Routing (Un-) Splittable Flow in Games with Player-Specific Linear Latency Functions*

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Abstract. In this work we study weighted *network congestion games* with *player-specific* latency functions where selfish *players* wish to route their *traffic* through a shared *network*. We consider both the case of *splittable* and *unsplittable* traffic. Our main findings are as follows:

- For routing games on parallel links with linear latency functions without a constant term we introduce two new potential functions for unsplittable and for splittable traffic respectively. We use these functions to derive results on the convergence to pure Nash equilibria and the computation of equilibria. We also show for several generalizations of these routing games that such potential functions do not exist.
- We prove upper and lower bounds on the price of anarchy for games with linear latency functions. For the case of unsplittable traffic the upper and lower bound are asymptotically tight.

1 Introduction

Motivation and Framework. Large scale communication networks, like e.g. the Internet, often lack a central regulation for several reasons. For instance the size of the network may be too large, or the users may be free to act according to their private interests. Such an environment – where users neither obey some central control instance nor cooperate with each other – can be modeled as a *non-cooperative game*. The concept of *Nash equilibria* [21] has become an important mathematical tool for analyzing non-cooperative games. A Nash equilibrium is a state in which no player can improve his private objective by unilaterally changing his strategy.

For a special class of non-cooperative games, now widely known as *congestion games*, Rosenthal [23] showed the existence of pure Nash equilibria with the help of a certain potential function. In a congestion game, the strategy set of each player is a subset of the power set of given resources and the private cost function of a player is defined as the sum (over the chosen resources) of functions in the number of players sharing this resource. An extension to congestion games in which the players have

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weights and thus different influence on the congestion of the resources are *weighted congestion games*. Weighted congestion games provide us with a general framework for modeling any kind of non-cooperative resource sharing problem. A typical resource sharing problem is that of routing. In a routing game the strategy sets of the players correspond to paths in a network. Routing games where the demand of the players cannot be split among multiple paths are also called *(weighted) network congestion games*.

Another model for selfish routing where traffic flows can be split arbitrarily – the *Wardrop model* – was already studied in the 1950's (see e.g. [4, 26]) in the context of road traffic systems. In a *Wardrop equilibrium* each player assigns its traffic in such a way that the latency experienced on all used paths is the same and minimum among all possible paths for the player. The Wardrop model can be understood as a special network congestion game with infinitely many players each carrying a negligible demand.

In order to measure the degradation of social welfare due to the selfish behavior, Koutsoupias and Papadimitriou [17] used a global objective function, usually termed as *social cost*. They defined the *price of anarchy* as the worst-case ratio between the value of social cost in a Nash equilibrium and that of some social optimum. Thus, the price of anarchy measures the extent to which non-cooperation approximates cooperation.

In weighted network congestion games, as well as in the Wardrop model, players have complete information about the system. However in many cases users of a routing network only have incomplete information about the system. Harsanyi [15] defined the Harsanyi transformation that transforms strategic games with incomplete information into *Bayesian* games where the players uncertainty is expressed in a probability distribution. A Bayesian routing game where players have incomplete information about each others traffic was introduced and studied by Gairing et al. [13]. Georgiou et al. [14] introduced a routing game where the players only have incomplete information about the vector that contains all edge latency functions. Each user's uncertainty about the latency functions is modelled with a probability distribution over a set of different possible latency function vectors. Georgiou et al. [14] showed, that such an incomplete information routing game can be transformed into a complete information routing game where the latency functions are *player-specific*. The resulting games with player-specific latency functions were earlier studied by Milchtaich [19]. Monderer [20] showed that games with player-specific latency functions are of particular importance since each game in strategic form is isomorphic to a congestion game with player-specific latency functions. In this paper we study routing games with player-specific latency functions for both splittable and unsplittable traffic.

Related Work. Routing Games: The class of *weighted congestion games* has been extensively studied (see [12] for a survey). Fotakis et al. [10] proved that a pure Nash equilibrium always exists if the latency functions are linear. For non-linear latency functions they showed, that a pure Nash equilibrium might not exist, even if there are only 2 players (this was also observed earlier by Libman and Orda [18]). For the class of weighted congestion games on parallel links a pure Nash equilibrium always exists, if all edge latency functions are non-decreasing. The *price of anarchy* was studied for congestion games with social cost defined as the total latency. For linear latency functions, it is exactly $\frac{5}{2}$ for unweighted [7] and $\frac{3+\sqrt{5}}{2}$ for weighted congestion games [2]. The exact price of anarchy is also known for polynomials with non-negative coefficients [1].

Inspired by the arisen interest in the price of anarchy Roughgarden and Tardos [25] re-investigated the *Wardrop model* and used the total latency as a social cost measure. In this context the price of anarchy was shown to be $\frac{4}{3}$ for linear latency functions [25] and $\Theta(\frac{d}{\ln d})$ for polynomials of degree at most d with non-negative coefficients [24]. If all latency functions are linear and do not include a constant, then every Wardrop equilibrium has optimum social cost [25]. Since a Wardrop equilibrium is a solution to a convex program it can be computed in polynomial time using the ellipsoid method of Khachyan [16]. This results also implies that the total latency is the same for all Wardrop equilibria. There are several papers (see e.g. [6, 8, 22]) studying games with a finite number of atomic players where each player can split its traffic over the available paths with the objective to minimize its latency. In this setting the price of anarchy is at most $\frac{3}{2}$ for linear latency functions [8].

Routing Games with Player-Specific Latency Functions: Weighted congestion games on parallel links with player-specific latency functions were studied by Milchtaich [19]. For the case of unweighted players and non-decreasing latency functions, Milchtaich showed that such games do in general not posses the finite improvement property but always admit a pure Nash equilibrium. In case of weighted players a pure Nash equilibrium might not exist, even for a game with 3 players and 3 edges (links) [19]. This is a tight result since such games possess the finite best-reply property in case of 2 players and the finite improvement property in case of 2 edges [19]. Georgiou et al. [14] studied the same class of games as Milchtaich but they only allowed linear latency functions without a constant term. They were able to prove upper bounds on the price of anarchy for both social cost defined as the maximum private cost of a player and social cost defined as the sum over the private cost of all players. Furthermore they presented a polynomial time algorithm to compute a pure Nash equilibrium in case of two edges.

Orda et al. [22] studied a splittable flow routing game with certain player-specific latency functions and a finite number of players each minimizing its latency. They showed that there is a unique Nash equilibrium for each game on parallel links. They also described a game on a more complex graph possessing two different Nash equilibria.

Contribution. In this work we generalize weighted network congestion games and the Wardrop model, to accommodate player-specific latency functions. Our main contributions are the definition of new potential functions and the extension of the techniques from [2, 7] to prove upper bounds on the price of anarchy also for games with player-specific latency functions. More specifically, we prove:

– For routing games on parallel links with linear latency functions without a constant term we introduce two new potential functions for unsplittable and for splittable traffic respectively.
  - In the case of unsplittable traffic we use our potential function to show that games with unweighted players possess the *finite improvement property*. We also show that games with weighted players do not possess the finite improvement property even if $n = 3$.
  - In the case of splittable traffic we show that our other convex potential function is minimized if and only if the corresponding assignment is an equilibrium. This result implies that an equilibrium can be computed in polynomial time.

  We also show for several generalizations of the above games that such potential functions do not exist.

– We prove upper and lower bounds on the price of anarchy for games with linear latency functions. For the case of unsplittable traffic the upper and lower bound are asymptotically tight.

Road Map. In Sect. 2 we define the games we consider. We present our results for unsplittable traffic in Sect. 3 and for splittable traffic in Sect. 4. Due to lack of space we have to omit many proofs.

2 Notation

For all $k \in \mathbb{N}$ denote $[k] = \{1, \ldots, k\}$. For a vector $\mathsf{v} = (v_1, \ldots, v_n)$ let $\mathsf{v}_{-\mathsf{i}} = (v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n)$ and $(\mathsf{v}_{-\mathsf{i}}, v_i') = (v_1, \ldots, v_{i-1}, v_i', v_{i+1}, \ldots, v_n)$.
Routing with Splittable Traffic. A *Wardrop game with player-specific latency functions* is a tuple $\Upsilon = (n, G, \mathbf{w}, Z, \mathbf{f})$. Here, n is the number of *players* and $G = (V, E)$ is an undirected *(multi)graph*. The vector $\mathbf{w} = (w_1, \ldots, w_n)$ defines for every player $i \in [n]$ its *traffic* $w_i \in \mathbb{R}^+$. For each player $i \in [n]$ the set $Z_i \subset 2^E$ consists of all possible routing paths in $G = (V, E)$ from some node $s_i \in V$ to some other node $t_i \in V$. Denote $Z = Z_1 \times \ldots \times Z_n$. Edge latency functions $\mathbf{f} = (f_{i\,e})_{i\in[n],e\in E}$ are player-specific and $f_{i\,e} : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is the non-negative, non-decreasing, and continuous *player-specific latency function* that player $i \in [n]$ assigns to edge $e \in E$. Notice that we are in the setting of the regular Wardrop game if $f_{ie} = f_{ke}$ for all $i, k \in [n]$, $e \in E$. In the majority of cases we consider *linear* player-specific latency functions $f_{ie}(u) = a_{ie} \cdot u + b_{ie}$ with $a_{ie}, b_{ie} \geq 0$. For a Wardrop game Υ with linear latency functions denote $\Delta(\Upsilon) = \max_{e\in E; i,k\in[n]}\{a_{i\,e}/a_{k\,e}; a_{i\,e} < \infty, a_{k\,e} < \infty\}$ with the understanding that $\frac{0}{0} = 1$ and $\frac{c}{0} = \infty$ if $c > 0$. $\Delta(\Upsilon)$ describes the maximum factor by which the slopes of the player-specific linear latency functions deviate. Note that $\Delta(\Upsilon)$ does not depend on the constants b_{ie} of the latency functions. We will use the term Wardrop game with *player-specific capacities* to denote a game where all latency functions are of the form $f_{i\,e}(u) = a_{i\,e} \cdot u, a_{i\,e} > 0$. In this case, we write $\mathbf{a}$ instead of $\mathbf{f}$ to denote the vector $\mathbf{a} = (a_{i\,e})_{i\in[n],e\in E}$. We will often consider games on a parallel link multi-graph $G = (V, E)$ that has two nodes $V = \{s, t\}$, $s = s_1 = \ldots s_n$, $t = t_1 = \ldots t_n$, and $|E|$ edges connecting these two nodes.
Strategies and Strategy Profiles. A player $i \in [n]$ can split its traffic w_i over the paths in Z_i. A (pure) *strategy* for player $i \in [n]$ is a tuple $\mathsf{x}_i = (x_{iR_i})_{R_i\in Z_i}$ with $\sum_{R_i\in Z_i} x_{iR_i} = w_i$ and $x_{iR_i} \geq 0$ for all $R_i \in Z_i$. Denote by $\mathcal{X}_i = \{\mathsf{x}_i \mid \mathsf{x}_i \text{ is a strategy for player } i\}$ the set of all strategies for player i. Note, that $\mathcal{X}_i$ is an infinite, compact and convex set. A *strategy profile* $\mathbf{x} = (\mathsf{x}_1, \ldots, \mathsf{x}_n)$ is an n-tuple of strategies for the players. Define $\mathcal{X} = \mathcal{X}_1 \times \ldots \times \mathcal{X}_n$ as the set of all possible strategy profiles.
Wardrop Equilibria. For a strategy profile $\mathbf{x}$ the load $\delta_e(\mathbf{x})$ on an edge $e \in E$ is given by $\delta_e(\mathbf{x}) = \sum_{i\in[n]} \sum_{R_i\in Z_i, R_i\ni e} x_{iR_i}$. A strategy profile $\mathbf{x}$ is a *Wardrop equilibrium*, if for every player $i \in [n]$, and every $R_i, R_i' \in Z_i$ with $x_{iR_i} > 0$ it holds that

$$\sum_{e\in R_i} f_{i\,e}(\delta_e(\mathbf{x})) \leq \sum_{e\in R_i'} f_{i\,e}(\delta_e(\mathbf{x})).$$

Observe that in a Wardrop equilibrium all flow paths of a player have equal latency. We can regard each player $i \in [n]$ as a service provider who has many clients each handling a negligible small amount of traffic. In a Wardrop equilibrium each service provider satisfies all his clients because none of them can improve its experienced latency.

Social Cost and Price of Anarchy. Associated with a game and a strategy profile $\mathbf{x}$ is the *social cost* $\mathsf{SC}(\mathbf{x})$ as a measure of social welfare:

$$\mathsf{SC}(\mathbf{x}) = \sum_{i\in[n]} \sum_{R_i\in Z_i} x_{iR_i} \sum_{e\in R_i} f_{i\,e}(\delta_e(\mathbf{x})).$$

This social cost is motivated by the interpretation as a game with infinitely many players with negligible demand and models the sum of the players latencies. The *optimum* associated with a game is defined by $\mathsf{OPT} = \min_{\mathbf{x}\in\mathcal{X}} \mathsf{SC}(\mathbf{x})$. The *price of anarchy*, also called *coordination ratio* and denoted PoA, is the maximum value, over all instances and Wardrop equilibria $\mathbf{x}$, of the ratio $\frac{\mathsf{SC}(\mathbf{x})}{\mathsf{OPT}}$.

Routing with Unsplittable Traffic. We also consider the case where players have to assign their traffic integrally to a single path. Denote such a *weighted network congestion game with player-specific latency functions* by $\Gamma = (n, G, \mathbf{w}, Z, \mathbf{f})$. The players are *unweighted* if they are all of traffic 1, i.e. $w_1 = \ldots = w_n = 1$. In this case we write $\mathbf{1}$ instead of $\mathbf{w}$. A pure strategy x_i for player $i \in [n]$ is a tuple $\mathsf{x}_i = (x_{iR_i})_{R_i\in Z_i}$ with $\sum_{R_i\in Z_i} x_{iR_i} = w_i$ and $x_{iR_i} \in \{0, w_i\}$ for all $R_i \in Z_i$. Alternatively, with a slight abuse of notation, we write $\mathbf{R} = (R_1, \ldots, R_n)$ where $R_i \in Z_i$, $1 \le i \le n$, to denote a strategy profile such that $x_{iR_i} = w_i$ for all $i \in [n]$. In this setting, $Z = Z_1 \times \ldots \times Z_n$ is the set of all pure strategy profiles. We define the *private cost* of player $i \in [n]$ as the sum over the player-specific latencies of all used edges: $\mathsf{PC}_i(\mathbf{R}) = \sum_{e\in R_i} f_{i\,e}(\delta_e(\mathbf{R}))$.

Given a pure strategy profile $\mathbf{R} = (R_1, \ldots, R_n)$ a *selfish step* of a player $i \in [n]$ is a deviation to strategy profile $(\mathbf{R}_{-i}, R_i')$ where $\mathsf{PC}_i(\mathbf{R}_{-i}, R_i') < \mathsf{PC}_i(\mathbf{R})$ and $R_i' \in Z_i$. Such a selfish step is a *greedy selfish step* if there is for player i no strategy $R_i'' \in Z_i$ such that $\mathsf{PC}_i(\mathbf{R}_{-i}, R_i'') < \mathsf{PC}_i(\mathbf{R}_{-i}, R_i')$.

A game Γ possesses the *finite best-reply property* if any sequence of greedy selfish steps is finite. If even any sequence of selfish steps is finite it possesses in addition the *finite improvement property*. Note, that the finite improvement property implies the finite best-reply property which again implies the existence of a pure Nash equilibrium.

We also consider *mixed strategies* $\mathsf{P_i}$ for the players. Then, $\mathsf{P_i} = (p(i, R_i))_{R_i\in Z_i}$ is a probability distribution over Z_i and $p(i, R_i)$ denotes the probability that player i chooses path R_i. A *mixed strategy profile* $\mathbf{P} = (\mathsf{P_1}, \ldots, \mathsf{P_n})$ is represented by an n tuple of mixed strategies. For a mixed strategy profile $\mathbf{P}$ denote $p(\mathbf{R}) = \prod_{i\in[n]} p(i, R_i)$ as the probability that the players choose the pure strategy profile $\mathbf{R} = (R_1, \ldots, R_n)$.

Nash Equilibrium, Social Cost, and Price of Anarchy. For a mixed strategy profile $\mathbf{P}$ the private cost of player $i \in [n]$ is $\mathsf{PC}_i(\mathbf{P}) = \sum_{\mathbf{R}\in Z} p(\mathbf{R}) \cdot \mathsf{PC}_i(\mathbf{R})$. For a pure strategy profile $\mathbf{R}$ the social cost $\mathsf{SC}(\mathbf{R})$ is defined as before whereas for a mixed strategy profile $\mathbf{P}$ the social cost is given by $\mathsf{SC}(\mathbf{P}) = \sum_{\mathbf{R}\in Z} p(\mathbf{R}) \cdot \mathsf{SC}(\mathbf{R})$. A strategy profile $\mathbf{P}$ is a Nash equilibrium if no player $i \in [n]$ can decrease its private cost PC_i if the other players stick to their strategies. More formally, $\mathbf{P} = (\mathsf{P_1}, \ldots, \mathsf{P_n})$ is a Nash equilibrium if $\mathsf{PC}_i(\mathbf{P}) \le \mathsf{PC}_i(\mathbf{P}_{-i}, \mathsf{P_i'})$ for all probability distributions $\mathsf{P_i'}$ over Z_i and for all $i \in [n]$. In the unsplittable setting the price of anarchy PoA is the worst-case ratio between the social cost of a mixed Nash equilibrium and that of some social optimum.

3 Results for Unsplittable Traffic

3.1 Unweighted Players: Finite Improvement Property

Milchtaich [19] showed that network congestion games on parallel links with player-specific latency functions and unweighted players do not possess the finite improvement property in general. In Theorem 1 we show that we achieve the finite improvement property if we restrict to player-specific capacities. In Theorem 2 we give counterexamples to show that a slight deviation from this model yields a loss of the finite improvement property. For the positive result in Theorem 1 we define for every strategy profile $\mathbf{R} = (R_1, \ldots, R_n)$ the following potential function:

$$\Phi(\mathbf{R}) = \prod_{i \in [n]} a_{i\,R_i} \cdot \prod_{e \in E} \delta_e(\mathbf{R})!$$

In contrast to all other potential functions we know, Φ does not contain any summation.

Theorem 1. *Every network congestion game on parallel links with unweighted players and player-specific capacities possesses the finite improvement property.*

Proof. Consider a selfish step $\mathbf{R} \to \mathbf{R}'$ of a player $i \in [n]$ from edge $j \in E$ to edge $k \in E$, i.e. $\mathbf{R} = (R_1, \ldots, R_{i-1}, j, R_{i+1}, \ldots, R_n)$ and $\mathbf{R}' = (R_1, \ldots, R_{i-1}, k, R_{i+1}, \ldots, R_n)$. If A denotes the common part of the expressions $\Phi(\mathbf{R})$ and $\Phi(\mathbf{R}')$ they can be written as $\Phi(\mathbf{R}') = A \cdot a_{i\,k} \cdot (\delta_k(\mathbf{R}) + 1)$ and $\Phi(\mathbf{R}) = A \cdot a_{i\,j} \cdot \delta_j(\mathbf{R})$. Since $\mathbf{R} \to \mathbf{R}'$ is a selfish step we have that $\mathsf{PC}_i(\mathbf{R}') = a_{i\,k} \cdot (\delta_k(\mathbf{R}) + 1) < a_{i\,j} \cdot \delta_j(\mathbf{R}) = \mathsf{PC}_i(\mathbf{R})$. Thus $\Phi(\mathbf{R}') = A \cdot \mathsf{PC}_i(\mathbf{R}') < A \cdot \mathsf{PC}_i(\mathbf{R}) = \Phi(\mathbf{R})$. The claim follows since the number of strategy profiles is finite. □

Theorem 2. *Network congestion games on a graph G with unweighted players and player-specific latency functions do (in general) not possess*

(a) the finite best-reply property if the game has 3 players, linear latency functions, and G is a parallel links graph.
(b) the finite improvement property if the game has 2 players, player-specific capacities, and G is a concatenation of 2 parallel link graphs connected in series.
(c) a pure Nash equilibrium if the game has 3 players, player-specific capacities, and all paths in G are of length at most 2.

3.2 Weighted Players: Finite Improvement Property

For weighted congestion games on parallel links with player-specific capacities Georgiou et al. [14] showed that a Nash equilibrium always exists in the case of 3 players. For arbitrary many players it is an open problem whether such a game still admits a pure Nash equilibrium or not. Theorem 3 implies that the finite improvement property can not be used to solve the open problem even if there are only 3 players. We would like to note that for the case of 2 players we can give a potential function showing that the finite improvement property is fulfilled.

Theorem 3. *There is a weighted congestion game on parallel links with 3 players and player-specific capacities that does not possess the finite improvement property.*

Proof. The 3 players of the game are of traffic $w_1 = 1$, $w_2 = 2$, and $w_3 = 79$. The player-specific capacities of the 11 edges are listed in this table ($\epsilon_1, \epsilon_2, \epsilon_3 > 0$ are small numbers we will discuss later):

j	1	2	3	4	5	6	7	8	9	10	11
a_{1j}	$\frac{3^8}{80}-\epsilon_3$	∞	1	$3-\epsilon_1$	$(3-\epsilon_1)^2$	$(3-\epsilon_1)^3$	$(3-\epsilon_1)^4$	$(3-\epsilon_1)^5$	$(3-\epsilon_1)^6$	$(3-\epsilon_1)^7$	$(3-\epsilon_1)^8$
a_{2j}	∞	1	$\frac{2}{3}-\epsilon_1$	$\frac{2^2}{3^2}-\epsilon_1$	$\frac{2^3}{3^3}-\epsilon_1$	$\frac{2^4}{3^4}-\epsilon_1$	$\frac{2^5}{3^5}-\epsilon_1$	$\frac{2^6}{3^6}-\epsilon_1$	$\frac{2^7}{3^7}-\epsilon_1$	$\frac{2^8}{3^8}-\epsilon_1$	∞
a_{3j}	1	∞	$\frac{80}{79}-\epsilon_1$	∞	∞	∞	∞	∞	∞	$\frac{80^2}{79^2}\cdot\frac{79}{81}-\epsilon_2$	$(\frac{80}{79}-\epsilon_1)^2$

Our cycle of selfish steps starts in the initial strategy profile $(3, 2, 1)$. We now perform 8 double-steps (A). A double-step (A) consists of a first step that moves player 2 from an edge j to the edge k player 1 is assigned to and a second step to an empty edge l that player 1 does. Both steps of a double-step (A) are selfish iff $a_{2k}/a_{2j} < \frac{2}{3}$ and $a_{1l}/a_{1k} < 3$. In each step of our 8 double-steps (A) the deviating player moves from edge t to edge $t+1$. After the double-steps (A) the strategy profile $(11, 10, 1)$ is reached. Notice that all 16 steps performed up to now are selfish since $\epsilon_1 > 0$.

The cycle continues with double-steps (B). A double-step (B) starts with a move of player 1 from edge j to the edge k used by player 3 followed by a step of player 3 to an empty edge l. Observe that (B) is a pair of selfish steps iff $a_{1k}/a_{1j} < \frac{1}{80}$ and $a_{3l}/a_{3k} < \frac{80}{79}$. We conduct 2 double-steps (B): (11,10,1) → (1,10,1) → (1,10,3) → (3,10,3) → (3,10,11). These steps are selfish if:

$$1 \cdot (3-\epsilon_1)^8 > 80 \cdot \left(\frac{3^8}{80} - \epsilon_3\right) \text{ i.e. } \epsilon_3 > \frac{3^8}{80} - \frac{(3-\epsilon_1)^8}{80} \text{ and} \tag{1}$$

$$1 \cdot \left(\frac{3^8}{80} - \epsilon_3\right) > 80 \cdot 1 \text{ i.e. } \epsilon_3 < \frac{3^8}{80} - 80. \tag{2}$$

Starting from the strategy profile (3,10,11) we proceed with a double-step (C) that moves player 3 to the edge 10 player 2 is assigned to and continues with a step of player 2 to the empty edge 2. This double-step consists of selfish steps iff $a_{3\,10}/a_{3\,11} < \frac{79}{81}$ and $a_{2\,2}/a_{2\,10} < \frac{81}{2}$. The double-step (C) is selfish if:

$$79 \cdot \left(\frac{80}{79} - \epsilon_1\right)^2 > 81 \cdot \left(\frac{80^2 \cdot 79}{79^2 \cdot 81} - \epsilon_2\right) \text{ i.e. } \epsilon_2 > \frac{80^2}{79 \cdot 81} - \frac{79}{81}\left(\frac{80}{79} - \epsilon_1\right)^2 \text{ and} \tag{3}$$

$$81 \cdot \left(\frac{2^8}{3^8} - \epsilon_1\right) > 2 \cdot 1 \text{ i.e. } \epsilon_1 < \frac{2^8}{3^8} - \frac{2}{81}. \tag{4}$$

The 11 double-steps explained up to now are followed by a final step that moves player 3 back to edge 1: (3,2,10) → (3,2,1). It is selfish if:

$$79 \cdot \left(\frac{80^2}{79^2} \cdot \frac{79}{81} - \epsilon_2\right) > 79 \cdot 1 \text{ i.e. } \epsilon_2 < \frac{80^2}{79 \cdot 81} - 1. \tag{5}$$

It is possible to select $\epsilon_1, \epsilon_2, \epsilon_3 > 0$ fulfilling (1) – (5). Thus the claim follows. □

3.3 General Networks and Linear Latency Functions: Price of Anarchy

In this section we study the price of anarchy for weighted congestion games with linear player-specific latency functions. To prove our upper bound we use similar techniques as Christodoulou and Koutsoupias [7] and Awerbuch et al. [2]. The proof is also based on the following technical lemma.

Lemma 1. *For all* $u, v \in \mathbb{R}_0^+$ *and* $c \in \mathbb{R}^+$ *we have* $v(u+v) \le c \cdot u^2 + \left(1 + \frac{1}{4c}\right) \cdot v^2$.

Theorem 4. *Let* Γ *be a weighted network congestion game with player-specific linear latency functions. Then,* $\mathsf{PoA} \le \frac{1}{2} \cdot [\Delta(\Gamma) + 2 + \sqrt{\Delta(\Gamma)(\Delta(\Gamma)+4)}]$.

Proof. Let $\mathbf{P} = (P_1, \ldots, P_n)$ be a mixed Nash equilibrium and let $\mathbf{Q}$ be a pure strategy profile with optimum social cost. Since $\mathbf{P}$ is a Nash equilibrium, player i cannot improve by switching from strategy P_i to Q_i. Thus,

$$\begin{aligned}
\mathsf{PC}_i(\mathbf{P}) \le \mathsf{PC}_i(\mathbf{P}_{-i}, Q_i) &= \sum_{\mathbf{R} \in Z} p(\mathbf{R}) \left[\sum_{e \in Q_i \cap R_i} f_{ie}(\delta_e(\mathbf{R})) + \sum_{e \in Q_i \setminus R_i} f_{ie}(\delta_e(\mathbf{R}) + w_i) \right] \\
&\le \sum_{\mathbf{R} \in Z} p(\mathbf{R}) \sum_{e \in Q_i} f_{ie}(\delta_e(\mathbf{R}) + \delta_e(\mathbf{Q})).
\end{aligned}$$

It follows that

$$\begin{aligned}
\mathsf{SC}(\mathbf{P}) &= \sum_{\mathbf{R} \in Z} p(\mathbf{R}) \sum_{i \in [n]} w_i \sum_{e \in R_i} f_{ie}(\delta_e(\mathbf{R})) = \sum_{i \in [n]} w_i \cdot \mathsf{PC}_i(\mathbf{P}) \\
&\le \sum_{i \in [n]} \sum_{\mathbf{R} \in Z} p(\mathbf{R}) \sum_{e \in Q_i} w_i \cdot f_{ie}(\delta_e(\mathbf{R}) + \delta_e(\mathbf{Q})) \\
&= \sum_{\mathbf{R} \in Z} p(\mathbf{R}) \sum_{e \in E} \sum_{i, Q_i \ni e} w_i \cdot [a_{ie}(\delta_e(\mathbf{R}) + \delta_e(\mathbf{Q})) + b_{ie}] \\
&= \sum_{\mathbf{R} \in Z} p(\mathbf{R}) \sum_{\substack{e \in E, \delta_e(\mathbf{Q}) > 0, \\ \delta_e(\mathbf{R}) > 0}} \frac{\sum_{i, Q_i \ni e} a_{ie} w_i}{\delta_e(\mathbf{Q})} \cdot \delta_e(\mathbf{Q}) \cdot (\delta_e(\mathbf{R}) + \delta_e(\mathbf{Q})) \\
&\quad + \sum_{\mathbf{R} \in Z} p(\mathbf{R}) \sum_{\substack{e \in E, \delta_e(\mathbf{Q}) > 0, \\ \delta_e(\mathbf{R}) = 0}} \delta_e(\mathbf{Q}) \cdot \sum_{i, Q_i \ni e} a_{ie} w_i + \sum_{\mathbf{R} \in Z} p(\mathbf{R}) \sum_{e \in E} \sum_{i, Q_i \ni e} w_i b_{ie}.
\end{aligned}$$

By Lemma 1 we get for $c \in \mathbb{R}^+$,

$$\begin{aligned}
\mathsf{SC}(\mathbf{P}) &\le \sum_{\mathbf{R} \in Z} p(\mathbf{R}) \sum_{\substack{e \in E, \delta_e(\mathbf{Q}) > 0, \\ \delta_e(\mathbf{R}) > 0}} \frac{\sum_{i, Q_i \ni e} a_{ie} w_i}{\delta_e(\mathbf{Q})} \cdot \left[\left(1 + \frac{1}{4c}\right) \cdot \delta_e(\mathbf{Q})^2 + c \cdot \delta_e(\mathbf{R})^2 \right] \\
&\quad + \sum_{\mathbf{R} \in Z} p(\mathbf{R}) \sum_{\substack{e \in E, \delta_e(\mathbf{Q}) > 0, \\ \delta_e(\mathbf{R}) = 0}} \delta_e(\mathbf{Q}) \cdot \sum_{i, Q_i \ni e} a_{ie} w_i + \sum_{\mathbf{R} \in Z} p(\mathbf{R}) \sum_{e \in E} \sum_{i, Q_i \ni e} w_i b_{ie} \\
&\le \left(1 + \frac{1}{4c}\right) \sum_{\mathbf{R} \in Z} p(\mathbf{R}) \sum_{e \in E} \left(\sum_{i, Q_i \ni e} a_{ie} w_i \right) \delta_e(\mathbf{Q}) + \sum_{\mathbf{R} \in Z} p(\mathbf{R}) \sum_{e \in E} \sum_{i, Q_i \ni e} w_i \cdot b_{ie} \\
&\quad + c \cdot \sum_{\mathbf{R} \in Z} p(\mathbf{R}) \sum_{\substack{e \in E, \delta_e(\mathbf{Q}) > 0, \\ \delta_e(\mathbf{R}) > 0}} \frac{\sum_{i, Q_i \ni e} a_{ie} w_i}{\delta_e(\mathbf{Q})} \cdot \delta_e(\mathbf{R})^2 \\
&\le \left(1 + \frac{1}{4c}\right) \cdot \mathsf{SC}(\mathbf{Q}) + c \cdot \sum_{\mathbf{R} \in Z} p(\mathbf{R}) \sum_{\substack{e \in E, \delta_e(\mathbf{Q}) > 0, \\ \delta_e(\mathbf{R}) > 0}} \frac{\sum_{i, Q_i \ni e} a_{ie} w_i}{\delta_e(\mathbf{Q})} \cdot \delta_e(\mathbf{R})^2.
\end{aligned}$$

Observe that $\frac{1}{\delta_e(\mathbf{Q})} \cdot \sum_{i, Q_i \ni e} a_{ie} w_i$ is a weighted average slope of latency functions for edge $e \in E$. With $\frac{a_{ie}}{a_{ke}} \leq \Delta(\Gamma)$ for all $i, k \in [n]$ with $a_{ie}, a_{ke} < \infty$ it follows that $\frac{1}{\delta_e(\mathbf{Q})} \cdot \sum_{i, Q_i \ni e} a_{ie} w_i \leq \Delta(\Gamma) \cdot \frac{1}{\delta_e(\mathbf{R})} \cdot \sum_{i, R_i \ni e} a_{ie} w_i$. We get,

$$\begin{aligned}
\mathsf{SC}(\mathbf{P}) &\leq \left(1 + \frac{1}{4c}\right) \cdot \mathsf{SC}(\mathbf{Q}) + c \cdot \sum_{\mathbf{R} \in Z} p(\mathbf{R}) \sum_{\substack{e \in E, \delta_e(\mathbf{Q}) > 0, \\ \delta_e(\mathbf{R}) > 0}} \Delta(\Gamma) \cdot \frac{\sum_{i, R_i \ni e} a_{ie} w_i}{\delta_e(\mathbf{R})} \cdot \delta_e(\mathbf{R})^2 \\
&\leq \left(1 + \frac{1}{4c}\right) \cdot \mathsf{SC}(\mathbf{Q}) + c \cdot \sum_{\mathbf{R} \in Z} p(\mathbf{R}) \sum_{e \in E} \Delta(\Gamma) \cdot \delta_e(\mathbf{R}) \cdot \sum_{i, R_i \ni e} a_{ie} w_i \\
&\leq \left(1 + \frac{1}{4c}\right) \cdot \mathsf{SC}(\mathbf{Q}) + c \cdot \Delta(\Gamma) \cdot \mathsf{SC}(\mathbf{P}).
\end{aligned}$$

Thus choosing $c = \frac{-\Delta(\Gamma) + \sqrt{\Delta(\Gamma)(\Delta(\Gamma)+4)}}{4\Delta(\Gamma)}$ yields

$$\frac{\mathsf{SC}(\mathbf{P})}{\mathsf{SC}(\mathbf{Q})} \leq \frac{4c+1}{4c(1 - c\Delta(\Gamma))} = \frac{\Delta(\Gamma) + 2 + \sqrt{\Delta(\Gamma)(\Delta(\Gamma)+4)}}{2}.$$

Since $\mathbf{P}$ is an arbitrary (mixed) Nash equilibrium the claim follows. □

Interestingly, we get with Theorem 4 an upper bound of $\frac{1}{2} \cdot (3 + \sqrt{5})$ in the case of $\Delta(\Gamma) = 1$ which matches the exact price of anarchy for weighted congestion games [2] even though our model still allows for player-specific constants $b_{ie} \neq b_{ke}$. We proceed with a lower bound on the price of anarchy that is asymptotically tight. Variations of the games used in the proof of the lower bound were also used in some recent papers to show lower bounds on the price of anarchy in different settings (see e.g. [3, 9, 11]).

Theorem 5. *For each $l \in \mathbb{N}$ and for each $\epsilon > 0$ there is a congestion game Γ on parallel links with unweighted players and player-specific capacities that possesses a pure Nash equilibrium $\mathbf{R}$ such that $\Delta(\Gamma) \geq l$ and $\mathsf{SC}(\mathbf{R})/\mathsf{OPT} \geq (1 - \epsilon) \cdot \Delta(\Gamma)$.*

The construction in the proof of Theorem 5 uses a large number of players. However, the price of anarchy is unbounded even for 2 player games.

Theorem 6. *For every $k \geq 1$ there is a weighted congestion game on parallel links with 2 players and player-specific capacities that possesses a pure Nash equilibrium $\mathbf{R}$ such that $\mathsf{SC}(\mathbf{R})/\mathsf{OPT} > k$.*

4 Results for Splittable Traffic

4.1 Parallel Links and Player-Specific Capacities: Existence of and Convergence to a Wardrop Equilibrium

In this section we consider Wardrop games on parallel links with player-specific capacities. For such a game and a strategy profile $\mathbf{x}$ define the following function:

$$\Psi(\mathbf{x}) = \sum_{i \in [n]} \sum_{e \in E} x_{i\,e} \cdot \ln(a_{i\,e}) + \sum_{\substack{e \in E, \\ \delta_e(\mathbf{x}) > 0}} \delta_e(\mathbf{x}) \cdot \ln(\delta_e(\mathbf{x})).$$

Note, that $\mathrm{e}^{\Psi(\mathbf{x})}$ has a similar form as the potential function Φ in Sect. 3. The next theorem shows that Ψ plays a similar role as the potential function Φ.

Theorem 7. *Let Υ be a Wardrop game on parallel links with player-specific capacities. Moreover let $\mathbf{x}$ be a strategy profile for Υ so that there exists a player $k \in [n]$, two edges $p, q \in E$, and some Λ, $0 < \Lambda \leq x_{k\,p}$ such that: $a_{k\,p} \cdot (\delta_p(\mathbf{x}) - \Lambda) \geq a_{k\,q} \cdot (\delta_q(\mathbf{x}) + \Lambda)$. Define a new strategy profile $\mathbf{y}$ by:*

$$y_{i\,j} = \begin{cases} x_{k\,p} - \Lambda & \text{if } i = k, j = p, \\ x_{k\,q} + \Lambda & \text{if } i = k, j = q, \\ x_{i\,j} & \text{otherwise.} \end{cases}$$

Then $\Psi(\mathbf{y}) < \Psi(\mathbf{x})$.

We now show that $\Psi(\mathbf{x})$ is minimized iff $\mathbf{x}$ is a Wardrop equilibrium.

Theorem 8. *Let Υ be a Wardrop game on parallel links with player-specific capacities. Moreover let $\mathbf{y}$ be a strategy profile for Υ. Then the following two conditions are equivalent:*

(a) $\Psi(\mathbf{y}) = \min_{\mathbf{x} \in \mathcal{X}} \Psi(\mathbf{x})$,
(b) $\mathbf{y}$ *is a Wardrop equilibrium.*

Proof. (a) ⇒ (b) follows immediately with Theorem 7. It is possible to show (b) ⇒ (a) with an argumentation based on the Karush-Kuhn-Tucker theorem (see [5]). □

Since Ψ is a convex function it follows with Theorem 8 that the ellipsoid method of Khachyan [16] can be used to compute a Wardrop equilibrium in time polynomial in the size of the instance and the number of bits of precision required.

4.2 Does there exist a convex potential function for a more general setting?

If a game can be described by a convex potential function then the set of Nash equilibria forms a convex set. In this section we show that no such convex function exists for general graphs with player-specific capacities (Theorem 9) whereas the existence remains an open problem for parallel links with strictly increasing player-specific latency functions (Theorem 10).

Theorem 9. *There is a Wardrop game Υ with player-specific capacities that possesses two Wardrop equilibria $\mathbf{x}$ and $\mathbf{y}$ where*

(a) $\delta_j(\mathbf{x}) \neq \delta_j(\mathbf{y})$ *for an edge* $j \in E$ *and* $\mathsf{SC}(\mathbf{x}) \neq \mathsf{SC}(\mathbf{y})$,
(b) the set of Wardrop equilibria for Υ does not form a convex set.

Theorem 10. *Let Υ be a Wardrop game on parallel links with strictly increasing player-specific latency functions. Let $\mathbf{x}$ and $\mathbf{y}$ be Wardrop equilibria for Υ. Then,*

(a) $\delta_j(\mathbf{x}) = \delta_j(\mathbf{y})$ *for all* $j \in E$ *and* $\mathsf{SC}(\mathbf{x}) = \mathsf{SC}(\mathbf{y})$,
(b) the set of Wardrop equilibria for Υ forms a convex set.

4.3 General Networks and Player-Specific Latency Functions: Existence of Wardrop Equilibria

Each Wardrop game possesses a Wardrop equilibrium (see [4]). It is possible to use Brouwer's fixed point theorem to prove the existence of equilibria for our more general class of games.

Theorem 11. *Every Wardrop game Υ with strictly increasing player-specific latency functions possesses a Wardrop equilibrium.*

4.4 General Networks and Linear Latency Functions: Price of Anarchy

In this section we give bounds on the price of anarchy. The proof of the upper bound uses the same technique as the proof of Theorem 4.

Theorem 12. *Let Υ be a Wardrop game with player-specific linear latency functions. Then,*

$$\mathsf{PoA} \leq \begin{cases} \frac{4}{4-\Delta(\Upsilon)} & \textit{if } \Delta(\Upsilon) \leq 2, \\ \Delta(\Upsilon) & \textit{otherwise.} \end{cases}$$

Theorem 13. *For each $n \in \mathbb{N}$ there is a Wardrop game Υ on 2 parallel links with n unweighted players and player-specific capacities that possesses a Wardrop equilibrium $\mathbf{x}$ such that $\Delta(\Upsilon) = n^2$ and $\mathsf{SC}(\mathbf{x})/\mathsf{OPT} \geq \frac{1}{4} \cdot \sqrt{\Delta(\Upsilon)}$.*

For the Wardrop model with linear latency functions, Roughgarden and Tardos [25] showed that the price of anarchy is exactly $\frac{4}{3}$. Theorem 12 with $\Delta(\Upsilon) = 1$ implies that the price of anarchy does not change even if the linear latency functions of the players have player-specific constants $b_{ie} \neq b_{ke}$. Although our upper bound is tight for $\Delta(\Upsilon) = 1$ there is for large $\Delta(\Upsilon)$ still a gap between the upper bound of $\Delta(\Upsilon)$ and the lower bound.

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