$=\; 1 - P[\,\forall\, x, y, z\ (\,x, xy\,) \notin W \ or\ (\,yz, z\,) \notin W\,]$$

$$\geq\; 1 - P[\,\forall\, x, y\ (x, xy) \notin W\,] - P[\,\forall\, y, z\ (yz, z) \notin W\,]$$

$$\geq\; 1 - 2(\,1-p\,)^{|S(n,k)|} \;\geq\; 1 - \exp(\,-\,p|S(n,k)| + \log 2\,) \tag{4}$$

Recalling that $|S(n,k)| = \Theta(\sqrt{N}\log N\,)$, if $p(N) = 1/o(\sqrt{N}\log N\,)$ then the probability in (4) approaches 1. □

Corollary 1: Let $\theta(\,n\,) = \sqrt{n}/\log n$.

a) If $f(n)$ is such that $\lim_{n\to\infty} f(n)/\theta(n) = 0$, i.e. $f(n) = o(\sqrt{n}/\log n)$ then

$$\lim_{N\to\infty} \frac{|\{\,W \subseteq B : |W| = f(N) \ and\ W\ has\ the\ Correspondence\ Property\,\}|}{|\{\,W \subseteq B : |W| = f(N)\,\}|} \;=\; 0$$

b) If $f(n)$ is such that $\lim_{n\to\infty} \theta(n)/f(n) = 0$, i.e. $f(n) = \omega(\sqrt{n}/\log n)$ then

$$\lim_{N\to\infty} \frac{|\{\,W \subseteq B : |W| = f(N) \ and\ W\ has\ the\ Correspondence\ Property\,\}|}{|\{\,W \subseteq B : |W| = f(N)\,\}|} \;=\; 1 \qquad \square$$

Now consider the following algorithm.

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Input: W = { ( x_1, y_1 ), . . . , ( x_m, y_m ) } ⊆ B(n, k)
Method:
x_xy_found := false; xy_x_found := false;
prefix_list := ∅; suffix_list := ∅;
for i := 1 to m loop
  if y_i = x_i z then
    -- A pair of the form ( x, xz )
    x_xy_found := true;
    suffix_list := suffix_list ∪ { z };
  end if;
  if x_i = zy_i then
    -- A pair of the form ( zy, y )
    xy_x_found := true;
    prefix_list := prefix_list ∪ { z };
  end if;
end loop;
if not ( x_xy_found or xy_x_found ) then
  -- W does not have a solution to the Correspondence Problem
  return 0;
elsif prefix_list ∩ suffix_list ≠ ∅ then
  -- W contains a pair ( x, xy ) and ( yz, z )
  -- (or a pair ( x, x ), which places ε, the empty word
  --  in both prefix_list and suffix_list)
  return 1;
else
  return ?;
end if;
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Algorithm 1: Partial Decision Procedure for the Correspondence Problem.

Corollary 2: If $f(n) = o(\sqrt{n}/\log n)$ Algorithm 1 correctly determines, in time polynomial in $|W|$, if $W \subseteq B$ has the Correspondence Property for almost all W containing at most $f(N)$ pairs.

If $f(n) = \omega(\sqrt{n}/\log n)$ Algorithm 1 correctly determines, in time polynomial in $|W|$, if $W \subseteq B$ has the Correspondence Property for almost all W containing at least $f(N)$ pairs.

Proof: Immediate from Corollary 1, since all the algorithm does is to verify if the necessary condition for a solution to exist holds, and if it does to check whether there are pairs of the form (x, xy) and (yz, z) present in W. Corollary 1 shows that the necessary condition fails to hold for almost W of size at most $o(\sqrt{N}/\log n)$ and that the sufficient condition holds for almost all W of size at least $\omega(\sqrt{N}/\log n)$. The algorithm fails to make a decision on a vanishingly small fraction of such instances. The polynomial time complexity is obvious. □

We now turn to our second example problem, for which our results are not quite as complete.

Definition 3: The (3, 2)-*Matrix Mortality Problem* takes as an instance a set of 3×3 integer-valued matrices $\mathbf{M} = \{ M_1, M_2, \ldots, M_r \}$ and returns 1 if there exists a finite sequence $< i_1, i_2, \ldots, i_k >$ of values $1 \le i_j \le r$ such that the (3, 2)-entry of the matrix product $\prod_{j=1}^{k} M_{i_j}$ is equal to 0. □

Fact 3: The (3, 2)-Matrix Mortality Problem is undecidable.

Proof: See Manna (1974), pp. 63-4; or Dunne (1992) pp. 93-95. The proof given is both texts is due to Floyd. □.

Let $M(n)$ (for $n \in \mathbf{N}$) denote the set of all 3×3 integer matrices whose entries, x, satisfy $-n \le x \le n$. Clearly, any instance, $\mathbf{M}$, of the Matrix Mortality Problem is a subset of $M(n)$ for some value of n. We can thus define the *Mortality Property* for subsets of $M(n)$, denoted $MP(n)$ by

$$MP(n) = \{ \mathbf{M} \subseteq M(n) : \textit{The Matrix Mortality Problem returns } 1 \textit{ on input } \mathbf{M} \}$$

It is obvious that the Mortality Property is monotone. For a given probability value, p, consider the random subset M_p of $M(n)$ chosen by including each matrix of $M(n)$ in M_p with probability p and excluding it with probability $1 - p$, all choices being made independently.

Theorem 2:

$$p(n) = \frac{1}{o(|M(n)|^{0.5(1+\log_9 6)})} \Rightarrow \lim_{n\to\infty} P[\, M_p \in MP(n)\,] = 1$$

Proof: Consider any matrix $X = [\, x_{i,j}\,]$ in $M(n)$. Any matrix Y whose second column $< y_{12}, y_{22}, y_{32} >$ is of the form $< -x_{32}, (x_{31} + x_{33}), -x_{32} >$ or $< -x_{32}, (x_{31} - x_{33}), x_{32} >$ is such that: the $(3,2)$-entry of XY equals

$$x_{31}\, y_{12} + x_{32}\, y_{22} + x_{33}\, y_{32} = \begin{cases} -x_{32}(x_{31} + x_{33}) + x_{32}(x_{31} + x_{33}) = 0 \\ or \\ -x_{32}(x_{31} - x_{33}) + x_{32}(x_{31} - x_{33}) = 0 \end{cases}$$

For any values, x_{31}, x_{33}, if $-n \le x_{31}, x_{33} \le n$ then at least one of $-n \le x_{31} + x_{33} \le n$, $-n \le x_{31} - x_{33} \le n$ is true. It follows that for any $X \in M(n)$ there are at least 6^n matrices $Y \in M(n)$ such that the $(3,2)$-entry of XY is equal to 0. Let us call such a pair $< X, Y >$ a *mortal pair*. We now have,

$$\begin{aligned} P[\, M_p \in MP(n)\,] &\ge P[\, \exists < X, Y > \in M_p : < X, Y > \textit{ is a mortal pair}\,] \\ &\ge 1 - P[\, M_p \textit{ contains no mortal pair}\,] \\ &\ge 1 - (1 - p^2)^{|\{ <X,Y> : <X,Y> \textit{ is a mortal pair} \}|} \\ &\ge 1 - \exp(-p^2\, |\{ < X, Y > : < X, Y > \textit{ is a mortal pair} \}|) \end{aligned}$$

For each of the 9^{2n+1} matrices, X, in $M(n)$ there are at least 6^{2n+1} matrices Y in $M(n)$ that could be used to form a mortal pair with X. Hence, the number of distinct mortal pairs is at least

$$9^{2n+1}\, 6^{2n+1} = |M(n)|^{1+\log_9 6}$$

and so, if $p(n)$ is as given by the Theorem statement, then then this probability approaches 1. □

Corollary 3: If $m(n) = \omega(|M(n)|^{0.5(1-\log_9 6)}) \approx \omega(|M(n)|^{0.0923})$, then almost all subsets of $M(n)$ containing at least $m(n)$ matrices have the Mortality Property.

Proof: Using the value of $p(n)$ established in the Theorem, the expected number of matrices in M_p, is $p\,|M(n)|$ which is the function give in the Corollary. □

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