Gallai Theorems Involving Domination Parameters

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Abstract

In 1959 Gallai [5] showed that the vertex independence number and the vertex covering number of a graph $G = (V, E)$ sum to $|V|$. Over the last twenty years, many results similar to Gallai's Theorem have been observed [3]. These theorems are referred to as "Gallai Theorems" and usually have the form: $\alpha + \beta = n$.

Slater [17] described several graph subset parameters using linear programs (LP) and integer programs. Gallai Theorems for the resulting parameters may be obtained by using the concepts of LP-duality and complementarity. Slater defines several of the parameters

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generated by various graph theoretic matrices, but leaves other parameters unstudied. In this paper, we take a closer look at some of those unstudied parameters and at some related parameters.

1 Introduction

Let $G = (V, E)$ be a simple graph on n vertices having m edges. For any vertex $v \in V$, the *open neighborhood* of v, denoted $N(v)$, is the set of all vertices adjacent to v. The *closed neighborhood* of v, denoted $N[v]$, is the set $N(v) \cup \{v\}$. For a subset $S \subseteq V$, the *open neighborhood of* S, denoted $N(S)$, is defined by $N(S) = \bigcup_{v \in S} N(v)$; similarly, we define $N[S] = \bigcup_{v \in S} N[v]$ to be the *closed neighborhood of* S.

A set $S \subseteq V$ is an *independent set* if no two vertices in S are adjacent, and D is a *dominating set* if $N[D] = V$. In other words, a set D is dominating if every vertex in V is either in D or adjacent to a vertex in D. The maximum and minimum sizes of a maximal (see Section 3 for an explanation of maximal) independent set are denoted $\beta_0(G)$ and $i(G)$, respectively. Also, we denote the maximum and minimum sizes of a minimal dominating set by $\Gamma(G)$ and $\gamma(G)$, respectively. When the graph is not in question, we simply write β_0, i, Γ, or γ.

In 1959 Gallai [5] published the following theorem which bears his name. In this theorem β_0, called the vertex independence number, is as defined above, and α_0, the vertex covering number, is the size of a smallest set $S \subseteq V$ such that every edge in the graph is incident with at least one vertex in S.

Theorem 1 (Gallai's Theorem) *For any graph G of order n having no isolates, $\alpha_0 + \beta_0 = n$.*

Over the last twenty years, many results similar to Gallai's Theorem have been observed [3]. These theorems are referred to as "Gallai Theorems" and usually have the form:

$$\alpha + \beta = n,$$

where α and β are integer-valued minimum or maximum functions corresponding to some property of a graph on n vertices. Refer to [7, 12, 10, 3] for examples of Gallai Theorems.

2 Using Linear and Integer Programming to Describe Graph Parameters

Following Slater's description [17] , we may define many graph subset parameters in terms of linear programs (LP) and integer programs (IP).

Graph theoretic minimization (maximization) problems can be modelled in terms of LP/IP problems using the adjacency matrix, A, the closed neighborhood matrix, N, and the incidence matrix, H. For instance, $S \subseteq V$ is a dominating set if $|N[v_i] \cap S| \geq 1$ for $1 \leq i \leq n$, and S is a (vertex) independent set if, for any $(v_i, v_j) \in E$, v_i and v_j are not both in S. The domination number, γ, and the vertex independence number, β_0, can be defined by the following IP's:

$$\begin{aligned} \gamma(G) = \text{minimize } & \sum_{i=1}^{n} x_i \\ \text{subject to } & \\ & Nx \geq \overline{1}_n \\ & x_i \in \{0,1\} \qquad 1 \leq i \leq n. \end{aligned}$$

$$\begin{aligned} \beta_0(G) = \text{maximize } & \sum_{i=1}^{n} x_i \\ \text{subject to } & \\ & H^T x \leq \overline{1}_m \\ & x_i \in \{0,1\} \qquad 1 \leq i \leq n. \end{aligned}$$

Let M be a $k \times h$ matrix $M = [m_{ji}]$ and let $r_j = \sum_{i=1}^{h} m_{ji}$ and $L_M = [r_1, r_2, \ldots, r_m]^T$. Also, let $c = (c_1, \ldots, c_h)$ be the h-tuple of objective function coefficients and $b = (b_1, \ldots, b_k)$ be the k-tuple of constraint coefficients. For any 0-1 matrix M and $Y \subseteq \mathbb{R}^+$, Slater shows that at most eight parameters for a given graph can be generated by looking at the duals and complementations of a given primal problem P. He calls this the Dual/Complementation 8-cycle.

$$\begin{array}{llcll} P\text{: min} & \displaystyle\sum_{i=1}^{h} c_i x_i & & \text{max} & \displaystyle\sum_{i=1}^{k} b_i x_i \\ \text{s.t.} & & \text{Dual} & \text{s.t.} & \\ & Mx \geq b & \longleftrightarrow & & M^T x \leq c \\ & x_i \in Y \quad 1 \leq i \leq h & & & x_i \in Y \quad 1 \leq i \leq k \\ & & & & \\ & \updownarrow \text{ Comp.} & & & \text{Comp. } \updownarrow \end{array}$$

$$
\begin{array}{llcll}
\max & \sum_{i=1}^{h} c_i x_i & & \min & \sum_{i=1}^{k} b_i x_i \\
\text{s.t.} & & & \text{s.t.} & \\
& Mx \leq L_M - b & & & M^T x \leq L_{M^T} - c \\
& x_i \in Y \quad 1 \leq i \leq h & & & x_i \in Y \quad 1 \leq i \leq k \\
& \updownarrow \text{Dual} & & & \text{Dual} \updownarrow \\
\min & \sum_{i=1}^{k} (L_M - b)_i x_i & & \max & \sum_{i=1}^{h} (L_{M^T} - c)_i x_i \\
\text{s.t.} & & & \text{s.t.} & \\
& M^T x \geq c & & & M^T x \leq b \\
& x_i \in Y \quad 1 \leq i \leq k & & & x_i \in Y \quad 1 \leq i \leq h \\
& \updownarrow \text{Comp.} & & & \text{Comp.} \updownarrow \\
\max & \sum_{i=1}^{k} (L_M - b)_i x_i & & \min & \sum_{i=1}^{h} (L_{M^T} - c)_i x_i \\
\text{s.t.} & & \text{Dual} & \text{s.t.} & \\
& M^T x \leq L_{M^T} - c & \longleftrightarrow & & Mx \geq L_M - b \\
& x_i \in Y \quad 1 \leq i \leq k & & & x_i \in Y \quad 1 \leq i \leq h
\end{array}
$$

In [17], Slater defines several of the parameters generated by using $M = A, H$, and N, but other parameters are left unstudied. In section 4, we take a closer look at some of these unstudied parameters.

First, we adopt notation similar to that of Slater. Let us abbreviate the following two linear programs as $(M, \min, c, b, Y)$ and $(M, \max, c, b, Y)$, respectively.

$$
\begin{array}{lll}
P_1 : \text{minimize} & \sum_{i=1}^{h} c_i x_i & \\
\quad \text{subject to} & & \\
& Mx \geq b & \\
& x_i \in Y & 1 \leq i \leq h
\end{array}
$$

$$
\begin{array}{lll}
P_2 : \text{maximize} & \sum_{i=1}^{k} c_i x_i & \\
\quad \text{subject to} & & \\
& Mx \leq b & \\
& x_i \in Y & 1 \leq i \leq k
\end{array}
$$

3 Definitions

Let P be a property of subsets of V. A set S having property P is called a *P-set*. We say that P is a *hereditary* property if every subset of a P-set S has property P. Likewise, P is *superhereditary* if every superset of S has property P. Also, a set S is a minimal (maximal) P-set if S has the property P and every $S' \subset S$ ($S' \supset S$) does not have the property P. Additionally, a set S is a 1-minimal (1-maximal) P-set if S has the property P and for every vertex $v \in S$ ($v \in (V - S)$), $S - \{v\}$ ($S \cup \{v\}$) is not a P-set. Certainly, minimal (maximal) P-sets are 1-minimal (1-maximal) P-sets, but the converse is not necessarily true. Haynes, Hedetniemi, and Slater [6] noted when the two concepts are equivalent.

Proposition 2 *Let G be a graph and let P_h be a hereditary property. Then a set S is a maximal P-set if and only if S is a 1-maximal P_h-set.*

Likewise, let P_s be a superhereditary property. Then a set S is a minimal P_s-set if and only if S is a 1-minimal P_s-set.

Various types of domination may be created by placing additional requirements on a dominating set. For example, if $S \subseteq V$ is a dominating set and if

i) the induced subgraph $\langle S \rangle$ is connected, then S is a *connected dominating set* (Sampathkumar and Walikar [14]);

ii) $\langle S \rangle$ contains no isolates, then we say that S is a *total dominating set* (Cockayne, Dawes, and Hedetniemi [2]);

iii) for all $u \in V - S$, there exists a vertex $v \in S$ such that $uv \in E$ and $deg(v) \geq deg(u)$, then S is called a *strong dominating set*, and S is a *weak dominating set* if, for all $u \in V - S$, there exists a vertex $v \in S$ such that $uv \in E$ and $deg(v) \leq deg(u)$ (Pushpa Latha and Sampathkumar [14]).

We introduce the following domination concept in this paper. If S is a dominating set such that $|N[x] \cap S| \geq deg(x)$ for all $x \in V$, then we say that S is a *degree-dominating set.* We also define $S \subseteq V$ to be a *$\{deg - k\}$-dominating set* if $|N[x] \cap S| \geq deg(x) - k + 1$ for all $x \in V$ and $k = 1, 2, \ldots, \delta$.

Notice that $\{deg - 1\}$-domination is our ordinary degree-domination.

The parameters $\Gamma_c, \Gamma_t, \Gamma_s, \Gamma_w, \Gamma_d, \Gamma_{d-k}$ and $\gamma_c, \gamma_t, \gamma_s, \gamma_w, \gamma_d, \gamma_{d-k}$ denote the maximum and minimum sizes of the various types of minimal dominating sets, that is, connected, total, strong, weak, degree-, and $\{deg - k\}$-dominating, respectively.

Additionally, note that each of the different domination properties is superhereditary.

A set $S \subseteq V(G)$ is a(n):

i) *restrained set* if $V - S$ has no isolated vertices;

ii) *co-connected set* if $V - S$ is connected;

iii) *enclaveless set* if, for all $v \in S$, $N(v) \cap (V - S) \neq \varnothing$ [18];

iv) *weak enclaveless set* if, for all $v \in S$, there exists a vertex $w \in V - S$ such that $vw \in E(G)$ and $deg(v) \leq deg(w)$;

v) *strong enclaveless set* if, for all $v \in S$, there exists a vertex $w \in V - S$ such that $vw \in E(G)$ and $deg(v) \geq deg(w)$;

vi) *2-packing* of G if for distinct $u, v \in S$, $d(u,v) > 2$;

vii) *k-restricted set* if for every $u \in V$, $|N[u] \cap S| \leq k$. Notice that a 1-restricted set $S \subseteq V$ is identical to a 2-packing of G.

Let Ψ_r and ψ_r denote the maximum and minimum cardinalities of a maximal restrained enclaveless set, respectively, and let Ψ_{cc} and ψ_{cc} denote the maximum and minimum sizes of a maximal co-connected enclaveless set of G, respectively. Let the maximum cardinality of a weak and strong dominating set be denoted by Ψ_w and Ψ_s, respectively; ψ_w (ψ_s) is the minimum cardinality of a maximal weak (strong) enclaveless set. We denote the maximum cardinality of a k-restricted set by $T_k(G)$; similarly, the minimum cardinality of a maximal k-restricted set is denoted by $\tau_k(G)$.

Meir and Moon [11] defined the *2-packing number* of G to be the maximum cardinality of a 2-packing of G, denoted $P_2(G)$. Similarly, the minimum cardinality of a maximal 2-packing of G is denoted by $p_2(G)$.

Notice that 2-packings, k-restricted, restrained enclaveless, and co-connected enclaveless sets are all hereditary properties.

4 Main Results

If a set S is a minimal P_1-set if and only if $V - S$ is a maximal P_2-set, then we say that properties P_1 and P_2 form a *Gallai pair*. Observe that P_1 is superhereditary if and only if P_2 is hereditary.

Theorem 3 *The following are Gallai pairs for a connected graph:*

(i) total domination and restrained enclaveless;

(ii) connected domination and co-connected enclaveless;

(iii) strong domination and weak enclaveless;

(iv) weak domination and strong enclaveless;

(v) $\{deg-k\}$*-domination and* k*-restricted.*

Proof

We prove only *(i)* as the proofs of *(ii)* - *(v)* are similar.

Suppose S_1 is a minimal total dominating set. Since S_1 is a dominating set, $V-S_1$ must be enclaveless, and, since S_1 is a total dominating set, $V-S_1$ is also restrained.

Conversely, suppose S_2 is a maximal restrained enclaveless set. Since S_2 is enclaveless, $V-S_2$ dominates S_2, and, since S_2 is also restrained, $V-S_2$ doesn't contain any isolates. Therefore, $V-S_2$ is a total dominating set.

Thus, we see that $V-S_1$ is actually a maximal restrained enclaveless set and $V-S_2$ is a minimal total dominating set.

□

It is easy to see that the sizes of the smallest minimal P_1-set and the largest maximal P_2-set sum to n; additionally, the sizes of the largest minimal P_1-set and the smallest maximal P_2-set sum to n. Thus, we have the following theorem. Note that *(vi)* follows from the fact that a 1-restricted set is simply a 2-packing.

Theorem 4 *For any connected graph* G *of order* n*,*

(i) $\gamma_t + \Psi_r = \Gamma_t + \psi_r = n$;

(ii) $\gamma_c + \Psi_{cc} = \Gamma_c + \psi_{cc} = n$;

(iii) $\gamma_s + \Psi_w = \Gamma_s + \psi_w = n$;

(iv) $\gamma_w + \Psi_s = \Gamma_w + \psi_s = n$;

(v) $\gamma_{d-k} + T_k = \Gamma_{d-k} + \tau_k = n$;

(vi) $\gamma_d + P_2 = \Gamma_d + p_2 = n$.

The total domination parameter γ_t is defined as $(A, \min, \overline{1}_n, \overline{1}_n, \{0,1\})$, and its complement, Ψ_r, is defined as $(A, \max, \overline{1}_n, D-\overline{1}_n, \{0,1\})$, where D is the $n \times 1$ degree vector (i.e., the i^{th} component of D is the degree of vertex v_i). The parameter γ_{d-k} is defined by the program $(N, \min, \overline{1}_n, \overline{k}_n, \{0,1\})$, and its complement, T_k, is defined as $(N, \max, \overline{1}_n, D-\overline{k}_n+\overline{1}_n, \{0,1\})$.

Not all of the parameters under consideration are associated with an obvious linear program. In particular, connected domination is difficult to describe by an LP. Also, neither strong nor weak domination is associated with an obvious linear program.

We can generalize Theorem 4 using the concept of *weighted domination.* Weighted domination parameters are studied in [13, 4, 9], for example.

To illustrate, we expand on Theorem 4 *(i).* Assign to every vertex v a positive integer-valued weight $w(v)$. For a set $S \subseteq V(G)$, we define $w(S) = \sum_{v \in S} w(v)$. We can now define the following parameters:

$\gamma_t^w(G) = min\ \{w(S) : S$ is a *total dominating* set of G $\}$.

$\Psi_r^w(G) = max\ \{w(S) : S$ is a *restrained enclaveless* set of G $\}$.

Note that γ_t^w can be formulated as the LP $(A, \min, w, \overline{1}_n, \{0,1\})$. Also, $\Psi_r^w(G)$ is formulated as the complement to the program for $\gamma_t^w(G)$ as $(A, \max, W, D - \overline{1}_n, \{0,1\})$.

Since total domination forms a Gallai pair with restrained enclaveless, we have the following corollary.

Corollary 5 *For any graph G with no isolates, $\gamma_t^w + \Psi_r^w = \Gamma_t^w + \psi_r^w = w(V)$.*

In the special case of weighted domination in which the weight function for vertices in a graph G is defined by $w(v) = 1$, we simply arrive at Theorem 4 *(v)*. If $w(v) = deg(v) - k$, the summation of the weights of all vertices in G is $w(V) = 2m - kn$, where $m = |E(G)|$ and $n = |V(G)|$, as usual.

Corollary 6 *For any graph G with no isolates and the weight function w defined for each vertex $v \in V(G)$ by $w(v) = deg(v) - k$, $k \in [0, 1, 2, \ldots, \delta - 1]$, $\gamma_t^w + \Psi_r^w = 2m - kn$.*

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