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All finitely axiomatizable tense logics of linear time flows are coNP-complete

Abstract. We prove that all finitely axiomatizable tense logics with temporal operators for 'always in the future' and 'always in the past' and determined by linear flows time are coNP-complete. It follows, for example, that all tense logics containing a density axiom of the form $\Box_F^{n+1}p \to \Box_F^n p$, for some $n \geq 0$, are coNP-complete. Additionally, we prove coNP-completeness of all $\bigcap$-irreducible tense logics. As these classes of tense logics contain many Kripke incomplete bimodal logics, we obtain many natural examples of Kripke incomplete normal bimodal logics which are nevertheless coNP-complete.

Keywords: NP-completeness, tense logic, temporal logic, computational complexity, frame incompleteness

1. Introduction

Temporal logics in the languages with operators $\Box_F$ for 'always in the future' and $\Box_P$ for 'always in the past' have been investigated in Philosophical Logic, Linguistics, and Computer Science, see for example [8, 5, 4]. One of the most important features of this language is its low computational complexity: validity is known to be coNP-complete for the tense logics determined by the class of all linear orderings, the natural numbers, the rationals and the reals. (With the exception of the natural numbers, those logics have the polynomial finite model property, see [7, 5, 11]. The first explicit proof of the coNP-completeness of the tense logic of the natural numbers seems to be in [3]. It is based on [9].) The aim of this note is to prove that the low computational complexity does not depend on any particular property (density, discreteness, etc) of the flow of time nor the finite model property or even Kripke completeness of the logic under consideration. In fact, we prove that every finitely axiomatizable and every $\bigcap$-irreducible tense logic extending the tense logic **Lin** of arbitrary linear orders is coNP-complete. It follows from general results of [11] regarding the finite axiomatizability of tense logics that this result covers, for example, all tense logics containing a density axiom of the form $\Box_F^{n+1}p \to \Box_F^n p$, for some $n \geq 0$, and therefore all tense logics of reflexive flows of time (i.e. logics in which $\Box_F$ and $\Box_P$ refer to

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the current moment as well) and all logics determined by dense strict linear orders like the rational numbers or real numbers. More examples can be found in [11]. Notice, however, that there are *non-finitely* axiomatizable tense logics of very high complexity since every tense logic determined by a class of frames of the form

$$\{\langle\{0,\ldots,m\},<\rangle \mid m \in M\},$$

where M is a set of natural numbers, is as complex as M itself (the argument will be provided below).

From the viewpoint of 'pure' modal logic the complexity result is of interest because it is the first general coNP-completeness result covering not only logics without the finite model property but also numerous Kripke-incomplete bimodal logics (actually, many of them do not even have any adequate lattice-complete BAOs, cf. [6]). Proofs of comparable general results on coNP-completeness by [10] (that all extensions of **S4.3** are coNP-complete) and [2, 1] (that all proper extensions of **S5** $\times$ **S5** are coNP-complete) were heavily based on the polynomial finite model property of all logics involved. The result also implies that all finitely axiomatizable unimodal logics containing **K4.3** are coNP-complete; the decidability of those logics has been proved in [12].

We briefly explain the idea of the proof. It is based on the result of [11] that all tense logics of linear time flows are determined by sequences of rather simple general tense frames (called atomic blocks). A typical example of an atomic block is the general frame consisting of an infinite sequence of irreflexive points followed by a reflexive point such that the set of possible values of propositional variables consists of all finite sets not containing the reflexive point and all cofinite sets containing the reflexive point. The logic determined by this frame does not have the finite model property. But it turns out that a formula φ is valid in this frame if and only if it is true in all models based on a finite frame (of polynomial size in the length of φ) consisting of a chain of irreflexive points followed by a reflexive point and in which no subformula of φ is true in exactly the reflexive point. The latter condition is easily seen to be decidable in coNP. Thus, although this logic does not have the polynomial finite model property it nevertheless is determined by models of polynomial size satisfying certain contraints. The main technical contribution of this paper shows that a similar argument applies to all sequences of atomic blocks.

2. Preliminaries

Formulas of propositional tense logic are built from propositional variables $p_1, p_2, \ldots$ using the boolean connectives $\wedge$, $\vee$, $\rightarrow$, and $\neg$ and the temporal operators $\Box_F$ ('always in the future'), $\Diamond_F$ ('eventually'), $\Box_P$ ('always in the past'), and $\Diamond_P$ ('at some moment in the past'). We interpret this language in *general tense frames* $\mathfrak{F} = \langle W, R, P\rangle$, where

- $\langle W, R\rangle$ is a linear ordering (i.e., R is transitive and connected:
$$\forall x \forall y(xRy \vee yRx \vee x = y))$$
- and P is a set of subsets of W closed under intersection, complement, and the operators
$$\Box_F^{\mathfrak{F}} X = \{x \in W \mid \forall y \in W(xRy \to y \in X)\}$$
and
$$\Box_P^{\mathfrak{F}} X = \{x \in W \mid \forall y \in W(xR^{-1}y \to y \in X)\}.$$

A valuation $\mathfrak{V}$ in $\mathfrak{F}$ maps propositional variables to elements of P. The satisfaction of a formula φ in a point $w \in W$ in the model $\mathfrak{M} = (\mathfrak{F}, \mathfrak{V})$, in symbols $(\mathfrak{M}, w) \models \varphi$, is defined as usual. φ is called *valid* in $\mathfrak{F}$ if it is true in every point of $\mathfrak{F}$ under every valuation. If $P = 2^W$, then we say that $\mathfrak{F}$ is *a tense frame* or *a Kripke frame* and write $\mathfrak{F} = \langle W, R\rangle$.

By **Lin** we denote the set of all formulas which are valid in all tense frames. A *tense logic* is a set of formulas containing **Lin** which is closed under modus ponens, substitution, and necessitation (from φ derive $\Box_P\varphi$ and $\Box_F\varphi$). A tense logic L is called *finitely axiomatizable* if there exists a formula φ such that L is the smallest tense logic containing φ; in this case we write $L = \mathbf{Lin} \oplus \varphi$.

All tense logics are determined by some class of general tense frames and, conversely, every class of general tense frames determines a tense logic. Well-known examples of tense logics are the logics determined by $\langle \mathbb{N}, <\rangle$, $\langle \mathbb{Q}, <\rangle$ and $\langle \mathbb{R}, <\rangle$. Call a tense logic L $\bigcap$-*irreducible* if, for any sequence $L_0, L_1 \ldots$ of tense logics it follows from $L = \bigcap_{i<\omega} L_i$ that $L = L_i$ for some $i < \omega$. Now we can formulate our main result:

THEOREM 2.1. *(i) All finitely axiomatizable tense logics are coNP-complete.*
(ii) All $\bigcap$-irreducible tense logics are coNP-complete.

Notice that there exist *non-finitely axiomatizable* tense logics of very high complexity: just consider, for any $M \subseteq \mathbb{N}$, the logic L_M determined by the class of frames
$$\{\langle \{1, \ldots, m\}, <\rangle \mid m \in M\}.$$
Set $\bot = p \wedge \neg p$ and $\top = p \vee \neg p$. Then the formula
$$\varphi_m = \Box_P \bot \wedge \Box_F^{m+1} \bot \wedge \Diamond_F^m \top$$
is satisfiable in a frame validating L_M iff $m + 1 \in M$.

3. Known results

We briefly remind the reader of the main results of [11] which are required for the proof of Theorem 2.1. They state that (a) every tense logic is axiomatizable using 'canonical' formulas and (b) every tense logic is determined by general frames which are composed from certain simple general frames. We will use the notation introduced in [13].

Given a finite sequence $\overline{\mathfrak{F}} = \langle \mathfrak{F}_i = \langle W_i, R_i, P_i \rangle : 1 \leq i \leq n \rangle$ of disjoint frames, we denote by $[\overline{\mathfrak{F}}] = \mathfrak{F}_1 \lhd \cdots \lhd \mathfrak{F}_n$ the ordered sum of them, i.e., the frame $\langle W, R, P \rangle$ in which

$$W = \bigcup_{i=1}^{n} W_i, \ R = \bigcup_{i=1}^{n} R_i \cup \bigcup_{1 \leq i < j \leq n} (W_i \times W_j)$$

and $P = \{X_1 \cup \cdots \cup X_n : X_i \in P_i\}$. Each finite frame can be represented then as the ordered sum $C_1 \lhd \cdots \lhd C_n$ of its clusters (i.e., sets of the form $\{w\} \cup \{v \mid wRv \text{ and } vRw\}$). A cluster is called *degenerate* iff it consists of a single reflexive point.

Fix two symbols m and j. A *cluster assignment* is a mapping $\mathbf{t} = (\mathbf{t}_1, \mathbf{t}_2)$ from the set of clusters of a finite frame into $\{\mathsf{m}, \mathsf{j}\} \times \{\mathsf{m}, \mathsf{j}\}$ such that every cluster consisting of an irreflexive point is mapped to (m, m) and no cluster consisting of more than one point is mapped to (j, j).

With every finite tense frame $\mathfrak{F} = \langle W, R \rangle = C_1 \lhd \cdots \lhd C_n$ with cluster assignment $\mathbf{t} = (\mathbf{t}_1, \mathbf{t}_2)$ we associate the formula[1]

$$\alpha(\mathfrak{F}, \mathbf{t}) = \neg \overline{\delta}(\mathfrak{F}, \mathbf{t}),$$

where

$$\overline{\delta}(\mathfrak{F}, \mathbf{t}) = \delta(\mathfrak{F}, \mathbf{t}) \wedge \Box_F \delta(\mathfrak{F}, \mathbf{t}) \wedge \Box_P \delta(\mathfrak{F}, \mathbf{t})),$$

and

$$\begin{aligned}
\delta(\mathfrak{F}, \mathbf{t}) \quad = \quad & \bigwedge \{p_x \to \neg p_y \mid x \neq y\} \wedge \\
& \bigwedge \{p_x \to \neg \Diamond_P p_y \mid \neg(yRx)\} \wedge \\
& \bigwedge \{p_x \to \Diamond_F p_y \mid \exists i \leq n \ (\mathbf{t}_1 C_i = \mathsf{m} \wedge x, y \in C_i \wedge xRy)\} \wedge \\
& \bigwedge \{p_x \to \Diamond_P p_y \mid \exists i \leq n \ (\mathbf{t}_2 C_i = \mathsf{m} \wedge x, y \in C_i \wedge xR^{-1}y)\} \wedge \\
& \bigvee \{p_y \mid y \in W\} \wedge \\
& \bigwedge \{\Diamond_F p_y \vee p_y \vee \Diamond_P p_y \mid y \in W\},
\end{aligned}$$

[1] Notice that these formulas are slightly different from the 'canonical' formulas presented for tense logics in [11] and [13]. It is not difficult to see, however, that they are satisfiability equivalent to each other. We have opted for this syntactic form since we believe it is more transparent.

where $p_x \neq p_y$ if $x \neq y$. To understand the meaning of these formulas, notice first that if $\mathbf{t}C = (\mathsf{m}, \mathsf{m})$ for all clusters C then $\mathfrak{G} \not\models \alpha(\mathfrak{F}, \mathbf{t})$ iff there exists a p-morphism from $\mathfrak{G}$ onto $\mathfrak{F}$. If $\mathbf{t}_i C = \mathsf{j}$ for some $i \in \{1, 2\}$ and some cluster C in $\mathfrak{F}$, then the formula $\alpha(\mathfrak{F}, \mathbf{t})$ can be refuted in frames of which $\mathfrak{F}$ is not necessarily a p-morphic image. In this case $\mathfrak{G} \not\models \alpha(\mathfrak{F}, \mathbf{t})$ iff there exist frames $\mathfrak{G}_i$, for $1 \leq i \leq n$, such that $\mathfrak{G} = \mathfrak{G}_1 \lhd \cdots \lhd \mathfrak{G}_n$ and $\mathfrak{G}_i \not\models \alpha(C_i, \mathbf{t} \restriction C_i)$ for all $1 \leq i \leq n$. So it suffices to consider $\mathfrak{G} \not\models \alpha(C, \mathbf{t})$ for a cluster C. Assume for simplicity that $\mathfrak{G}$ is a tense frame. *Case 1:* $\mathbf{t}C = (\mathsf{j}, \mathsf{j})$. Then $\mathfrak{G} \not\models \alpha(C, \mathbf{t})$ iff $|\mathfrak{G}| \geq |C|$. *Case 2:* $\mathbf{t}C = (\mathsf{m}, \mathsf{j})$. Then C is non-degenerate and $\mathfrak{G} \not\models \alpha(C, \mathbf{t})$ iff either $\mathfrak{G}$ contains an R-final cluster of cardinality $\geq |C|$ or it has no R-final point at all. *Case 3:* $\mathbf{t}C = (\mathsf{j}, \mathsf{m})$. This is the mirror image of Case 2. *Case 4:* $\mathbf{t}C = (\mathsf{m}, \mathsf{m})$. If C is an irreflexive point then $\mathfrak{G}$ is an irreflexive point as well whenever $\mathfrak{G} \not\models \alpha(C, \mathbf{t})$. If C is non-degenerate and $\mathfrak{G} \not\models \alpha(C, \mathbf{t})$ then $\mathfrak{G}$ satisfies the conditions of Cases 2 and 3.

The length $l(\mathfrak{F})$ of a finite frame $\mathfrak{F} = C_1 \lhd \cdots \lhd C_n$ is n. The cluster size $cl(\mathfrak{F})$ of $\mathfrak{F}$ is the size of the maximal cluster in $\mathfrak{F}$. For a formula φ, we denote by $sub(\varphi)$ the closure of the set of subformulas of φ under single negation. $l(\varphi)$ will denote the cardinality of the set $sub(\varphi)$.

The following result is a straightforward extension of Theorem 1 in [11].

THEOREM 3.1. *There is an algorithm which, given a formula φ, returns formulas*

$$\alpha(\mathfrak{F}_1, \mathbf{t}_1), \ldots, \alpha(\mathfrak{F}_n, \mathbf{t}_n)$$

such that $\mathbf{Lin} \oplus \varphi = \mathbf{Lin} \oplus \alpha(\mathfrak{F}_1, \mathbf{t}_1) \oplus \cdots \oplus \alpha(\mathfrak{F}_n, \mathbf{t}_n)$ *and* $l(\mathfrak{F}_i) \leq 2l(\varphi) + 1$ *and* $cl(\mathfrak{F}_i) \leq l(\varphi)$.

Axiomatizations of standard tense logics based on this Theorem can be found in [13]. Notice that in Theorem 1 of [11] the bound on the size of the clusters of the $\mathfrak{F}_i$ was exponential in the size of the number of variables of φ. It is straightforward to see, however, that in the proof of Theorem 1 in [11] on page 155 it is not necessary to add any points to a cluster which intersects with g_0. From this observation one obtains the upper bound provided here. Notice also that the length of φ in [11] is the number of subformulas of φ. In this paper it is the number of elements of $sub(\varphi)$ which contains also single negations of subformulas of φ.

We now introduce the building blocks of the frames required for a general completeness result for tense logics.

DEFINITION 3.2 (Atomic blocks). (1) Denote by ⓚ the non-degenerate cluster with $k > 0$ points, by $\bullet$ the irreflexive point and by $\circ$ (or ①) the reflexive point.

(2) Let $\omega^{<}(0)$ be the strictly ascending chain $\langle \omega, < \rangle$ of natural numbers, $\omega^{<}(1)$ the chain $\langle \omega, \leq \rangle$, $\omega^{<}(2)$ the ascending chain of natural numbers in which precisely the even points are reflexive, $\omega^{<}(3)$ the chain in which precisely the multiples of 3 are reflexive, and so on; $\omega^{>}(n)$ is the mirror image of $\omega^{<}(n)$.

(3) Let $\mathfrak{C}(0, \textcircled{1}) = \langle \omega^{<}(0) \lhd \textcircled{1}, P\rangle$, where P consists of all cofinite sets containing $\textcircled{1}$ and their complements. We generalize this construction to chains $\omega^{<}(n)$ and clusters $\textcircled{k}$. Namely, for $0 < n < \omega$, $k > 1$ and $\textcircled{k} = \{a_0, \ldots, a_{k-1}\}$, we put

$$\mathfrak{C}(n, \textcircled{k}) = \langle \omega^{<}(n) \lhd \textcircled{k}, P\rangle,$$

where P is generated using intersection and complementation from the set of all finite subsets of $\omega^{<}(n)$ together with the sets $\{X_i : 0 \leq i \leq k-1\}$, for $X_i = \{a_i\} \cup \{kj+i : j \in \omega\}$, $0 \leq i \leq k-1$.

We call the cluster $\textcircled{k}$ in $\mathfrak{C}(n, \textcircled{k})$ the (m, j)-cluster of $\mathfrak{C}(n, \textcircled{k})$.

(4) For $n < \omega$, let $\mathfrak{C}(\textcircled{k}, n)$ denote the mirror image of $\mathfrak{C}(n, \textcircled{k})$. We call the cluster $\textcircled{k}$ in $\mathfrak{C}(\textcircled{k}, n)$ the (j, m)-cluster of $\mathfrak{C}(\textcircled{k}, n)$.

(5) Let $\mathfrak{C}(0, \textcircled{1}, 0) = \langle \omega^{<}(0) \lhd \textcircled{1} \lhd \omega^{>}(0), P\rangle$, where P consists of all cofinite sets containing $\textcircled{1}$ and their complements.

We call the cluster $\textcircled{1}$ in $\mathfrak{C}(0, \textcircled{1}, 0)$ the (j, j)-cluster of $\mathfrak{C}(0, \textcircled{1}, 0)$.

Notice that logics determined by a frame of the form $\mathfrak{C}(n, \textcircled{k})$, $k \geq 2$, are Kripke incomplete, see [11]. By $\mathfrak{B}^*$ we denote the set of all finite sequences $\overline{\mathfrak{G}}$ of frames of the form (1) or (3)-(5) in Definition 3.2. Clusters in $[\overline{\mathfrak{G}}]$ which were called t-clusters in their blocks will be called t-clusters in $[\overline{\mathfrak{G}}]$ as well. Clusters C which are members of the set of possible values P of $[\overline{\mathfrak{G}}]$ will be called (m, m)-clusters.

The following result is Theorem 12 of [11].

THEOREM 3.3. *Every $\bigcap$-irreducible tense logic is determined by a frame of the form $[\overline{\mathfrak{G}}]$ for some $\overline{\mathfrak{G}} \in \mathfrak{B}^*$.*

Notice that to prove coNP-completeness of all $\bigcap$-irreducible tense logics it is thus sufficient to show that, given some $[\overline{\mathfrak{G}}]$ with $\overline{\mathfrak{G}} \in \mathfrak{B}^*$, it is decidable in non-deterministic polynomial time whether a formula φ is satisfiable in $[\overline{\mathfrak{G}}]$.

Given a sequence $\overline{\mathfrak{G}}$ in $\mathfrak{B}^*$, we denote by $w(\overline{\mathfrak{G}})$ the maximal n such that a frame of the form $\mathfrak{C}(n, \textcircled{k})$ or $\mathfrak{C}(\textcircled{k}, n)$ occurs in $\overline{\mathfrak{G}}$. If no such frame occurs in $\overline{\mathfrak{G}}$, then set $w(\overline{\mathfrak{G}}) = 0$.

The following completeness result is proved in [11], Theorem 14 (i) (modulo the reduction in the size of clusters discussed above):

THEOREM 3.4. *Suppose*

$$L = \mathbf{Lin} \oplus \alpha(\mathfrak{F}_1, \mathbf{t}_1) \oplus \cdots \oplus \alpha(\mathfrak{F}_n, \mathbf{t}_n)$$

and $\varphi \notin L$. Then $\neg\varphi$ is satisfiable in a frame $\mathfrak{G} = \mathfrak{G}_1 \lhd \cdots \lhd \mathfrak{G}_m \in [\mathcal{B}^*]$ such that

(a) $\mathfrak{G} \models L$;

(b) $m \leq (2l(\varphi) + 1) \times (2 + \max\{l(\mathfrak{F}_i) \mid 1 \leq i \leq n\})$;

(c) $w(\mathfrak{G}) \leq 2 + \max\{l(\mathfrak{F}_i) \mid 1 \leq i \leq n\}$;

(d) $cl(\mathfrak{G}) \leq l(\varphi)$.

Notice that we obtain coNP-completeness of L if we can show that

- it can be checked in non-deterministic polynomial time whether a formula φ is satisfiable in a frame $\mathfrak{G}$ satisfying constraints (b)-(d) of Theorem 3.4;
- it can be checked in polynomial time (in the length of φ) whether a frame $\mathfrak{G}$ satisfying the constraints (b)-(d) validates L.

CoNP-hardness follows obviously from the fact that all those logics contain classical propositional calculus.

4. NP-completeness

We first prove a lemma which will allow us to check in non-deterministic polynomial time whether a formula φ is satisfiable in a given frame $[\overline{\mathfrak{G}}]$, where $\overline{\mathfrak{G}} \in \mathfrak{B}^*$. The lemma will state that a formula φ is satisfiable in $[\overline{\mathfrak{G}}]$ iff it is satisfiable under a 'good' valuation in a finite subframe of $[\overline{\mathfrak{G}}]$ whose size is polynomial in φ.

Given a finite frame $\mathfrak{F} = C_1 \lhd \cdots \lhd C_n$, we write

$$((C_1, \mathbf{t}C_1) \lhd \cdots \lhd (C_n, \mathbf{t}C_n))$$

for $(\mathfrak{F}, \mathbf{t})$. In addition, if

$$(\mathfrak{F}, \mathbf{t}) = \overbrace{(\mathfrak{G}, \mathbf{t}') \lhd \cdots \lhd (\mathfrak{G}, \mathbf{t}')}^{n \text{ times}},$$

we will denote it by $(\mathfrak{F}, \mathbf{t}) = [(\mathfrak{G}, \mathbf{t}')]^n$.

Now define the m-reduct $r_m(\mathfrak{G})$ of $\overline{\mathfrak{G}} \in \mathcal{B}^*$ as follows.

Definition 4.1 (m-reduct, canonical embedding). (1) For $\mathfrak{F}$ a cluster, $r_m(\mathfrak{F}) = (\mathfrak{F}, \mathbf{t})$, where $\mathbf{t}$ assigns (m, m) to the cluster.

(2) $r_m(\mathfrak{C}(0, \textcircled{k})) = [(\bullet, (\mathsf{m}, \mathsf{m}))]^{m+1} \lhd (\textcircled{k}, (\mathsf{m}, \mathsf{j}))$. The embedding r_m from the underlying frame of $r_m(\mathfrak{C}(0, \textcircled{k}))$ to $\mathfrak{C}(0, \textcircled{k})$ which maps $[(\bullet, (\mathsf{m}, \mathsf{m}))]^{m+1}$ onto an initial segment of $\omega^{<}(0)$ and $\textcircled{k}$ onto $\textcircled{k}$ is called the *canonical embedding* of $r_m(\mathfrak{C}(0, \textcircled{k}))$ into $\mathfrak{C}(0, \textcircled{k})$.

(3) For $n > 0$, $r_m(\mathfrak{C}(n, \textcircled{k})) = [(\circ, (\mathsf{m}, \mathsf{m})) \lhd [(\bullet, (\mathsf{m}, \mathsf{m}))]^{n-1}]^{m+1} \lhd (\textcircled{k}, (\mathsf{m}, \mathsf{j}))$. The canonical embedding r_m from $r_m(\mathfrak{C}(n, \textcircled{k}))$ into $\mathfrak{C}(n, \textcircled{k})$ is defined as in (2) above.

(4) For $n \geq 0$, the m-reduct of $\mathfrak{C}(\textcircled{k}, n)$ as well as the canonical embedding r_m are the the mirror images of the definition under (2) and (3).

(5) $r_m(\mathfrak{C}(0, \textcircled{1}, 0)) = [(\bullet, (\mathsf{m}, \mathsf{m}))]^{m+1} \lhd (\circ, (\mathsf{j}, \mathsf{j})) \lhd [(\bullet, (\mathsf{m}, \mathsf{m}))]^{m+1}$. The canonical embedding r_m from the underlying frame of $r_m(\mathfrak{C}(0, \textcircled{1}, 0))$ to $\mathfrak{C}(0, \textcircled{1}, 0)$ maps the initial $[(\bullet, (\mathsf{m}, \mathsf{m}))]^{m+1}$ to the initial segment of $\mathfrak{C}(0, \textcircled{1}, 0)$, $\textcircled{1}$ to $\textcircled{1}$, and the final $[(\bullet, (\mathsf{m}, \mathsf{m}))]^{m+1}$ to the final segment of $\mathfrak{C}(0, \textcircled{1}, 0)$.

(6) For $\overline{\mathfrak{F}} = \langle \mathfrak{F}_1, \ldots, \mathfrak{F}_n \rangle \in \mathcal{B}^*$, the m-reduct is defined as follows:

$$r_m(\overline{\mathfrak{F}}) = r_m(\mathfrak{F}_1) \lhd \cdots \lhd r_m(\mathfrak{F}_m).$$

The *canonical embedding* r_m of $r_m(\overline{\mathfrak{F}})$ into $[\overline{\mathfrak{F}}]$ is defined componentwise.

DEFINITION 4.2 (Good valuation). Consider a formula φ and a finite frame $\mathfrak{F} = C_1 \lhd C_2 \cdots \lhd C_n$ with a type assignment $\mathbf{t} = (\mathbf{t}_1, \mathbf{t}_2)$. A valuation $\mathfrak{V}$ in $\mathfrak{F}$ is called *good for* φ and $\mathbf{t}$ iff the following hold for every $\psi \in sub(\varphi)$ and every cluster C_i of $\mathfrak{F}$:

- if $\mathbf{t}_1 C_i = \mathsf{j}$ and $C_i \cap \mathfrak{V}(\psi) \neq \emptyset$, then there exists C_j such that $j > i$ and $C_j \cap \mathfrak{V}(\psi) \neq \emptyset$;
- if $\mathbf{t}_2 C_i = \mathsf{j}$ and $C_i \cap \mathfrak{V}(\psi) \neq \emptyset$, then there exists C_j such that $j < i$ and $C_j \cap \mathfrak{V}(\psi) \neq \emptyset$.

If φ is of the form $\overline{\delta}(\mathfrak{F}, \mathbf{s})$ for some finite $\mathfrak{F}$, then $\mathfrak{V}$ is called *canonically good* for φ if those two conditions hold for every *propositional variable* in φ.

LEMMA 4.3. *(a) The following conditions are equivalent, for every ψ, $\overline{\mathfrak{G}} \in \mathfrak{B}^*$, and $m \geq 2l(\psi) + 1$:*

- *ψ is satisfiable in $[\overline{\mathfrak{G}}]$;*
- *ψ is satisfiable in $r_m(\overline{\mathfrak{G}})$ under a valuation which is good for ψ and the type assignment of $r_m(\overline{\mathfrak{G}})$.*

(b) The following conditions are equivalent, for every $\overline{\delta}(\mathfrak{F}, \mathbf{t})$, $\overline{\mathfrak{G}} \in \mathfrak{B}^*$, and $m \geq 2cl(\mathfrak{F})l(\mathfrak{F})$:

- *$\overline{\delta}(\mathfrak{F}, \mathbf{t})$ is satisfiable in $[\overline{\mathfrak{G}}]$;*
- *$\overline{\delta}(\mathfrak{F}, \mathbf{t})$ is satisfiable in $r_m(\overline{\mathfrak{G}})$ under a valuation which is canonically good for $\overline{\delta}(\mathfrak{F}, \mathbf{t})$ and the type assignment of $r_m(\overline{\mathfrak{G}})$.*

PROOF. (a) Suppose $(\mathfrak{M}, w) \models \psi$, where $\mathfrak{M} = ([\overline{\mathfrak{G}}], \mathfrak{V})$ is a model based on $[\overline{\mathfrak{G}}]$. Select for any $\chi \in sub(\psi)$ which is satisfied at some point in $\mathfrak{M}$ a maximal point x_{max} and a minimal point x_{min} such that for $x \in \{x_{max}, x_{min}\}$, $(\mathfrak{M}, x) \models \chi$. Denote by X the set of all these points. Then $|X| \leq 2l(\psi)$. Clearly, we find an embedding e of the underlying frame of $r_m(\overline{\mathfrak{G}})$ into $[\overline{\mathfrak{G}}]$ such that

- any cluster C of $r_m(\overline{\mathfrak{G}})$ of type t is mapped onto a t-cluster D of $[\overline{\mathfrak{G}}]$;
- X is a subset of the range $ran(e)$ of e.

Now take a valuation $\mathfrak{V}'$ in $r_m(\overline{\mathfrak{G}})$ defined as $\mathfrak{V}'(p) = e^{-1}(\mathfrak{V}(p))$. It is not difficult to prove by induction for every $\chi \in sub(\psi)$ and point x in $r_m(\overline{\mathfrak{G}})$

$$((r_m(\overline{\mathfrak{G}}), \mathfrak{V}'), x) \models \chi \text{ iff } (([\overline{\mathfrak{G}}], \mathfrak{V}), e(x)) \models \chi.$$

Therefore, $\mathfrak{V}'$ is good for the type assignment of $r_m(\overline{\mathfrak{G}})$. Moreover, let x be a maximal point such that $(\mathfrak{M}, x) \models \psi$. Then ψ is true in $e^{-1}(x)$ in $(r_m(\overline{\mathfrak{G}}), \mathfrak{V}')$.

Conversely, suppose that ψ is satisfied in a point w of $r_m(\overline{\mathfrak{G}})$ under a valuation $\mathfrak{V}$ which is good for the type assignment of $r_m(\overline{\mathfrak{G}})$. Take the canonical embedding r_m of $r_m(\overline{\mathfrak{G}})$ into $[\overline{\mathfrak{G}}]$. We define a valuation $\mathfrak{V}'$ in $[\overline{\mathfrak{G}}]$ as follows: for every v in the domain of $[\overline{\mathfrak{G}}]$ and propositional variable p set $v \in \mathfrak{V}'(p)$ iff there exists $v' \in \mathfrak{V}(p)$ such that $v = r_m(v')$ or v is not in the range of r_m but

- there exists $\mathfrak{C}(n, ⓚ)$ in $\overline{\mathfrak{G}}$ and $a_i \in ⓚ$ such that $v \in X_i$ and $a_i \in \mathfrak{V}(p)$, or
- there exists $\mathfrak{C}(ⓚ, n)$ in $\overline{\mathfrak{G}}$ and $a_i \in ⓚ$ such that $v \in X_i$ and $a_i \in \mathfrak{V}(p)$, or
- there exists $\mathfrak{C}(0, ①, 0))$ in $\overline{\mathfrak{G}}$ with v in its domain such that $① \in \mathfrak{V}(p)$.

Using the condition that $\mathfrak{V}$ is good for the type assignment of $r_m(\overline{\mathfrak{G}})$, it is not difficult to prove by induction for every $\chi \in sub(\psi)$ and point x in $r_m(\overline{\mathfrak{G}})$

$$((r_m(\overline{\mathfrak{G}}), \mathfrak{V}), x) \models \chi \text{ iff } (([\overline{\mathfrak{G}}], \mathfrak{V}'), r_m(x)) \models \chi.$$

Therefore, ψ is satisfied in $r_m(w)$ in $([\overline{\mathfrak{G}}], \mathfrak{V}')$.

(b) can be proved analogously. ■

We now provide a lemma from which it follows that validity of a formula $\alpha(\mathfrak{F}, \mathbf{t})$ in a frame $[\overline{\mathfrak{G}}]$, where $\overline{\mathfrak{G}} \in \mathfrak{B}^*$, can be checked in polynomial time in the parameters of $\overline{\mathfrak{G}}$.

Definition 4.4 (type morphism). Assume $(\mathfrak{F}, \mathbf{t})$, $\mathfrak{F} = \langle W, R \rangle$, and $(\mathfrak{F}', \mathbf{t}')$, $\mathfrak{F}' = \langle W', R' \rangle$, are two frames with types and f is a mapping from the set of clusters of $\mathfrak{F}$ onto set of clusters of $\mathfrak{F}'$. We say that f is a *type morphism* iff the following conditions are satisfied:

1. For any two clusters C_1 and C_2 in $\mathfrak{F}$, $C_1 R C_2$ implies $f(C_1) R' f(C_2)$.
2. If D is a degenerate cluster in $\mathfrak{F}'$, then $f^{-1}(D)$ contains only a single degenerate cluster.
3. If $D = ⓚ$ is a cluster in $\mathfrak{F}'$, $\mathbf{t}'_1 D = \mathsf{m}$ and C is the R-largest cluster in $f^{-1}(D)$, then C is a cluster $\geq k$ and $\mathbf{t}_1 C = \mathsf{m}$.
4. If $D = ⓚ$ is a cluster in $\mathfrak{F}'$ and $\mathbf{t}'_2 D = \mathsf{m}$ and C is the R-smallest cluster in $f^{-1}(D)$, then C is a cluster $\geq k$ and $\mathbf{t}_2 C = \mathsf{m}$.
5. Suppose $D = ⓚ$ is a cluster in $\mathfrak{F}'$, $\mathbf{t}'_1 D = \mathsf{j}$, and $\bigcup f^{-1} D = C_1 \lhd C_2 \cdots \lhd C_n$. Then the maximal final sequence $C_m \lhd C_{m+1} \lhd \cdots \lhd C_n$ such that $\mathbf{t}_1 C_i = \mathsf{m}$ for $m \leq i \leq n$ contains at least k points.

6. Suppose $D = ⓚ$ is a cluster in $\mathfrak{F}'$, $\mathbf{t}'_2 D = \mathsf{j}$, and $\bigcup f^{-1}D = C_1 \lhd C_2 \cdots \lhd C_n$. Then the maximal initial sequence $C_1 \lhd C_2 \lhd \cdots \lhd C_m$ in $\bigcup f^{-1}D$ such that $\mathbf{t}_2 C_i = \mathsf{m}$ for $1 \leq i \leq m$ contains at least k points.

LEMMA 4.5. *The following conditions are equivalent, for every* $\overline{\mathfrak{G}} \in \mathfrak{B}^*$*, every finite frame with types* $(\mathfrak{F}, \mathbf{t})$*, and every* $m \geq 2l(\mathfrak{F})cl(\mathfrak{F})$*:*

- $[\overline{\mathfrak{G}}]$ *validates* $\alpha(\mathfrak{F}, \mathbf{t})$*;*
- *There does not exist a type morphism from* $r_m(\overline{\mathfrak{G}})$ *onto* $(\mathfrak{F}, \mathbf{t})$*.*

PROOF. By Lemma 4.3 (b) it is sufficient to show that the following two conditions are equivalent:

- $\overline{\delta}(\mathfrak{F}, \mathbf{t})$ is satisfiable under a canonically good valuation in $r_m(\overline{\mathfrak{G}})$;
- there exists a type morphism from $r_m(\overline{\mathfrak{G}})$ onto $(\mathfrak{F}, \mathbf{t})$.

Suppose first that $\overline{\delta}(\mathfrak{F}, \mathbf{t})$ is satisfied in $r_m(\overline{\mathfrak{G}})$ under a canonically good valuation $\mathfrak{V}$. Define a mapping f from the set of clusters of $r_m(\overline{\mathfrak{G}})$ onto the set of clusters of $\mathfrak{F}$ as follows: $f(C) = D$ iff there exists $x \in D$ and $y \in C$ such that p_x is satisfied in y under $\mathfrak{V}$. We claim that f is the required type morphism:

- f is a partial function: suppose there are x_1 and x_2 in different clusters from $\mathfrak{F}$ such that there are y_1 and y_2 in the same cluster of $r_m(\overline{\mathfrak{G}})$ and p_{x_1} is satisfied in y_1 and p_{x_2} is satisfied in y_2. If $y_1 = y_2$, then this contradicts the first conjunct of $\delta(\mathfrak{F}, \mathbf{t})$. Otherwise $p_{x_1} \wedge \Diamond_P p_{x_2}$ and $p_{x_2} \wedge \Diamond_P p_{x_1}$ are both satisfied. This contradicts the second conjunct of $\delta(\mathfrak{F}, \mathbf{t})$.
- f is defined for every cluster in $r_m(\overline{\mathfrak{G}})$: this follows from the fifth conjunct of $\delta(\mathfrak{F}, \mathbf{t})$.
- f is surjective: this follows from the last conjunct of $\delta(\mathfrak{F}, \mathbf{t})$ since each p_x is satisfied in at least one point of $r_m(\overline{\mathfrak{G}})$.
- Condition (i) follows from the second conjunct of $\delta(\mathfrak{F}, \mathbf{t})$.
- Condition (ii) follows from the second conjunct of $\delta(\mathfrak{F}, \mathbf{t})$.
- Condition (iii) follows from the third conjunct of $\delta(\mathfrak{F}, \mathbf{t})$.
- Condition (iv). This follows from the fourth conjunct of $\delta(\mathfrak{F}, \mathbf{t})$.
- Conditions (v) and (vi) follow from the condition that $\mathfrak{V}$ is a canonically good valuation.

Conversely, suppose f is a type morphism from $r_m(\overline{\mathfrak{G}})$ onto $(\mathfrak{F}, \mathbf{t})$. Denote by $\mathbf{s} = (\mathbf{s}_1, \mathbf{s}_2)$ the type assignment of $r_m(\overline{\mathfrak{G}})$.

Define a valuation $\mathfrak{V}$ in $r_m(\overline{\mathfrak{G}})$ as follows: let D be a cluster in $\mathfrak{F}$ and let $C_1 \lhd C_2 \lhd \cdots \lhd C_n$ be the interval consisting of the clusters mapped to D. If $D = \{x\}$, for some x, then set $\mathfrak{V}(p_x) = \bigcup_{1 \le i \le n} C_i$. Otherwise we distinguish three cases:

- $\mathbf{t}D = (\mathsf{m}, \mathsf{m})$. In this case, by conditions (iii) and (iv), C_1 and C_n are of cardinality $\geq |D|$. Hence define $\mathfrak{V}(p_x)$, $x \in D$, in such a way that they form a partition of C_1, C_2 and $\bigcup_{1 \le i \le n} C_i$.
- $\mathbf{t}D = (\mathsf{m}, \mathsf{j})$. In this case, by condition (iii), C_n is of cardinality $\geq |D|$. By condition (vi), there exists $m \leq n$ such that $C_1 \lhd C_2 \lhd \cdots \lhd C_m$ has cardinality $\geq |D|$ and $\mathbf{s}_2 C_i = \mathsf{m}$ for all $1 \leq i \leq m$. But then define $\mathfrak{V}(p_x)$, $x \in D$, in such a way that they form a partition of C_n, $C_1 \lhd C_2 \lhd \cdots \lhd C_m$, and $C_1 \lhd C_2 \lhd \cdots \lhd C_n$.
- $\mathbf{t}D = (\mathsf{j}, \mathsf{m})$. Take the mirror image of the definition above using (iv) and (v).

It is not difficult to see that $\mathfrak{V}$ is a canonically good valuation and that $\overline{\delta}(\mathfrak{F}, \mathbf{t})$ is satisfied in $r_m(\overline{\mathfrak{G}})$. ■

LEMMA 4.6. *Assume that* $(\mathfrak{G}, \mathbf{t})$ *is of size* m *and* $(\mathfrak{G}', \mathbf{t}')$ *is of size* n. *It can be checked in time polynomial in* m *whether there exists a type morphism from* $(\mathfrak{G}, \mathbf{t})$ *onto* $(\mathfrak{G}', \mathbf{t}')$

PROOF. As follows from the definition of type morphism, every type morphism induces a distinct partition of $\mathfrak{G}$ into $\leq n$ intervals. So it is enough to examine all such divisions of $\mathfrak{G}$. But the number of these partitions is bounded by m^n. ■

PROOF. of Theorem 2.1 (ii). As was already observed, every $\bigcap$-irreducible tense logic is determined by $[\overline{\mathfrak{G}}]$, for some $\overline{\mathfrak{G}} \in \mathcal{B}^*$. Suppose now that $\overline{\mathfrak{G}} \in \mathcal{B}^*$ is given and we want to check whether ψ is satisfiable in $[\overline{\mathfrak{G}}]$. Let $m = 2(l(\psi) + 1)$. By Lemma 4.3, ψ is satisfiable in $[\overline{\mathfrak{G}}]$ iff it is satisfiable under a ψ-good valuation in $r_m(\overline{\mathfrak{G}})$. The cardinality of $r_m(\overline{\mathfrak{G}})$ is polynomial in $l(\psi)$. Now choose non-deterministically any valuation $\mathfrak{V}$ in $r_m(\overline{\mathfrak{F}})$ and check in polynomial time whether it is good for ψ and whether ψ is satisfied under $\mathfrak{V}$. ■

PROOF. of Theorem 2.1 (i). Suppose

$$L = \mathbf{Lin} \oplus \alpha(\mathfrak{F}_1, \mathbf{t}_1) \oplus \cdots \oplus \alpha(\mathfrak{F}_n, \mathbf{t}_n)$$

is given. Set $r_1 = \max\{l(\mathfrak{F}_i) \mid 1 \leq i \leq n\}$ and $r_2 = \max\{cl(\mathfrak{F}_i) \mid 1 \leq i \leq n\}$. Given a formula ψ, choose non-deterministically $\overline{\mathfrak{G}} \in \mathcal{B}^*$ satisfying the constraints (b)-(d) of Theorem 3.4 and a valuation $\mathfrak{V}$ in $r_m(\overline{\mathfrak{G}})$, where

$$m = \max\{2r_1r_2, 2l(\psi) + 1\}.$$

Notice that the size of $r_m(\overline{\mathfrak{G}})$ is polynomial in ψ. Now it can be checked in polynomial time whether $\mathfrak{V}$ is good for ψ and ψ is satisfied. Moreover, by Lemma 4.6 it can be checked in polynomial time whether any of $(\mathfrak{F}_i, \mathbf{t}_i)$ is a type-morphic image of $r_m(\overline{\mathfrak{G}})$. In other words, by Lemma 4.5 and 4.3, we can decide in non-deterministic polynomial time whether ψ is satisfiable in a frame satisfying the constraints (a)-(d) of Theorem 3.4. But then, by Theorem 3.4, L is coNP-complete. ■

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