# On the Membership of Invertible Diagonal and Scalar Matrices

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**Abstract**

In this paper we consider decidability questions that are related to the membership problem in matrix semigroups. In particular we consider the membership of a given invertible diagonal matrix in a matrix semigroup and then a scalar matrix, which has a separate geometric interpretation. Both problems have been open for any dimensions and are shown to be undecidable in dimension 4 with integral matrices by a reduction of the Post Correspondence Problem (PCP). Although the idea of PCP reduction is standard for such problems, we suggest a new coding technique to cover the case of diagonal matrices.

*Key words:* Membership problem, Matrix semigroups, Diagonal matrix, Undecidability, Post correspondence problem

## 1 Introduction

In this paper we consider decidability questions that are related to the membership problem in matrix semigroups. The membership problem for a semigroup with only one generator ("is a matrix $B$ a power of a matrix $A$") was known as the "orbit problem" and was shown to be decidable (in polynomial time) by Kannan and Lipton in 1986 [7]. The most natural generalization of the the "orbit problem" is the membership problem for matrix semigroups, given by a list of generators.

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**Problem 1 The membership problem.** *Let $S$ be a given finitely generated semigroup of $n \times n$ matrices from $\mathbb{Q}^{n\times n}$. Determine whether a matrix $M$ belongs to $S$. In other words, determine whether there exists a sequence of matrices $M_1, M_2, \ldots, M_k$ in $S$ such that $M_1 M_2 \cdots M_k = M$.*

Paterson [9] showed that the problem is undecidable even for $3 \times 3$ integral matrices when he considered a special case of the membership problem for matrix semigroups - the *mortality problem* (determination of whether the zero matrix belongs to a matrix semigroup). It was shown in [6] that the mortality problem is undecidable even for a case of two generators where the dimension of the matrices is at least 24. The most general class of matrix semigroups with a decidable membership problem that is currently known is a class of row-monomial matrices over a commutative semigroup [8].

The current research in this area is focused on long standing open problems in low dimensions such as freeness, membership or vector reachability problems for $2 \times 2$ matrices and on the problems that are open in any dimension, like the membership problem for the identity matrix in a matrix semigroup [1]. A related problem, also open at the moment, asks whether the semigroup S contains a diagonal matrix [2]. In this context, we consider the following problem:

**Problem 2** *Given a multiplicative semigroup $S$ generated by a finite set of $n \times n$ integer matrices and an invertible diagonal matrix $M_{\mathrm{D}}$. Decide whether the semigroup $S$ contains the matrix $M_{\mathrm{D}}$.*

In this paper we show that above problem is undecidable in dimension 4. As a corollary of the above fact we show that the membership of any given scalar matrix $k \cdot I$ where $k \neq 0, \pm 1$, is also undecidable for a $4 \times 4$ rational matrix semigroup. One of the interpretations for a scalar matrix is geometric scaling that alters the size of an object. So checking the membership of a scalar matrix gives us an answer to the following geometric problem:

**Problem 3 Scaling problem.** *Given a finite set of linear transformations with integer coefficients and a scalar $k \in \mathbb{Z} \setminus \{0, \pm 1\}$. Decide whether it is possible to have a combination of such transformations that enlarges the size of an object by $k$ times.*

We show a particular scalar for which Problem 3 is undecidable. Both Problem 2 and Problem 3 are shown to be undecidable by a reduction of the Post Correspondence Problem (PCP) to the membership problem in an integral matrix semigroup. The idea of PCP reduction is quite common in algorithmic matrix problems [6,9,10], but in this paper we use a different technique. In particular, in the process of coding the words and indices, we design a set of matrices in such a way that only the correct order of matrix products leads to a particular matrix in a semigroup. In other words we design the semigroup

generator set as a set of "tiles" that should be connected in the product with appropriate order, otherwise the product preserves some parts that cannot be reduced later.

## 2 Notations and Definitions

In what follows we use traditional denotations $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Z}^+$ and $\mathbb{Q}$ for the sets of naturals, integers, positive integers and rationals, respectively.

A *semigroup* is a pair $(S, \cdot)$, where $S$ is a set and $\cdot$ is an associative binary operation on $S$. A semigroup $(S, \cdot)$ is generated by a set $A$ of its elements iff every element of $S$ is a finite product $a_{i_1} a_{i_2} \cdots a_{i_k}$ where $a_{i_j} \in A$. The set of $n \times n$ matrices over integers is denoted by $\mathbb{Z}^{n \times n}$. It is clear that the identity element for a semigroup $(\mathbb{Z}^{n \times n}, \cdot)$ is the identity matrix that we denote by $I_n$ (or $I$). Submatrix $M^{j_1,\ldots,j_l}_{i_1,\ldots,i_k}$ of a matrix $M$ is just the matrix formed from the selected rows $i_1, \ldots, i_k$ and columns $j_1, \ldots, j_l$

$$\mathbf{M} = \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{pmatrix} \qquad \mathbf{M}^{j1..jl}_{i1..1k} = \begin{pmatrix} a_{i_1,j_1} & \cdots & a_{i_1,j_l} \\ \vdots & \ddots & \vdots \\ a_{i_k,j_1} & \cdots & a_{i_k,j_l} \end{pmatrix}.$$

We denote an empty word by $\varepsilon$. The concatenation of two strings $w$ and $v$ is a string obtained by appending the symbols of $v$ to the right end of $w$, that is, if $w = a_1 a_2 \cdots a_n$ and $v = b_1 b_2 \cdots b_n$ then the concatenation of $w$ and $v$, denoted by $w \cdot v$ or $wv$, is $a_1 a_2 \cdots a_n b_1 b_2 \cdots b_n$. We denote a word $\underbrace{aa \cdots a}_{k}$ by $a^k$. The reverse of a string is obtained by writing the symbols in reverse order; if $w$ is a string as shown above, then its reverse $w^{\mathrm{R}}$ is $a_n \cdots a_2 a_1$. The inverse of a character $a$ is written $a^{-1}$ and is the unique character such that $a \cdot a^{-1}$ is equal to the identity element. For any word $w$, we define $\mathrm{suff}_n(w)$ to be the *suffix* of length $n$ from the word $w$.

We also define a notation for use with words called an *inverse palindrome*. This is a word in which the second half is equal to the reverse of the first half of the word using inverse elements. For example if $w$ is an inverse palindrome, then it can be written $w = z \cdot \overline{z^R} = z \cdot (z)^{-1} = \varepsilon$ for some word $z$ where $\overline{z}$ denotes the same word, $z$, but using *inverse elements*. For example $w = w_1 w_2 w_3 \overline{w_3 w_2 w_1} = \varepsilon$ is an inverse palindrome.

*2.1 Two Mappings Between Words and Matrices*

Now we introduce two mappings $\psi$ and $\varphi$ that give us an embedding from words to matrices. Let us consider the mapping $\psi$ between $\{0,1\}^*$ and $2 \times 2$ matrices:

$$\psi : \varepsilon \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \quad \psi : 0 \mapsto \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = M_0 \quad \psi : 1 \mapsto \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = M_1$$

$$\psi : w_1 w_2 \cdots w_r \mapsto M_{w_1} M_{w_2} \cdots M_{w_r}.$$

It is a well known fact that the mapping $\psi$ is an isomorphism between $\{0,1\}^+$ and elements of the matrix semigroup generated by $2 \times 2$ matrices $M_0$ and $M_1$. Since for every matrix with non-zero determinant there is only one unique inverse matrix, we can also define a similar mapping $\varphi$. It can be defined using inverse matrices of the semigroup generator. Mapping $\varphi$ is also an isomorphism between $\{0,1\}^+$ and elements of the matrix semigroup generated by $2 \times 2$ matrices $\{M_0^{-1}, M_1^{-1}\}$ :

$$\varphi : \varepsilon \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \quad \varphi : 0 \mapsto \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} = M_0^{-1} \quad \varphi : 1 \mapsto \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} = M_1^{-1}$$

$$\varphi : w_1 w_2 \cdots w_r \mapsto M_{w_1}^{-1} M_{w_2}^{-1} \cdots M_{w_r}^{-1}.$$

Note that the mappings from $w \in \{0,1\}^+$ to $w^{\mathrm{R}} \in \{0,1\}^+$ and from $\psi(u)$ to $\varphi(u^{\mathrm{R}})$ are bijective. Another very useful property of these mapping is that they can be used to define a free semigroup and group:

**Proposition 4** *[4,12] The semigroup and group generated by the pair of matrices $\psi(0)$, $\psi(1)$ is free.*

Moreover since the semigroup generated by $\{\psi(0), \psi(1)\}$ is free we can express an equality on words in terms of matrix equality:

**Lemma 5** *[10] Given two words $u, v \in X^*$, $u = v$ iff $\varphi(u^{\mathrm{R}}) = (\psi(v))^{-1}$.*

From Lemma 5 and the fact that matrices $\psi(w)$ and $\varphi(w)$ have inverse matrices (elements) for any word $w \in \{0,1\}^*$, the following lemma holds:

**Lemma 6** *[10] Given a binary alphabet $X$, a finite sequence of pairs of words in $X^*$:*

$$(u_1, v_1), \ldots, (u_k, v_k)$$

*and a finite sequence of indices $i_1, i_2, \ldots, i_m$ with $i_j \in \{1..k\}$. The word $u = u_{i_1} \cdots u_{i_m}$ is equal to the word $v = v_{i_1} \cdots v_{i_m}$ if and only if*

$$\varphi(u^{\mathrm{R}}) \cdot \psi(v) = I.$$

### *2.2 Reduced words and their cyclic permutation*

Let $\Gamma = \{0, 1, 0^{-1}, 1^{-1}\}$ where $0 \equiv \varphi(0)$, $1 \equiv \varphi(1)$, $0^{-1} \equiv \psi(0)$ and $1^{-1} \equiv \psi(1)$. For any word, $u$, such that

$$u = y_1 y_2 \cdots y_n, \quad (y_i \in \Gamma)$$

we say that $u$ is a reduced word if $y_i \neq y_{i+1}^{-1}$, $(1 \leq i < n)$, i.e $u$ does not contain a subword of the type $yy^{-1}$ for any character $y \in \Gamma$.

Let $P$ be the semigroup generated by the matrices $\{\varphi(0), \varphi(1), \psi(0), \psi(1)\}$ shown above, using multiplication as the associative operator. We define an isomorphism, $\omega$, from $P$ to its reduced word in $\Gamma$, i.e. $\omega : P \to \Gamma^*$. As an example, $\omega(\varphi(0)\psi(1)\psi(0)) = 01^{-1}0^{-1}$.

We shall also define $|\omega(X)|$ to be the number of characters in $\omega(X)$ for any matrix X. It is clear that $\omega(I) = \varepsilon$ therefore $|\omega(I)| = 0$. Let $\omega_k(X)$ denote the kth symbol of $\omega(X)$ and $\omega_F(X)$ denote the final symbol of $\omega(X)$.

For any sequence $T = (t_1, t_2, \cdots, t_n)$ we define a $k$-cyclic permutation (or k-cyclic shift) to be the result of moving all elements to the right $k$ times with elements overflowing from the right being inserted to the left. Thus if we shift this sequence $k$ places to the right we get the sequence $\mathrm{suff}_k(T) \cdot T \cdot \mathrm{suff}_k(T)^{-1}$ where $0 \leq k \leq n$.

## 3 PCP Encoding

In this section we show the idea of reducing the Post Correspondence Problem to the membership problem for an invertible diagonal matrix. Post's correspondence problem (in short, PCP) is formulated as follows:

*Given a finite alphabet $X$ and a finite set of pairs of words in $X^* \times X^*$: $\{(u_1, v_1), \ldots, (u_k, v_k)\}$. Is there a finite sequence of indices $i_1 i_2 \ldots i_m$ with $i_j \in \{1..k\}$, such that $u_{i_1} u_{i_2} \cdots u_{i_m} = v_{i_1} v_{i_2} \cdots v_{i_m}$?*

PCP(n) denotes the same problem with a sequence of $n$ pairs. Without loss of generality we assume that the alphabet $X$ is binary [11].

Let us define a class of semigroups that has an undecidable membership problem for invertible diagonal matrices. For an instance of the PCP with $n$ pairs of words we define a generator $G$ containing $4n+2$ different matrices as follows:

(1) For each pair $(u_i, v_i)$, we will create four matrices:
- Matrix of type 1: $A_{1,2}^{1,2} = \varphi(u^R), A_{3,4}^{3,4} = \varphi(0^i 1), A_5^5 = 1$
- Matrix of type 2: $B_{1,2}^{1,2} = \varphi(u^R), B_{3,4}^{3,4} = \varphi(0^i 1), B_5^5 = 2$
- Matrix of type 3: $C_{1,2}^{1,2} = \psi(v), C_{3,4}^{3,4} = \psi(0^i 1), C_5^5 = 1$
- Matrix of type 4: $D_{1,2}^{1,2} = \psi(v), D_{3,4}^{3,4} = \psi(0^i 1), D_5^5 = 3$

(2) A single matrix, $J$, where $J_{1,2}^{1,2} = I, J_{3,4}^{3,4} = \varphi(1), J_5^5 = 5$

(3) A single matrix, $K$, where $K_{1,2}^{1,2} = I, K_{3,4}^{3,4} = \psi(1), K_5^5 = 7$

We assign zero to all matrix elements not defined above. Now we state the reduction lemma and give an example of such an encoding below.

**Lemma 7** *An instance of PCP has a solution iff the corresponding semigroup $S$ with generator $G$ defined as above contains the matrix $M_D$*

$$M_D = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 210 \end{pmatrix}.$$

**Example 8** *Given an instance of the PCP, $P = (\,(101, 1), (0, 01010)\,)$. We will construct an example of how our coding will represent this problem in a semigroup and what a solution to the problem will look like.*

*We are given two separate 'tiles' and need to construct a solution to the PCP. We can see that $P_1, P_2, P_1$ is one such solution. We will have a semigroup generator $G$ of $((4n)+2) = 10$ matrices $G_1, G_2, \ldots, G_{10}$ where:*

| *Matrix Number* | *Word part* | *Index Part* | *Factor part* |
|---|---|---|---|
| *Matrix 1 :* | ${G_1}_{1,2}^{1,2} = \varphi(101)$ | ${G_1}_{3,4}^{3,4} = \varphi(01)$ | ${G_1}_{5,5}^{5,5} = 1$ |
| *Matrix 2 :* | ${G_2}_{1,2}^{1,2} = \varphi(101)$ | ${G_2}_{3,4}^{3,4} = \varphi(01)$ | ${G_2}_{5,5}^{5,5} = 2$ |
| *Matrix 3 :* | ${G_3}_{1,2}^{1,2} = \varphi(0)$ | ${G_3}_{3,4}^{3,4} = \varphi(001)$ | ${G_3}_{5,5}^{5,5} = 1$ |
| *Matrix 4 :* | ${G_4}_{1,2}^{1,2} = \varphi(0)$ | ${G_4}_{3,4}^{3,4} = \varphi(001)$ | ${G_4}_{5,5}^{5,5} = 2$ |
| *Matrix 5 :* | ${G_5}_{1,2}^{1,2} = \psi(1)$ | ${G_5}_{3,4}^{3,4} = \psi(01)$ | ${G_5}_{5,5}^{5,5} = 1$ |
| *Matrix 6 :* | ${G_6}_{1,2}^{1,2} = \psi(1)$ | ${G_6}_{3,4}^{3,4} = \psi(01)$ | ${G_6}_{5,5}^{5,5} = 3$ |
| *Matrix 7 :* | ${G_7}_{1,2}^{1,2} = \psi(01010)$ | ${G_7}_{3,4}^{3,4} = \psi(001)$ | ${G_7}_{5,5}^{5,5} = 1$ |
| *Matrix 8 :* | ${G_8}_{1,2}^{1,2} = \psi(01010)$ | ${G_8}_{3,4}^{3,4} = \psi(001)$ | ${G_8}_{5,5}^{5,5} = 3$ |
| *Matrix 9(J) :* | ${G_9}_{1,2}^{1,2} = I$ | ${G_9}_{3,4}^{3,4} = \varphi(1)$ | ${G_9}_{5,5}^{5,5} = 5$ |
| *Matrix 10(K) :* | ${G_{10}}_{1,2}^{1,2} = I$ | ${G_{10}}_{3,4}^{3,4} = \psi(1)$ | ${G_{10}}_{5,5}^{5,5} = 7$ |

*All other matrix elements are equal to* 0. *As stated before, a sequence from* $P$ *giving a solution of PCP is 1,2,1 thus we can define the following matrix product* $\Omega = G_9 \cdot G_2 \cdot G_3 \cdot G_1 \cdot G_{10} \cdot G_6 \cdot G_7 \cdot G_5$. *We will now show this gives the requires form of a matrix.*

*Consider the word part of the matrix first,* $W(\Omega) = I \cdot \varphi(101) \cdot \varphi(0) \cdot \varphi(101) \cdot I \cdot \psi(1) \cdot \psi(01010) \cdot \psi(1) = I$. *Now consider the index part,* $Ind(\Omega) = \varphi(1) \cdot \varphi(01) \cdot \varphi(001) \cdot \varphi(01) \cdot \psi(1) \cdot \psi(01) \cdot \psi(001) \cdot \psi(01) = I$. *Finally we have the factorization part as a (integer) product,* $Fac(\Omega) = 5 \cdot 2 \cdot 1 \cdot 1 \cdot 7 \cdot 3 \cdot 1 \cdot 1 = 210$. *This is indeed a matrix is of the required form and is a solution of the above PCP instance.*

In the next section we prove Lemma 7 by showing the correctness of the presented reduction.

### *3.1 Correctness of the reduction*

Let $S$ be a semigroup that is constructed by the above rules for an instance of the PCP problem. We start by showing the word equation coding in submatrix $M_{1,2}^{1,2}$. Given a sequence of pairs of words in a binary alphabet $A = \{0, 1\}$ :

$$(u_1, v_1), \ldots, (u_n, v_n).$$

Let us construct the sequence of pairs of $2 \times 2$ matrices using two mappings $\varphi$ and $\psi$: $(\varphi(u_1), \psi(v_1)), \ldots, (\varphi(u_n), \psi(v_n))$.

Instead of equation $u = v$ we would like to consider a concatenation of two words $u^R \cdot v$ that is a palindrome in the case where $u = v$. Now we show a matrix interpretation of this concatenation. We associate $2 \times 2$ matrix $Y$ with a word $w$ of the form $u^R \cdot v$. Initially we can think that $Y$ is an identity matrix corresponding to an empty word. The extension of a word $w$ by a new pair of

words $(u_r, v_r)$ (i.e. that gives us $w' = u_r^R \cdot w \cdot v_r$) corresponds to the following matrix multiplication

$$Y_{w'} = Y_{u_r^R \cdot w \cdot v_r} = \varphi(u_r^R) \times Y_w \times \psi(v_r) \tag{1}$$

According to Lemma 6 $u = u_{i_1} \cdots u_{i_m} = v_{i_1} \cdots v_{i_m} = v$ for a finite sequence of indexes $i_1 i_2 \ldots i_m$ with $i_j \in \{1..k\}$ if and only if $\varphi(u^R) \times \psi(v)$ is equal to the identity matrix. So the question of the word equality can be reduced to the problem of finding a sequence of pairwise matrix multiplications that gives us the identity matrix. Note that not only an inverse palindrome but also all its cyclic permutations are equal to the identity element.

**Lemma 9** *Any $k$-cyclic permutation of an inverse palindrome of length $n$ is a concatenation of two distinct (i.e. non-overlapping) inverse palindromes when $1 \leq k < n$.*

**PROOF.** We are given a word $w = w_1 w_2 \cdots w_n$ which is an inverse palindrome (i.e. it can be written as $w = z \cdot z^{-1}$).

For a $k$-cyclic shift of $w$ we will get a word of the form:

$$w' = \text{suff}_k(w) \cdot w \cdot \text{suff}_k(w)^{-1} \qquad | \quad 1 \leq k < n$$

In an inverse palindrome, element $w_1$ is inverse to $w_n$ and $w_2$ is inverse to $w_{n-1}$ etc. we can see that any cyclic permutation simply changes the order of the multiplication from left to right (i.e. $w_1 \cdot w_n$ becomes $w_n \cdot w_1$). Each time we shift right, the first sub-word increases by size 2 whilst the sub-word on the right decreases by size 2. Thus any cyclic shift of an inverse palindrome gives either one or two inverse palindromes (depending whether $k = n$). $\square$

Now by the definition of an inverse palindrome, each opposite pair from the centre outwards is inverse to each other and thus in any such word all elements cancel to give $w = \varepsilon$. For any k-cyclic shift, we get one or two sub words which are inverse palindromes. In terms of matrices, this means all such cyclic permutations produce the identity matrix.

Since we cannot control the order of a matrix product in the semigroup we cannot directly apply the idea of pairwise multiplication. So we show that it is possible to avoid the pairwise matrix multiplications problem by increasing the dimension from 2 to 5 using the idea of relative matrices for index encoding.

The idea is to design such associated "tiles" for the above matrices that they disallow any products that cannot be represented as pairwise multiplications.

In particular, a sequence of "tiles" in an incorrect order preserves some parts that cannot be reduced later.

For each matrix $M \in S$ we can guarantee such a property by using two specially designed submatrices of five dimensional matrices, $M_{3,4}^{3,4}$ and $M_5^5$. It is easy to see that the submatrix $M_5^5$ controls the exact number of appearances of auxiliary matrices and the minimum number of main matrices to avoid an empty word solution. This is achieved by assigning unique prime values to some matrices and by employing the fundamental theorem of arithmetic regarding prime factorization.

We will now prove that the index coding will only result in the identity matrix in the case that the matrix multiplication is of the correct form. The initial conditions of the following lemma are satisfied by the prime factorization in the last diagonal element of each matrix. Note also that the $M, N$ matrices correspond to $J_{3,4}^{3,4}$ , $K_{3,4}^{3,4}$ respectively and $U_i, V_i$ matrices are the same subparts of matrices $A, B, C, D$ of generator $G$ from Lemma 7.

**Lemma 10** *Let $S$ be a set containing matrices $M = \varphi(1)$, $N = \psi(1)$, $U_i = \varphi(0^i1)$ and $V_i = \psi(0^i1)$ where $1 \leq i \leq n$. Let $P$ be a set of matrices where each member of $P$ is the product of at least one $U$ and $V$ matrix and exactly one $M$ and $N$ matrix from set $S$. The identity matrix is a member of the set $P$ iff it is a cyclic permutation of the following sequence:*

$$M \cdot U_{i_1} \cdot U_{i_2} \cdots U_{i_n} \cdot N \cdot V_{i_n} \cdots V_{i_2} \cdot V_{i_1}$$

**PROOF.** $\Leftarrow$ From the definition of the lemma, we know there is 1 matrix of type $M$, 1 matrix of type $N$ and at least 1 matrix of type $U$ and $V$. Thus we will prove by induction that any multiplication of the above forms gives the identity element.

Let us prove the base case, when $n = 1$:

$$M \cdot U_{i_1} \cdot N \cdot V_{i_1} = \varphi(1) \cdot \varphi(0^{i_1}1) \cdot \psi(1) \cdot \psi(0^{i_1}1) = I$$

Let us consider a matrix multiplication of the form $U_{n+1} \cdot N \cdot V_{n+1}$:

$$\varphi(0^{n+1}1) \cdot \psi(1) \cdot \psi(0^{n+1}1) = \psi(1) = N \tag{2}$$

We now assume the inductive hypothesis that for any $n$:

$$M \cdot U_{i_1} \cdots U_{i_n} \cdot N \cdot V_{i_n} \cdots V_{i_1} = I$$

But since we showed in (2) that $U_{n+1} \cdot N \cdot V_{n+1} = N$, we can substitute this into the above expression to get the same result for $n$+1. Thus this product gives the identity matrix by the principle of induction.

We now prove that if the above form is equal to the identity matrix, all cyclic permutations are aswell.

We can clearly see that in terms of atomic matrices, the given form of the matrix product is an inverse palindrome as defined in lemma 9. This can be seen by looking at an example:

$$M \cdot U_{i_1} \cdot U_{i_2} \cdot N \cdot V_{i_2} \cdot V_{i_1} = \varphi(10^{i_1}10^{i_2}1) \cdot \psi(10^{i_2}10^{i_1}1)$$

But as shown in lemma 9, any cyclic permutation of an inverse palindrome is equal to two smaller inverse palindromes (except for the trivial case where we cycle a multiple of the number of matrices and get the original word back). Further, we showed that inverse palindromes are clearly equal to the identity element. Thus any cyclic shift of the above sequence gives the identity matrix.

$\Rightarrow$ We now move to the reverse direction of the proof; proving that all identity elements must be a cyclical permutation of the following form:

$$M \cdot U_{i_1} \cdot U_{i_2} \cdots U_{i_n} \cdot N \cdot V_{i_n} \cdots V_{i_2} \cdot V_{i_1}$$

We have four different matrix "types". We shall show that these elements cannot produce the identity element in any way other than the above form.

We first consider only $U$ and $V$ matrices in a product. We define a sequence of matrices by $(Y_{i_1}, Y_{i_2}, \cdots, Y_{i_n})$ where $Y_i \in \{U_i, V_i\}$. For any k, $\omega_1(U_k) = 0$ and $\omega_F(U_k) = 1$. Similarly, $\omega_1(V_k) = 0^{-1}$ and $\omega_F(V_k) = 1^{-1}$. Therefore any multiplication of these matrices will not have any consecutive inverse pairs since $1 \cdot 0^{-1} \neq I$ and $1^{-1} \cdot 0 \neq I$. More formally,

$$\left|\omega\left(\prod_{k=1}^{n} Y_{i_k}\right)\right| = \sum_{k=1}^{n} |\omega(Y_{i_k})| \quad , Y_k \in \{U_k, V_k\} \tag{3}$$

We will now prove that if we have a matrix sequence containing any consecutive products $U_i \cdot V_j$ (similarly for $V_j \cdot U_i$), it will never be able to be fully cancelled using just one $M$ and $N$ matrix:

Let $X = U_i \cdot V_j$ where $1 \leq i, j \leq n$. Given that $X = \varphi(0^i 1) \cdot \psi(0^j 1)$ and $\omega(X) = 0^i 1 (0^{-1})^j 1^{-1}$, we can see $\omega_1(X) = 0$. Since for all $Y \in S, \omega_F(Y) = 1$ or $1^{-1}$, no matrix can be pre-multiplied to reduce the length of $\omega(X)$. Given that $\omega_F(X) = 1^{-1}$, we can *only* post multiply by $M$ to reduce the length of $\omega(X)$ because $\omega_1(M) = 1$. Therefore $U_i \cdot V_j \cdot M = \varphi(0^i 1) \cdot \psi(0^j)$ and $\omega(U_i \cdot V_j \cdot M) = 0^i 1 (0^{-1})^j$. Again, no matrix can be pre-multiplied, but we can post-multiply by some $U_k$ since only $\omega_1(U) = 0$. We have three cases, where $(k < j)$, $(k = j)$ and $(k > j)$ giving matrices $\varphi(0^i 1) \cdot \psi(0^{j-k}) \cdot \varphi(1)$, $\varphi(0^i 11)$ and $\varphi(0^i 1) \cdot \varphi(0^{k-j} 1)$ respectively. Since $\omega_F$ of these three matrices equals 1 it can only cancel with an $N$ matrix. In all cases, there are now either 0 or $0^{-1}$ symbols on the right of the product. It can be seen however that further multiplications by the remaining $U, V$ matrices will not fully cancel any of these products.

A similar argument holds for $X = V_j \cdot U_i$. Therefore if any matrix sequence contains consecutive elements $U_i, V_j$ or $V_j, U_i$, its product cannot equal the identity matrix.

We now consider a matrix $M$ in a product. We have four cases to consider.

(1) Given $\omega(U_i \cdot M) = 0^i 11$, we see that it cannot be reduced to zero size because only $N$ cancels with the final symbol leaving $U_i$ and (3) shows that using only $U$ and $V$ matrices never gives the identity matrix.
(2) $M \cdot U_i$ is of the correct form as shown in the first part of the proof.
(3) $V_i \cdot M$ is of the correct form as shown in the first part of the proof.
(4) Given $\omega(M \cdot V_i) = 1(0^{-1})^i 1^{-1}$. We cannot post-multiply by any remaining matrix type to reduce the number of matrix elements. We can pre-multiply this product by $N$ but this again leaves only $U$ and $V$ matrices. Pre-multiplying by $V_j$ gives a product: $\omega(V_j \cdot M \cdot V_i) = (0^{-1})^{(j+i)} 1^{-1}$, equal to $V_{j+i}$ which cannot reduce to zero length because the $M$ matrix has been used and only $M$ cancels with the last element.

Since each matrix type has an inverse, the same situation occurs with the $N$ matrix. Therefore any matrix product containing $(U_i \cdot M)$, $(M \cdot V_i)$, $(V_i \cdot N)$ or $(N \cdot U_i)$ will never be able to result in the identity element.

Thus the sequence must of the following form to produce the identity matrix:

$$\cdots V_{i_1} \cdot M \cdot U_{j_1} \cdots U_{j_n} \cdot N \cdot V_{k_m} \cdots V_{k_1} \cdot M \cdot U_{l_1} \cdots U_{l_p} \cdot N \cdot V_{h_q} \cdots$$

This pattern can repeat indefinitely, but since we only have a single $M$ and $N$ matrix:

$$V_{i_m} \cdots V_{i_1} \cdot M \cdot U_{j_1} \cdots U_{j_n} \cdot N \cdot V_{k_p} \cdots V_{k_1} | \, m, p \in \mathbb{Z}^+, n \in \mathbb{N} \tag{4}$$

$$U_{i_m} \cdots U_{i_1} \cdot N \cdot V_{j_1} \cdots V_{j_n} \cdot M \cdot U_{k_p} \cdots U_{k_1} | \, m, p \in \mathbb{Z}^+, n \in \mathbb{N} \tag{5}$$

Since $\varphi(1)$ is only inverse to $\psi(1)$ and each $U$,$V$ matrix sequence contains exactly one of these matrices, the number of $U$ matrices must equal the number of $V$ matrices, thus $m + p = n$.

For equation (4), let us define two sub-sequences of matrices $\alpha_1, \alpha_2$ such that

$$\alpha_1 = (V_{i_m}, \cdots, V_{i_1}, M, U_{j_1}, \cdots, U_{j_m})$$

$$\alpha_2 = (U_{j_{(n-p+1)}}, \cdots, U_{j_n}, N, V_{k_p}, \cdots, V_{k_1})$$

Assume that equation (4) is equal to the identity element. Now assume by contradiction that $\alpha_1 \neq M$. There is an equal number of $U$ and $V$ matrices and the $U$ matrices follow the $M$ matrix therefore the last matrix in the sequence must be a $U$ matrix. $\omega_F$ of any $U$ is always 1. If $\alpha_2 = N$ then it will cancel with this element but leave a non-identity element in $\alpha_1$(since $\alpha_1 \neq M$). Thus $\alpha_2 \neq N$ and since it has an equal number of $U$ and $V$ matrices and the $U$ matrices precede all $V$ matrices, the first matrix in the sequence must also be some $U$ matrix. But as shown in equation (3) the product of $U$ matrices only increases the size of the sequence. The reverse argument also holds if $\alpha_2 \neq N$. Thus the resulting matrix cannot be the identity element unless $\alpha_1 = M$ and $\alpha_2 = N$.

Define $\alpha_1[i]$ to be the $i$'th element of the sequence. Now we prove that $\alpha_1[(m+1)-k] \cdot \alpha_1[(m+1)+k] = I$ where $(0 \leq k \leq m)$. Let us assume by contradiction that their exists some $k$ where $\alpha_1[(m+1)-k] \cdot \alpha_1[(m+1)+k] \neq I$, i.e. 2 opposite elements who are not inverse to each other. Therefore we have:

$$V_a \cdot M \cdot U_b = \psi(0^a 1) \cdot \varphi(1) \cdot \varphi(0^b 1) \quad | \, a \neq b$$

If $a > b$ then it will give a matrix $\psi(0^{a-b}) \cdot \varphi(1)$. Since $M$ has been used and $N$ is not in this sequence however, we only have $U$ and $V$ matrices which equation (3) shows cannot reduce the length of $|\omega(\psi(0^{a-b}) \cdot \varphi(1))|$ (and $|M| = 1$). If $a < b$ then it gives a matrix $\varphi(0^{b-a}1)$ which is equal to $U_{(b-a)}$. Again, since $M$ has been used, there are only $U$ and $V$ matrices left which cannot reduce to $M$ giving a contradiction.

Similarly for $\alpha_2$ we get opposite matrices produced which gives the same result that all opposite matrices from the central element ($N$) are inverse.

Thus equation (4) given above must be of the form:

$$V_{i_m} \cdots V_{i_1} \cdot M \cdot U_{i_1} \cdots U_{i_m} \cdots U_{i_n} \cdot N \cdot V_{i_n} \cdots V_{i_{n-m+1}}$$

We can clearly see that this is a cyclic permutation of the form given in the first part of the proof and the form of equation (5) is a cyclic permutation of

equation (4), thus both must be equal to the identity matrix. □

The proof of above lemma ends the proof of reduction from Lemma 7 since the PCP has a solution if and only if the semigroup $S$ contains the matrix $M_D$. Thus the following Theorem holds:

**Theorem 11** *Problem 2 is undecidable in dimension five.*

### *3.2 Reduction to lower dimensions*

Now we can reduce the dimensions used and state some corollaries.

**Corollary 12** *Problem 2 is undecidable in dimension* 4.

**PROOF.** The element $M_5^5$, in our previous construction, is a scalar value and is commutative in all matrices since all other elements along row and column 5 are zero. Therefore we can multiply submatrix $M_{3,4}^{3,4}$ by $M_5^5$ and we will still preserve this value across the multiplication without changing the structure of multiplications of $M_{3,4}^{3,4}$. □

**Corollary 13** *Problem 3 is undecidable for linear transformations defined by a finite set of integral* $4 \times 4$ *matrices.*

**PROOF.** In order to prove the undecidability of Problem 3 we can show that the scalar matrix $M = 210 \cdot I_4$ is undecidable. We use the same idea as we did for Problem 2 with the only difference being that we extend the generator of the semigroup by the following matrix $R$:

$$\begin{pmatrix} 210 & 0 & 0 & 0 \\ 0 & 210 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

It is easy to see that the above matrix commutes with all other matrices in the semigroup, since the submatrix $M_{1,2}^{1,2}$ is a scalar matrix and $M_{3,4}^{3,4}$ is the identity matrix. On the other hand we cannot use more than one copy of matrix $R$ since the determinant of the top left block $M_{1,2}^{1,2}$ of any matrix from the semigroup that uses more than one copy of $R$ will be more than $210^2$. So

the matrix $M = 210 \cdot I_4$ is reachable if and only if the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 210 & 0 \\ 0 & 0 & 0 & 210 \end{pmatrix}$$

is reachable and does not use $R$, that in turn is undecidable. $\square$

In fact we can prove an even stronger claim that membership of any non-unimodular scalar matrix over rationals is undecidable in dimension four.

**Corollary 14** *Given a semigroup $S$ generated by a finite set of $n \times n$ matrices over rationals and a scalar $k \in \mathbb{Q}$ such that $k \neq \pm 1$. It is undecidable to check whether the scalar matrix $k \cdot I$ belongs to $S$ for any $n \geq 4$.*

**PROOF.** We use Lemma 7 to show the undecidability of the membership problem for a scalar matrix $k \cdot I_4$ by repeating the proof of Corollary 13 and introducing another matrix $R$ that is now equal to

$$\begin{pmatrix} k & 0 & 0 & 0 \\ 0 & k & 0 & 0 \\ 0 & 0 & \frac{k}{210} & 0 \\ 0 & 0 & 0 & \frac{k}{210} \end{pmatrix}.$$

$\square$

## 4 Conclusion and Some Remarks

In this paper we have proven that the membership problem of a particular invertible diagonal matrix is undecidable for a $4 \times 4$ integral matrix semigroup. Then as a corollary of this fact we have shown that the membership of a particular invertible scalar matrix is undecidable in dimension 4 for an integral matrix semigroup. Moreover we have shown that in dimension 4 the membership of any non-unimodular scalar matrix is undecidable for a rational matrix semigroup. Recently, our results were improved in [5], where the authors reduced the number of matrices needed in the generator from 30 to just 14. The same problems for lower dimensions and the membership problem for an arbitrary diagonal matrix in a matrix semigroup are still open.

Unfortunately we have not been able to extend this result to cover the membership of the identity matrix in a semigroup. The reason for this is that we use a particular encoding which relies upon the fact that certain matrices (which we denoted $M, N$) only appear in a correct solution once. If we are allowed to use these matrices more than once, we can get a sequence of cycles, none of which contain a correct solution to PCP, but when multiplied together, they *appear* to. By using a prime number coding in the determinant of these matrices, we can avoid this problem of multiple cycles, at the expense of not being able to encode the identity matrix. This remains an open problem (for any dimension greater than 3) and it was shown to be decidable for $2 \times 2$ integer matrix semigroups [3].

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