Vizing Theorem

Theorem 2.2 *For every simple graph $\chi'(G) \leq \Delta + 1$.*

Proof. Let G be the input graph. We present an algorithm that colors the edges of G using at most $\Delta + 1$ colors.

The algorithm has the following framework.

Algorithm Edge-coloring
Input: a graph G on n vertices with maximum degree Δ
Output: an edge colouring of G with $\Delta + 1$ colours
(1) let G_0 be the empty graph on n vertices
/* edges of G are $e_1, \ldots, e_m$ */
(2) **for** $i = 1$ **to** m
extend to colouring of G_{i-1} to colour $G_i = G_{i-1} \cup \{e_i\}$

We need to explain how the graph G_i can be coloured with at most $\Delta + 1$ colors. Inductively, suppose that we have coloured the edges of G_{i-1} using at most $\Delta + 1$ colors. Now $G_i = G_{i-1} \cup \{e_i\}$ where suppose $e_i = (v_1, w)$.

CASE 1 If both vertices v_1 and w miss[2] a common color c, then we simply color e_i with c and we obtain a valid coloring for the graph G_i.

CASE 2. We suppose that there is no color that is missed by both v_1 and w. Let $c_1 \in \text{Missed}(v_1)$ and $c_0 \in \text{Missed}(w)$. Since $c_1 \notin \text{Missed}(w)$ there is an edge (v_2, w) colored with c_1. Now if v_2 and w have a common missed colour, we stop. Otherwise, let $c_2 \in \text{Missed}(v_2)$ and $c_2 \notin \text{Missed}(w)$. Let (v_3, w) be the edge coloured c_2.

We thus construct a "fan" (see Figure 2) that consists of h neighbours $v_1, \ldots, v_h$ of w and $h-1$ different colours $c_1, \ldots, c_{h-1}$, such that:

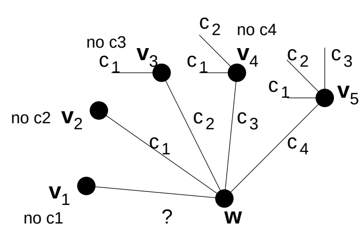

Figure 2: A fan in the execution of the algorithm implicit in the proof of Vizing's theorem.

c1 for all $j = 1, \ldots, h-1$, v_j misses c_j and (v_{j+1}, w) is colored c_j;

c2 none of $v_1, \ldots, v_{h-1}$ have a common missed color with w;

c3 for all $j = 2, \ldots, h-1$, v_j does not misses $c_1, \ldots, c_{j-1}$.

Notice that requirement **c3** is true of v_2 (v_2 does not miss c_1, just because (v_2, w) takes this colour in G_{i-1}) but in general it implies that $j-2$ edges incident to v_j are coloured $c_1, \ldots, c_{j-2}$.

There are three possible sub-cases.

SubCase 1. the vertex v_h satisfies **c3** and v_h has no common missed color with w. Then we can expand the fan (there must be (v_{h+1}, w) colored c_h). Since the degree of w is finite, SubCase 1 must fail at some stage and one of the following two other cases should happen.

SubCase 2. the vertex v_h has a common missed color c_0 with w (see Figure 3).

Then we change the coloring of the fan by coloring (v_h, w) with c_0, and coloring (v_i, w) with c_i for $i = 1, \ldots, h-1$. It is easy to verify that this gives a valid edge coloring for the graph G_i.

[2]We say that a vertex u in a graph misses a color c if no edge incident on u is colored c. Note that each vertex **must** miss at least one color.

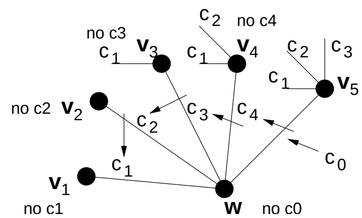

Figure 3: Solution to SubCase 2.

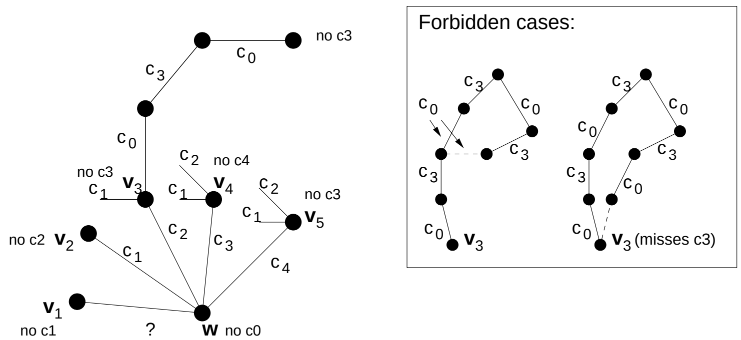

Figure 4: SubCase 3.1 with $u_t \neq w$.

SubCase 3. the vertex v_h misses a color c_s, $1 \leq s \leq h-1$. Let c_0 be a color missed by w. We start from the vertex v_s. Since v_s has no common missed color with w, there is an edge (v_s, u_1) colored with c_0. Now if u_1 does not miss c_s, there is an edge (u_1, u_2) colored with c_s, now we look at vertex u_2 and see if there is an edge colored with c_0, and so on. By this, we obtained a path P_s whose edges are alternatively colored by c_0 and c_s. The path has the following properties:

p1 the path P_s must be finite and simple since each vertex of the graph G_{i-1} has at most two edges colored with c_0 and c_s;

p2 the path P_s cannot be a cycle since the vertex v_s misses the color c_s; and

p3 the vertex w is not an interior vertex of P_s since w misses the color c_0 (it can only be u_t).

Let $P_s = \{v_s, u_1, \ldots, u_t\}$, where v_s misses color c_s, u_t misses either c_s or c_0.

If $u_t \neq w$, then interchange the colors c_0 and c_s on the path to make vertex v_s miss c_0. Then color (v_s, w) with c_0 and color (v_j, w) with c_j, for $j = 1, \ldots, s-1$.

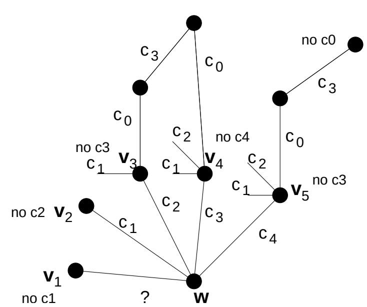

Figure 5: SubCase 3.2: $u_t = w$.

If $u_t = w$, we must have $u_{t-1} = v_{s+1}$ (see Figure 5). Then we grow a $c_0 - c_s$ path P_h starting from v_h which also misses color c_s. P_h must be finite and simple. Moreover, P_h cannot end at w. Therefore, we interchange colors c_0 and c_s on P_h to make v_h miss c_0. Then color (v_h, w) with c_0 and (v_j, w) with c_j for $j = 1, \ldots, h-1$.

It is also easy to see that this process can be implemented by a polynomial time algorithm. We leave the detailed implementation of this process to the interested reader. ■

Exercise. Find an example graph on which the algorithm implicit in the last proof performs the Kempe interchange (with colours stored as integers, chosen in increasing order and edges colored starting from those adjacent to v_1 then to v_2 and so on). Petersen graphs (example of non-factorizable graphs) are $\Delta + 1$ colorable but do not seem to require any color interchange.