Fusions of Modal Logics Revisited

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Abstract

The fusion $L_l \otimes L_r$ of two normal modal logics formulated in languages with disjoint sets of modal operators is the smallest normal modal logic containing $L_l \cup L_r$. This paper proves that decidability, interpolation, uniform interpolation, and Halldén-completeness are preserved under forming fusions of normal polyadic polymodal logics. Those problems remained open in [Fine & Schurz [3]] and [Kracht & Wolter [10]]. The paper defines the fusion $\vdash_l \otimes \vdash_r$ of two classical modal consequence relations and proves that decidability transfers also in this case. Finally, these results are used to prove a general decidability result for modal logics based on superintuitionistic logics.

Given two logical system L_1 and L_2 it is natural to ask whether the fusion (or join) $L_1 \otimes L_2$ of them inherits the common properties of both L_1 and L_2. Let us consider some examples: (i) It is known that the first order theory of one equivalence relation has the finite model property and is decidable. However, the first order theory of two equivalence relations does not have the finite model property and is in fact undecidable (see Janiczak [7]). This result shows that even if we know the first order properties of the individual relations of a theory, there may be no algorithm to determine the purely logical consequences of these properties. (ii) Various positive and negative results are known for joins of term rewriting systems (TRSs) whose vocabularies are disjoint. For example, the join of two TRSs is confluent iff the two TRSs are confluent but there are complete TRSs whose join is not complete (see e.g. Klop [8]). In fact, the literature on TRSs shows how useful the study of joins of systems can be. (iii) In contrast to first order theories the join of two decidable equational theories in disjoint languages is decidable as well. This was proved by Pigozzi in [12]. So we observe interesting differences between logical systems by investigating the behavior of joins.

To form the join of two modal logics (in languages with disjoint sets of modal operators) is – in a sense – a generalization of forming the join of two equational theories in disjoint languages. Namely, it is well-known that each modal logic corresponds to an equational theory of boolean algebras with operators. So the join of two modal logics corresponds to the join of equational theories of boolean algebras with operators. However, in this case the equational theories are not in disjoint languages since we have common symbols: the boolean operations conjunction and negation.

The first who discussed fusions of modal logics was S. Thomason. In [15] he proved that fusions of modal logics are conservative extensions of their unimodal fragments by using the fact that two countably infinite atomless boolean algebras are isomorphic (cf. e.g. [9]). About ten years later the transfer of properties under forming fusions was investigated in detail in [3], [10], and [14]. Some results where also obtained in [6] and [4]. All those papers, however, are technically based on Kripke semantics and prove general results only for logics which are complete with respect to Kripke frames. For instance, transfer of decidability and interpolation for Kripke-complete logics was proved independently in [3] and [10], but the method did not give any positive result for incomplete logics. In this paper we shall combine Thomason's use of the $\aleph_0$-categoricity of atomless boolean algebras with some techniques introduced in [10] to prove that decidability, interpolation and uniform interpolation in the sense of Pitts [13] transfer in general. The paper does not use Kripke semantics but only algebraic methods.

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1 Syntax

A *modal similarity type* $S = \langle F, \rho \rangle$ consists of a set F of modal operators and a map $\rho : F \to \omega$ assigning to each $\mathsf{f} \in F$ a finite arity $\rho(\mathsf{f}) \in \omega$. The propositional modal language $\mathcal{L}(S)$ over S is defined in the usual way by using countably many propositional variables, the operators in F and the boolean connectives $\wedge, \vee, \to, \leftrightarrow, \perp, \top$.

Let us fix a modal similarity type $S = \langle F, \rho \rangle$. A *S-consequence relation* $\vdash$ is a finitary structural consequence relation[1] over $\mathcal{L}(S)$ with the following properties.

- $\vdash \varphi$, for each classical tautology φ.
- $p, p \to q \vdash q$.

[1]Recall that a finitary structural consequence relation $\vdash$ over a propositional language $\mathcal{L}$ is a relation $\vdash$ between sets of formulas and formulas satisfying

- $\varphi \in \Gamma \Rightarrow \Gamma \vdash \varphi$.
- If $\Gamma_1 \subseteq \Gamma_2$ and $\Gamma_1 \vdash \varphi$, then $\Gamma_2 \vdash \varphi$.
- If $\Gamma_1 \vdash \Gamma_2$ and $\Gamma_2 \vdash \varphi$, then $\Gamma_1 \vdash \varphi$.
- $\Gamma \vdash \varphi \Rightarrow s\Gamma \vdash s\varphi$, for all substitutions s.
- If $\Gamma \vdash \varphi$, then there exist a finite subset Γ' of Γ with $\Gamma' \vdash \varphi$.

- For all $\mathsf{f} \in F$,

$$p_1 \leftrightarrow q_1, \ldots, p_{\rho(\mathsf{f})} \leftrightarrow q_{\rho(\mathsf{f})} \vdash \mathsf{f}(p_1, \ldots, p_{\rho(\mathsf{f})}) \leftrightarrow \mathsf{f}(p_1, \ldots, p_{\rho(\mathsf{f})}).$$

We call a subset Λ of $\mathcal{L}(S)$ a *S-logic* iff there exists a S-consequence relation $\vdash$ such that $\Lambda = \Lambda(\vdash)$, where

$$\Lambda(\vdash) = \{\varphi \in \mathcal{L}(S) : \ \emptyset \vdash \varphi\}.$$

Conversely, denote by $\mathrm{cons}(\Lambda)$ the set of all S-consequence relations $\vdash$ satisfying $\Lambda = \Lambda(\vdash)$. It is known that $\mathrm{cons}(\Lambda)$ contains a smallest and a largest consequence relation: the consequence relation $\vdash_\Lambda$ which is defined by $\Gamma \vdash_\Lambda \varphi$ iff φ is derivable from $\Gamma \cup \Lambda$ by using the rules above and the consequence relation $\vdash^a_\Lambda$ determined by the set of all Λ-admissible rules, i.e. rules $\varphi_1, \ldots, \varphi_k/\psi$ under which Λ is closed.

Call a S-logic Λ *normal* if the following holds for all $\mathsf{f} \in F$ and $1 \leq i \leq \rho(\mathsf{f})$:

- $\mathsf{f}(p_1, \ldots, p_i \vee q_i, \ldots, p_{\rho(\mathsf{f})}) \leftrightarrow \mathsf{f}(p_1, \ldots, p_i, \ldots, p_{\rho(\mathsf{f})}) \vee \mathsf{f}(p_1, \ldots, q_i, \ldots, p_{\rho(\mathsf{f})}) \in \Lambda$.
- $\neg\mathsf{f}(p_1, \ldots, p_{i-1}, \bot, p_{i+1}, \ldots, p_{\rho(\mathsf{f})}) \in \Lambda$.

This definition is a natural generalization of the well-known notion of a normal modal logic when all modal operators are unary. Let us now consider two disjoint modal similarity types S_l and S_r and take the language $\mathcal{L} = \mathcal{L}(S_l \cup S_r)$. Then the *fusion*

$$\vdash_l \otimes \vdash_r$$

of a S_l-consequence relation $\vdash_l$ and a S_r-consequence relation $\vdash_r$ is the smallest $S_l \cup S_r$-consequence relation containing $\vdash_l \cup \vdash_r$. Correspondingly, the *fusion*

$$\Lambda_l \otimes \Lambda_r$$

of a S_l-logic Λ_l and a S_r-logic Λ_r is the smallest $S_l \cup S_r$-logic containing $\Lambda_l \cup \Lambda_r$. Here l abbreviates *the logic on the left* and r abbreviates *the logic on the right*. In what follows we shall assume that the similarity types $S_l = \langle E, l\rangle$ and $S_r = \langle G, r\rangle$ are fixed and disjoint and that $\vdash_l$, Λ_l, $\vdash_r$ and Λ_r are formulated in $\mathcal{L}(S_l)$ and $\mathcal{L}(S_r)$, respectively. Fusions of consequence relations and fusions of logics are connected as follows.

Theorem 1 *(1) For all $\vdash_l$ and $\vdash_r$,*

$$\Lambda(\vdash_l \otimes \vdash_r) = \Lambda(\vdash_l) \otimes \Lambda(\vdash_r).$$

(2) For all modal logics Λ_l and Λ_r,

$$\Lambda_l \otimes \Lambda_r = \Lambda(\vdash^a_{\Lambda_l} \otimes \vdash^a_{\Lambda_r}).$$

Certainly (2) is a consequence of (1) and obviously $\Lambda(\vdash_l \otimes \vdash_r) \supseteq \Lambda(\vdash_l) \otimes \Lambda(\vdash_r)$. The proof of the other inclusion is not so easy and will be delivered in the section *Decidability of the consequence relation.*

2 Semantics

S-consequence relations (and S-logics) are interpreted in *S-algebras*, i.e., algebras

$$\mathcal{A} = \langle A, \vee, -, \top, \bot, \langle \mathsf{f}^{\mathcal{A}} : \mathsf{f} \in F \rangle \rangle$$

in which the *boolean reduct* $\langle A, \vee, -, \top, \bot \rangle$ is a boolean algebra and the $\mathsf{f}^{\mathcal{A}}$, $\mathsf{f} \in F$, are functions of arity $\rho(\mathsf{f})$. A *valuation* v in $\mathcal{A}$ is a homomorphism from the algebra of formulas $\mathcal{L}(S)$ into $\mathcal{A}$. Quite often we shall specify a valuation v only for a certain set V of propositional variables. In all those cases it is assumed that v is defined arbitrarily but fixed for all the variables not in V. With $\mathcal{A}$ we associate a (not always finitary) consequence relation $\models_{\mathcal{A}}$ defined by

$$\Gamma \models_{\mathcal{A}} \varphi \Leftrightarrow (\forall v)(v(\Gamma) \subseteq \{\top\} \Rightarrow v(\varphi) = \top).$$

Define for a class of S-algebras $\mathbf{Q}$ the consequence relation

$$\models_{\mathbf{Q}} = \bigcap \{\models_{\mathcal{A}} : \mathcal{A} \in \mathbf{Q}\}$$

and call it the *consequence relation determined by* $\mathbf{Q}$. Correspondingly define

$$\mathsf{Log}\mathcal{A} = \{\varphi : \emptyset \models_{\mathcal{A}} \varphi\} \text{ and } \mathsf{Log}\mathbf{Q} = \bigcap \{\mathsf{Log}\mathcal{A} : \mathcal{A} \in \mathbf{Q}\},$$

and call $\mathsf{Log}\mathbf{Q}$ the *logic determined by* $\mathbf{Q}$. Conversely, we put

$$\mathsf{Alg} \vdash = \{\mathcal{A} : \vdash_{\mathcal{A}} \supseteq \vdash\}.$$

Members of $\mathsf{Alg} \vdash$ are called $\vdash$ - algebras. For a S-logic Λ we put

$$\mathsf{Alg}\Lambda = \{\mathcal{A} : \mathcal{A} \models \Lambda\},$$

where $\mathcal{A} \models \Lambda$ abbreviates $\models_{\mathcal{A}} \varphi$, for all $\varphi \in \Lambda$. The following result is well-known and easy to prove. (For information on varieties and quasivarieties consult e.g. [11].)

Theorem 2 *For each class* $\mathbf{Q}$ *of S-algebras the following conditions are equivalent.*

1. *There exists a S-consequence relation* $\vdash$ *with* $\mathbf{Q} = \mathsf{Alg} \vdash$.
2. $\mathbf{Q}$ *is a quasivariety.*

Also the following conditions are equivalent.

1. *There exists a S-logic* Λ *such that* $\mathbf{Q} = \mathsf{Alg}\Lambda$.
2. $\mathbf{Q}$ *is a variety.*

Call an element a of a boolean algebra $\mathcal{A}$ an *atom* if $a \neq \bot$ and

$$\{x \in A : x \leq a\} = \{\bot, a\}.$$

$\mathcal{A}$ is called *atomless* iff $\mathcal{A}$ contains no atoms. In what follows we shall call a S-algebra $\mathcal{A}$ a *c.i.a. algebra* iff the boolean reduct of $\mathcal{A}$ is a countably infinite atomless boolean algebra. Denote the class of c.i.a. algebras in $\mathsf{Alg}\vdash$ by $\mathsf{Atg}\vdash$ and denote the class of c.i.a. algebras in $\mathsf{Alg}\Lambda$ by $\mathsf{Atg}\Lambda$. The following result states that S-consequences are determined by c.i.a. algebras.

Theorem 3 (1) *For each S-consequence relation* $\vdash$,

$$\vdash \; = \; \models_{\mathsf{Alg}\vdash} \; = \; \models_{\mathsf{Atg}\vdash} .$$

(2) *For each S-logic* Λ,

$$\Lambda = \mathsf{LogAlg}\Lambda = \mathsf{LogAtg}\Lambda.$$

Proof. (1) It suffices to show that for $\Gamma \not\vdash \varphi$ there exists an $\mathcal{A} \in \mathsf{Atg}\vdash$ such that $\Gamma \not\models_{\mathcal{A}} \varphi$. Certainly we may assume that for each formula ψ there is a propositional variable p which is not in Γ and not in ψ. Define a congruence relation on the algebra of formulas $\mathcal{L}(S)$ by putting

$$\chi_1 \sim \chi_2 \Leftrightarrow \Gamma \vdash \chi_1 \leftrightarrow \chi_2$$

Now it is well-known (and easy to check) that $\mathcal{A} = \mathcal{L}(S)/{\sim} \in \mathsf{Alg}\vdash$ and that $\mathcal{A}$ refutes $\Gamma \vdash \varphi$. It remains to show that $\mathcal{A}$ is countably infinite and atomless. Clearly $\mathcal{A}$ is countably infinite whenever it is atomless. So it suffices to show that $\mathcal{A}$ is atomless. Denote by $[\chi]$ the equivalence class containing χ. Let $\psi \in \mathcal{L}(S)$ with $[\psi] \neq \bot$ and take a propositional variable p which is not in Γ and not in ψ. It follows immediately that

$$\bot < [\psi] \wedge [p] < [\psi].$$

So $[\psi]$ is not an atom of $\mathcal{A}$ and we conclude that $\mathcal{A}$ is atomless. (2) is proved similarly. $\dashv$

Suppose that $\mathbf{Q}_l$ is a class of S_l-algebras and $\mathbf{Q}_r$ is a class of S_r-algebras. The *fusion*

$$\mathbf{Q}_l \otimes \mathbf{Q}_r$$

of $\mathbf{Q}_l$ and $\mathbf{Q}_r$ is the class of those $S_l \cup S_r$-algebras whose S_l-reducts are in $\mathbf{Q}_l$ and whose S_r-reducts are in $\mathbf{Q}_r$. The following model theoretic characterization of fusions follows immediately from Theorem 3.

Theorem 4 *(1) For all consistent* $\vdash_l$ *and* $\vdash_r$,

$$\mathsf{Alg}(\vdash_l) \otimes \mathsf{Alg}(\vdash_r) = \mathsf{Alg}(\vdash_l \otimes \vdash_r).$$

(2) For all consistent Λ_l *and* Λ_r,

$$\mathsf{Alg}(\Lambda_l) \otimes \mathsf{Alg}(\Lambda_r) = \mathsf{Alg}(\Lambda_l \otimes \Lambda_r).$$

3 Basic results

We first recall some basic facts on atomless boolean algebras. (See e.g. [9] for more details.) For a boolean algebra $\mathcal{A} = \langle A, \vee, -, \top, \bot \rangle$ and $a \in \mathcal{A}$ recall that the algebra

$$\mathcal{A}_a = \langle \{a \wedge b : b \in A\}, \vee^a, -^a, a, \bot \rangle$$

is a boolean algebra as well which is called the *relative* of $\mathcal{A}$. Here we put $b_1 \vee^a b_2 = b_1 \vee b_2$ and $-^a b = a \wedge -b$. A finite set $\{a_i : i \in I\}$ is called a *partition* of $\mathcal{A}$ iff it is *pairwise disjoint*, i.e. $a_i \wedge a_j \neq \bot$, for all $i, j \in I$, and $\bigvee \{a_i : i \in I\} = \top$. The following is well-known.

Lemma 5 *Suppose that $\{a_i : i \in I\}$ is a partition of $\mathcal{A}$. Then*

$$\sigma : \mathcal{A} \to \prod \langle \mathcal{A}_{a_i} : i \in I \rangle,$$

defined by $\sigma(a) = \langle a \wedge a_i : i \in I \rangle$, is a surjective isomorphism.

Clearly $b \leq a$ is an atom in $\mathcal{A}$ iff it is an atom in $\mathcal{A}_a$. So we have

Lemma 6 *If $\mathcal{A}$ is atomless, then $\mathcal{A}_a$ is atomless, for each $a \in A$ with $a \neq \bot$.*

In what follows we shall use the following conventions for mappings. If mappings $\sigma_i : A_i \to B_i$, $i \in I$, are given, then $\sigma = \prod \langle \sigma_i : i \in I \rangle$ denotes the mapping from $\prod \langle A_i : i \in I \rangle$ into $\prod \langle B_i : i \in I \rangle$ which is defined by putting $\sigma \langle a_i : i \in I \rangle = \langle \sigma_i(a_i) : i \in I \rangle$. For a sequence of valuations v_n in algebras $\mathcal{A}_n$, $n \in I$, however, we denote by $v = \prod \langle v_n : n \in I \rangle$ the valuation of $\prod \langle \mathcal{A}_n : n \in I \rangle$ which is defined by putting $v(p) = \langle v_n(p) : n \in I \rangle$.

Proposition 7 *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are c.i.a. boolean algebras and $\{a_i : i \in I\}$ and $\{b_i : i \in I\}$ are partitions of $\mathcal{A}$ and $\mathcal{B}$, respectively. Then there exists an isomorphism σ from $\mathcal{A}$ onto $\mathcal{B}$ such that*

$$\sigma(a_i) = b_i,$$

for all $i \in I$.

Proof. By Lemma 6, the algebras $\mathcal{A}_{a_i}$, $\mathcal{B}_{b_i}$, $i \in I$, are c.i.a.. Hence there are isomorphisms σ_i from $\mathcal{A}_{a_i}$ onto $\mathcal{B}_{b_i}$. We get an isomorphism

$$\sigma = \prod \langle \sigma_i : i \in I \rangle : \prod \langle \mathcal{A}_{a_i} : i \in I \rangle \to \prod \langle \mathcal{B}_{b_i} : i \in I \rangle$$

By Lemma 5, σ is as required. $\dashv$

The second part of the following theorem was already proved in [15].

Theorem 8 *(1) For all consistent $\vdash_l$ and $\vdash_r$ the fusion $\vdash_l \otimes \vdash_r$ is a conservative extension of both $\vdash_l$ and $\vdash_r$. (2) For all consistent Λ_l and Λ_r the fusion $\Lambda_l \otimes \Lambda_r$ is a conservative extension of both Λ_l and Λ_r.*

Proof. (1) Put $\vdash = \vdash_l \otimes \vdash_r$. Suppose that $\Gamma; \varphi \subseteq \mathcal{L}(S_l)$ and $\Gamma \nvdash_l \varphi$. By Theorem 3 there exists

$$\mathcal{A} = \langle A, \vee, -, \top, \bot, \langle \mathsf{f}^{\mathcal{A}} : \mathsf{f} \in E \rangle \rangle \in \mathsf{Atg} \vdash_l$$

and a valuation v in $\mathcal{A}$ such that $v(\Gamma) \subseteq \{\top\}$ and $v(\varphi) \neq \top$. Take any

$$\mathcal{B} = \langle B, \vee^{\mathcal{B}}, -^{\mathcal{B}}, \top^{\mathcal{B}}, \bot^{\mathcal{B}}, \langle \mathsf{g}^{\mathcal{B}} : \mathsf{g} \in G \rangle \rangle \in \mathsf{Atg} \vdash_r .$$

Such an algebra exists since $\vdash_r$ is consistent. Now the boolean reducts of $\mathcal{A}$ and $\mathcal{B}$ are isomorphic. Hence we may assume that $A = B$ and that the boolean operations of $\mathcal{A}$ and $\mathcal{B}$ coincide, i.e. $\vee = \vee^{\mathcal{B}}$ and $- \ = \ -^{\mathcal{B}}$. But then

$$\mathcal{D} = \langle A, \vee, -, \top, \bot, \langle \mathsf{f}^{\mathcal{A}} : \mathsf{f} \in E \rangle, \langle \mathsf{g}^{\mathcal{B}} : \mathsf{g} \in G \rangle \rangle \in \mathsf{Alg} \vdash$$

refutes $\Gamma \vdash \varphi$. (2) is proved similarly. $\dashv$

4 Decidability of the consequence relation

For each formula φ of the form $\mathsf{t}(\varphi_1, \dots, \varphi_{\rho(\mathsf{t})}) \in \mathcal{L}(S_l \cup S_r)$, $\mathsf{t} \in E \cup G$, we reserve a new variable q_φ which will be called the *surrogate* of φ. We assume that the surrogate variables are different from our original set of variables. If $\varphi \in \mathcal{L}(S_l \cup S_r)$ then $\mathrm{var}^p(\varphi)$ denotes the set of variables in φ which are not surrogates. For a formula φ without surrogate variables denote by $\varphi^l \in \mathcal{L}(S_l)$ the formula which results from φ when all occurences of formulas $\mathsf{g}(\varphi_1, \dots, \varphi_{\rho(\mathsf{g})})$, $\mathsf{g} \in G$, which are not within the scope of a $\mathsf{g} \in G$ are replaced by their surrogate variable $q_{\mathsf{g}(\varphi_1, \dots, \varphi_{\rho(\mathsf{g})})}$. For a set Γ of formulas put $\Gamma^l = \{\varphi^l : \varphi \in \Gamma\}$ and define ϕ^r as well as Γ^r correspondingly. For instance, if S_l consists of two operators f_1 and f_2 of arity 1 and S_r consists of one operator g of arity 1, then

$$(\mathsf{g}p \wedge \mathsf{f}_1 p \wedge \mathsf{g}(p \wedge \mathsf{f}_2 p))^l = q_{\mathsf{g}p} \wedge \mathsf{f}_1 p \wedge q_{\mathsf{g}(p \wedge \mathsf{f}_2 p)}.$$

$$(\mathsf{g}p \wedge \mathsf{f}_1 p \wedge \mathsf{g}(p \wedge \mathsf{f}_2 p))^r = \mathsf{g}p \wedge q_{\mathsf{f}_1 p} \wedge \mathsf{g}(p \wedge q_{\mathsf{f}_2 p}).$$

Denote by $\mathrm{sf}(\Gamma)$ the set of subformulas of formulas in Γ and by $\mathrm{sf}^{\,l}(\Gamma)$ the set of variables of formulas in Γ as well as all subformulas of $\mathsf{g}(\varphi_1, \dots \varphi_{r(\mathsf{g})}) \in \mathrm{sf}(\Gamma)$, $\mathsf{g} \in G$. Formally we can define

$$\mathrm{sf}^{\,l}(\Gamma) = \mathrm{sf}\{\psi : q_\psi \in \mathrm{var}(\Gamma^l)\} \cup \mathrm{var}^p(\Gamma).$$

Define $\mathrm{sf}^{\,r}(\Gamma)$ correspondingly. Suppose now that $\Gamma \subseteq \mathcal{L}(S_l \cup S_r)$ is a finite set of formulas closed under subformulas. Define the *consistency-set* of Γ by

$$\mathcal{C}(\Gamma) = \{\psi_c : c \subseteq \Gamma\},$$

where for $c \subseteq \Gamma$,

$$\psi_c = \bigwedge \langle \chi : \chi \in c \rangle \wedge \bigwedge \langle \neg \chi : \chi \in \Gamma - c \rangle.$$

We abbreviate $\mathcal{C}l(\Gamma) = \mathcal{C}(\mathrm{sf}^{\,l}(\Gamma))$ and $\mathcal{C}r(\Gamma) = \mathcal{C}(\mathrm{sf}^{\,r}(\Gamma))$.

Theorem 9 *Suppose that $\vdash_l$ and $\vdash_r$ are consequence relations in S_l and S_r, respectivel, and that $\varphi, \psi \in \mathcal{L}(S_l \cup S_r)$. Put $\vdash = \vdash_l \otimes \vdash_r$. The following conditions are equivalent.*

1. *$\varphi \nvdash \psi$.*
2. *There exists $D \subseteq \mathcal{C}l(\{\varphi, \psi\})$ such that*
$$\varphi^l, (\bigvee D)^l \nvdash_l \psi^l \tag{1}$$
and, for all $\chi \in D$,
$$\varphi^l, (\bigvee D)^l \nvdash_l \neg\chi^l \text{ and } (\bigvee D)^r \nvdash_r \neg\chi^r \tag{2}$$
3. *There exists $D \subseteq \mathcal{C}r(\{\varphi, \psi\})$ such that $\varphi^r, (\bigvee D)^r \nvdash_r \psi^r$ and, for all $\chi \in D$,*
$$\varphi^r, (\bigvee D)^r \nvdash_r \neg\chi^r \text{ and } (\bigvee D)^l \nvdash_l \neg\chi^l.$$

If D satisfies 2., then $\varphi, \bigvee D \nvdash \psi$ and $\bigvee D \nvdash \neg\chi$, for all $\chi \in D$.

Proof. 2. $\Rightarrow$ 1. Take a $D \subseteq \mathcal{C}l(\{\phi, \psi\})$ satisfying (1) and (2). By Theorem 3 for each $\chi \in D \cup \{\neg\psi\}$ there exists a $\mathcal{A}_\chi \in \mathsf{Atg} \vdash_l$ and a valuation v_χ in $\mathcal{A}_\chi$ such that
$$v_\chi((\bigvee D)^l \wedge \varphi^l) = \top \text{ and } v_\chi(\chi^l) > \bot.$$
Put
$$\mathcal{A} = \prod \langle \mathcal{A}_\chi : \chi \in D \cup \{\neg\psi\}\rangle$$
and define a valuation v in $\mathcal{A}$ by
$$v^l = \prod \langle v_\chi : \chi \in D \cup \{\neg\psi\}\rangle.$$
We have $\mathcal{A} \in \mathsf{Atg} \vdash_l$, $v^l(\varphi^l) = \top$, $v^l(\psi^l) \neq \top$ and the set
$$\{v^l(\chi^l) : \chi \in D\}$$
is a partition of $\mathcal{A}$. On the other hand we get in a similar way from $(\bigvee D)^r \nvdash_r \neg\chi^r$, for all $\chi \in D$, a $\mathcal{B} \in \mathsf{Atg} \vdash_r$ with a valuation v^r such that the set
$$\{v^r(\chi^r) : \chi \in D\}$$
is a partition of $\mathcal{B}$. By Proposition 7 there exists an isomorphism σ from the boolean reduct of $\mathcal{B}$ onto the boolean reduct of $\mathcal{A}$ such that
$$\sigma(v^r(\chi^r)) = v^l(\chi^l),$$
for all $\chi \in D$. Hence we may assume, by identifying B with A, that we have an algebra
$$\mathcal{D} = \langle A, \vee, -, \top, \bot, \langle \mathsf{f}^{\mathcal{D}} : \mathsf{f} \in E\rangle, \langle \mathsf{g}^{\mathcal{D}} : \mathsf{g} \in G\rangle\rangle \in \mathsf{Alg} \vdash$$

with two valuation v^l and v^r satisfying

$$v^r(\chi^r) = v^l(\chi^l),$$

for all $\chi \in D$, and such that v^l still has the properties mentioned above. By using the properties of $\mathcal{C}l(\{\varphi,\psi\})$ it is now easily shown that for all $\alpha \in \mathrm{sf}^{\,l}(\{\phi,\psi\})$,

$$\begin{aligned} v^l(\alpha^l) &= \bigvee\langle v^l(\chi^l) : \chi \in D,\ \alpha \text{ is a conjunct of } \chi\rangle \\ &= \bigvee\langle v^r(\chi^r) : \chi \in D,\ \alpha \text{ is a conjunct of } \chi\rangle \\ &= v^r(\alpha^r). \end{aligned}$$

We define a new valuation v in $\mathcal{D}$ by putting, for all non-surrogate variables p in φ, ψ,

$$v(p) = v^l(p)\ (= v^r(p)).$$

Claim. $v(\alpha) = v^l(\alpha^l)\ (= v^r(\alpha^r))$, for all $\alpha \in \mathrm{sf}^{\,l}(\{\phi,\psi\})$.

The proof of this claim, which is almost trivial now, is by induction on the subformulas of α. The interesting steps are for $\alpha = \mathsf{f}(\varphi_1,\ldots,\varphi_{\rho(\mathsf{f})})$ and $\alpha = \mathsf{g}(\varphi_1,\ldots,\varphi_{\rho(\mathsf{g})})$, for $\mathsf{f} \in E$ and $\mathsf{g} \in G$. Let us consider f. Then

$$\begin{aligned} v(\mathsf{f}(\varphi_1,\ldots,\varphi_{\rho(\mathsf{f})})) &= \mathsf{f}^{\mathcal{D}}(v(\varphi_1),\ldots,v(\varphi_{\rho(\mathsf{f})})) \\ &= \mathsf{f}^{\mathcal{D}}(v^l(\varphi_1^l),\ldots,v^l(\varphi_{\rho(\mathsf{f})}^l)) \\ &= v^l(\mathsf{f}(\varphi_1,\ldots,\varphi_{\rho(\mathsf{f})})). \end{aligned}$$

The case $\alpha = \mathsf{g}(\varphi_1,\ldots,\varphi_{\rho(\mathsf{g})})$ is dual by using the induction hypothesis for v^r.

Clearly, by the Claim above, $v(\varphi) = v^l(\varphi^l) = \top$ and $v(\psi) = v^l(\psi^l) \neq \top$. Hence $\phi \not\vdash \psi$, as required.

1. $\Rightarrow$ 2. Suppose that $\varphi \not\vdash \psi$. There is an $\mathcal{A} \in \mathsf{Alg}\vdash$ with a valuation v such that $v(\varphi) = \top$ and $v(\psi) \neq \top$. Put

$$D = \{\chi \in \mathrm{sf}^{\,l}(\{\varphi,\psi\}) : v(\chi) > \bot\}.$$

Certainly D is as required in 2.

The equivalence of 1. and 3. can be proved in the same way. ⊣

Corollary 10 *Suppose that $\vdash_l$ and $\vdash_r$ are consistent. Then $\vdash_l \otimes \vdash_r$ is decidable iff both $\vdash_r$ and $\vdash_l$ are decidable.*

Proof. Put $\vdash\ = \vdash_l \otimes \vdash_r$. One direction follows immediately from the fact that $\vdash$ is a conservative extension of both $\vdash_l$ and $\vdash_r$. The other direction follows from Theorem 9 and the observation that $\mathcal{C}l(\{\varphi,\psi\})$ is finite. ⊣

Corollary 11 *Suppose that $\mathbf{Q}_l$ and $\mathbf{Q}_r$ are nontrivial quasivarieties. Then the set of quasi-identities valid in $\mathbf{Q}_l \otimes \mathbf{Q}_r$ is recursive iff both the set of quasi-identies valid in $\mathbf{Q}_l$ and the set of quasi-identies valid in $\mathbf{Q}_r$ are recursive.*

Define for a consequence relation $\vdash$ in $\mathcal{L}(S_l \cup S_r)$ the following subsets of $\mathcal{C}l(\{\varphi\})$ and $\mathcal{C}r(\{\varphi\})$:

$$\Sigma_{l(\vdash)}(\varphi) = \{\chi : \chi \in \mathcal{C}l(\varphi),\ \emptyset \not\vdash \neg\chi\} \text{ and } \Sigma_{r(\vdash)}(\varphi) = \{\chi : \chi \in \mathcal{C}r(\varphi),\ \emptyset \not\vdash \neg\chi\}.$$

Correspondingly, put for a logic Λ,

$$\Sigma_{l(\Lambda)}(\varphi) = \Sigma_{l(\vdash_\Lambda)}(\varphi) \text{ and } \Sigma_{r(\Lambda)}(\varphi) = \Sigma_{r(\vdash_\Lambda)}(\varphi).$$

Notice that $\vdash \bigvee \Sigma_{l(\vdash)}(\varphi)$ and that $\bigvee \Sigma_{l(\vdash)}(\varphi) \not\vdash \neg\chi$, for all $\chi \in \Sigma_{l(\vdash)}(\varphi)$. Thus, if $\not\vdash \varphi$ then

$$(\bigvee \Sigma_{l(\vdash)}(\varphi))^l \not\vdash \neg\chi^l \text{ and } (\bigvee \Sigma_{l(\vdash)}(\varphi))^l \not\vdash \varphi^l \text{ and } (\bigvee \Sigma_{l(\vdash)}(\varphi))^r \not\vdash \varphi^r,$$

for all $\chi \in \Sigma_{l(\vdash)}(\varphi)$. So we obtain from Theorem 9 by putting $D = \Sigma_{l(\vdash)}(\varphi)$ the following

Corollary 12 *Suppose that $\vdash_l$ and $\vdash_r$ are consistent and $\varphi \in \mathcal{L}(S_l \cup S_r)$. Put $\vdash = \vdash_l \otimes \vdash_r$. Then the following conditions are equivalent.*

1. $\emptyset \vdash \varphi$.
2. $(\bigvee \Sigma_{l(\vdash)}(\varphi))^l \vdash_l \varphi^l$.
3. $(\bigvee \Sigma_{r(\vdash)}(\varphi))^r \vdash_r \varphi^r$.

We shall first use Corollary 12 to prove Theorem 1. In a certain sense all the formulas in $\Sigma_{r(\vdash)}(\varphi)$ or all the formulas in $\Sigma_{l(\vdash)}(\varphi)$ are less complex than φ itself so that the corollary above allows to prove results by induction on a measure of this complexity, namely the *alternation depth*. First define the *left-alternation-depth* of φ, $a^l(\varphi) \in \omega$. It is the length of a longest sequence $\langle \mathsf{g}_1, \mathsf{f}_1, \mathsf{g}_2, \mathsf{f}_2, \ldots \rangle$ such that

$$\mathsf{g}_1(\ldots \mathsf{f}_1(\ldots \mathsf{g}_2(\ldots (\mathsf{f}_2(\ldots\ldots))))$$

is in φ and $\mathsf{g}_1, \mathsf{g}_2, \ldots \in G$ and $\mathsf{f}_1, \mathsf{f}_2, \ldots \in E$. Correspondingly the *right-alternation-depth* $a^r(\varphi) \in \omega$ is the length of a longest sequence $\langle \mathsf{f}_1, \mathsf{g}_1, \mathsf{f}_2, \mathsf{g}_2, \ldots \rangle$ such that

$$\mathsf{f}_1(\ldots \mathsf{g}_1(\ldots \mathsf{f}_2(\ldots (\mathsf{g}_2(\ldots\ldots))))$$

is in φ and $\mathsf{g}_1, \mathsf{g}_2, \ldots \in G$ and $\mathsf{f}_1, \mathsf{f}_2, \ldots \in E$. The alternation depth of φ is $a(\varphi) = a^l(\varphi) + a^r(\varphi)$. The following Lemma is easily shown.

Lemma 13 *For all $\varphi \in \mathcal{L}(S_l \cup S_r)$ which contain a modal operator,*

$$a(\varphi) > a(\chi),\ \textit{for all } \chi \in \mathcal{C}l(\varphi),$$

or

$$a(\varphi) > a(\chi),\ \textit{for all } \chi \in \mathcal{C}r(\varphi).$$

Proof of Theorem 1. It remains to show the following implication: $\varphi \in \Lambda(\vdash_l \otimes \vdash_r) \Rightarrow \varphi \in \Lambda(\vdash_l) \otimes \Lambda(\vdash_r)$. The proof is by induction on $a(\varphi)$. The case $a(\varphi) = 0$ is trivial. Now suppose that $a(\varphi) = n > 0$ and that the implication holds for all χ with $a(\chi) < n$. Then, by Lemma 13, we may assume w.l.o.g. that $a(\varphi) > a(\chi)$, for all $\chi \in \mathcal{C}l(\varphi)$. So, by induction hypothesis,

$$\Sigma_{l(\vdash_l \otimes \vdash_r)}(\varphi) = \Sigma_{l(\Lambda(\vdash_l)\otimes\Lambda(\vdash_r))}(\varphi).$$

Hence

$$(\bigvee \Sigma_{l(\vdash_l \otimes \vdash_r)}(\varphi))^l \vdash_l \varphi^l \Leftrightarrow (\bigvee \Sigma_{l(\Lambda(\vdash_l)\otimes\Lambda(\vdash_r))}(\varphi))^l \vdash_l \varphi^l$$

and we conclude (with Corollary 12) that $\varphi \in \Lambda(\vdash_l \otimes \vdash_r)$ iff $\varphi \in \Lambda(\vdash_l) \otimes \Lambda(\vdash_r)$. $\dashv$

5 Decidability of normal logics

This section deals with the problem whether $L_l \otimes L_r$ is decidable whenever both L_l and L_r are decidable. Here we found an answer in the positive only for normal S-logics, the question of transfer of decidability for subnormal logics remains open. So in this section we assume that all logics and algebras are normal. Corollary 12 shows already the way we choose to prove transfer of decidability.

The smallest normal S-logic is denoted by $\mathbf{K}(S)$. For each $\mathsf{f} \in F$ define a new modal operator $\mathsf{f}_\Box$ by putting

$$\mathsf{f}_\Box(p_1, \ldots, p_{\rho(\mathsf{f})}) = \bigwedge\langle \neg\mathsf{f}(\top, \ldots, \neg p_i, \ldots, \top) : 1 \le i \le \rho(\mathsf{f})\rangle.$$

Then we have for all $\mathsf{f} \in F$

- $\mathsf{f}_\Box(p_1 \wedge q_1, \ldots, p_{\rho(\mathsf{f})} \wedge q_{\rho(\mathsf{f})}) \leftrightarrow (\mathsf{f}_\Box(p_1, \ldots, p_{\rho(\mathsf{f})}) \wedge \mathsf{f}_\Box(q_1, \ldots, q_{\rho(\mathsf{f})})) \in \mathbf{K}(S)$.
- $\mathsf{f}_\Box(\top, \ldots, \top) \in \mathbf{K}(S)$.

Define, for each $\varphi \in \mathcal{L}(S)$, the formula

$$\Box_S\varphi = \bigwedge\langle \mathsf{f}_\Box(\varphi, \ldots, \varphi) : \mathsf{f} \in F\rangle$$

and put inductively

$$\Box_S^{\le 0}\varphi = \varphi \text{ and } \Box_S^{\le m+1}\varphi = \Box_S^{\le m}\varphi \wedge \Box_S^{m+1}\varphi.$$

The crucial and easily proved property of $\Box_S$ is stated in the following

Lemma 14 *For all normal S-algebras $\mathcal{A}$ and all $\mathsf{f} \in F$ and $c, a_1, b_1 \ldots, a_{\rho(\mathsf{f})}, b_{\rho(\mathsf{f})} \in A$, if*

$$a_i \wedge c = b_i \wedge c,$$

for all $1 \le i \le \rho(\mathsf{f})$, then

$$\mathsf{f}^{\mathcal{A}}(a_1, \ldots, a_{\rho(\mathsf{f})}) \wedge \Box_S^{\le 1} c = \mathsf{f}^{\mathcal{A}}(b_1, \ldots, b_{\rho(\mathsf{f})}) \wedge \Box_S^{\le 1} c.$$

One can also show the following deduction theorem.

Proposition 15 *Suppose that Λ is a normal S-logic. Then*

$$\varphi \vdash_\Lambda \psi \Leftrightarrow (\exists m \in \omega)(\Box_S^{\leq m}\varphi \to \psi \in \Lambda).$$

The proof is left to the reader, however, since we shall not use this proposition. But it motivates our next steps. For suppose that Λ_r and Λ_l are decidable and assume that we have an algorithm for deciding $\alpha \in \Lambda_l \otimes \Lambda_r$, for all α with $a(\alpha) < n$. Take a φ with $a(\varphi) = n$. By Lemma 13 we may assume w.l.o.g. that $a(\chi) < a(\varphi)$, for all $\chi \in \mathcal{C}l(\varphi)$. Hence we can construct $\Sigma_l(\varphi)$. Now, if we can effectively determine $m \in \omega$ such that

$$(\bigvee \Sigma_r(\varphi))^l \vdash_{\Lambda_l} \varphi^l \Leftrightarrow \Box_{S_l}^{\leq m}(\bigvee \Sigma_l(\varphi))^l \to \varphi^l \in \Lambda_l,$$

then we can decide whether $\varphi \in \Lambda_r \otimes \Lambda_l$, by Corollary 12. Here and in what follows $\Sigma_l(\varphi) = \Sigma_{l(\Lambda_l \otimes \Lambda_r)}(\varphi)$ and $\Sigma_r(\varphi) = \Sigma_{r(\Lambda_l \otimes \Lambda_r)}(\varphi)$. It will turn out that the *left-depth* and the *right-depth* of φ, in symbols $d^l(\varphi)$ and $d^r(\varphi)$, are as required for m. Define inductively

$$\begin{array}{rcl}
d^l(p) & = & 0 \\
d^l(\varphi \wedge \psi) & = & \max\{d^l(\varphi), d^l(\psi)\} \\
d^l(\neg\varphi) & = & d^l(\varphi) \\
d^l(\mathsf{f}(\varphi_1, \ldots, \varphi_{l(\mathsf{f})})) & = & \max\{d^l(\varphi_1), \ldots, d^l(\varphi_{l(\mathsf{f})})\} + 1, \text{ for } \mathsf{f} \in F \\
d^l(\mathsf{g}(\varphi_1, \ldots, \varphi_{\rho(\mathsf{g})})) & = & \max\{d^l(\varphi_1), \ldots, d^l(\varphi_{\rho(\mathsf{g})})\}, \text{ for } \mathsf{g} \in E
\end{array}$$

$d^r(\varphi)$ is defined correspondingly.

Theorem 16 *Suppose that Λ_l and Λ_r are consistent normal logics. Let $\varphi \in \mathcal{L}(S_l \cup S_r)$ and put $m = d^l(\varphi)$, $n = d^r(\varphi)$. Then the following conditions are equivalent.*

1. *$\varphi \in \Lambda_l \otimes \Lambda_r$.*
2. *$\Box_{S_l}^{\leq m}(\bigvee \Sigma_l(\varphi))^l \to \varphi^l \in \Lambda_l$.*
3. *$\Box_{S_r}^{\leq n}(\bigvee \Sigma_r(\varphi))^r \to \varphi^r \in \Lambda_r$.*

In what follows we shall write $\Box_l$ and $\Box_r$ for $\Box_{S_l}$ and $\Box_{S_r}$, respectively. For the proof of Theorem 16 we shall need two Lemmas.

Lemma 17 *Suppose that $\Box_l^{\leq m}(\bigvee \Sigma_l(\varphi))^l \to \varphi^l \notin \Lambda_l$. Then there exists $\mathcal{A} \in \mathsf{Atg}\Lambda_l$ such that there is a valuation v^l and a sequence $\langle a_n : 0 \leq n \leq m\rangle$ satisfying*

(a1) $a_m \leq \Box_l^{\leq 1} a_{m-1} \leq \Box_l^{\leq 1} a_{m-2} \leq \ldots \leq \Box_l^{\leq 1} a_0$.

(a2) $a_n \leq v(\Box_l^{\leq n}(\bigvee \Sigma_l(\varphi))^l)$, *for all* $0 \leq n \leq m$.

(a3) $a_m \wedge v(\neg\varphi^l) \neq \bot$.

(a4) *The set* $\{v^l(\chi^l) \wedge a_m : \chi \in \Sigma_l(\varphi)\}$ *is a partition of* $\mathcal{A}_{a_m}$.

(a5) *The sets* $\{v^l(\chi^l) \wedge (a_n - a_{n+1}) : \chi \in \Sigma_l(\varphi)\}$ *are partitions of* $\mathcal{A}_{(a_n - a_{n+1})}$, *for all* $n < m$.

Proof. Certainly there is a $\mathcal{B} \in \mathsf{Atg}\Lambda_l$ and a valuation v such that

$$v(\neg\varphi^l \wedge \Box_l^{\leq m}(\bigvee \Sigma_l(\varphi))^l) > \bot.$$

We put for $0 \leq n \leq m$,

$$b_n = v(\Box_l^{\leq n}(\bigvee \Sigma_l(\varphi))^l).$$

Now take for each $n \leq m$ an algebra $\mathcal{A}_n \in \mathsf{Atg}\Lambda_l$ with a valuation w_n such that

$$\{w_n(\chi^l) : \chi \in \Sigma_l(\varphi)\}$$

is a partition of $\mathcal{A}_n$. Put

$$\mathcal{A} = \mathcal{B} \times \prod\langle \mathcal{A}_n : n \leq m\rangle,\ v^l = v \times \prod\langle w_n : n \leq m\rangle$$

and put for $0 \leq n \leq m$,

$$a_n = \langle b_n, \underbrace{\overbrace{\bot, \ldots, \bot}^{n}, \top, \ldots, \top}_{m+1}\rangle.$$

We show that $\langle a_n : 0 \leq n \leq m\rangle$ and v^l are as required. (a1) follows from

$$a_n \wedge \Box_l a_n = \langle b_{n+1}, \underbrace{\overbrace{\bot, \ldots, \bot}^{n}, \top, \ldots, \top}_{m+1}\rangle.$$

(a2) follows from $v^l(\Box^{\leq n}(\bigvee \Sigma_l(\varphi)^l)) = \langle b_n, \top, \ldots, \top\rangle$ and (a3) follows from

$$a_m \wedge v^l(\neg\varphi^l) = \langle b_m \wedge v(\neg\varphi^l), \bot, \ldots, \bot, \top \wedge w_m(\neg\varphi^l)\rangle$$

and $b_m \wedge v(\neg\varphi^l) > \bot$.

(a4) $v^l(\chi_1^l) \wedge a_m$ and $v^l(\chi_2^l) \wedge a_m$ are disjoint for different χ_1 and χ_2, by the definition of $\Sigma_l(\varphi)$. By the definition of a_m we have $v^l(\chi^l) \wedge a_m > \bot$, for all $\chi \in \Sigma_l(\varphi)$, and it is also clear that $\bigvee\{v^l(\chi) \wedge a_m : \chi \in \Sigma_l\} = a_m$. (a5) is proved similarly to (a4) and is left to the reader. ⊣

Lemma 18 *Suppose that* $\Box_l^{\leq m}(\bigvee \Sigma_l(\varphi))^l \to \varphi^l \notin \Lambda_l$. *Then there exists* $\mathcal{A} \in \mathsf{Atg}\Lambda_l \otimes \Lambda_r$ *such that there are valuations* v^l *and* v^r *and a sequence* $\langle a_n : 0 \leq n \leq m\rangle$ *satisfying the conditions (a1), ..., (a5) and*

(a6) $v^l(\chi^l) \wedge a_0 = v^r(\chi^r) \wedge a_0$, *for all* $\chi \in \Sigma_l(\varphi)$.

(a7) For all $\mathsf{g} \in G$, *all* $0 \leq n \leq m$, *and all* $b_1, \dots, b_{r(\mathsf{g})} \in A$,

$$\mathsf{g}^{\mathcal{A}}(b_1, \dots, b_{r(\mathsf{g})}) \wedge a_n = \mathsf{g}^{\mathcal{A}}(b_1 \wedge a_n, \dots, b_{r(\mathsf{g})} \wedge a_n).$$

Proof. For each $n \leq m$ take an $\mathcal{A}_n \in \mathsf{Atg}\Lambda_r$ and a valuation v_n such that

$$\{v_n(\chi^r) : \chi \in \Sigma_l(\varphi)\}$$

is a partition of $\mathcal{A}_n$. Also take an arbitrary $\mathcal{A}_{-1} \in \mathsf{Atg}\Lambda_r$ and an arbitrary valuation v_{-1} of $\mathcal{A}_{-1}$. Define a valuation v^r of the product $\mathcal{B} = \prod\langle \mathcal{A}_n : -1 \leq n \leq m\rangle$ by putting

$$v^r = \prod\langle v_n : -1 \leq n \leq m\rangle.$$

Take an $\mathcal{A} \in \mathsf{Atg}\Lambda_l$ with a valuation v^l and a sequence $\langle a_n : 0 \leq n \leq m\rangle$ so that the conditions (a1), ..., (a5) are satisfied. Put $a_{-1} = \top^{\mathcal{A}} - a_0$ and assume w.l.o.g. that $a_{-1} \neq \bot$. Now there are surjective boolean isomorphisms

$$\sigma_m : \mathcal{A}_m \to \mathcal{A}_{a_m}$$

$$\sigma_n : \mathcal{A}_n \to \mathcal{A}_{a_n - a_{n+1}},$$

such that

$$\sigma_m(v^r(\chi^r)) = v^l(\chi^l) \wedge a_m \text{ and } \sigma_n(v^r(\chi^r)) = v^l(\chi^l) \wedge (a_n - a_{n+1}).$$

for all $n < m$ and all $\chi \in \Sigma_l(\varphi)$. Take also an arbitrary boolean isomorphism σ_{-1} from $\mathcal{A}_{-1}$ onto $\mathcal{A}_{a_{-1}}$. We get a boolean isomorphism σ from $\mathcal{B}$ onto $\mathcal{A}$ by putting

$$\sigma = \prod\langle \sigma_n : -1 \leq n \leq m\rangle : \prod\langle \mathcal{A}_n : -1 \leq n \leq m\rangle \to \mathcal{A}.$$

Using this isomorphism we can identify B with A in the obvious way and get the required algebra $\mathcal{D}$. (a6) is satisfied by the properties of σ. (a7) follows from the fact that the S_r-reduct of $\mathcal{D}$ is isomorphic to a product of S_r-algebras based on the relative boolean algebras $\mathcal{D}_{a_n}$ and $\mathcal{D}_{\top - a_n}$, for all $0 \leq n \leq m$. $\dashv$

Proof of Theorem 16.

1. $\Rightarrow$ 2. Suppose that $\Box_l^{\leq m}(\bigvee \Sigma_l(\varphi))^l \to \varphi^l \notin \Lambda_l$. Take $\mathcal{A} \in \mathsf{Atg}\Lambda_l \otimes \Lambda_r$ and valuations v^l and v^r and a sequence $\langle a_n : 0 \leq n \leq m\rangle$ satisfying the conditions (a1), ..., (a7).

By using the properties of $\Sigma_l(\{\varphi\})$ and (a4), (a5), (a6) it is easily shown that for all $\alpha \in \mathrm{sf}^{\,l}(\{\phi\})$

$$\begin{aligned} a_0 \wedge v^l(\alpha^l) &= a_0 \wedge \bigvee\langle v^l(\chi^l) : \chi \in \Sigma_l(\varphi),\ \alpha \text{ is a conjunct of } \chi\rangle \\ &= a_0 \wedge \bigvee\langle v^r(\chi^r) : \chi \in \Sigma_l(\varphi),\ \alpha \text{ is a conjunct of } \chi\rangle \\ &= a_0 \wedge v^r(\alpha^r). \end{aligned}$$

We define a new valuation v in $\mathcal{A}$ by putting for all variables p in φ,

$$v(p) = v^l(p).$$

Claim. For $0 \leq k \leq m$ and all $\alpha \in \mathrm{sf}(\{\varphi\})$ such that $d^l(\alpha) \leq k$:

$$a_k \wedge v(\alpha) = a_k \wedge v^l(\alpha^l).$$

The proof of this claim is by induction on k.

Suppose that $k = 0$. The proof is by induction on the subformulas of α for $d^l(\alpha) = 0$. For propositional variables this follows from the definition. The boolean steps are trivial. So, assume that $\alpha = \mathsf{g}(\varphi_1, \ldots, \varphi_{r(\mathsf{g})})$, for a $\mathsf{g} \in G$. Notice that there do not occur $\mathsf{f} \in E$ in φ_i since $d^l(\alpha) = 0$. Hence $\alpha = \alpha^r$ and the equality $v(\alpha) \wedge a_0 = v^r(\alpha^r) \wedge a_0$ follows immediately. Hence $v(\alpha) \wedge a_0 = v^l(\alpha^l) \wedge a_0$ since $v^r(\alpha^r) \wedge a_0 = v^l(\alpha^l) \wedge a_0$.

From k to $k+1$. The proof is again by induction on the subformulas of α for $d^l(\alpha) \leq k+1$. The interesting steps are for

$$\alpha = \mathsf{f}(\varphi_1, \ldots, \varphi_{\rho(\mathsf{f})}) \text{ and } \alpha = \mathsf{g}(\varphi_1, \ldots, \varphi_{\rho(\mathsf{g})})$$

for $\mathsf{f} \in E$ and $\mathsf{g} \in G$. Let us first consider f. By induction hypothesis, we have $a_k \wedge v(\varphi_i) = a_k \wedge v^l(\varphi_i^l)$, for all $1 \leq i \leq \rho(\mathsf{f})$. Hence, by Lemma 14,

$$\begin{aligned}
\Box_l^{\leq 1} a_k \wedge v(\alpha) &= \Box_l^{\leq 1} a_k \wedge \mathsf{f}^{\mathcal{A}}(v(\varphi_1), \ldots, v(\varphi_{\rho(\mathsf{f})})) \\
&= \Box_l^{\leq 1} a_k \wedge \mathsf{f}^{\mathcal{A}}(v^l(\varphi_1^l), \ldots, v^l(\varphi_{\rho(\mathsf{f})}^l)) \\
&= \Box_l^{\leq 1} a_k \wedge v^l(\alpha^l)
\end{aligned}$$

Now $a_{k+1} \wedge v(\alpha) = a_{k+1} \wedge v^l(\alpha^l)$ follows from $a_{k+1} \leq \Box_l^{\leq 1} a_k$, i.e. condition (a1).

Assume $\alpha = \mathsf{g}(\varphi_1, \ldots, \varphi_{r(\mathsf{g})})$. We know, by induction hypothesis, $a_{k+1} \wedge v(\varphi_i) = a_{k+1} \wedge v^r(\varphi_i^r)$. Hence

$$\mathsf{g}^{\mathcal{A}}(a_{k+1} \wedge v(\varphi_1), \ldots, a_{k+1} \wedge v(\varphi_{\rho(\mathsf{g})})) = \mathsf{g}^{\mathcal{A}}((a_{k+1} \wedge v^r(\varphi_1^r), \ldots, a_{k+1} \wedge v^r(\varphi_{\rho(\mathsf{g})}^r)).$$

Using condition (a7) we conclude

$$a_{k+1} \wedge \mathsf{g}^{\mathcal{A}}(v(\varphi_1), \ldots, v(\varphi_{\rho(\mathsf{g})})) = a_{k+1} \wedge \mathsf{g}^{\mathcal{A}}(v^r(\varphi_1^r), \ldots, v^r(\varphi_{\rho(\mathsf{g})}^r)),$$

which gives $a_{k+1} \wedge v(\alpha) = a_{k+1} \wedge v^r(\alpha^r)$ and we conclude $a_{k+1} \wedge v(\alpha) = a_{k+1} \wedge v^l(\alpha^l)$ since $a_{k+1} \wedge v^l(\alpha^l) = a_{k+1} \wedge v^r(\alpha^r)$.

2. $\Rightarrow$ 1. is clear.

3. $\Leftrightarrow$ 1. can be proved similarly. $\dashv$

We get the following Corollary, as explained above.

Corollary 19 *Suppose that Λ_r and Λ_l are consistent normal logics. Then $\Lambda_l \otimes \Lambda_r$ is decidable iff both Λ_l and Λ_r are decidable.*

By $\mathbf{var}\phi$ ($\mathbf{var}\Gamma$) we denote the set of variables in φ (in formulas in Γ). Recall that a logic Λ has the *interpolation property* iff for all $\varphi \to \psi \in \Lambda$ there exists a formula χ in $\mathbf{var}\varphi \cap \mathbf{var}\psi$ such that both $\varphi \to \chi \in \Lambda$ and $\chi \to \psi \in \Lambda$. Λ is Halldén-complete iff $\varphi \vee \psi \in \Lambda$ implies $\varphi \in \Lambda$ or $\psi \in \Lambda$ whenever φ and ψ have no common propositional variables.

Corollary 20 *(i) Suppose that Λ_r and Λ_l are normal modal logics with the interpolation property. Then $\Lambda_l \otimes \Lambda_r$ has the interpolation property. (ii) Suppose that Λ_r and Λ_l are normal modal logics which are Halldén-complete. Then $\Lambda_l \otimes \Lambda_r$ is Halldén-complete.*

Proof. The proof uses Theorem 16 in the same way as this was done in [10] for Kripke-complete logics. $\dashv$

We say that a modal logic Λ has *uniform interpolation* if for any formula φ and variables $\vec{q} = \{q_1, \ldots, q_k\}$ there exists a *uniform interpolant* $\exists\vec{q}\varphi$ for φ, i.e.,

- $\varphi \to \exists\vec{q}\varphi \in \Lambda$,
- $\mathbf{var}\exists\vec{q}\varphi \subseteq \mathbf{var}\varphi - \vec{q}$,
- $\exists\vec{q}\varphi \to \psi \in \Lambda$ whenever $\varphi \to \psi \in \Lambda$ and $\mathbf{var}\psi \cap \vec{q} = \emptyset$.

Pitts [13] proved that intuitionistic propositional logic has uniform interpolation. [5] and [16] prove that **K**, provability logic **GL** and Grzegorzcyk's system **Grz** have uniform interpolation but that **S4** lacks it. It is easily proved that a normal modal logic Λ has uniform interpolation whenever it has interpolation and $\mathsf{Alg}\Lambda$ is locally finite, i.e. each finitely generated algebra in $\mathsf{Alg}\Lambda$ is finite. (Take as the uniform interpolant for φ the conjunction over all interpolants for $\varphi \to \psi$, $\varphi \to \psi \in \Lambda$.) Hence e.g. **S5** has uniform interpolation.

To prove that $\Lambda_l \otimes \Lambda_r$ has uniform interpolation whenever both Λ_l and Λ_r have uniform interpolation we require the following observations. Let

$$\nabla(\varphi, \psi) = \{\varphi_1 \to \neg\psi_1 : \varphi_1 \in \Sigma_l(\varphi), \psi_1 \in \Sigma_l(\psi), \varphi_1 \to \neg\psi_1 \in \Lambda_l \otimes \Lambda_r\}.$$

Then $\bigvee \Sigma_l(\varphi \to \psi)$ equals (modulo boolean transformations) $\bigvee \Sigma_l(\varphi) \wedge \bigvee \Sigma_l(\psi) \wedge \bigwedge \nabla(\varphi, \psi)$. Now the proof of Theorem 16 is easily extended to show the following, for any two formulas φ, ψ: $\varphi \to \psi \in \Lambda_l \otimes \Lambda_r$ if and only if, for $n_1 = d^l(\varphi)$ and $n_2 = d^l(\psi)$,

$$(\dagger) \quad (\Box_l^{\leq n_1}(\bigvee \Sigma_l(\varphi))^l \wedge \Box_l^{\leq n_2}(\bigvee \Sigma_l(\psi))^l \wedge \Box_l^{\leq n_1}(\bigwedge \nabla(\varphi, \psi))^l) \to (\varphi \to \psi)^l \in \Lambda_l.$$

Theorem 21 *$\Lambda = \Lambda_l \otimes \Lambda_r$ has uniform interpolation whenever both Λ_l and Λ_r have uniform interpolation.*

Proof. Let us fix $\vec{q} = \{q_1, \ldots, q_k\}$. We prove by induction on $a(\varphi)$ that there exists a uniform interpolant $\exists\vec{q}\varphi$ for φ. This is clear if φ contains no modal operators. Assume now that φ contains modal operators and that uniform interpolants exist for all χ with $a(\varphi) > a(\chi)$. We may assume that $a(\varphi) > a(\chi)$, for all $\chi \in \mathcal{C}l(\varphi)$, and take uniform interpolants $\exists\vec{q}\chi$ for those χ. Let, for $n_1 = d^l(\varphi)$,

$$Q = \varphi \wedge \Box_l^{\leq n_1} \bigvee \Sigma_l(\varphi) \wedge \Box_l^{\leq n_1} \bigwedge\{\chi \to \exists\vec{q}\chi : \chi \in \Sigma_l(\varphi)\},$$

$$\vec{r} = \vec{q} \cup \{q_\alpha : \alpha = \mathsf{g}(\chi_1, \ldots, \chi_{\rho(\mathsf{g})}) \in \ \mathrm{sf}(\varphi),\ \mathsf{g} \in G,\ \vec{q} \cap \mathbf{var}\alpha \neq \emptyset\}.$$

Λ_l has uniform interpolation. Thus we can take a uniform interpolant $\exists\vec{r}Q^l$ for Q^l in the logic Λ_l. Certainly there exists a (uniquely determined) formula $\exists\vec{q}\varphi$ such that

$$\exists\vec{r}Q^l = (\exists\vec{q}\varphi)^l.$$

Note that $\vec{q} \cap \mathbf{var}\exists\vec{q}\varphi = \emptyset$ by the definition of $\vec{r}$. We show that $\exists\vec{q}\varphi$ is a uniform interpolant for φ. We have $Q^l \to \exists\vec{r}Q^l \in \Lambda_l$. Thus $Q \to \exists\vec{q}\varphi \in \Lambda$ and so $\varphi \to \exists\vec{q}\varphi \in \Lambda$ since $Q \leftrightarrow \varphi \in \Lambda$. Assume now that $\varphi \to \psi \in \Lambda$ and $\mathbf{var}\psi \cap \vec{q} = \emptyset$. We show $\exists\vec{q}\varphi \to \psi \in \Lambda$. It follows from (†) that, for $n_2 = d^l(\psi)$, $Q^l \to R^l \in \Lambda_l$, where

$$R = (\Box_l^{\leq n_2} \bigvee \Sigma_l(\psi) \wedge \Box_l^{\leq n_1} \bigwedge\{\exists\vec{q}\varphi_1 \to \neg\psi_1 : \varphi_1 \to \neg\psi_1 \in \nabla(\varphi, \psi)\}) \to \psi.$$

So $\exists\vec{r}Q^l \to R^l \in \Lambda_l$ since $\vec{r} \cap \mathbf{var}R^l = \emptyset$. But then $\exists\vec{q}\varphi \to R \in \Lambda$ and so $\exists\vec{q}\varphi \to \psi \in \Lambda$ since $\psi \leftrightarrow R \in \Lambda$. $\dashv$

It follows that e.g. $\mathbf{K}(S)$ and $\mathbf{S5} \otimes \mathbf{S5}$ have uniform interpolation. (That $\mathbf{K}(S)$ has uniform interpolation was first proved in [1].)

6 An Application

Recall that a superintuitionistic logic L is a subset of the propositional language $\mathcal{L}_I$ with connectives $\to$, $\wedge$, $\vee$, $\top$, $\bot$ which contains intuitionistic logic and is closed under modus ponens and substitutions. Benote by **Int** intuitionistic logic and denote by $\mathbf{Int} + \Gamma$ the smallest superintuitionistic logic containing Γ. Denote by $\mathcal{L}_\Box$ the language $\mathcal{L}$ with a new connective $\Box$. A *superintuitionistic modal logic* is a subset of $\mathcal{L}_\Box$ which contains **Int**,

$$\Box(p \wedge q) \leftrightarrow \Box p \wedge \Box q \text{ and } \Box\top$$

and which is closed under modus ponens, substitutions, and $p \to q / \Box p \to \Box q$. For information on superintuitionistic modal logics of this type consult e.g. [17]. Denote by **IntK** the smallest superintuitionistic modal logic and denote by $\Lambda \oplus \Gamma$ the smallest superintuitionistic modal logic containing Λ and Γ. We are going to prove the following extension of [[17], Theorem 13].

Theorem 22 *Suppose that a superintuitionistic logic* $\mathbf{Int}+\Gamma$ *is decidable. Then the logics* $\mathbf{IntK} \oplus \Gamma$ *and* $\mathbf{IntK} \oplus \Gamma \oplus \Box p \to p$ *are decidable as well.*

We give a sketch of the proof only, since it is similar to the proof of [[17], Theorem 13]. In [17] the Gödel translation of intuitionistic formulas into modal formulas is extended to a translation t of $\mathcal{L}_\Box$ into the bimodal language with operators $\Box_I$ and $\Box_M$. Namely, t is defined inductively by putting

$$\begin{array}{rcl} t(p) & = & \Box_I p, \\ t(\bot) & = & \Box_I \bot, \\ t(\varphi \circ \psi) & = & \Box_I(t(\varphi) \circ t(\psi)), \text{ for } \circ \in \{\wedge, \vee, \rightarrow\}, \\ t(\Box\varphi) & = & \Box_I \Box_M t(\varphi). \end{array}$$

Denote for a normal modal logic Λ and a set of formulas Γ by $\Lambda \oplus \Gamma$ the smallest normal modal logic containing Λ and Γ. A normal bimodal logic Λ is called a BM-companion of a superintuitionistic modal logic L if

$$L = \{\varphi \in \mathcal{L}_\Box : t(\varphi) \in \Lambda\}.$$

Clearly L is decidable whenever a BM-companion of L is decidable. Hence it suffices to show that all the logics defined in the theorem have decidable BM-companions. It is proved in [17] for all $\Gamma \subseteq \mathcal{L}_I$

- $(\mathbf{S4} \oplus t(\Gamma)) \otimes \mathbf{K}$ is a BM-companion of $\mathbf{IntK} \oplus \Gamma$.
- $(\mathbf{S4} \oplus t(\Gamma)) \otimes (\mathbf{K} \oplus \Box_M p \rightarrow p)$ is a BM-companion of $\mathbf{IntK} \oplus \Gamma \oplus \Box p \rightarrow p$.

Hence it suffices to prove that $(\mathbf{S4} \oplus t(\Gamma)) \otimes \mathbf{K}$ as well as $(\mathbf{S4} \oplus t(\Gamma)) \otimes (\mathbf{K} \oplus \Box_M p \rightarrow p)$ are decidable whenever $\mathbf{Int} + \Gamma$ is decidable. But it is shown in [18] that $\mathbf{S4} \oplus t(\Gamma)$ is decidable whenever $\mathbf{Int} + \Gamma$ is decidable and both $\mathbf{K}$ as well as $\mathbf{K} \oplus \Box p \rightarrow p$ are known to be decidable. Hence all the fusions are decidable.

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