Properties of tense logics

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Abstract

Based on the results of [11] this paper delivers uniform algorithms for deciding whether a finitely axiomatizable tense logic has

- the finite model property,
- is complete with respect to Kripke semantics,
- is strongly complete with respect to Kripke semantics,
- is d-persistent,
- is r-persistent.

It is also proved that a tense logic is strongly complete iff the corresponding variety of bimodal algebras is complex and that a tense logic is d-persistent iff it is complete and its Kripke frames form a first order definable class. From this we obtain many natural non-d-persistent tense logics whose corresponding varieties of bimodal algebras are complex.
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1 Introduction

Together with [12] this paper shows that for nearly all interesting properties P there exist algorithms which decide whether a finitely axiomatizable tense logic enjoys P. It is to be emphasized that we also investigate tense logics with respect to the properties introduced above in order to get some insight into those properties in general. This is indeed the case since tense logics turn out to be a class of polymodal logics in which we can already observe most of the interesting phenomena which occur in the more general framework of polymodal logic.

We use the notation from [11]. Recall that a tense logic is a normal bimodal logic which contains the bimodal logic **Lin**.t of the class of weak orderings (i.e. sets with a relation R which is transitive and connected). We assume that tense logics are formulated in the propositional language $\mathcal{L}_\omega$ with ω propositional variables and modal operators $\Box^+$ and $\Box^-$. A tense frame is a structure $\mathcal{G} = \langle g, R^+, R^-, A\rangle$ such that $R^- = (R^+)^{-1}$, $\langle g, R^+\rangle$ is a weak ordering, and $A \subseteq 2^g$ is closed under boolean operations and

$$\Box a = \{x \in g | (\forall y \in g)(xSy \Rightarrow y \in a)\},$$

for $(\Box, S) = (\Box^+, R^+), (\Box^-, R^-)$. (We shall often abbreviate $\langle g, R^+, R^-, A\rangle$ by writing $\langle g, R, A\rangle$, where $R = R^+$). It is well known that each tense logic is determined by tense frames. $\mathcal{G}$ is a Kripke frame iff $A = 2^g$ and we denote it by $\langle g, R\rangle$ or simply by g. The class of Kripke frames is denoted by Fr. (Consult [11] for a detailed discussion of the semantics for tense logics).

We shall now introduce those properties of tense logics which will be investigated in this paper. A tense logic Λ has the finite model property iff it is determined by finite tense frames. Λ is complete (with respect to Kripke semantics) iff Λ is determined by Kripke frames, i.e. iff, for all $\phi \notin \Lambda$, there exists a Kripke frame g with $g \models \Lambda$ such that $g \not\models \phi$. Call a set $\Gamma \subseteq \mathcal{L}_\omega$ of formulas Λ-consistent iff $\perp$ is not derivable from $\Gamma \cup \Lambda$ by using modus ponens. Now Λ is called strongly complete (alias compact) iff each Λ-consistent set Γ is satisfiable in a Kripke-frame for Λ. (As usual, Γ is called satisfiable in a Kripke frame g iff there exists a valuation V and $x \in g$ such that $\langle g, V, x\rangle \models \Gamma$). Consult e.g. [8], [9], [3] for results on as well as discussion of strong completeness. A tense logic Λ is called elementary iff it is complete and the Kripke frames for Λ form a first order definable class (in the language with equality and a relational symbol R). Consult e.g. [2] and [1] for correspondence theory.

The following concepts will be introduced in the general setting of normal bimodal logics. Thus we shall use the notion of a (bimodal) frame in the usual sense and do not restrict attention to rooted frames. (Note that the restriction to rooted frames is implicit in our definition of tense frames). Recall that an algebra $\mathcal{A} = \langle A, \cap, -, \Box^+, \Box^-, 1\rangle$ is a bimodal algebra if $\langle A, \cap, -, 1\rangle$ is a boolean algebra and $\Box 1 = 1$ and $\Box(a \cap b) = \Box a \cap \Box b$, for all $a, b \in \mathcal{A}$ and $\Box = \Box^+, \Box^-$. The class $\mathcal{V}(\Lambda)$ is the variety (equational class) of bimodal algebras $\mathcal{A}$ with $\mathcal{A} \models \phi = 1$, for all $\phi \in \Lambda$ (cf. [4]). For a frame $\mathcal{G} = \langle g, A\rangle$ the set A forms a bimodal algebra $\mathcal{G}^+ = \langle A, \cap, -, \Box^+, \Box^-, g\rangle$. The underlying Kripke frame of $\mathcal{G}$ is $\kappa\mathcal{G} = g$. (In [11] $\kappa\mathcal{G}$ is denoted by $\mathcal{G}_+$). Following Goldblatt [5] we call a variety $\mathcal{V}$ of bimodal algebras complex iff, for each $\mathcal{A} \in \mathcal{V}$, there exists a frame $\mathcal{G}$ with $(\kappa\mathcal{G})^+ \in \mathcal{V}$ such that $\mathcal{G}^+ \simeq \mathcal{A}$. Consult [5] also for a motivation of this definition. Strongly complete logics Λ are closely related to complex varieties. Call a normal modal logic Λ κ-complete, $\kappa \geq \omega$ a cardinal, iff each Λ-consistent set Γ in the language $\mathcal{L}_\kappa$ with κ propositional variables is satisfiable in a Kripke frame g with $g \models \Lambda$. (All the undefined terms here are the canonical extensions of the notation for $\mathcal{L}_\omega$). Correspondingly, call a variety $\mathcal{V}$ κ-complex iff, for each $\mathcal{A} \in \mathcal{V}$ with $|\mathcal{A}| \leq \kappa$, there exists a frame $\mathcal{G}$ with $(\kappa\mathcal{G})^+ \in \mathcal{V}$ such that $\mathcal{G}^+ \simeq \mathcal{A}$. The following is proved in [10]. (Independently, this result was proved by Goldblatt, cf. [6].)

Theorem 1 *For all infinite cardinals κ and normal (poly-) modal logics Λ, Λ is κ-complete iff $\mathcal{V}(\Lambda)$ is κ-complex.*

In general, it is an open problem whether the cardinality conditions can be dispensed with, that is, it is not known whether $\mathcal{V}(\Lambda)$ is complex whenever Λ is strongly complete. However, for tense logics we shall show in the present paper (cf. Theorem 13) that the two notions are equivalent.

A frame $\mathcal{G} = \langle g, R^+, R^-, A\rangle$ is refined iff

$$[\mathrm{ref}_1]\ x = y \Leftrightarrow \forall a \in A(x \in a \Leftrightarrow y \in a),$$

$$[\mathrm{ref}_2]\ xSy \Leftrightarrow \forall a \in A(x \in \Box a \Rightarrow y \in a),$$

for $(\Box, S) = (\Box^+, R^+), (\Box^-, R^-)$. Call a logic Λ r-persistent iff $\kappa\mathcal{G} \models \Lambda$ whenever $\mathcal{G} \models \Lambda$ and $\mathcal{G}$ is refined. [7] is an extensive study of refined frames and r-persistent logics and [2] shows that all r-persistent logics are elementary. (Note that [2] calls r-persistent logics natural). A frame $\mathcal{G}$ is called descriptive if it is refined and $\bigcap U \neq \emptyset$, for all ultrafilters U in $\langle A, \cap, -, g\rangle$. The class of descriptive frames is denoted by Dfr. It is well known that, for each algebra $\mathcal{A}$, there exists a (uniquely) determined descriptive frame $\mathcal{A}_+$ with $\mathcal{A} \simeq (\mathcal{A}_+)^+$ (cf. [4]). A logic Λ is called d-persistent iff $\kappa(\mathcal{A}_+) \models \Lambda$, for each $\mathcal{A} \in \mathcal{V}(\Lambda)$; equivalently, Λ is d-persistent iff $\kappa\mathcal{G} \models \Lambda$ whenever $\mathcal{G} \models \Lambda$ and $\mathcal{G} \in \mathsf{Dfr}$. (We note that, by a result of [7], d-persistency coincides with canonicity. Recall that a normal modal logic Λ is called canonical if $\kappa((\mathcal{F}_\Lambda(\beta)_+) \models \Lambda$, for all cardinals β. Here $\mathcal{F}_\Lambda(\beta)$ denotes the β-generated free algebra in $\mathcal{V}(\Lambda)$). Although, in general, d-persistency does not imply elementarity (cf. [2]), this turns out to be true for tense logics (cf. Theorem 21). Thus we shall show, for all tense logics Λ,

$$\Lambda \text{ is r-persistent}$$

$$\Downarrow$$

$$\Lambda \text{ is d-persistent } (\Leftrightarrow \Lambda \text{ is elementary})$$

$$\Downarrow$$

$$\Lambda \text{ is strongly complete } (\Leftrightarrow \mathcal{V}(\Lambda) \text{ is complex})$$

$$\Downarrow$$

$$\Lambda \text{ is complete}$$

No other implication holds for all tense logics. The failure of an implication from the third line to the second line is the interesting case. In [3] it is shown that the logic $\mathbf{R}.t$ of the reals $\langle \mathbf{R}, <\rangle$ is strongly complete. It will follow immediately from Theorem 13 below that $\mathbf{R}.t$ is not d-persistent. Moreover, also by Theorem 13 below, $\mathbf{R}.t$ and many other natural tense logics are examples of non-d-persistent logics Λ such that $\mathcal{V}(\Lambda)$ is complex. The problem whether such logics exist remained open in [5].

Both our notation as well as the technique of many proofs are based on [11]. (In what follows [11] is called Part I). We assume that the reader has at hand Part I so that we do not introduce all the notation once more.

2 Notation and basic results

We shall briefly introduce only the basic notation from Part I. Recall that an interval in a Kripke frame $\langle g, R\rangle$ is a non-empty subset I of g such that $\{z \in g | xRzRy\} \subseteq I$, for all $x, y \in g$. An interval in a tense frame $\mathcal{G}$ is an interval I in $\kappa\mathcal{G}$ such that $I \in \mathcal{G}^+$. Let I be an interval in a tense frame $\mathcal{G} = \langle g, R, A\rangle$ and define a set of basic interval types as follows.

- I is a joker if I is just an interval, we write $I \in \mathsf{Jok}$.
- $I \in \mathsf{C(0)}$ iff I is a degenerate cluster.
- $I \in [J, \rightarrow]$ iff all $x \in I$ have a proper R-successor in I.
- $I \in [J, \mathsf{C(m)}]$, $m > 0$, iff I has a non degenerate cluster of size $\geq m$ to the right.
- $I \in [J, \rightarrow^n]$, $n \geq 1$, iff $I \in [J, \rightarrow]$ or $I \in [J, \mathsf{C(n)}]$.

The meaning of other terms, for instance $I \in [\mathsf{C(n)}, J]$ or $I \in [\leftarrow^n, \rightarrow]$, are defined in the obvious way. If T is a set of interval types then precisely the expressions

$$\{\exists^p \vec{\mathsf{S}} : \vec{\mathsf{S}} = \langle \mathsf{S}_i | 1 \leq i \leq n\rangle \text{ is a sequence of interval types from } \mathsf{T}\}$$

belong to L_{T}. A sequence of intervals $\vec{I} = \langle I_i | 1 \leq i \leq n\rangle$ in a frame $\mathcal{G}$ satisfies $\vec{\mathsf{S}} = \langle \mathsf{S}_i : 1 \leq i \leq n\rangle$, in symbols

$$\mathcal{G}, \vec{I} \models \vec{\mathsf{S}},$$

iff $g = (I_1 \lhd \ldots \lhd I_n)$ and $I_i \in \mathsf{S}_i$, for all $1 \leq i \leq n$. We say that $\exists^p \vec{\mathsf{S}}$ holds in $\mathcal{G}$, in symbols $\mathcal{G} \models \exists^p \vec{\mathsf{S}}$, iff there exists a partition $\vec{I} = \langle I_i | 1 \leq i \leq n\rangle$ of $\mathcal{G}$ with $\mathcal{G}, \vec{I} \models \vec{\mathsf{S}}$. Put $\neg L_{\mathsf{T}} = \{\neg P | P \in L_{\mathsf{T}}\}$. Then $\mathcal{G} \models \neg\exists^p \vec{\mathsf{S}}$ is an abbrevation for $\mathcal{G} \not\models \exists^p \vec{\mathsf{S}}$. For $\Phi \subseteq \neg L_{\mathsf{T}}$ or $\Phi \subseteq \mathcal{L}_\omega$ and $\mathsf{K} \subseteq \mathsf{Fr} \cup \mathsf{Dfr}$ put

$$\mathsf{Mod}_{\mathsf{K}}(\Phi) = \{\mathcal{G} \in \mathsf{K} | \forall P \in \Phi(\mathcal{G} \models P)\}$$

and call $\mathsf{Mod}_{\mathsf{K}}(\Phi)$ the class of frames for Φ in K. The following result from Part I will be basic for our investigation. Put

$$\mathsf{M} = \{\mathsf{Jok}, \mathsf{C(0)}\} \cup \{[J, \rightarrow^m] | m > 0\} \cup \{[\leftarrow^m, J] | m > 0\} \cup \{[\leftarrow^m, \rightarrow^m] | m > 0\}.$$

Theorem 2 *There is an effective procedure which gives for $\phi \in \mathcal{L}_\omega$ a finite set $N(\phi)$ of modal interval types such that*

$$\mathsf{Mod}_{\mathsf{Dfr}\cup\mathsf{Fr}}(\phi) = \mathsf{Mod}_{\mathsf{Dfr}\cup\mathsf{Fr}}\{\neg\exists^p(\vec{\mathsf{S}}) | \vec{\mathsf{S}} \in N(\phi)\}.$$

Conversely, there is an effective procedure which gives for each $\vec{\mathsf{S}} \in \mathsf{M}$ a tense formula ϕ with

$$\mathsf{Mod}_{\mathsf{Dfr}\cup\mathsf{Fr}}(\phi) = \mathsf{Mod}_{\mathsf{Dfr}\cup\mathsf{Fr}} \neg\exists^p(\vec{\mathsf{S}}).$$

With this result at hand we shall work in the language $\neg L_{\mathsf{M}}$ in order to solve the problems introduced. Nearly all the notation for tense logics is easily translated from $\mathcal{L}_\omega$ to $\neg L_{\mathsf{M}}$. First, we simply say that a subset Λ of $\neg L_{\mathsf{M}}$ is a tense logic iff

$$(\mathsf{Mod}_{\mathsf{Dfr}\cup\mathsf{Fr}}(\Lambda) \subseteq \mathsf{Mod}_{\mathsf{Dfr}\cup\mathsf{Fr}}(P)) \Rightarrow P \in \Lambda,$$

for all $P \in \neg L_{\mathsf{M}}$. The smallest tense logic containing a set $\Phi \subseteq \neg L_{\mathsf{M}}$ is denoted by $\mathbf{Lin}.t \oplus \Phi$. For $\mathcal{G} \in \mathsf{Fr} \cup \mathsf{Dfr}$ define the theory of $\mathcal{G}$ by putting $\mathsf{Th}\mathcal{G} = \{P \in \neg L_{\mathsf{M}} | \mathcal{G} \models P\}$.

Now we are going to introduce the frames we shall work with. The class of frames introduced in Part I will not be sufficient for dealing with d-persistency, r-persistency and strong completeness. So we add another type of frames. First recall that, for an ordinal α, we denote by $\alpha^<$ the frame (or the empty set) $\langle \alpha, <, > \rangle$. $\alpha^>$ denotes the mirror image of $\alpha^<$. We shall sometimes denote by $\omega^<(0)$ the frame $\omega^<$ and denote by $\omega^>(0)$ the frame $\omega^>$. Now define for $n > 0$ and $m \in \omega$

$$\omega^<(n) = \langle \omega, < \cup\{(ni, ni) | i \in \omega\}, > \cup\{(ni, ni) | i \in \omega\}\rangle,$$

$$m^<(n) = \langle \{0, \dots, m-1\}, < \cup\{(ni, ni) | ni < m\}, > \cup\{(ni, ni) | ni < m\}\rangle.$$

The mirror images of $\omega^<(n)$ and $m^<(n)$ are denoted by $\omega^>(n)$ and $m^>(n)$ respectively. Recall that the class of basic Kripke frames is

$$\mathbf{BK} = \{\omega^<(n) : n \in \omega\} \cup \{\omega^>(n) : n \in \omega\} \cup \{C(n) : n \in \omega\}.$$

Definition 3 *Suppose that* $C(m) = \{x_0, \dots, x_{m-1}\}$.
(1) For $n \in \omega$ *and* $m > 0$ *with* $m > 1$ *if* $n > 0$ *define the general tense frame*

$$\mathbf{G}[n, C(m)] = \langle \omega^<(n) \lhd C(m), A \rangle.$$

Here A *is the tense algebra generated by* $\{b_i | 0 \leq i \leq m-1\}$ *in* $(\omega^<(n) \lhd C(m))$ *with*

$$b_i = \{x_i\} \cup \{mj + i | j \in \omega\},\ 0 \leq i \leq m-1.$$

(2) $\mathbf{G}[C(m), n]$ *is the mirror image of* $\mathbf{G}[n, C(m)]$. *The class of frames defined under (1) and (2) is denoted by* $\mathbf{Df}$.
(3) Define for $m > 0$ *and* $n_1, n_2 \in \omega$ *with* $m > 1$ *if* $n_1 + n_2 > 0$,

$$\mathbf{G}[n_1, C(m), n_2] = \langle \omega^<(n_1) \lhd C(m) \lhd \omega^>(n_2), A \rangle,$$

where A *is the tense algebra generated by* $\{b_i : 0 \leq i \leq m-1\}$ *with*

$$b_i = \{x_i\} \cup \{(nj + i, 0) : j \in \omega\} \cup \{(n, j + i, 1) : j \in \omega\},$$

for $i < m$. *(Here we assume that* $\omega^<(m_1)$ *and* $\omega^>(m_2)$ *are based on the sets* $\omega \times \{0\}$ *and* $\omega \times \{1\}$, *respectively). The class of frames defined under (3) is denoted by* $\mathbf{Dl}$.

The class **Df** has already been introduced in Part I. In this paper we shall need the following classes

$$\mathbf{BD}_0 = \mathbf{Df} \cup \{C(m) : m \in \omega\} \text{ and } \mathbf{DS} = \mathbf{BD}_0 \cup \mathbf{Dl}.$$

In Part I we discussed the definition of **Df** and the reason why the case $m = 1$ and $n > 0$ is omitted in the definition of **Df**. In a similar way we here omit the case $n_1 + n_2 \neq 0$ and $m = 1$ for **Dl** since in this case we would get a non-finitely generated algebra $(\mathbf{G}[m_1, C(1), m_2])^+$. There will be no need for those frames.

Proposition 4 *$[\vec{\mathcal{G}}]$ is descriptive and $[\vec{\mathcal{G}}]^+$ is finitely generated, for all $\vec{\mathcal{G}} \in \mathcal{S}\mathbf{DS}$.*

Proof. A straightforward proof shows that the $[\vec{\mathcal{G}}]$ are descriptive. Assume that $\vec{\mathcal{G}} = \langle \mathcal{G}_i : 1 \leq i \leq n \rangle$. Then $[\vec{\mathcal{G}}]$ is generated by

$$\{g_i : 1 \leq i \leq n\} \cup \{a_{ij} : 1 \leq j \leq n_i, 1 \leq i \leq n\},$$

if $\{a_{ij} : 1 \leq j \leq n_i\}$ generates $\mathcal{G}_i^+$, for $1 \leq i \leq n$. The $\mathcal{G}_i^+$ are finitely generated, by definition. $\dashv$

The following Lemma plays a role similar to Lemmas 11 and 13 of Part I. Define for $m \in \omega$ and $n > 0$, the frame $\mathbf{G}^+[C(n), m]$ by putting

$$\mathbf{G}^+[C(n), m] = \mathbf{G}[C(n), m] \lhd C(0)^{m-1} \lhd \mathbf{G}[m, C(n), m],$$

for $m > 1$, and

$$\mathbf{G}^+[C(n), m] = \mathbf{G}[C(n), m] \lhd \mathbf{G}[m, C(n), m],$$

otherwise. $\mathbf{G}^+[m, C(n)]$ is the mirror image of $\mathbf{G}^+[C(n), m]$. Put

$$\mathbf{G}^2[m_1, C(n), m_2] = \mathbf{G}[m_1, C(n), m_2] \lhd C(0)^{m_2-1} \lhd \mathbf{G}[m_2, C(n), m_2],$$

for $m_2 > 1$, and

$$\mathbf{G}^2[m_1, C(n), m_2] = \mathbf{G}[m_1, C(n), m_2] \lhd \mathbf{G}[m_2, C(n), m_2],$$

otherwise. Also put

$$(m_1, m_2)^k = (m_1(k+1))^<(m_1) \lhd C(0)^{m_1-1} \lhd (m_2(k+1))^>(m_2),$$

for $m_1 > 1$, and

$$(m_1, m_2)^k = (m_1(k+1))^<(m_1) \lhd (m_2(k+1))^>(m_2),$$

otherwise. Recall the following definitions. For $\vec{\mathsf{S}} \in \mathsf{M}$ denote by $l(\vec{\mathsf{S}})$ the length of $\vec{\mathsf{S}}$ and put

$$\mathrm{Cl}(\vec{\mathsf{S}}) = \begin{cases} 1 & : \text{ no } \to^m \text{ and no } \leftarrow^m \text{ occurs in } \vec{\mathsf{S}}, \\ \max\{m : \to^m \text{ or } \leftarrow^m \text{ occurs in } \vec{\mathsf{S}}\} & : \text{ otherwise.} \end{cases}$$

Now define, for $\alpha, \beta \leq \omega$ and $\mathcal{G} \in \mathsf{Dfr} \cup \mathsf{Fr}$,

$$\mathsf{M}(\alpha, \beta) = \{\vec{\mathsf{S}} \in \mathsf{M} | l(\vec{\mathsf{S}}) \leq \alpha, \mathrm{Cl}(\vec{\mathsf{S}}) \leq \beta\} \text{ and } \mathsf{Th}_{\alpha,\beta}\mathcal{G} = \{P \in \neg L_{\mathsf{M}(\alpha,\beta)} | \mathcal{G} \models P\}.$$

Lemma 5 *For all $m, m_1, m_2 \in \omega$ and $n > 0$,*

1. $\mathsf{Th}_{k,\omega}\mathbf{G}[m_1, C(n), m_2] = \mathsf{Th}_{k,\omega}(m_1, m_2)^k$, *for $k > 0$.*
2. $\mathsf{Th}\mathbf{G}[m_1, C(n_1), m_2] = \mathsf{Th}\mathbf{G}[m_1, C(n_2), m_2)]$, *for $n_1, n_2 > 0$.*
3. $\mathsf{Th}\mathbf{G}[C(n), m] = \mathsf{Th}\mathbf{G}^+[C(n), m]$.
4. $\mathsf{Th}\mathbf{G}[m_1, C(n), m_2] = \mathsf{Th}\mathbf{G}^2[m_1, C(n), m_2]$.

Proof. Similar to the proof of Lemma 13 in Part I. $\dashv$

Lemma 5 is of basic importance in what follows. It says how we can manipulate frames without effect on the validity of formulas. Certainly this is useful when we are interested in the representation of algebras by frames. Another easy but important consequence of this Lemma is

Theorem 6 $\mathsf{Th}[\vec{\mathcal{G}}]$ *is decidable, for all $\vec{\mathcal{G}} \in \mathcal{S}\mathbf{DS}$, and* $\mathsf{Th}[\vec{g}]$ *is decidable, for all $\vec{g} \in \mathcal{S}\mathbf{BK}$.*

Proof. (1) follows from Theorem 14 of Part I and Lemma 5 (1) above (by application of Proposition 5 of Part I). (2) is Theorem 18 (4) of Part I. $\dashv$

Often it will be helpful to have the following notation. A *gap* γ in a frame $\langle g, R\rangle$ is a pair $\langle I_1, I_2\rangle$ of disjoint intervals with $g = I_1 \cup I_2$. We shall identify objects $\langle I_1, I_2\rangle$ with the gap

$$\gamma = \langle \{x \in g : (\exists y \in I_1) xRy\} \cup I_1, I_2 \cup \{x \in g : (\exists y \in I_2) yRx\}\rangle,$$

if I_1 and I_2 are disjoint intervals and $I_1 \cup I_2$ is an interval. For two gaps $\gamma_0 = \langle I_1, I_2\rangle$ and $\gamma_1 = \langle I_3, I_4\rangle$ in $\langle g, R\rangle$ we write $\gamma_0 R \gamma_1$ if $I_1 \subset I_3$. In this case we define the interval $[\gamma_0, \gamma_1]$ by putting

$$[\gamma_0, \gamma_1] = I_2 \cap I_3.$$

Hence, each finite set of gaps $G = \{\gamma_0 \; R \ldots R \; \gamma_n\}$ gives rise to a partition

$$\mathrm{par}[G] = \{[\infty, \gamma_0], [\gamma_0, \gamma_1], \ldots, [\gamma_{n-1}, \gamma_n], [\gamma_n, \infty]\}.$$

Conversely, given a partition $\vec{I} = \langle I_i : 1 \leq i \leq n\rangle$ of $\langle g, R\rangle$ we put

$$\mathrm{gap}(\vec{I}) = \{\langle I_i, I_{i+1}\rangle : i < n\}.$$

For a finite set of partitions $\{\vec{I^i} : i \leq n\}$ put

$$\bigvee \langle \vec{I^i} : i \leq n\rangle = \mathrm{par}[\{\bigcup \{\mathrm{gap}(\vec{I^i}) : i \leq n\}].$$

We shall need the following paramaters for talking about sequences $\vec{\mathcal{G}} \in \mathcal{S}\mathbf{DS}$. The definitions are canonical extensions of the definitions in Part I for **BD**.

- $\mathrm{Cl}(\vec{\mathcal{G}}) \in \omega$ is the cardinality of the maximal cluster in $[\vec{\mathcal{G}}]$.

- $\mathrm{R}(\vec{\mathcal{G}}) \in \omega$ denotes the number of $\mathcal{G}_i$ in $\vec{\mathcal{G}}$ which are not irreflexive points.
- $\mathrm{IR}(\vec{\mathcal{G}}) \in \omega$ is the length of the maximal infix subsequence of the form $(C(0))^n$ in $\vec{\mathcal{G}}$.
- $\mathrm{S}(\vec{\mathcal{G}}) \in \omega$ is the maximal n such that there is a $\mathcal{G}_i$ in $\vec{\mathcal{G}}$ with $\mathcal{G}_i = \mathbf{G}[n, C]$ or $\mathcal{G}_i = \mathbf{G}[C, n]$ or $\mathcal{G}_i = \mathbf{G}[n, C, k]$ or $\mathcal{G}_i = \mathbf{G}[k, C, n]$.

Clearly there exist only finitely many sequences $\vec{\mathcal{G}} \in \mathcal{S}\mathbf{DS}$ where all those parameters are bounded. The following Proposition will be basic for most of the decision procedures.

Proposition 7 *Suppose that $\mathcal{H}' = \mathcal{H}_1 \lhd \mathcal{G} \lhd \mathcal{H}_2 \in \mathsf{Dfr}$ and that $\Phi \subseteq \neg L_{\mathsf{M}(m,n)}$ with $\mathcal{H}' \models \Phi$. Let $\mathbf{H}$ be a set of frames with*

$$\mathcal{H}_1 \lhd \mathcal{H} \lhd \mathcal{H}_2 \not\models \Phi,$$

for all $\mathcal{H} \in \mathbf{H}$. Then there exist $\vec{\mathcal{F}}_1, \vec{\mathcal{F}}_2 \in \mathcal{S}\mathbf{BD}_0$ with $R(\vec{\mathcal{F}}_1) + R(\vec{\mathcal{F}}_2) \leq 4m|\mathbf{H}|$, $Cl(\vec{\mathcal{F}}_j) \leq n$, $S(\vec{\mathcal{F}}_j) \leq m+2$, and $IR(\vec{\mathcal{F}}_j) \leq m+1$, for $j \in \{1,2\}$, such that $[\vec{\mathcal{F}}_1] \lhd \mathcal{G} \lhd [\vec{\mathcal{F}}_2] \models \Phi$ and

$$[\vec{\mathcal{F}}_1] \lhd \mathcal{H} \lhd [\vec{\mathcal{F}}_2] \not\models \Phi,$$

for all $\mathcal{H} \in \mathbf{H}$.

Proof. For each $\mathcal{H} \in \mathbf{H}$ take a partition $\vec{I}_{\mathcal{H}} = \langle I_i^{\mathcal{H}} : 1 \leq i \leq n_{\mathcal{H}} \rangle$ of $\mathcal{H}_1 \lhd \mathcal{H} \lhd \mathcal{H}_2$ such that

$$\mathcal{H}_1 \lhd \mathcal{H} \lhd \mathcal{H}_2, \vec{I}_{\mathcal{H}} \models \vec{\mathsf{S}},$$

for a $\vec{\mathsf{S}}$ with $\neg\exists^p \vec{\mathsf{S}} \in \Phi$. We consider the restrictions of the $\vec{I}_{\mathcal{H}}$ to $\mathcal{H}_j$, defined by

$$\mathcal{H}_j \cap \vec{I}_{\mathcal{H}} := \langle h_j \cap I_i^{\mathcal{H}} : 1 \leq i \leq n_{\mathcal{H}} \rangle,$$

for $j = 1, 2$. Now form, for $j = 1, 2$,

$$\vec{I}^j = \bigvee \langle \mathcal{H}_j \cap \vec{I}_{\mathcal{H}} : \mathcal{H} \in \mathbf{H} \rangle.$$

Then $l(\vec{I}^1) + l(\vec{I}^2) \leq m|\mathbf{H}|$. Suppose that $\vec{I}^j = \langle I_i^j : 1 \leq i \leq n_j \rangle$, for $j = 1, 2$, and fix an I_i^j. It follows immediately from the proof of Theorem 14 (1) in Part I that there exists $\vec{\mathcal{F}}_i^j \in \mathcal{S}\mathbf{BD}_0$ with $\mathrm{R}(\vec{\mathcal{F}}_i^j) \leq 4$, $\mathrm{S}(\vec{\mathcal{F}}_i^j) \leq m+2$, $\mathrm{IR}(\vec{\mathcal{F}}_i^j) \leq m+1$, and $\mathrm{Cl}(\vec{\mathcal{F}}_i^j) \leq n$, such that

$$\mathsf{Th}_{m,n}\mathcal{H}_{I_i^j} \subseteq \mathsf{Th}_{m,n}[\vec{\mathcal{F}}_i^j],$$

$$(\forall \mathsf{S} \in \mathsf{M}(m,n))(\mathcal{H}_{I_i^j} \in \mathsf{S} \Rightarrow [\vec{\mathcal{F}}_i^j] \in \mathsf{S}).$$

Thus, by Proposition 5 of Part I, the sequences $\vec{\mathcal{F}}_j = \langle \vec{\mathcal{F}}_j^i : 1 \leq i \leq n_j \rangle$, $j = 1, 2$, prove the Theorem up to the condition on $\mathrm{IR}(\vec{\mathcal{F}}^j)$. We only know $\mathrm{IR}(\vec{\mathcal{F}}_j) \leq 2(m+1)$. But, by Lemma 13 (2) of Part I, we can reduce sequences $C(0)^l$, $l > m+1$, in $\vec{\mathcal{F}}_j$ to sequences $C(0)^{m+1}$. ⊣

3 Complex varieties and strong completeness

Since $\neg L_{\mathsf{M}}$ is not talking about bimodal algebras and not even about arbitrary frames for tense logics but only about rooted frames we need the following easily proved translation of the concept of complex varieties. Call a tense logic Λ (κ-) complex iff $\mathcal{V}(\Lambda)$ is (κ-) complex.

Proposition 8 *Let κ be an infinite cardinal. For each tense logic Λ, Λ is (κ-) complex iff, for each descriptive tense frame $\mathcal{G}$ (with $|\mathcal{G}^+| \leq \kappa$) with $\mathcal{G} \models \Lambda$ there exists a tense frame $\mathcal{H}$ such that $\mathcal{G}^+ \simeq \mathcal{H}^+$ and $\kappa\mathcal{H} \models \Lambda$.*

We are going to show that a tense logic Λ is complex iff all Λ-frames of the form $[\vec{\mathcal{G}}]$, $\vec{\mathcal{G}} \in \mathcal{S}\mathbf{DS}$, can be represented by Kripke frames for Λ. First we need some more information on gaps in descriptive tense frames. A gap $\gamma = \langle I_1, I_2 \rangle$ in $\langle g, R \rangle$ is

- *simple* iff $I_1 \in [J, \mathsf{C}(\mathsf{n})]$ and $I_2 \in [\mathsf{C}(\mathsf{m}), J]$, for some $m, n \in \omega$. The set of simple gaps in $\langle g, R \rangle$ is denoted by $\mathrm{SG}(\langle g, R \rangle)$.
- *left-proper* iff $I_1 \in [J, \rightarrow]$.
- *right-proper* iff $I_2 \in [\leftarrow, J]$.
- *proper* iff it is both left and right-proper.

The set of non-simple gaps in $\langle g, R \rangle$ is denoted by $\mathrm{PG}(\langle g, R \rangle)$. Applied to frames in $\mathbf{DS}$ we see that $\kappa\mathbf{G}[C(m), n]$ contains precisely one non-simple gap which is right-proper. The frame $\kappa\mathbf{G}[m_1, C(n), m_2]$ contains two non-simple gaps, one is right-proper and the other is left proper.

Lemma 9 *Suppose that $\mathcal{G}$ is descriptive. Then $\kappa\mathcal{G}$ contains no proper gap.*

Proof. Assume that $\gamma = \langle I_0, I_1 \rangle$ with $I_0 \in [J, \rightarrow]$, $I_1 \in [\leftarrow, J]$. Certainly there exists a set of intervals H in $\mathcal{G}$ with $I_i \cap I \neq \emptyset$, for all $I \in H$, such that $I_i \cap \bigcap H = \emptyset$, for $i = 0, 1$. But then $\bigcap H = \emptyset$ and $\bigcap H' \neq \emptyset$, for all finite subsets H' of H. Hence $\mathcal{G}$ is not descriptive. $\dashv$

Thus, for each non-simple gap $\gamma = \langle I_1, I_2 \rangle$ in $\kappa\mathcal{G}$, $\mathcal{G}$ descriptive, there is a cluster C such that $I_1 = \{x \in g : xRC\}$ or $I_2 = \{x \in g : CRx\}$. We call C the cluster corresponding to γ. Recall the definition of the canonical type assignment $\mathbf{c}$ for $\mathcal{G}$ from Part I. In terms of this notion we have, by Theorem 3 of Part I, that $\mathbf{c}^+C = \mathbf{j}$ if C corresponds to a right-proper gap and $\mathbf{c}^-C = \mathbf{j}$ iff C corresponds to a left-proper gap. If $\gamma = \langle I_1, I_2 \rangle$ is a simple gap in $\kappa\mathcal{G}$ then $I_1 \in \mathcal{G}^+$. Thus, both I_1 and I_2 are intervals in $\mathcal{G}$.

For an algebra $\mathcal{A}$, we are going to describe the frames $\mathcal{H}$ with $\mathcal{H}^+ \simeq \mathcal{A}$. The replacement of non-simple gaps is the basic operation. For a set of gaps G in $\langle g, R \rangle$ and a set of (mutually disjoint) objects $\{h_\gamma : \gamma \in G\}$ which are Kripke frames or empty sets we denote by

$$\langle g, R\rangle[h_\gamma/\gamma : \gamma \in G]$$

the frame resulting when all gaps $\gamma, \gamma \in G$, are replaced by h_γ in $\langle g, R\rangle$.

Suppose that $\mathcal{G} = \langle g, R, A\rangle$ is a descriptive frame and $G \subseteq \mathrm{PG}(\langle g, R\rangle)$. Let $\{h_\gamma : \gamma \in G\}$ be a set of Kripke frames. For each γ, $\gamma \in G$, take an x_γ from the cluster C_γ corresponding to γ. Define a set B by putting $b \in B$ iff

$$(\exists a \in A)(b = a \cup \bigcup\{h_\gamma : x_\gamma \in a, \gamma \in G\}).$$

Lemma 10 $\mathcal{G}^+ \simeq \langle\langle g, R\rangle[h_\gamma/\gamma : \gamma \in G], B\rangle^+$.

Proof. We show that $a \mapsto a \cup \bigcup\{h_\gamma : x_\gamma \in a, \gamma \in G\}$, $a \in A$, defines an isomorphism from $\mathcal{G}^+$ onto $\langle\langle g, R\rangle[h_\gamma/\gamma : \gamma \in G], B\rangle^+$. This is clear for the boolean reducts. The proof for $\Diamond^+$ and $\Diamond^-$ is symmetric. So it suffices to show that, for all $a \in A$,

$$\Diamond_1^+(a \cup \bigcup\{h_\gamma : x_\gamma \in a, \gamma \in G\}) = \Diamond_2^+ a \cup \bigcup\{h_\gamma : x_\gamma \in \Diamond_2^+ a, \gamma \in G\},$$

where $\Diamond_1^+$ is computed in $\langle g, R\rangle[h_\gamma/\gamma : \gamma \in G]$ and $\Diamond_2^+$ is computed in $\langle g, R\rangle$. But this equation is an immediate consequence of the description of non-simple gaps in descriptive frames. $\dashv$

Mostly it will be sufficient to replace gaps by frames from the following set

$$\mathbf{Fill}_m = \{n^< \lhd C(1) \lhd n^< : n \leq m+1\} \cup \{n^< : n \leq m+1\}.$$

The following lemma explains why this is so.

Lemma 11 *For each finite frame h there exists $m \in \omega$ and $g \in \mathbf{Fill}_m$ with $\mathsf{Th}h \subseteq \mathsf{Th}g$. For each finite frame h and $m \in \omega$ there exists $g \in \mathbf{Fill}_m$ with $\mathsf{Th}_{m,\omega}h \subseteq \mathsf{Th}_{m,\omega}g$.*

Proof. Follows from Lemmas 10 and 13 (2) of Part I. $\dashv$

Lemma 12 *Let $\Lambda = \mathbf{Lin}.t \oplus \Phi$ with $\Phi \subseteq \neg L_{\mathsf{M}(m,n)}$. Suppose that Λ is not complex. Then there exists a $\vec{\mathcal{G}} \in \mathcal{S}\mathbf{DS}$ with $[\vec{\mathcal{G}}] \models \Lambda$ but*

$$(\kappa[\vec{\mathcal{G}}])[h_\gamma/\gamma : \gamma \in PG(\kappa[\vec{\mathcal{G}}])] \not\models \Lambda,$$

for all $\{h_\gamma : \gamma \in PG(\kappa[\vec{\mathcal{G}}])\} \subseteq \mathbf{Fill}_m$.

Proof. By Proposition 8 there exists a descriptive tense frame $\mathcal{H} = \langle h, S, B\rangle$ with $\mathcal{H} \models \Lambda$ such that $\kappa\mathcal{F} \not\models \Lambda$, for each $\mathcal{F}$ with $\mathcal{F}^+ \simeq \mathcal{H}^+$. Hence, by Lemma 10,

$$(\kappa\mathcal{H})[h_\gamma/\gamma : \gamma \in \mathrm{PG}(\kappa\mathcal{H})] \not\models \Phi,$$

for all $\{h_\gamma : \gamma \in \mathrm{PG}(\kappa\mathcal{H})\} \subseteq \mathbf{Fill}_m$.

Some notation. For a partition $\vec{I} = \langle I_i : 1 \le i \le r\rangle$ of $(\kappa\mathcal{H})[h_\gamma/\gamma : \gamma \in \mathrm{PG}(\kappa\mathcal{H})]$ denote by $\vec{I}_\mathcal{H}$ the partition $\vec{I}_\mathcal{H} = \langle I_i \cap h : 1 \le i \le r\rangle$ of $\kappa\mathcal{H}$. Denote by $\mathbf{c}$ the canonical type assignment for $\mathcal{H}$.

Certainly there exists a finite set $D_1 \subseteq \mathrm{PG}(\kappa\mathcal{H})$ and a finite set $D_2 \subseteq \mathrm{SG}(\kappa\mathcal{H})$ such that, for all $s = \{h_\gamma : \gamma \in D_1\} \subseteq \mathbf{Fill}_m$, there exists $\vec{\mathsf{S}}$ with $\neg\exists^p\vec{\mathsf{S}} \in \Phi$ and a partition $\vec{I}(s)$ of $\kappa\mathcal{H}[h_\gamma/\gamma : \gamma \in D_1]$ with

$$\mathrm{gap}(\vec{I}(s)_\mathcal{H}) \subseteq D_1 \cup D_2$$

and

$$(\kappa\mathcal{H})[h_\gamma/\gamma : \gamma \in D_1], \vec{I}(s) \models \vec{\mathsf{S}}.$$

We may assume that for any two gaps $\gamma_1, \gamma_2 \in D_1$ with $\gamma_1 S \gamma_2$ such that there exists a simple gap γ in $\kappa\mathcal{H}$ with $\gamma_0 S\gamma S\gamma_2$ there already exists a $\gamma \in D_2$ with $\gamma_0 S\gamma S\gamma_2$. (The only possibility that there is no simple gap γ with $\gamma_0 S\gamma S\gamma_1$ is that γ_0 is left-proper and γ_1 is right-proper such that the cluster C corresponding to γ_0 coincides with the cluster C corresponding to γ_1. In this case $\mathbf{c}C = (\mathbf{j}, \mathbf{j})$, for this cluster C. This situation causes no special problems and is covered by the construction in case (4.) and (5.) below).

Denote by $\vec{I} = \langle I_i : 1 \le i \le k\rangle$ the partition $\mathrm{par}(D_2)$ defined by D_2 in $\kappa[\vec{\mathcal{H}}]$. All gaps in D_2 are simple. Hence $\vec{I}$ is also a partition of $\mathcal{H}$, i.e. $I_i \in \mathcal{H}^+$, for $1 \le i \le k$. Consider an interval I_i from $\vec{I}$. Depending on I_i we shall construct a sequence $\vec{\mathcal{G}^i} \in \mathcal{S}\mathbf{DS}$ with

1. $\mathsf{Th}[\vec{\mathcal{G}^i}] \supseteq \mathsf{Th}\mathcal{H}_{I_i}$,
2. $\mathcal{H}_{I_i} \in \mathsf{S} \Rightarrow [\vec{\mathcal{G}^i}] \in \mathsf{S}$, for all $\mathsf{S} \in \mathsf{M}(m, n)$,

but so that in $\kappa[\vec{\mathcal{G}^i}]$ we still find the required non-simple gaps.

Case 1. No gap γ from D_1 is in I_i. Similar to the proof of Proposition 7 in Part I we find a sequence $\vec{\mathcal{G}^i} \in \mathcal{S}\mathbf{BD}_0$ satisfying (1.) and (2.).

Case 2. A right-proper gap γ from D_1 is in I_i and $\mathbf{c}^-C = \mathbf{m}$, for the cluster C corresponding to γ. Similar to the proof of Proposition 7 in Part I we find a

$$\vec{\mathcal{H}^i} \in \mathcal{S}(\mathbf{BD}_0 \cup \{\mathbf{G}[C(1), k] : k > 0\})$$

such that a $\mathbf{G}[C(m_1), m_2]$ occurs in $\vec{\mathcal{H}^i}$, for some $m_2, m_1 \in \omega$ with $m_1 = |C|$ if C is finite and $m_1 = n$, otherwise, and such that $\vec{\mathcal{H}^i}$ satisfies (1.) and (2.). Replace all $\mathbf{G}[C(1), l]$, $l > 0$, in $\vec{\mathcal{H}^i}$ by $\mathbf{G}[l, C(2), l] \in \mathbf{Dl}$ and denote the result by $\vec{\mathcal{G}^i}$. Clearly $\vec{\mathcal{G}^i}$ still satisfies (2.). By Lemma 5 (1.), $\vec{\mathcal{G}^i}$ satisfies (1.).

Case 3. A left-proper gap γ from D_1 is in I_i and $\mathbf{c}^+C = \mathbf{m}$, for the cluster C corresponding to γ. Construct $\vec{\mathcal{G}^i}$ as the mirror image of $\vec{\mathcal{G}^i}$ in case 2.

Case 4. A right-proper gap γ from D_1 is in I_i and $\mathbf{c}^- C = \mathbf{j}$, for the cluster C corresponding to γ. Similar to the proof of Proposition 7 in Part I we find a $\vec{\mathcal{G}^i} \in \mathcal{S}\mathbf{DS}$ such that a $\mathbf{G}[m_1, C(n), m_2] \in \mathbf{Dl}$ occurs in $\vec{\mathcal{G}^i}$ and $\vec{\mathcal{G}^i}$ satisfies (1.) and (2.). (Here Lemma 5 (1.) is useful).

Case 5. A right-proper gap γ from D is in I_i and $\mathbf{c}^+ C = \mathbf{j}$, for the cluster C corresponding to γ_j. Construct $\vec{\mathcal{G}^i}$ as the mirror image of $\vec{\mathcal{G}^i}$ in case 4.

Put $\vec{\mathcal{G}} = \langle \vec{\mathcal{G}^i} : 1 \le i \le k \rangle$. Then $\mathsf{Th}[\vec{\mathcal{G}}] \supseteq \mathsf{Th}\mathcal{H}$, by Proposition 5 of Part I and (1.) above. Hence $[\vec{\mathcal{G}}] \models \Lambda$. Moreover, by (2.) and the construction of the $\vec{\mathcal{G}^i}$,

$$(\kappa[\vec{\mathcal{G}}])[h_\gamma/\gamma : \gamma \in \mathrm{PG}(\kappa[\vec{\mathcal{G}}])] \not\models \Lambda,$$

for all $\{h_\gamma : \gamma \in \mathrm{PG}(\kappa[\vec{\mathcal{G}}])]\} \subseteq \mathbf{Fill}_m$. $\dashv$

For $\mathcal{G} \in \mathbf{DS}$ and $k \in \omega$ define a set of Kripke frames $\tau_k\mathcal{G}$ by putting

$$\tau_k\mathcal{G} = \begin{cases} \{\mathcal{G}\} & : \ \mathcal{G} \text{ finite} \\ \{\omega^<(n)\} \cup \{\omega^<(n) \lhd h \lhd C(m) : h \in \mathbf{Fill}_k\} & : \ \mathcal{G} = \mathbf{G}[n, C(m)], \\ \{\omega^>(n)\} \cup \{C(m) \lhd h \lhd \omega^>(n) : h \in \mathbf{Fill}_k\} & : \ \mathcal{G} = \mathbf{G}[C(m), n], \\ \{\omega^<(m_1) \lhd h \lhd \omega^>(m_2) : h \in \mathbf{Fill}_k\} & : \ \mathcal{G} = \mathbf{G}[m_1, C(m), m_2], \end{cases}$$

and put, for a sequence $\vec{\mathcal{G}} = \langle \mathcal{G}_i : 1 \le i \le r \rangle \in \mathcal{S}\mathbf{DS}$,

$$\tau_k\vec{\mathcal{G}} = \{\langle f_i : 1 \le i \le r \rangle : f_i \in \tau_k\mathcal{G}_i, 1 \le i \le r\}.$$

Note that the set $\tau_k\mathcal{G}$ nearly coincides with the set of frames $\kappa\mathcal{G}[h_\gamma/\gamma : \gamma \in \mathrm{PG}(\kappa\mathcal{G})]$, $\{h_\gamma : \gamma \in \mathrm{PG}(\kappa\mathcal{G})\} \subseteq \mathbf{Fill}_k$, for $\mathcal{G} \in \mathbf{DS}$. Recall that an algebra $\mathcal{A}$ is simple if it has no non-trivial congruences. We are in a position now to characterize complex tense logics.

Theorem 13 *Suppose that $\Lambda = \mathbf{Lin}.t \oplus \Phi$ with $\Phi \subseteq \neg L_{\mathsf{M}(m,n)}$. The following conditions are equivalent.*

1. *Λ is complex.*
2. *Λ is strongly complete.*
3. *There exists a tense frame $\mathcal{G}$ with $\kappa\mathcal{G} \models \Lambda$ and $\mathcal{G}^+ \simeq \mathcal{A}$, for all finitely generated simple algebras $\mathcal{A} \in \mathcal{V}(\Lambda)$.*
4. *For all $\vec{\mathcal{G}} \in \mathcal{S}\mathbf{DS}$ with $[\vec{\mathcal{G}}] \models \Lambda$ there exists $\vec{f} \in \tau_m\vec{\mathcal{G}}$ with $[\vec{f}] \models \Lambda$.*
5. *For all $\vec{\mathcal{G}} \in \mathcal{S}\mathbf{DS}$ with $[\vec{\mathcal{G}}] \models \Lambda$ there exists a set $\{h_\gamma : \gamma \in PG(\kappa[\vec{\mathcal{G}}])\} \subseteq \mathbf{Fill}_m$ with*

$$(\kappa[\vec{\mathcal{G}}])[h_\gamma/\gamma : \gamma \in PG(\kappa[\vec{\mathcal{G}}])] \models \Lambda.$$

Proof. (5) $\Rightarrow$ (1) is Lemma 12.

(4) $\Rightarrow$ (5). Suppose that (4) holds and suppose that $\vec{\mathcal{G}} = \langle \mathcal{G}_i : 1 \leq i \leq r\rangle \in \mathcal{S}\mathbf{DS}$ with $[\vec{\mathcal{G}}] \models \Lambda$. Put, for $1 \leq i \leq r$,

$$\mathcal{F}_i = \begin{cases} \mathcal{G}_i & : \ \mathcal{G}_i \text{ finite}, \\ \mathbf{G}^+[C(k), l] & : \ \mathcal{G}_i = \mathbf{G}[C(k), l], \\ \mathbf{G}^+[l, C(k)] & : \ \mathcal{G}_i = \mathbf{G}[l, C(k)], \\ \mathbf{G}^2[m_1, C(k), m_2] & : \ \mathcal{G}_i = \mathbf{G}[m_1, C(k), m_2]. \end{cases}$$

By Lemma 5, $\vec{\mathcal{F}} \models \Lambda$. It is easy (but a bit tedious) to check that for each $\vec{f} \in \tau_m \vec{\mathcal{F}}$ there exists a $k \in \omega$ and $\{h_\gamma : \gamma \in \mathrm{PG}(\kappa[\vec{\mathcal{G}}])\} \subseteq \mathbf{Fill}_k$ with

$$\mathsf{Th}[\vec{f}] \subseteq \mathsf{Th}(\kappa([\vec{\mathcal{G}}])[h_\gamma/\gamma : \gamma \in \mathrm{PG}(\kappa[\vec{\mathcal{G}}])]).$$

But then, by Lemma 11, for each $\vec{f} \in \tau_m \vec{\mathcal{F}}$ there exists $\{h_\gamma : \gamma \in \mathrm{PG}(\kappa[\vec{\mathcal{G}}])\} \subseteq \mathbf{Fill}_m$ with

$$\mathsf{Th}_{m,\omega}[\vec{f}] \subseteq \mathsf{Th}_{m,\omega}(\kappa([\vec{\mathcal{G}}])[h_\gamma/\gamma : \gamma \in \mathrm{PG}(\kappa[\vec{\mathcal{G}}])]).$$

By (4), there is an $\vec{f} \in \tau_m \vec{\mathcal{F}}$ with $[\vec{f}] \models \Lambda$. Hence there is a set $\{h_\gamma : \gamma \in \mathrm{PG}(\kappa[\vec{\mathcal{G}}])\} \subseteq \mathbf{Fill}_m$ with $\kappa([\vec{\mathcal{G}}])[h_\gamma/\gamma : \gamma \in \mathrm{PG}(\kappa[\vec{\mathcal{G}}])] \models \Lambda$.

(3) $\Rightarrow$ (4). Assume that (3) holds and suppose that $\vec{\mathcal{G}} \in \mathcal{S}\mathbf{DS}$ with $[\vec{\mathcal{G}}] \models \Lambda$. It is easy to check that $[\vec{\mathcal{G}}]^+$ is simple. By Proposition 4, $[\vec{\mathcal{G}}]^+$ is finitely generated. Hence (3) applies to $\mathcal{A} = [\vec{\mathcal{G}}]^+$. Thus there exists a $\mathcal{H} = \langle h, S, B\rangle$ with $\kappa\mathcal{H} \models \Lambda$ and $\mathcal{H}^+ \simeq [\vec{\mathcal{G}}]^+$. Certainly we may assume that all the atoms in $\mathcal{H}^+$ are of the form $\{x\}$ for an $x \in h$. Thus, we may assume there is a sequence $\vec{h} = \langle h_i : 1 \leq i \leq r\rangle$ with $\langle h, S\rangle = [\vec{h}]$ and such that

- $h_i = \mathcal{G}_i$, whenever $\mathcal{G}_i$ is finite,
- $h_i = \omega^<(m_1) \lhd f_i \lhd \omega^>(m_2)$, for an arbitrary (possibly empty) f_i, whenever $\mathcal{G}_i = \mathbf{G}[m_1, C(k), m_2]$.
- $h_i = \omega^<(l) \lhd f_i$, for some f_i with $f_i = \emptyset$ or $f_i \in [J, \to^k]$, whenever $\mathcal{G}_i = \mathbf{G}[l, C(k)]$.
- $h_i = f_i \lhd \omega^>(l)$, for some f_i with $f_i = \emptyset$ or $f_i \in [\leftarrow^k, J]$, whenever $\mathcal{G}_i = \mathbf{G}[C(k), l]$.

Let $1 \leq i \leq r$. By Theorem 18 (1) of Part I, we have that $f_i = \emptyset$ or, for all $\mathsf{S} \in \mathsf{M}$ with $f_i \in \mathsf{S}$, there exists $\vec{g} \in \mathcal{S}\mathbf{BK}$ with $\mathsf{Th} f_i \subseteq \mathsf{Th}[\vec{g}]$ and $[\vec{g}] \in \mathsf{S}$. Using this fact it is straightforward to show that there exists $l \in \omega$ and $h_i^1 \in \tau_l \mathcal{G}_i$ such that $\mathsf{Th} h_i \subseteq \mathsf{Th} h_i^1$. Hence, by Lemma 11 and Proposition 5 of Part I, there exists a $h_i^2 \in \tau_m \mathcal{G}_i$ with

$$\mathsf{Th}_{m,\omega} h_i \subseteq \mathsf{Th}_{m,\omega} h_i^1 = \mathsf{Th}_{m,\omega} h_i^2.$$

But then, $[\vec{h}^2] \models \Lambda$ and $\vec{h}^2 \in \tau_m \vec{\mathcal{G}}$, for $\vec{h}^2 := \langle h_i^2 : 1 \leq i \leq r\rangle$.

(1) $\Rightarrow$ (3) is clear. Hence (1), (3), (4), and (5) are equivalent. The equivalence (1) $\Leftrightarrow$ (2) follows from (1) $\Leftrightarrow$ (3) with Theorem 1. $\dashv$

Theorem 14 *$\{\Phi \subseteq \neg L_{\mathsf{M}}$ finite $: \mathbf{Lin}.t \oplus \Phi$ is strongly complete $\}$ is recursive. Moreover, $\Lambda = \mathbf{Lin}.t \oplus \Phi$, $\Phi \subseteq \neg L_{\mathsf{M}(m,n)}$, is strongly complete iff, for all $\vec{\mathcal{G}} \in \mathcal{S}\mathbf{DS}$ with $[\vec{\mathcal{G}}] \models \Lambda$ and $Cl(\vec{\mathcal{G}}) \leq n$, $S(\vec{\mathcal{G}}) \leq m+2$, $IR(\vec{\mathcal{G}}) \leq m+1$, and $R(\vec{\mathcal{G}}) \leq 1 + 4m(1+(m+1)(m+2))$, there exists a $\vec{f} \in \tau_m \vec{\mathcal{G}}$ with $[\vec{f}] \models \Lambda$.*

Proof. By Theorem 6 it suffices to show the moreover part. The direction from left to right follows from Theorem 13. Conversely suppose that Λ is not strongly complete. By Theorem 13, there is a $\vec{\mathcal{H}} = \langle \mathcal{H}_i : 1 \leq j \leq r \rangle \in \mathcal{S}\mathbf{DS}$ with $[\vec{\mathcal{H}}] \models \Lambda$ such that $[\vec{f}] \not\models \Lambda$, for all $\vec{f} \in \tau_m \vec{\mathcal{H}}$. For $\mathcal{G} \in \mathbf{DS}$ define a set of frames $t^n_m \mathcal{G}$ by putting

$$t^n_m \mathcal{G} = \begin{cases} \{\mathcal{G}\} & : \ \mathcal{G} \text{ finite}, \\ \{\mathbf{G}[l, C(n)]\} \cup \{\mathbf{G}[l, C(n)] \lhd h \lhd C(k) : h \in \mathbf{Fill}_m\} & : \ \mathcal{G} = \mathbf{G}[l, C(k)], \\ \{\mathbf{G}[C(n), l]\} \cup \{C(k) \lhd h \lhd \mathbf{G}[C(n), l] : h \in \mathbf{Fill}_m\} & : \ \mathcal{G} = \mathbf{G}[C(k), l], \\ \{\mathbf{G}[m_1, C(n)] \lhd h \lhd \mathbf{G}[C(n), m_2] : h \in \mathbf{Fill}_m\} & : \ \mathcal{G} = \mathbf{G}[m_1, C(k), m_2] \end{cases}$$

By Lemma 17 of Part I there is a bijection $\pi : \tau_m \mathcal{G} \to t^n_m \mathcal{G}$ such that

1. $\mathsf{Th}_{\omega,n} h = \mathsf{Th}_{\omega,n} \pi(h)$, for all $h \in \tau_m \mathcal{G}$.
2. Also, $\mathsf{Th}\mathcal{H} \subseteq \mathsf{Th}\mathcal{G}$, for all $\mathcal{H} \in t^n_m \mathcal{G}$.

Note that each frame $\mathcal{F} \in t^n_m \mathcal{G}$ is of the form $[\vec{\mathcal{F}}]$, for some $\vec{\mathcal{F}} \in \mathcal{S}\mathbf{DS}$. Thus, by (1.) and (2.) we may assume that there is an infinite $\mathcal{H}_i$ in $\vec{\mathcal{H}}$ such that

$$\mathcal{H}^1 \lhd \mathcal{H} \lhd \mathcal{H}^2 \not\models \Lambda,$$

for $\mathcal{H}^1 = (\mathcal{H}_1 \lhd \ldots \lhd \mathcal{H}_{i-1})$ and $\mathcal{H}^2 = (\mathcal{H}_{i+1} \lhd \ldots \lhd \mathcal{H}_r)$, and all $\mathcal{H} \in t^n_m \mathcal{H}_i$. (We briefly justify this assumption. Assume that there is no $\vec{\mathcal{H}} \in \mathcal{S}\mathbf{DS}$ with

$$\text{(i) } [\vec{\mathcal{H}}] \models \Lambda \text{ and (ii) } [\vec{f}] \not\models \Lambda, \text{ for all } \vec{f} \in \tau_m \vec{\mathcal{H}},$$

in which such a $\mathcal{H}_i$ occurs. Put $\vec{\mathcal{H}}(0) = \vec{\mathcal{H}}$ for some $\vec{\mathcal{H}}$ with (i) and (ii). Certainly there is an infinite $\mathcal{H}_i$ in $\vec{\mathcal{H}}$. Also, there is a sequence $\vec{\mathcal{F}}^1$ with $[\vec{\mathcal{F}}^1] \in t^n_m \mathcal{H}_i$ such that $\mathcal{H}^1 \lhd [\vec{\mathcal{F}}^1] \lhd \mathcal{H}^2 \models \Lambda$. Denote the sequence resulting by replacing $\mathcal{H}_i$ by $[\vec{\mathcal{F}}^1]$ in $\vec{\mathcal{H}}(0)$ by $\vec{\mathcal{H}}(1)$. Then $\vec{\mathcal{H}}(1)$ also satisfies (i) and (ii). We go on with $\vec{\mathcal{H}}(1)$ and form $\vec{\mathcal{H}}(2)$, $\vec{\mathcal{H}}(3)$ etc. in the same way. Finally we get a $\vec{\mathcal{H}}(j) \in \mathcal{S}\mathbf{DS}$ with $k \geq n$, for all $\mathbf{G}[C(k), l]$ and $\mathbf{G}[l, C(k)]$ in $\vec{\mathcal{H}}(j)$, and in which there does not occur a frame from $\mathbf{Dl}$. But then $[\vec{f}] \models \Lambda$, for a $\vec{f} \in \tau_m \vec{\mathcal{H}}(j)$, by Lemma 17 of Part I. This contradicts condition (ii) for $\vec{\mathcal{H}}(j)$).

By Proposition 7, there exist $\vec{\mathcal{F}}_1, \vec{\mathcal{F}}_2 \in \mathcal{S}\mathbf{BD}_0$ with $\mathrm{R}(\vec{\mathcal{F}}_1) + \mathrm{R}(\vec{\mathcal{F}}_2) \leq 4m|\mathbf{H}|$, $\mathrm{Cl}(\vec{\mathcal{F}}_j) \leq n$, $\mathrm{S}(\vec{\mathcal{F}}_j) \leq m+2$, $\mathrm{IR}(\vec{\mathcal{F}}_j) \leq m+1$, for $j \in \{1,2\}$, such that

$$[\vec{\mathcal{F}}_1] \lhd \mathcal{H}_i \lhd [\vec{\mathcal{F}}_2] \models \Phi \text{ and } [\vec{\mathcal{F}}_1] \lhd \mathcal{H} \lhd [\vec{\mathcal{F}}_2] \not\models \Phi,$$

for all $\mathcal{H} \in \mathbf{H} = t^n_m \mathcal{H}_i$. We have $|\mathbf{H}| \leq 1 + (m+1)(m+2)$. Certainly we may assume that $\mathrm{Cl}(\mathcal{H}_i) \leq n$, $\mathrm{S}(\mathcal{H}_i) \leq m+2$. Then the sequence $\vec{\mathcal{G}} = \langle \vec{\mathcal{F}}_1, \mathcal{H}_i, \vec{\mathcal{F}}_2 \rangle$ satisfies the cardinality conditions and, by (1.) and (2.), $[\vec{f}] \not\models \Lambda$, for all $\vec{f} \in \tau_m \vec{\mathcal{G}}$. $\dashv$

Examples. Put, for $n > 0$, $\mathbf{gap}_n = \neg \exists^p \langle [J, \rightarrow^n], [\leftarrow^n, J] \rangle$, and define

- $\mathbf{Gap}_n = \mathbf{Lin}.t \oplus \mathbf{gap}_n$,
- $\mathbf{Gap}_n^+ = \mathbf{Gap}_n \oplus \neg\exists^p \mathsf{C}(0) \oplus \neg\exists^p[\mathsf{Jok}, \mathsf{C}(0)] \oplus \neg\exists^p[\mathsf{C}(0), \mathsf{Jok}]$,
- $\Theta_n = \mathbf{Gap}_n^+ \oplus \mathbf{Dens}_2$, where $\mathbf{Dens}_2$ is the $\neg L_{\mathsf{M}}$-formula corresponding to the density axiom $\Box^+\Box^+ p \to \Box^+ p$.

$\mathbf{Dens}_2$ and $\mathbf{gap} = \mathbf{gap}_1$ are discussed in Part I where is it also shown that $\Theta_1 = \mathsf{Th}\langle \mathbf{R}, < \rangle$, $\mathbf{R}$ the set of reals.

Corollary 15 *For all $n > 0$, $\mathbf{Gap}_n$, $\mathbf{Gap}_n^+$ and Θ_n are strongly complete and complex but not d-persistent.*

Proof. Let Λ be one of those logics and suppose that $\vec{\mathcal{G}} \in \mathcal{S}\mathbf{DS}$ with $[\vec{\mathcal{G}}] \models \Lambda$. It is readily checked that

$$\kappa([\vec{\mathcal{G}}])[1^</\gamma : \gamma \in \mathrm{PG}(\kappa[\vec{\mathcal{G}}])] \models \Lambda.$$

Thus Λ is strongly complete, by Theorem 13. Certainly $\mathbf{G}[2, C(n), 2] \models \Lambda$, for all $n > 1$. There is an $m > 1$ with $\mathbf{gap}_m \in \Lambda$. But then $\kappa\mathbf{G}[2, C(m), 2] \not\models \Lambda$. Hence, by Proposition 4, Λ is not d-persistent. $\dashv$

It is well known and easy to check that d-persistency transfers under forming joins of logics, i.e., if Λ_1 and Λ_2 are d-persistent, then so is $\Lambda_1 \sqcup \Lambda_2$. We are now going to show that this does not hold for strong completeness.

Recall from Part I the axiomatization the logic $\mathbf{Lin}.t \oplus \mathbf{RDisc}$ determined by the class of weak orderings in which each irreflexive point has an immediate irreflexive successor. This logic is d-persistent, hence complex and strongly complete.

Corollary 16 *(1) There are strongly complete logics Λ_1 and Λ_2 such that $\Lambda_1 \sqcup \Lambda_2$ is not strongly complete. (2) There are complex varieties $\mathcal{V}_1$ and $\mathcal{V}_2$ such that $\mathcal{V}_1 \cap \mathcal{V}_2$ is not complex.*

Proof. (1) It suffices to show that $\Lambda = \mathbf{Gap}_1 \sqcup (\mathbf{Lin}.t \oplus \mathbf{RDisc})$ is not strongly complete. But $\mathbf{G}[0, C(2)] \models \Lambda$ and $(\omega^<(0) \lhd h \lhd C(2)) \not\models \Lambda$, for all $h \in \mathbf{Fill}_m$, $m \in \omega$. By Theorem 13, Λ is not strongly complete. (2) Put $\mathcal{V}_1 = \mathcal{V}(\mathbf{Gap}_1)$ and $\mathcal{V}_2 = \mathcal{V}(\mathbf{Lin}.t \oplus \mathbf{RDisc})$. $\dashv$

Let us finally give a simple example of a complete but not strongly complete tense logic, namely $\Lambda = \mathsf{Th}(\langle \mathbf{Z}, < \rangle)$, where $\mathbf{Z}$ denotes the set of integers. A finite axiomatization of Λ is delivered in Part I. Now we have

$$(\mathbf{G}[C(1), 0] \lhd \mathbf{G}[0, C(1)]) \models \Lambda \text{ but } (C(1) \lhd f \lhd C(1)) \not\models \Lambda,$$

for all tense frames f. Thus, by Theorem 13 (5.), Λ is not strongly complete.

4 d-persistency and elementarity

Again we need an easily proved simplification of the notion of d-persistency.

Proposition 17 *A tense logic Λ is d-persistent iff $\kappa\mathcal{G} \models \Lambda$, whenever $\mathcal{G}$ is a descriptive tense frame validating Λ.*

In order to prove that each d-persistent tense logic is elementary we introduce another language for talking about intervals in Kripke frames. Consider the following interval types.

$$\begin{array}{rcl} \mathsf{Q} & = & \{[\mathsf{C(n)},\mathsf{C(m)}] | n, m \in \omega\} \cup \{\mathsf{C(0)}\} \\ & \cup & \{[\mathsf{C(n)}, \rightarrow] | n \in \omega\} \\ & \cup & \{[\leftarrow, \mathsf{C(n)}] | n \in \omega\} \\ & \cup & \{[\leftarrow, \rightarrow]\} \end{array}$$

Lemma 18 *Let $P \in \neg L_{\mathsf{M}}$. Then there exist $Q_1, \ldots, Q_n \in \neg L_{\mathsf{Q}}$ such that*

$$g \models P \Leftrightarrow (\forall i \leq n)(g \models Q_i),$$

for all Kripke frames g.

Proof. Certainly it suffices to show that for each $\mathsf{S} \in \mathsf{M}$ there exists a set Q_{S} of interval types in Q such that

$$g \models \neg\exists^p \mathsf{S} \Leftrightarrow (\forall \mathsf{B} \in Q_{\mathsf{S}})(g \models \neg\exists^p \mathsf{B}).$$

For instance, for $\mathsf{S} = [J, \rightarrow^m]$ the set

$$Q_{\mathsf{S}} = \{[\mathsf{C(0)},\mathsf{C(m)}], [\mathsf{C(0)},\rightarrow], [\mathsf{C(1)},\mathsf{C(m)}], [\mathsf{C(1)},\rightarrow], [\leftarrow,\rightarrow]\}$$

is as required. All the other cases are similar.$\dashv$

For an interval type $\mathsf{B} \in \mathsf{Q}$ denote by $L(\mathsf{B})$ and $R(\mathsf{B})$ the symbol to the left and the symbol to the right in B, respectively. Put

$$L_{\mathsf{NG}} = \{\exists^p \langle \mathsf{B}_i : 1 \leq i < n \rangle \in L_{\mathsf{Q}} : \neg(\exists i \leq n)(R(\mathsf{B}_i) = \rightarrow \wedge L(\mathsf{B}_{i+1}) = \leftarrow)\}.$$

Lemma 19 *Suppose that $Q \in \neg L_{\mathsf{NG}}$. Then $\mathsf{Mod}_{\mathsf{Fr}} Q$ is first order definable.*

Proof. Obvious.$\dashv$

Lemma 20 *Suppose that $\kappa[\vec{\mathcal{G}}] \models \Lambda$, for all $\vec{\mathcal{G}} \in \mathcal{S}\mathbf{DS}$ with $[\vec{\mathcal{G}}] \models \Lambda$. Then $\mathsf{Mod}_{\mathsf{Fr}}\Lambda$ is definable in $\neg L_{\mathsf{NG}}$.*

Proof. Suppose that $\mathsf{Mod}_{\mathsf{Fr}}\Lambda$ is not definable in $\neg L_{\mathsf{NG}}$. Put

$$\Gamma = \{\neg Q \in \neg L_{\mathsf{NG}} : (\forall g)(g \models Q \Rightarrow g \not\models \Lambda)\}.$$

We have $M \supset \mathsf{Mod}_{\mathsf{Fr}}\Lambda$, for $M = \mathsf{Mod}_{\mathsf{Fr}}\Gamma$. Moreover $h \in M - \mathsf{Mod}_{\mathsf{Fr}}\Lambda$ iff

1. there exists $\neg\exists^p\vec{\mathsf{S}} \in \Lambda$ such that $h \models \exists^p\vec{\mathsf{S}}$.
2. $\neg\exists^p\vec{\mathsf{S}} \notin \Lambda$, for all $\vec{\mathsf{S}} \in \mathsf{M}$ such that there is a partition $\vec{I}$ of h with $h, \vec{I} \models \vec{\mathsf{S}}$ such that $\mathrm{gap}(\vec{I})$ contains no proper gap.

Take an $\langle f, R\rangle \in M - \mathsf{Mod}_{\mathsf{Fr}}\Lambda$ and a partition $\vec{I} = \langle I_i : 1 \leq i \leq r\rangle$ of $\langle f, R\rangle$ such that $\langle f, R\rangle, \vec{I} \models \vec{\mathsf{S}}$, for some $\vec{\mathsf{S}}$ with $\neg\exists^p\vec{\mathsf{S}} \in \Lambda$. We may assume that the size of clusters in $\langle f, R\rangle$ is bounded by $\mathrm{Cl}(\vec{\mathsf{S}})$. We show that, for $1 \leq i \leq r$, there is a sequence $\vec{h}^i \in \mathcal{S}\mathbf{BK}$ with

(a) $f_{I_i} \in \mathsf{B} \Rightarrow [\vec{h}^i] \in \mathsf{B}$, for all $\mathsf{B} \in \mathsf{Q}$.

(b) For all $\vec{\mathsf{S}} \in \mathsf{M}$, if $[\vec{h}^i], \vec{J}^1 \models \vec{\mathsf{S}}$ and $\mathrm{gap}(\vec{J}^1)$ contains no proper gaps then the exists a partition $\vec{J}^2$ of f_{I_i} such that $\mathrm{gap}(\vec{J}^2)$ contains no proper gaps and so that $f_{I_i}, \vec{J}^2 \models \vec{\mathsf{S}}$.

(c) $\mathsf{Th}[\vec{h}^i] \supseteq \mathsf{Th} f_{I_i}$.

Note that it follows from (a) that $[\vec{h}^i] \in \mathsf{S}_i$. One has to distiguish the cases $I_i \in \mathsf{B} = [\mathsf{C}(0)]$, $I_i \in \mathsf{B} = [\mathsf{C}(\mathsf{m}), \mathsf{C}(\mathsf{n})]$, where $m, n \in \omega$ are maximal with this property, $I_i \in \mathsf{B} = [\leftarrow, \mathsf{C}(\mathsf{n})]$, where $n \in \omega$ is maximal with this property, $I_i \in \mathsf{B} = [\mathsf{C}(\mathsf{n}), \rightarrow]$, where $n \in \omega$ is maximal with this property, and $I_i \in \mathsf{B} = [\leftarrow, \rightarrow]$. In all those cases there exists a $\vec{h}^i$ satisfying (b) and (c) with $[\vec{h}^i] \in \mathsf{B}$. The proof is similar to the proof of Proposition 7 in Part I. Some care has to be taken in order to satisfy condition (b). Instead of delivering the tedious detailed proof we give two instructive examples. Assume

$$f_{I_i} = \langle \mathbf{Q}, < \rangle,$$

where $\mathbf{Q}$ is the set of rational numbers. Then the frame $(\omega^>(1) \lhd \omega^<(1))$ satisfies all the conditions required for $\vec{h}^i$ with the exception of (b). (b) is not satisfied since

$$(\omega^>(1) \lhd \omega^<(1)) \models \exists^p\langle [J, \rightarrow^1], [\leftarrow^1, J]\rangle,$$

but $\langle \mathbf{Q}, <\rangle, \langle I_1, I_2\rangle \models \langle [J, \rightarrow^1], [\leftarrow^1, J]\rangle$ only if $I_1 \in [J, \rightarrow]$ and $I_2 \in [\leftarrow, J]$. Here the frame $(\omega^<(2) \lhd C(0) \lhd \omega^>(2))$ is as required.

Now assume

$$f_{I_i} = C(1) \lhd \omega^>(0) \lhd \omega^<(0) \lhd \omega^>(0) \lhd \omega^<(0) \lhd \omega^>(0) \lhd \omega^<(0) \ldots$$

Then the frame $\omega^<(1)$ satisfies all the conditions for $\vec{h}^i$ with the exception of (b). Here a counterexample for (b) is the sequence

$$\vec{\mathsf{S}} = \langle [\leftarrow^1, J], [J, \rightarrow^1], [\leftarrow^1, J]\rangle.$$

The frame $C(1) \lhd \omega^>(0) \lhd \omega^<(0)$ is as required.

Put $\vec{g} = \langle g_i : 1 \leq i \leq k \rangle := \langle \vec{h}^i : 1 \leq i \leq r \rangle$ and

$$D_1 = \{1 \leq i \leq k : g_i = \omega^<(m_1) \text{ and } g_{i+1} = \omega^>(m_2) \text{ for some } m_1, m_2 \in \omega\},$$

$$D_2 = \{1 \leq i \leq k : i, i-1 \notin D_1\},$$

and $n = \max\{\mathrm{Cl}(\vec{\mathfrak{S}}), 2\}$. Now define, for $i \in D_1 \cup D_2$,

$$\mathcal{G}_i = \begin{cases} g_i & : \ g_i \text{ finite}, \\ \mathbf{G}[l, C(n)] & : \ g_i = \omega^<(l),\ i \in D_2, \\ \mathbf{G}[C(n), l] & : \ g_i = \omega^>(l),\ i \in D_2, \\ \mathbf{G}[m_1, C(n), m_2] & : \ g_i = \omega^<(m_1),\ g_{i+1} = \omega^>(m_2), \end{cases}$$

and put $\vec{\mathcal{G}} = \langle \mathcal{G}_i : i \in D_1 \cup D_2 \rangle$. It follows from condition (b) for $\vec{h}^i$, $1 \leq i \leq r$, and the construction of $\vec{\mathcal{G}}$ that $[\vec{\mathcal{G}}] \models \Lambda$. (Case four in the definition of $\mathcal{G}_i$ is essential). On the other hand, by condition (a) for $\vec{h}^i$ and the construction of $\vec{\mathcal{G}}$, we have $\kappa[\vec{\mathcal{G}}] \not\models \Lambda$. Details are left to the reader. ⊣

Theorem 21 *The following conditions are equivalent, for each tense logic* Λ.

1. $\kappa[\vec{\mathcal{G}}] \models \Lambda$, *for all* $\vec{\mathcal{G}} \in \mathcal{S}\mathbf{DS}$ *with* $[\vec{\mathcal{G}}] \models \Lambda$.
2. Λ *is complete and* $\mathsf{Mod_{Fr}}\Lambda$ *is definable in* $\neg L_{\mathsf{NG}}$.
3. Λ *is elementary.*
4. Λ *is d-persistent.*

Proof. (1) $\Rightarrow$ (2) is Lemma 20. (2) $\Rightarrow$ (3) is Lemma 19. (3) $\Rightarrow$ (4) is proved in [2] and (4) $\Rightarrow$ (1) is trivial. ⊣

Theorem 22 $\{\Phi \subseteq \neg L_{\mathsf{M}}$ *finite* $: \mathbf{Lin}.t \oplus \Phi$ *is d-persistent* $\}$ *is recursive. Moreover,* $\Lambda = \mathbf{Lin}.t \oplus \Phi$, $\Phi \subseteq \neg L_{\mathsf{M}(m,n)}$, *is d-persistent iff, for all* $\vec{\mathcal{G}} \in \mathcal{S}\mathbf{DS}$ *with* $[\vec{\mathcal{G}}] \models \Lambda$ *and* *Cl*$(\vec{\mathcal{G}}) \leq n$, *S*$(\vec{\mathcal{G}}) \leq m+2$, *IR*$(\vec{\mathcal{G}}) \leq m+1$ *and* *R*$(\vec{\mathcal{G}}) \leq 1+4m$, *we have* $\kappa[\vec{\mathcal{G}}] \models \Lambda$, *as well.*

Proof. By Theorem 6, it suffices to show the moreover part. The direction from left to right follows from Proposition 4. Now suppose that Λ is not d-persistent. By Theorem 21, there is a $\vec{\mathcal{H}} = \langle \mathcal{H}_i : 1 \leq i \leq r \rangle \in \mathcal{S}\mathbf{DS}$ with $[\vec{\mathcal{H}}] \models \Lambda$ but $\kappa[\vec{\mathcal{H}}] \not\models \Lambda$. Put, for $\mathcal{G} \in \mathbf{DS}$ and $k > 0$,

$$s_k\mathcal{G} = \begin{cases} \{\mathcal{G}\} & : \ \mathcal{G} \text{ finite}, \\ \mathbf{G}[n, C(k)] \lhd C(m) & : \ \mathcal{G} = \mathbf{G}[n, C(m)], \\ C(m) \lhd \mathbf{G}[C(k), n] & : \ \mathcal{G} = \mathbf{G}[C(m), n], \\ \mathbf{G}[m_1, C(k)] \lhd C(m) \lhd \mathbf{G}[C(k), m_2] & : \ \mathcal{G} = \mathbf{G}[m_1, C(m), m_2]. \end{cases}$$

By Lemma 17 of Part I,

$$\mathsf{Th}_{\omega,n}\kappa\mathcal{G} = \mathsf{Th}_{\omega,n}s_n\mathcal{G},$$

for all $\mathcal{G} \in \mathcal{S}\mathbf{DS}$. Hence we may assume that there exists a $\mathcal{H}_i$ such that $\mathcal{H}^1 \lhd s_n\mathcal{H}_i \lhd \mathcal{H}^2 \not\models \Lambda$, for $\mathcal{H}^1 = (\mathcal{H}_1 \lhd \ldots \lhd \mathcal{H}_{i-1})$ and $\mathcal{H}^2 = (\mathcal{H}_{i+1} \lhd \ldots \lhd \mathcal{H}_r)$. By Proposition 7 there exist $\vec{\mathcal{F}}_1, \vec{\mathcal{F}}_2 \in \mathcal{S}\mathbf{BD}_0$ with $\mathrm{R}(\vec{\mathcal{F}}_1)+\mathrm{R}(\vec{\mathcal{F}}_2) \leq 4m$, $\mathrm{Cl}(\vec{\mathcal{F}}_j) \leq n$, $\mathrm{S}(\vec{\mathcal{F}}_j) \leq m+2$, and $\mathrm{IR}(\vec{\mathcal{F}}_j) \leq m+1$, for $j \in \{1,2\}$, such that

$$[\vec{\mathcal{F}}_1] \lhd \mathcal{G}_i \lhd [\vec{\mathcal{F}}_2] \models \Phi \text{ and } [\vec{\mathcal{F}}_1] \lhd t_n\mathcal{H}_i \lhd [\vec{\mathcal{F}}_2] \not\models \Phi.$$

(Here we put $\mathbf{H} = \{s_n\mathcal{H}_i\}$). We may assume that $\mathrm{Cl}(\mathcal{H}_i) \leq n$, $\mathrm{S}(\mathcal{H}_i) \leq m+2$. But then the sequence $\vec{\mathcal{G}} = \langle \vec{\mathcal{F}}_1, \mathcal{H}_i, \vec{\mathcal{F}}_2 \rangle$ satisfies the cardinality conditions and $\kappa[\vec{\mathcal{G}}] \not\models \Lambda$. $\dashv$

5 r-persistency

In [7] it is shown that each refined frame $\mathcal{H}$ is, in a certain sense, embeddable into a descriptive frame $\mathcal{G}$ with $\mathcal{H}^+ \simeq \mathcal{G}^+$. (Intuitively, this is achieved by adding precisely those ultrafilters U in $\mathcal{H}^+$ to h for which $\bigcap U = \emptyset$). The precise formulation is as follows.

Proposition 23 *Suppose that $\mathcal{H}^+ \simeq \mathcal{G}^+$, where $\mathcal{H} = \langle h, S, A \rangle$ is descriptive and $\mathcal{G}$ is refined. Then there exists $\mathcal{H}_1 = \langle h_1, S_1, A_1 \rangle$, where $h_1 \subseteq h$, $S_1 = S \cap h_1^2$ and $A_1 = \{a \cap h_1 : a \in A\}$, such that $\mathcal{G} \simeq \mathcal{H}_1$ and such that $a \mapsto a \cap h_1$, $a \in A$, defines an isomorphism from $\mathcal{H}$ onto $\mathcal{H}_1$. Moreover, for the canonical type assignment $\mathbf{c}$ for $\mathcal{H}$ and all clusters C in $\mathcal{H}$,*

- *$C \subseteq h_1$ or $C \cap h_1$ infinite, for $\mathbf{c}C = (\mathbf{m}, \mathbf{m})$.*
- *$C \subseteq h_1$ or $C \cap h_1$ is infinite or $C \cap h_1 = \emptyset$, for $\mathbf{c}C \in \{(\mathbf{m}, \mathbf{j}), (\mathbf{j}, \mathbf{m})\}$.*
- *$C \cap h_1$ is arbitrary, for $\mathbf{c}C = (\mathbf{j}, \mathbf{j})$.*

Proof. The first part is shown in [7] and the moreover part is an immediate consequence of the first part and the description of clusters in descriptive tense frames. $\dashv$

For refined frames we have to take some care when working with L_{M} instead of $\mathcal{L}_\omega$. Theorem 2 does not hold for refined frames. (For example, the theories of $\mathbf{G}[0, C(1)]$ and

$$\mathcal{G} := \langle \omega^{<}(0), \{a \subseteq \omega : a \text{ finite or cofinite }\} \rangle$$

in $\mathcal{L}_\omega$ coincide but $\mathbf{G}[0, C(1)] \notin [J, \rightarrow^2]$ and $\mathcal{G} \in [J, \rightarrow^2]$). So, we shall not evaluate formulas from $\neg L_{\mathsf{M}}$ in frames which are not in $\mathsf{Dfr} \cup \mathsf{Fr}$ but define the expression $\mathcal{G} \models \Lambda$ as an abbrevation of $(\mathcal{G}^+)_+ \models \Lambda$. This should not lead to confusion since we shall use it only two times.

Corollary 24 *Suppose that Λ is not r-persistent. Then there exists a descriptive frame $\mathcal{H}$ with $\mathcal{H} \models \Lambda$ and a refined frame $\mathcal{H}_1 = \langle h_1, S_1, A_1 \rangle$ with $\kappa\mathcal{H}_1 \not\models \Lambda$ and $h_1 \subseteq h$, $S_1 = S \cap h_1^2$ and $A_1 = \{a \cap h_1 : a \in A\}$, such that $a \mapsto a \cap h_1$, $a \in A$, defines an isomorphism from $\mathcal{H}$ onto $\mathcal{H}_1$ and such that for the canonical type assignment $\mathbf{c}$ for $\mathcal{H}$ and all clusters C in $\mathcal{H}$,*

- *$C \subseteq h_1$, for $\mathbf{c}C = (\mathbf{m},\mathbf{m})$.*
- *$C \subseteq h_1$ or $C \cap h_1 = \emptyset$, for $\mathbf{c}C \in \{(\mathbf{m},\mathbf{j}),(\mathbf{j},\mathbf{m})\}$.*
- *$C \cap h_1$ arbitrary, for $\mathbf{c}C = (\mathbf{j},\mathbf{j})$.*

Proof. Suppose that Λ is not r-persistent. There exists a refined tense frame $\mathcal{G}$ with $\kappa\mathcal{G} \not\models \Lambda$ and $\mathcal{G} \models \Lambda$. Certainly there exists a descriptive frame $\mathcal{H} = \langle h, S, A\rangle$ with $\mathcal{H}^+ \simeq \mathcal{G}^+$. Take a frame $\mathcal{H}_1' = \langle h_1', S_1', A_1'\rangle$ which satiefies the condition of Proposition 23. Now put, for C a cluster in h, $C_1 = h_1' \cap C$ if $h_1' \cap C$ is finite, and $C_1 = C$, otherwise. Put

$$h_1 = h_1' \cup \bigcup\{C_1 : C \text{ cluster in } h\},$$

and $\mathcal{H}_1 = \langle h_1, S \cap h_1^2, \{a \cap h_1 : a \in A\}\rangle$. Only the size of infinite clusters is manipulated. So it is straightforward to show that $\mathcal{H}_1$ is as required. $\dashv$

Put, for $\mathcal{G} \in \mathbf{DS}$,

$$s\mathcal{G} = \begin{cases} \{\mathcal{G}\} & : \mathcal{G} \text{ finite}, \\ \{\omega^<(n)\} \cup \{\kappa\mathcal{G}\} & : \mathcal{G} = \mathbf{G}[n, C(m)], \\ \{\omega^>(n)\} \cup \{\kappa\mathcal{G}\} & : \mathcal{G} = \mathbf{G}[C(m), n], \\ \{\omega^<(m_1) \lhd \omega^>(m_2)\} \cup \{\omega^<(m_1) \lhd C(m) \lhd \omega^>(m_2)\} & : \mathcal{G} = \mathbf{G}[m_1, C(m), m_2], \end{cases}$$

and define, for $\vec{\mathcal{G}} = \langle \mathcal{G}_i : 1 \le i \le r\rangle \in \mathcal{S}\mathbf{DS}$,

$$s\vec{\mathcal{G}} = \{\langle f_i : 1 \le i \le r\rangle : f_i \in s\mathcal{G}_i, 1 \le i \le r\}.$$

Note that, for $\vec{\mathcal{G}} \in \mathcal{S}\mathbf{DS}$ and $\mathcal{H} = [\vec{\mathcal{G}}]$, the frames in $\{[\vec{f}] : \vec{f} \in s\vec{\mathcal{G}}\}$ basically coincide with the frames of the form $\kappa\mathcal{H}_1$, for $\mathcal{H}_1$ satisfying the conditions of Corollary 24 with respect to $\mathcal{H}$.

Theorem 25 *The following properties are equivalent, for $\Lambda = \mathbf{Lin}.t \oplus \Phi$, $\Phi \subseteq \neg L_{\mathsf{M}(m,n)}$.*

1. *Λ is r-persistent.*
2. *$[\vec{f}] \models \Lambda$, for all $\vec{f} \in s\vec{\mathcal{G}}$ with $\vec{\mathcal{G}} \in \mathcal{S}\mathbf{DS}$ and $[\vec{\mathcal{G}}] \models \Lambda$.*
3. *$[\vec{f}] \models \Lambda$, for all $\vec{f} \in s\vec{\mathcal{G}}$ with $\vec{\mathcal{G}} \in \mathcal{S}\mathbf{DS}$ and $Cl(\vec{\mathcal{G}}) \le n$, $S(\vec{\mathcal{G}}) \le m+2$, $IR(\vec{\mathcal{G}}) \le m+1$, $R(\vec{\mathcal{G}}) \le 4m$ and $[\vec{\mathcal{G}}] \models \Lambda$.*

The set $\{\Phi \subseteq \neg L_{\mathsf{M}}$ finite : $\mathbf{Lin}.t \oplus \Phi$ is r-persistent $\}$ is recursive.

Proof. (1) $\Rightarrow$ (2). Consider $\langle h, S\rangle = [\vec{f}]$, for a $\vec{f} \in s\vec{\mathcal{G}}$ with $[\vec{\mathcal{G}}] \models \Lambda$. For $[\vec{\mathcal{G}}] = \langle g, R, A\rangle$ we may assume that $h \subseteq g$ and $S = R \cap h^2$. Put $B = \{a \cap h : a \in A\}$ and $\mathcal{H} = \langle h, S, B\rangle$. It is easy to check that $a \mapsto a \cap h$, $a \in A$, defines an isomorphism from $\mathcal{G}^+$ onto $\mathcal{H}^+$. Clearly $\mathcal{H}$ is refined. Hence, by (1), $\langle h, S\rangle \models \Lambda$.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1). (Sketch) Suppose that Λ is not r-persistent. Take a descriptive frame $\mathcal{H}$ with $\mathcal{H} \models \Lambda$ and a refined frame $\mathcal{H}_1$ satisfying the conditions of Corollary 24. So we have $\kappa\mathcal{H}_1 \models \neg\exists^p\vec{\mathsf{S}}$, for a $\neg\exists^p\vec{\mathsf{S}} \in \Phi$. Similar to a lot of previous examples and especially to the proof of Proposition 7 of Part I we can reduce $\mathcal{H}_1$ in such a way that we get a $\vec{\mathcal{G}} \in \mathcal{S}\mathbf{DS}$ satisfying the cardinality conditions of (3) such that there is an $\vec{f} \in s\vec{\mathcal{G}}$ with $[\vec{f}] \models \neg\exists^p\vec{\mathsf{S}}$. We leave the construction to the reader. $\dashv$

A simple example of a d-persistent tense logic which is not r-persistent is the logic $\mathbf{Mk}.t$ defined in Part I. It is easily verified that this logic is d-persistent. On the other hand, $\mathbf{Mk}.t$ is not r-persistent since

$$\mathbf{G}[0, C(1)] \models \mathbf{Mk}.t \text{ but } \omega^{<}(0) \not\models \mathbf{Mk}.t.$$

6 The finite model property

Recall from Part I the definition of the k-diagram $\mathrm{dg}_k\vec{\mathcal{G}} \in L_{\mathsf{M}}$ of sequences $\vec{\mathcal{G}} \in \mathbf{BD}_0$. It has the properties

$$\mathcal{H} \models \mathrm{dg}_k[\vec{\mathcal{G}}] \Rightarrow \mathsf{Th}_{k,\omega}(\mathcal{H}) \subseteq \mathsf{Th}_{k,\omega}[\vec{\mathcal{G}}],$$
$$[\vec{\mathcal{G}}] \models \mathrm{dg}_k[\vec{\mathcal{G}}],$$

for all $\mathcal{H}$ (cf. Proposition 8, Part I).

Lemma 26 *Let $m \in \omega$. A finitely axiomatizable logic Λ is determined by a class M of frames iff, for all $\vec{\mathcal{G}} \in \mathcal{S}\mathbf{BD}_0$ with $[\vec{\mathcal{G}}] \models \Lambda$ and all $k \geq m$, there exists $\mathcal{H} \in M$ with $\mathcal{H} \models \Lambda$ and $\mathcal{H} \models dg_k\vec{\mathcal{G}}$.*

Proof. By Theorem 14 (1) of Part I, each finitely axiomatizable logic is determined by a class of frames $\{[\vec{\mathcal{G}}] : \vec{\mathcal{G}} \in Q\}$, with $Q \subseteq \mathcal{S}\mathbf{BD}_0$. Now the Lemma follows with Proposition 8 of Part I. $\dashv$

For $\mathcal{G} \in \mathbf{BD}_0$ and $k, l \in \omega$ define a set of frames $\sigma_{lk}\mathcal{G}$ by putting

$$\sigma_{lk}\mathcal{G} = \begin{cases} \{\mathcal{G}\} & : \ \mathcal{G} \text{ finite}, \\ \{n(k+1)^{<}(n) \lhd h \lhd C(m) : h \in \mathbf{Fill}_l\} & : \ \mathcal{G} = \mathbf{G}[n, C(m)],\ n > 0, \\ \{(k+1)^{<}(0) \lhd h \lhd C(m) : h \in \mathbf{Fill}_l\} & : \ \mathcal{G} = \mathbf{G}[0, C(m)], \\ \{C(m) \lhd h \lhd n(k+1)^{<}(n) : h \in \mathbf{Fill}_l\} & : \ \mathcal{G} = \mathbf{G}[C(m), n],\ n > 0, \\ \{C(m) \lhd h \lhd (k+1)^{<}(0) : h \in \mathbf{Fill}_l\} & : \ \mathcal{G} = \mathbf{G}[C(m), 0], \end{cases}$$

and define, for a sequence $\vec{\mathcal{G}} = \langle \mathcal{G}_i : 1 \le i \le r \rangle \in \mathcal{S}\mathbf{BD}_0$

$$\sigma_{lk}\vec{\mathcal{G}} = \{\langle f_i : 1 \le i \le r \rangle : f_i \in \sigma_{lk}\mathcal{G}_i, 1 \le i \le r\} \text{ and } \sigma_k\vec{\mathcal{G}} = \bigcup\{\sigma_{lk}\mathcal{G} : l \in \omega\}.$$

Lemma 27 *For all $\vec{\mathcal{G}} \in \mathcal{S}\mathbf{BD}_0$ and all finite frames h, $h \models dg_k\vec{\mathcal{G}}$ iff there exists $\vec{f} \in \sigma_k\vec{\mathcal{G}}$ with $\mathsf{Th}h \subseteq \mathsf{Th}[\vec{f}]$.*

Proof. We have $[\vec{f}] \models \mathrm{dg}_k\vec{\mathcal{G}}$, for all $\vec{f} \in \sigma_k\vec{\mathcal{G}}$. Conversely, suppose that $h \models \mathrm{dg}_k\vec{\mathcal{G}}$, for a finite frame h. Assume that $\mathrm{dg}_k\mathcal{G}_i = \exists^p\vec{\mathsf{S}}_i$, for $1 \le i \le r$, and that $\vec{I} = \langle I_i : 1 \le i \le r \rangle$ is a partition of h such that $h_{I_i} \models \exists^p\vec{\mathsf{S}}_i$, $1 \le i \le r$. The Lemma follows if there exist $f_i \in \sigma_k\mathcal{G}_i$ with $\mathsf{Th}h_{I_i} \subseteq \mathsf{Th}f_i$, for $1 \le i \le r$. This is clear if $\mathcal{G}_i$ is finite. Suppose that $\mathcal{G}_i = \mathbf{G}[n, C(m)], n > 0$. Certainly we may assume that

$$h_{I_i} = n(k+1)^{<}(n) \lhd h' \lhd C(m),$$

for some finite frame h'. But, by Lemma 11, there exists a $l \in \omega$ and a $g \in \mathbf{Fill}_l$ with $\mathsf{Th}h' \subseteq \mathsf{Th}g$. Hence

$$\mathsf{Th}h_{I_i} \subseteq \mathsf{Th}(n(k+1)^{<}(n) \lhd g \lhd C(m)),$$

and the frame to the right is in $\sigma_k\mathcal{G}_i$. The cases $\mathcal{G}_i = \mathbf{G}[0, C(m)]$ and $\mathcal{G}_i = \mathbf{G}[C(m), n]$, $n \in \omega$, are considered similar. $\dashv$

Theorem 28 *$\{\Phi \subseteq \neg L_{\mathsf{M}}$ finite : $\mathbf{Lin}.t \oplus \Phi$ has the fmp $\}$ is a recursive set. Moreover, for $\Lambda = \mathbf{Lin}.t \oplus \Phi$, $\Phi \subseteq \neg L_{\mathsf{M}(m,n)}$, the following conditions are equivalent.*

1. *Λ has the fmp.*
2. *For all $\vec{\mathcal{G}} \in \mathcal{S}\mathbf{BD}_0$ with $[\vec{\mathcal{G}}] \models \Lambda$ and all $k \ge m$ there there exists $\vec{f} \in \sigma_k\vec{\mathcal{G}}$ with $[\vec{f}] \models \Lambda$.*
3. *For all $\vec{\mathcal{G}} \in \mathcal{S}\mathbf{BD}_0$ with $[\vec{\mathcal{G}}] \models \Lambda$ and $Cl(\vec{\mathcal{G}}) \le n$, $S(\vec{\mathcal{G}}) \le m+2$, $IR(\vec{\mathcal{G}}) \le m+1$, and $R(\vec{\mathcal{G}}) \le 1 + 4m(m+1)(m+2)$, there exists $\vec{f} \in \sigma_{mm}\vec{\mathcal{G}}$ with $[\vec{f}] \models \Lambda$.*

Proof. By Theorem 6 is suffices to show the moreover part. We have

(a) $((\exists\vec{f} \in \sigma_k\vec{\mathcal{G}})([\vec{f}] \models \Lambda) \Leftrightarrow (\exists\vec{f} \in \sigma_{mm}\vec{\mathcal{G}})([\vec{f}] \models \Lambda))$, for all $\vec{\mathcal{G}} \in \mathcal{S}\mathbf{BD}_0$ and all $k \ge m$.

(b) $\mathsf{Th}_{m,\omega}[\vec{f}] \subseteq \mathsf{Th}_{m,\omega}[\vec{\mathcal{G}}]$, for all $\vec{f} \in \sigma_{mm}\vec{\mathcal{G}}$ and all $\vec{\mathcal{G}} \in \mathcal{S}\mathbf{BD}_0$.

(a) follows from Lemma 13 of Part I and (b) follows from $[\vec{f}] \models \mathrm{dg}_m\vec{\mathcal{G}}$, for all $\vec{f} \in \sigma_m\vec{\mathcal{G}}$.

(1.) $\Leftrightarrow$ (2.) follows from Lemma 26 and Lemma 27 and (2.) $\Rightarrow$ (3.) follows immediately from (a). It remains to show (3.) $\Rightarrow$ (2.). Suppose that there is a $\vec{\mathcal{G}} = \langle \mathcal{G}_i : 1 \le i \le r \rangle \in$

$\mathcal{S}\mathbf{BD}_0$ and a $k \geq m$ such that $[\vec{f}] \not\models \Lambda$, for all $\vec{f} \in \sigma_k\vec{\mathcal{G}}$. By (a), $[\vec{f}] \not\models \Lambda$, for all $\vec{f} \in \sigma_{mm}\vec{\mathcal{G}}$. We may assume, by (b), that there is an infinite $\mathcal{G}_i$ in $\vec{\mathcal{G}}$ such that

$$\mathcal{H}_1 \lhd f \lhd \mathcal{H}_2 \not\models \Lambda,$$

for $\mathcal{H}_1 = (\mathcal{G}_1 \lhd \ldots \lhd \mathcal{G}_{i-1})$ and $\mathcal{H}_2 = (\mathcal{G}_{i+1} \lhd \ldots \lhd \mathcal{G}_r)$ and all $f \in \sigma_{mm}\mathcal{G}_i$. (The justification of this assumption is similar to the proof of Theorem 14). By Proposition 7, there exist $\vec{\mathcal{F}}_1$, $\vec{\mathcal{F}}_2 \in \mathcal{S}\mathbf{BD}_0$ with $\mathrm{R}(\vec{\mathcal{F}}_1)+\mathrm{R}(\vec{\mathcal{F}}_2) \leq 4m|\mathbf{H}|$, $\mathrm{Cl}(\vec{\mathcal{F}}_j) \leq n$, $\mathrm{S}(\vec{\mathcal{F}}_j) \leq m+2$, and $\mathrm{IR}(\vec{\mathcal{F}}_j) \leq m+1$, for $j \in \{1,2\}$, such that

$$[\vec{\mathcal{F}}_1] \lhd \mathcal{G}_i \lhd [\vec{\mathcal{F}}_2] \models \Phi \text{ and } [\vec{\mathcal{F}}_1] \lhd f \lhd [\vec{\mathcal{F}}_2] \not\models \Phi,$$

for all $f \in \mathbf{H} := \sigma_{mm}\mathcal{G}_i$. We have $|\mathbf{H}| \leq (m+1)(m+2)$. Certainly we may assume that $\mathrm{Cl}(\mathcal{G}_i) \leq n$, $\mathrm{S}(\mathcal{G}_i) \leq m+2$. Thus $\vec{\mathcal{H}} = \langle \vec{\mathcal{F}}_1, \mathcal{G}_i, \vec{\mathcal{F}}_2 \rangle$ refutes (3.), i.e. $\vec{\mathcal{H}}$ satisfies the cardinality conditions of (3.) and there does not exist a $\vec{f} \in \sigma_{mm}\vec{\mathcal{H}}$ such that $[\vec{f}] \models \Lambda$. $\dashv$

7 Completeness

For $\mathcal{G} \in \mathbf{BD}_0$ and $l, k \in \omega$ define a set of frames $\gamma_{lk}\mathcal{G}$ by putting

$$\gamma_{lk}\mathcal{G} = \begin{cases} \{\mathcal{G}\} & : \mathcal{G} \text{ finite,} \\ \sigma_{lk}\mathcal{G} \cup \{\omega^<(n)\} \cup \{\omega^<(n) \lhd g \lhd C(m) : g \in \mathbf{Fill}_l\} & : \mathcal{G} = \mathbf{G}[n, C(m)], \\ \sigma_{lk}\mathcal{G} \cup \{\omega^>(n)\} \cup \{C(m) \lhd g \lhd \omega^>(n) : g \in \mathbf{Fill}_l\} & : \mathcal{G} = \mathbf{G}[C(m), n], \end{cases}$$

and put, for a sequence $\vec{\mathcal{G}} = \langle \mathcal{G}_i : 1 \leq i \leq r \rangle \in \mathcal{S}\mathbf{BD}_0$,

$$\gamma_{lk}\vec{\mathcal{G}} = \{\langle f_i : 1 \leq i \leq r : f_i \in \gamma_{lk}\mathcal{G}_i, 1 \leq i \leq r\} \text{ and } \gamma_k\vec{\mathcal{G}} = \bigcup\{\gamma_{lk}\mathcal{G} : l \in \omega\}.$$

Lemma 29 *For all $\vec{\mathcal{G}} \in \mathcal{S}\mathbf{BD}_0$ and all Kripke frames h, if $h \models dg_k\vec{\mathcal{G}}$ then there exists $\vec{f} \in \gamma_k\vec{\mathcal{G}}$ with $\mathsf{Th}_{k,\omega}h \subseteq \mathsf{Th}_{k,\omega}[\vec{f}]$.*

Proof. Suppose that $h \models \mathrm{dg}_k\vec{\mathcal{G}}$, for a Kripke frame h. Let $\mathrm{dg}_k\mathcal{G}_i = \exists^p\vec{\mathsf{S}}_i$, for $1 \leq i \leq r$, and assume that $\langle I_i : 1 \leq i \leq r \rangle$ is a partition of h such that $h_{I_i} \models \exists^p\vec{\mathsf{S}}_i$, $1 \leq i \leq r$. The Lemma follows if there exist $f_i \in \gamma_k\mathcal{G}_i$ with $\mathsf{Th}_{k,\omega}f_i \supseteq \mathsf{Th}_{k,\omega}h_{I_i}$, for all $1 \leq i \leq r$. This is clear if $\mathcal{G}_i$ is finite. Suppose that $\mathcal{G}_i = \mathbf{G}[n, C(m)], n > 0$. Certainly we may assume that

$$(a)\ h_{I_i} = n(k+1)^<(n) \lhd f^1 \lhd C(m) \text{ or } (b)\ h_{I_i} = n(k+1)^<(n) \lhd f^2,$$

where f^1 and f^2 are Kripke frames and $f^2 \in [J, \rightarrow]$. If (b) holds then $\mathsf{Th}_{k,\omega}h_{I_i} \subseteq \mathsf{Th}_{k,\omega}\omega^<(n)$ and $\omega^<(n) \in \gamma_k\mathcal{G}_i$. Now assume (a).

Case 1. f^1 is finite. Then the statement follows as in Lemma 27.

Case 2. f^1 is infinite. By Theorem 18 of Part I, t here exists a sequence $\vec{f} = \langle f_i : 1 \leq i \leq t\rangle \in \mathcal{S}\mathbf{BK}$ with $\mathsf{Th}f^1 \subseteq \mathsf{Th}[\vec{f}]$. Suppose that there is a minimal j such that $f_j = \omega^{>}(m)$, for some $m \in \omega$. Then

$$\mathsf{Th}h_{I_i} \subseteq \mathsf{Th}(n(k+1)^{<}(n) \lhd [\langle f_i : 1 \leq i \leq j-1\rangle] \lhd C(m)).$$

Thus, we may assume that $\vec{f}$ contains no $\omega^{>}(m)$ since $\langle f_i : 1 \leq i \leq j-1\rangle$ contains no $\omega^{>}(m)$. Now take a maximal $j \leq t$ such that $f_j = \omega^{<}(m)$, for some $m \in \omega$. Then

$$\mathsf{Th}h_{I_i} \subseteq \mathsf{Th}(\omega^{<}(n) \lhd [\langle f_i : j+1 \leq i \leq t\rangle] \lhd C(m)])$$

and $[\langle f_i : j+1 \leq i \leq t\rangle]$ is finite. Case 2 is reduced to Case 1.

The cases $\mathcal{G}_i = \mathbf{G}[0, C(m)]$ and $\mathcal{G}_i = \mathbf{G}[C(m), n]$ are treated similar. $\dashv$

Theorem 30 *$\{\Phi \subseteq \neg L_{\mathsf{M}}$ finite : $\mathbf{Lin}.t \oplus \Phi$ is complete $\}$ is a recursive set. Moreover, for $\Lambda = \mathbf{Lin}.t \oplus \Phi$, $\Phi \subseteq \neg L_{\mathsf{M}(m,n)}$, the following conditions are equivalent.*

1. *Λ is complete.*
2. *For all $\vec{\mathcal{G}} \in \mathcal{S}\mathbf{BD}_0$ with $[\vec{\mathcal{G}}] \models \Lambda$ and $k \geq m$ there exists $\vec{f} \in \gamma_k\vec{\mathcal{G}}$ with $[\vec{f}] \models \Lambda$.*
3. *For all $\vec{\mathcal{G}} \in \mathcal{S}\mathbf{BD}_0$ with $[\vec{\mathcal{G}}] \models \Lambda$ and $Cl(\vec{\mathcal{G}}) \leq n$, $S(\vec{\mathcal{G}}) \leq m+2$, $IR(\vec{\mathcal{G}}) \leq m+1$, and $R(\vec{\mathcal{G}}) \leq 1 + 4m(1+2(m+1)(m+2))$, there exists $\vec{f} \in \gamma_{mm}\vec{\mathcal{G}}$ with $[\vec{f}] \models \Lambda$.*

Proof. By Theorem 6 is suffices to show the moreover part. Similar to the proof of Theorem 28 we obtain

(a) $((\exists \vec{f} \in \gamma_k\vec{\mathcal{G}})([\vec{f}] \models \Lambda) \Leftrightarrow (\exists \vec{f} \in \gamma_{mm}\vec{\mathcal{G}})([\vec{f}] \models \Lambda))$, for all $\vec{\mathcal{G}} \in \mathcal{S}\mathbf{BD}_0$ and all $k \geq m$.

(b) $\mathsf{Th}_{m,\omega}[\vec{f}] \subseteq \mathsf{Th}_{m,\omega}\vec{\mathcal{G}}$, for all $\vec{f} \in \gamma_{mm}\vec{\mathcal{G}}$ and all $\vec{\mathcal{G}} \in \mathcal{S}\mathbf{BD}_0$.

(1) $\Leftrightarrow$ (2) follows from Lemma 26 and Lemma 29 and the fact that $[\vec{f}] \models \mathrm{dg}_k\vec{\mathcal{G}}$, for all $\vec{f} \in \gamma_k\vec{\mathcal{G}}$. (2) $\Rightarrow$ (3) follows immediately from (a). (3) $\Rightarrow$ (2). Suppose that there is a $\vec{\mathcal{G}} = \langle \mathcal{G}_i : 1 \leq i \leq r\rangle \in \mathcal{S}\mathbf{BD}_0$ and a $k \geq m$ such that $[\vec{f}] \not\models \Lambda$, for all $\vec{f} \in \gamma_k\vec{\mathcal{G}}$. By (a), $[\vec{f}] \not\models \Lambda$, for all $\vec{f} \in \gamma_{mm}\vec{\mathcal{G}}$. Put, for $\mathcal{G} \in \mathbf{BD}_0$,

$$g_{mn}\mathcal{G} = \sigma_{mm}\mathcal{G} \cup t^n_m\mathcal{G}.$$

Now the proof is basically a combination of the proofs of Theorems 14 and 28. There is a bijection $\pi : \gamma_{mm}\mathcal{G} \to g_{mn}\mathcal{G}$ such that $\mathsf{Th}_{m,n}f = \mathsf{Th}_{m,n}\pi(f)$, for all $f \in \gamma_{mm}\mathcal{G}$. Hence, by (b), we may assume that there is an infinite $\mathcal{G}_i$ such that for all $\mathcal{H} \in g_{mn}\mathcal{G}$,

$$\mathcal{H}_1 \lhd \mathcal{H} \lhd \mathcal{H}_2 \not\models \Lambda,$$

for $\mathcal{H}_1 = (\mathcal{G}_1 \lhd \ldots \lhd \mathcal{G}_{i-1})$ and $\mathcal{H}_2 = (\mathcal{G}_{i+1} \lhd \ldots \lhd \mathcal{G}_r)$. By Proposition 7 there exist $\vec{\mathcal{F}}_1$, $\vec{\mathcal{F}}_2 \in \mathcal{S}\mathbf{BD}_0$ with $\mathrm{R}(\vec{\mathcal{F}}_1) + \mathrm{R}(\vec{\mathcal{F}}_2) \leq 4m|\mathbf{H}|$, $\mathrm{Cl}(\vec{\mathcal{F}}_j) \leq n$, $\mathrm{S}(\vec{\mathcal{F}}_j) \leq m+2$, $\mathrm{IR}(\vec{\mathcal{F}}_j) \leq m+1$, for $j \in \{1,2\}$ such that

$$[\vec{\mathcal{F}}_1] \lhd \mathcal{G}_i \lhd [\vec{\mathcal{F}}_2] \models \Phi \text{ and } [\vec{\mathcal{F}}_1] \lhd \mathcal{H} \lhd [\vec{\mathcal{F}}_2] \not\models \Phi,$$

for all $\mathcal{H} \in \mathbf{H} := g_{mn}\mathcal{G}_i$. Again we may assume that $\mathrm{Cl}(\mathcal{G}_i) \leq n$ and $\mathrm{S}(\mathcal{G}_i) \leq m+2$. Thus $\langle \vec{\mathcal{F}}_1, \mathcal{G}_i, \vec{\mathcal{F}}_2 \rangle$ refutes (3). $\dashv$

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