Efficient Sequential Algorithms, Comp309

University of Liverpool

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Part 4: NP-Completeness

References: T. H. Cormen, C. E. Leiserson, R. L. Rivest **Introduction to Algorithms**, Second Edition. MIT Press (2001). "NP-completeness"

C.H. Papadimitriou, **Computational Complexity** Addison-Wesley (1993).

Revision from Comp202

A decision problem is a computational problem for which the output is either yes or no.

The input to a computational problem is encoded as a finite binary string s of length $|s|$.

For a decision problem X, $L(X)$ denotes the set of (binary) strings (inputs) for which the algorithm should output "yes". We refer to $L(X)$ as a language. We say that an algorithm A accepts a language $L(X)$ if A outputs "yes" for each $s \in L(X)$ and outputs "no" for every other input.

Revision from Comp202

The complexity class P is the set of all decision problems X (or languages $L(X)$) that can be solved in polynomial time.

That is, there is an algorithm A that accepts language $L(X)$. The amount of time that algorithm A takes on input s is at most $p(|s|)$ where $p(n)$ is of the form $p(n) = n^k$ for a some constant k. ($p(n)$ is a polynomial in n).

The complexity class EXP is the set of all decision problems X (or languages $L(X)$) that can be solved in exponential time.

That is, there is an algorithm A that accepts language $L(X)$. The amount of time that algorithm A takes on input s is at most $p(|s|)$ where $p(n)$ is a function of the form $p(n) = 2^{n^k}$ for some constant k.

Space complexity classes

PSPACE is the set of all decision problems X (or languages $L(X)$) that can be solved in polynomial space.

That is, there is an algorithm A that accepts language $L(X)$. The amount of computer memory that algorithm A uses on input s is at most $p(|s|)$ where $p(n)$ is a polynomial in n.

More revision: Nondeterministic computation

An algorithm that guesses some number of non-deterministic bits during its execution is called a non-deterministic algorithm.

We say that a non-deterministic algorithm A accepts a string s if there exists a choice of non-deterministic bits that causes algorithm A to output "yes" with input s. Otherwise, we say that A does not accept s.

We say that a non-deterministic algorithm A accepts a language $L(X)$ if A accepts every string $s \in L(X)$ and no other strings.

The complexity class NP is the set of all decision problems X (or languages $L(X)$) that can be non-deterministically accepted in polynomial time.

That is, there is a non-deterministic algorithm A that accepts language $L(X)$. The amount of time that algorithm A takes on input s is at most $p(|s|)$ where $p(n)$ is a polynomial in n.

It is easy to see that $\mathrm{P} \subseteq \mathrm{NP}$

If $L(X)$ is accepted by a polynomial-time algorithm A then it is also accepted by a non-deterministic algorithm in polynomial time.

The non-deterministic algorithm doesn't have to make non-deterministic choices — it can just simulate algorithm A.

Polynomial-time reducibility

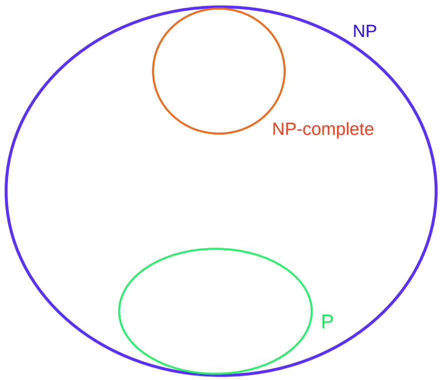
We say that a language L, defining some decision problem, is polynomial-time reducible to a language M (written $L \overset{\text{poly}}{\rightarrow} M$) if there is a polynomial-time-computable function f that takes as input a binary string s and outputs a binary string $f(s)$ so that $s \in L$ iff $f(s) \in M$.

As you saw in Comp202, If $L_1 \overset{\text{poly}}{\rightarrow} L_2$ and $L_2 \overset{\text{poly}}{\rightarrow} L_3$ then $L_1 \overset{\text{poly}}{\rightarrow} L_3$.

NP-completeness

We say that a language M, defining some decision problem, is NP-hard if every langauge $L \in \mathrm{NP}$ is polynomial-time reducible to M.

We say that a language M is NP-complete if M is in NP and M is NP-hard.



The Cook-Levin Theorem is that the problem SAT is NP-complete.

Name: SAT **Instance:** A Boolean formula F
Question: Does F have a satisfying assignment?

Recall that a **Boolean formula** is an expression like

$$(x_{25} \wedge x_{12}) \vee \neg(\neg x_{70} \vee (\neg x_3 \wedge x_{34}))$$

made up of the constants *true* and *false,* propositional variables x_i, parentheses and the connectives $\wedge$, $\vee$, $\neg$, $\Rightarrow$, $\Leftrightarrow$. An assignment of the *truth-values* true and false to the variables is satisfying if it makes the formula evaluate to true.

If a language M is NP-hard and $M \overset{\text{poly}}{\rightarrow} L$ then L is NP-hard.

Thus, to show that a language L is NP-complete, we do the following.

1. Show that L is in NP, and
2. Take some NP-hard problem M and find a polynomial-time reduction from M to L.

Make sure you don't go the wrong direction!

We will now show that some problems are NP-complete.

3-Conjunctive Normal Form Satisfiability (3-CNF)

- **Input:** A boolean formula F expressed as an AND of clauses in which each clause is the OR of exactly three distinct literals.
- **Output:** Is there an assignment of boolean values to the variables which causes F to evaluate to *true*?

$$F = (\neg y_1 \vee \neg x_1 \vee y_1) \wedge (\neg y_1 \vee x_1 \vee \neg y_2) \wedge (\neg y_1 \vee x_1 \vee y_2)$$

Note that y_1 and $\neg y_1$ are distinct literals.

3-CNF is in NP.

The non-deterministic algorithm "guesses" a satisfying assignment then checks in polynomial time that the guess is a satisfying assignment for F.

To show that 3-CNF is NP-complete, we take some NP-complete problem, say SAT, and find a polynomial-time reduction from SAT to 3-CNF.

We will show that there is a polynomial-time computable function f that takes as input an input F of SAT and outputs an input $f(F)$ of 3-CNF so that $f(F)$ is a "yes" instance of 3-CNF iff F is a "yes" instance of SAT.

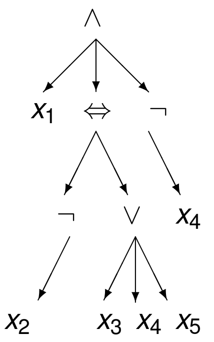
The transformation from F to $f(F)$

Step 1: Transform F into a formula F' which is the AND of clauses, each of which has at most 3 literals.

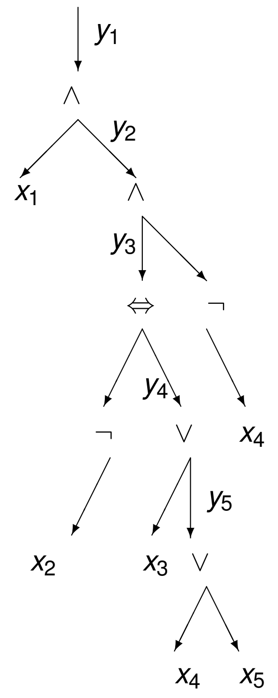
First, parse F

$F = x_1 \wedge (\neg x_2 \Leftrightarrow (x_3 \vee x_4 \vee x_5)) \wedge \neg x_4$

$F = x_1 \wedge (\neg x_2 \Leftrightarrow (x_3 \vee x_4 \vee x_5)) \wedge \neg x_4$



Now use the associativity of $\wedge$ and $\vee$ to form an equivalent tree in which every node has at most 2 children.



Now label the parent-edge out of every internal node (on the previous slide) by a new variable.

Rewrite the formula as an equation.

$$
\begin{aligned}
F' = y_1 \quad & \wedge \quad (y_1 \Leftrightarrow (x_1 \wedge y_2)) \\
& \wedge \quad (y_2 \Leftrightarrow (y_3 \wedge \neg x_4)) \\
& \wedge \quad (y_3 \Leftrightarrow (\neg x_2 \Leftrightarrow y_4)) \\
& \wedge \quad (y_4 \Leftrightarrow (x_3 \vee y_5)) \\
& \wedge \quad (y_5 \Leftrightarrow (x_4 \vee x_5))
\end{aligned}
$$

We have now transformed F into a formula F' which is the AND of clauses, each of which has at most 3 literals. F' is satisfiable iff F is.

Note that the transformation from F to F' can be implemented in polynomial time. Each connective in F introduces at most one variable and one clause to F' to $|F'|$ is at most a polynomial in $|F|$.

The transformation from F to $f(F)$

Step 2: Transform F' into a formula F'' which is the AND of clauses, each of which is the OR of at most 3 literals.

We will use a truth table to transform each clause of F' to the AND of at most 8 clauses which are algebraically equivalent.

For example, take this clause of F': $y_1 \Leftrightarrow (x_1 \wedge y_2)$

y_1	x_1	y_2	result
1	1	1	1
1	1	0	0
1	0	1	0
1	0	0	0
0	1	1	0
0	1	0	1
0	0	1	1
0	0	0	1

The first 0 in the result column of the truth table says you can't have $y_1 x_1 \neg y_2$ so insert the first clause below.

$$
\begin{gathered}
(\neg y_1 \vee \neg x_1 \vee y_2) \wedge (\neg y_1 \vee x_1 \vee \neg y_2) \wedge \\
(\neg y_1 \vee x_1 \vee y_2) \wedge (y_1 \vee \neg x_1 \vee \neg y_2)
\end{gathered}
$$

Having done steps 1 and 2 we have now shown how to transform F into a formula F'' which is the AND of clauses, each of which is the OR of at most 3 literals.

F'' is satisfiable iff F is.

The transformation can be accomplished in polynomial time.

The transformation from F to $f(F)$

Step 3: Transform F'' into a formula F''' which is the AND of clauses, each of which is the OR of exactly 3 literals. Let $f(F) = F'''$.

Transform a 2-literal clause like this, using a new variable p.

$$(x \vee y) \Rightarrow (x \vee y \vee p) \wedge (x \vee y \vee \neg p)$$

Transform a 1-literal clause like this, using new variables p and q.

$$x \Rightarrow (x \vee p \vee q) \wedge (x \vee p \vee \neg q) \wedge (x \vee \neg p \vee q) \wedge (x \vee \neg p \vee \neg q)$$

We have shown that there is a polynomial-time computable function f that takes as input an input F of SAT and outputs an input $f(F)$ of 3-CNF so that $f(F)$ is a "yes" instance of 3-CNF iff F is a "yes" instance of SAT.

This is a polynomial-time reduction from SAT to 3-CNF.

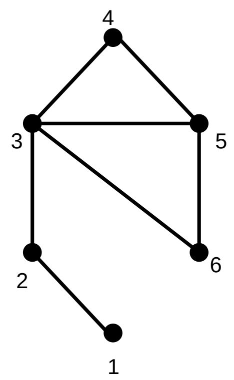
Since SAT is NP-hard, we conclude that 3-CNF is NP-hard.

We already showed that 3-CNF is in NP, so we conclude that 3-CNF is NP-complete.

Another computational problem

Clique

- **Input:** An undirected graph G and an integer j
- **Output:** Is there a set of j vertices of G, each pair of which is connected by an edge?



Clique is in NP

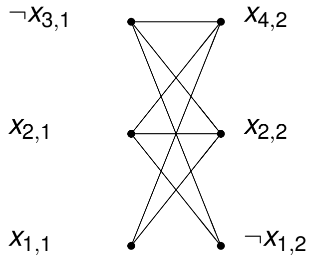
The non-deterministic algorithm “guesses” a set of j vertices then checks in polynomial time to see whether each pair is connected by an edge.

3-CNF $\stackrel{\text{poly}}{\rightarrow}$ Clique

Let F be an input to 3-CNF. We show how to transform it to into an input (G, j) of Clique such that G has a j-clique iff F is satisfiable.

Let j = number of clauses in F. For every clause $C_r = (x_1 \vee x_2 \vee \neg x3)$, introduce vertices $x_{1,r}$, $x_{2,r}$ and $\neg x_{3,r}$. Introduce edges *between* vertices in different clauses, **unless** they are the negation of each other. For example...

$(x_1 \vee x_2 \vee \neg x3) \wedge (\neg x_1 \vee x_2 \vee x_4)$



Suppose we have a satisfying assignment. We can choose one “true” literal from each of the j clauses, and that gives us a clique.

Similarly, we can turn a clique into a satisfying assignment.

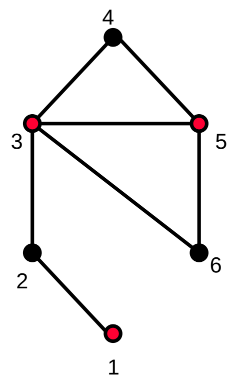
Note that the transformation takes polynomial time.

We have shown that Clique is NP-complete.

Another computational problem, familiar from our work on matchings

Vertex Cover

- **Input:** An undirected graph G and an integer k
- **Output:** Is there a set U of k vertices of G such that for every edge (u, v) of G, at least one of u and v is in U?



Vertex Cover is in NP

The non-deterministic algorithm “guesses” a set of k vertices then checks in polynomial time to see whether every edge is covered.

Clique $\stackrel{\text{poly}}{\to}$ Vertex Cover

Let $G = (V, E)$ and j be an input to clique. We show how to transform it to into an input (G', k) of Vertex Cover such that G' has a vertex cover of size k iff G has a clique of size j.

Method: Let $\overline{E} = \{(u, v) \mid (u, v) \notin E\}$ and $G' = (V, \overline{E})$ and $k = |V| - j$.

If U is a clique then $V - U$ covers all non-edges (and vice-versa).

This is a polynomial-time transformation, so we have shown that vertex cover is NP-complete.

One last computational problem (this one is pretty tricky!)

Subset Sum

- **Input:** A set S of non-negative integers and a non-negative integer t.
- **Output:** Is there a subset of S whose elements sum to t?

Example: $S = \{1, 3, 5\}$. What about $t = 4$? What about $t = 2$?

Subset Sum is in NP

The non-deterministic algorithm “guesses” the subset and checks that its elements sum to t.

Vertex Cover $\overset{\text{poly}}{\to}$ Subset Sum

Let $G = (V, E)$ and k be an input to vertex cover. We show how to transform it to an input S, t of subset sum such that G has a vertex cover of size k iff S has a subset that sums to t.

Notation: Let $V = \{v_0, \ldots, v_{n-1}\}$. Let $E = \{e_0, \ldots, e_{m-1}\}$.

The (polynomial-time) transformation:

For $i \leftarrow 0$ to $n-1$
    $x_i \leftarrow 4^m$
    For $j \leftarrow 0$ to $m-1$
        If e_j is incident on v_i
            $x_i \leftarrow x_i + 4^j$
For $j \leftarrow 0$ to $m-1$
    $y_j \leftarrow 4^j$
$S \leftarrow \{x_0, \ldots, x_{n-1}, y_0, \ldots, y_{m-1}\}$
$t \leftarrow k4^m + \sum_{j=0}^{m-1} 2 \cdot 4^j$
Return S and t

We claim that if G has a size-k vertex cover then S has a subset that sums to t.

- Start with a size-k vertex cover.
- Let S' contain x_is for vertices in the cover and y_js for edges incident **once** on cover.
- Sum of x_is in S' is $k4^m$.
- Edge incident twice on cover contributes $2 \cdot 4^j$ to x's
- Edge incident once on cover contributes 4^j to xs and 4^j to y's.
- Elements in S' sum to t.

We claim that if S has a subset that sums to t then G has a size-k vertex cover.

- Start with S' which sums to t.
- Each e^j contributes at most $2 \cdot 4^j$ to xs and 4^j to ys.
- The e^js do not contribute to the $k4^m$ in t.
- S' has k x_is.
- These k vertices are a vertex cover because each e_j contributes exactly $2 \cdot 4^j$ to t but only 4^j of this can come from y_j so it must be adjacent to one of the vertices in S'.

We have shown that Subset Sum is NP-complete.