A Note on Concepts and Distances

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Abstract

We combine the description logic $\mathcal{ALC}$ with the metric logics defined in [15]. Entities that are conceived of as abstract points in the realm of $\mathcal{ALC}$ are given a spatial extension via an 'extension relation,' connecting abstract points in the domain of an $\mathcal{ALC}$-model to points in a metric space. Conversely, regions in the metric space are connected via the converse 'extension relation' to certain points in the $\mathcal{ALC}$-model. We prove the decidability of the satisfiability problem for the resulting hybrid formalism, give a few examples, and discuss further extensions of the ideas introduced.

1 Introduction

Motivating example

Everybody knows that only two things can be worse than flat-hunting (in London—only one). You visit the $n+1$st flat offered by your estate agent and see that it's too far from your college, there are no shops around, a telephone is missing, the neighbors go to the nearest tube station by taxi, and their kids buzz like bees. But you asked her so many times not to offer flats like this! You even wrote down your constraints:

(A) The house should not be too far from the college, not more than 5 miles.

(B) The house should be close to shops, say, within 1 mile.

(C) There should be a 'green zone' around the house, at least within 2 miles.

(D) There must be a sports center around, and moreover, all sports centers of the district should be reachable on foot, i.e., they should be within, say, 3 miles.

(E) Public transport should easily be accessible: whenever you are not more than 8 miles away from home, the nearest bus stop or tube station should be reachable within 1 mile.

(F) The house should have a telephone.

(G) The neighbors shouldn't have children.

And answering her question "Are supermarkets shops?" you added

(H) All supermarkets are shops.

Even a computer could have done her job better!

Formalization

Constraints (A)–(H) contain two kinds of knowledge. (F)–(H) can be classified as conceptual knowledge which is captured by almost any description logic, in particular, $\mathcal{ALC}$:

$$\begin{aligned}
&\text{(F)}\ \textit{house} : \exists \mathsf{has}.\mathsf{Telephone}\\
&\text{(G)}\ \textit{house} : \forall \mathsf{neighbour}.\forall \mathsf{child}.\bot\\
&\text{(H)}\ \mathsf{Supermarket} \sqsubseteq \mathsf{Shop}
\end{aligned}$$

(A)–(E) speak about distances and can be represented in the logic $\mathcal{MS}$ of metric spaces introduced in [15]:

$$\begin{aligned}
&\text{(A)}\ \textit{house} \sqsubseteq \mathsf{E}_{\leq 5}\textit{college}\\
&\text{(B)}\ \textit{house} \sqsubseteq \mathsf{E}_{\leq 1}\mathsf{Shop}\\
&\text{(C)}\ \textit{house} \sqsubseteq \mathsf{E}_{\leq 2}\mathsf{Green_zone}\\
&\text{(D)}\ \textit{house} \sqsubseteq (\mathsf{E}_{\leq 3}\mathsf{Sports_center}) \sqcap (\mathsf{A}_{>3}\neg\mathsf{Sports_center})\\
&\text{(E)}\ \textit{house} \sqsubseteq \mathsf{A}_{\leq 8}\mathsf{E}_{\leq 1}\mathsf{Public_transport}
\end{aligned}$$

Here, the object names *house* and *college* are interpreted as singletons in a metric space $\mathfrak{D} = \langle D, \delta \rangle$,[1] set variables Shop, Green_zone, etc. as subsets of D, and, for every non-negative rational number α and every $X \subseteq D$,

$$\begin{aligned}
\mathsf{E}_{\leq \alpha}X &= \{x \in D \mid \exists y \in D \ : \ \delta(x,y) \leq \alpha \wedge y \in X\}\\
\mathsf{A}_{\leq \alpha}X &= \{x \in D \mid \forall y \in D \ : \ \delta(x,y) \leq \alpha \to y \in X\}\\
\mathsf{A}_{> \alpha}X &= \{x \in D \mid \forall y \in D \ : \ \delta(x,y) > \alpha \to y \in X\}
\end{aligned}$$

[1] I.e., D is a non-empty set and δ is a distance function from $D \times D$ into $\mathbb{R}_+$ satisfying three conditions: $\delta(x,y) = \delta(y,x)$, $\delta(x,y) = 0$ iff $x = y$, and $\delta(x,y) + \delta(y,z) \geq \delta(x,z)$.

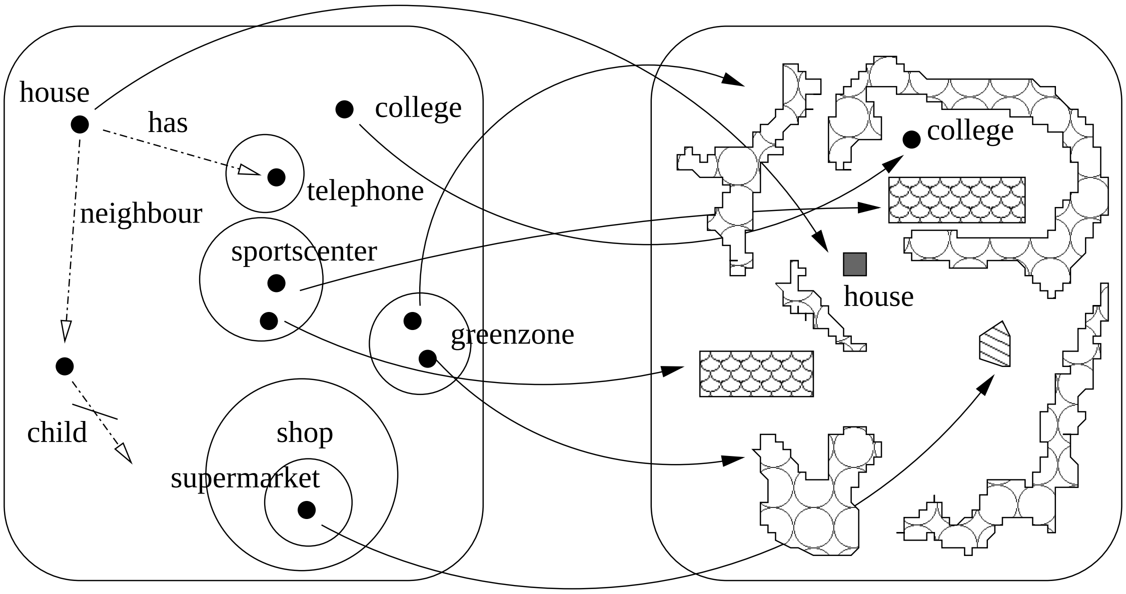

Figure 1: Combined model for the metric description logic.

Thus, the problem is to *combine* these two KR formalisms. Semantically, the combination can be very simple (see Fig. 1). We just take an $\mathcal{ALC}$-model $\mathfrak{I} = \left\langle \Delta, \cdot^{\mathfrak{I}} \right\rangle$, a metric space $\mathfrak{D} = \langle D, \delta \rangle$, i.e., an $\mathcal{MS}$-model, and associate with some elements in Δ subsets of D—the spatial 'regions' occupied by objects represented by these elements. In other words, we define a binary relation, say, $\rightsquigarrow$ on $\Delta \times D$. Those elements a of Δ for which $\{x \in D : a \rightsquigarrow x\} = \emptyset$ can be called *abstract*—there is no sense to think of them as occupying space. Elements that are not abstract can be called *localizable*.

If C is an $\mathcal{ALC}$-concept, then its *spatial extension* $\varsigma(C)$ in the combined model is defined as

$$\varsigma(C) = \{x \in D \mid \exists a \in C^{\mathfrak{I}}\ a \rightsquigarrow x\}.$$

For example, $\varsigma(\mathsf{Shop})$ represents the space occupied by all shops. On the other hand, given a set term t from $\mathcal{MS}$ interpreted as a subset of D, we can define its *conceptual extension* $\varsigma^{-}(t)$ by taking

$$\varsigma^{-}(t) = \{a \in \Delta \mid \exists x \in t^{\mathfrak{D}}\ a \rightsquigarrow x\}.$$

Thus, $\varsigma^{-}(t)$ is the concept containing all those localizable elements that have a non-empty intersection with 'region' t. For instance, the concept $\varsigma^{-}\mathsf{E}_{\leq 1}\varsigma(\mathsf{Shop})$ comprises all localizable elements from which at least one shop can be reached within 1 mile.

Using the constructors $\boldsymbol{\varsigma}$ and $\boldsymbol{\varsigma}^-$ connecting $\mathcal{ALC}$- and $\mathcal{MS}$-models, we can represent constraints (A)–(H) as the concept $\mathsf{Good\_house}$ defined by the following knowledge base:

$$
\begin{aligned}
\mathsf{Good\_house} &= \mathsf{House} \sqcap \mathsf{Well\_located} \sqcap \exists \mathsf{has.Telephone} \sqcap \forall \mathsf{neighbor}.\forall \mathsf{child}.\bot \\
\mathsf{Well\_located} &= \boldsymbol{\varsigma}^-\big(\mathsf{E}_{\leq 5}\boldsymbol{\varsigma}(\mathit{college}) \sqcap \mathsf{E}_{\leq 1}\boldsymbol{\varsigma}(\mathsf{Shop}) \sqcap \mathsf{E}_{\leq 2}\boldsymbol{\varsigma}(\mathsf{Green\_zone}) \sqcap \\
&\qquad \mathsf{E}_{\leq 3}\boldsymbol{\varsigma}(\mathsf{Sports\_center}) \sqcap \mathsf{A}_{>3}\boldsymbol{\varsigma}(\neg\mathsf{Sports\_center}) \sqcap \\
&\qquad \mathsf{A}_{\leq 8}\mathsf{E}_{\leq 1}\boldsymbol{\varsigma}(\mathsf{Public\_transport})\big) \\
\mathsf{Supermarket} &\sqsubseteq \mathsf{Shop}
\end{aligned}
$$

We now give rigorous definitions of the syntax and semantics of the combined language we used in the example above.

2 Combining description and metric logics

The definition of the syntax of $\mathcal{ALC}(\mathcal{MS})$ is rather straightforward: we simply join the languages of $\mathcal{ALC}$ and $\mathcal{MS}$ and bridge them by means of the spatial and conceptual extension functions ς and ς^-.

Syntax

The alphabet of the *metric description logic* $\mathcal{ALC}(\mathcal{MS})$ consists of

- the primitive symbols of $\mathcal{ALC}$, i.e.,
  - a list $A_0, A_1, \dots$ of concept names,
  - a list $R_0, R_1, \dots$ of role names,
  - a list $c_0, c_1, \dots$ of object names,
  - the Booleans $\sqcap$, $\neg$, $\top$, and the existential role restrictions $\exists R_i$;
- the primitive symbols of $\mathcal{MS}$, i.e.,
  - a list $X_0, X_1, \dots$ of set variables,
  - a list $n_0, n_1, \dots$ of nominals,
  - the set term constructors $\mathsf{E}_{\leq\alpha}$ and $\mathsf{E}_{>\alpha}$, for all $\alpha \in \mathbb{Q}_+$, and the Booleans $\cap$ and $\neg$;
- the spatial and conceptual extension constructors $\boldsymbol{\varsigma}$ and $\boldsymbol{\varsigma}^-$, respectively.

Concepts and set terms of $\mathcal{ALC}(\mathcal{MS})$ are defined inductively as follows:

- all concept names A_i are concepts;

- all set variables X_i and nominals n_i are set terms;
- if C, D are concepts, R a role name, and t a set term, then

$$C \sqcap D, \quad \neg C, \quad \exists R.C, \quad \varsigma^{-}(t)$$

are concepts;
- if s, t are set terms, C a concept, c an object name, and $\alpha \in \mathbb{Q}_{+}$, then

$$t \cap s, \quad \neg t, \quad \mathsf{E}_{\leq\alpha} t, \quad \mathsf{E}_{>\alpha} t, \quad \varsigma(C), \quad \varsigma(c)$$

are set terms.

Formulas of $\mathcal{ALC}(\mathcal{MS})$ are Boolean combinations of atomic formulas of the form

$$c : C, \quad cRd, \quad C \sqsubseteq D, \quad t \sqsubseteq s,$$

where C, D are arbitrary concepts and t, s arbitrary set terms, respectively.

Semantics

As is shown in Fig. 1, each model of the constructed language consists of a standard $\mathcal{ALC}$-model, a metric space model for $\mathcal{MS}$ and a relation between their domains interpreting the spatial and conceptual extension functions ς and ς^{-}. Thus, an $\mathcal{ALC}(\mathcal{MS})$-*model* is a triple of the form $\mathfrak{M} = \langle \mathfrak{I}, \mathfrak{D}, \rightsquigarrow \rangle$, in which

- $\mathfrak{I} = \left\langle \Delta, R_0^{\mathfrak{I}}, \dots, A_0^{\mathfrak{I}}, \dots, c_0^{\mathfrak{I}}, \dots \right\rangle$ is an $\mathcal{ALC}$-model, where Δ is a non-empty set, the *object domain* of $\mathfrak{M}$, $R_i^{\mathfrak{I}}$ are binary relations on Δ interpreting the role names, $A_i^{\mathfrak{I}}$ subsets of Δ interpreting the concept names, and $c_i^{\mathfrak{I}}$ are elements of Δ interpreting the object names;
- $\mathfrak{D} = \left\langle D, \delta, X_0^{\mathfrak{D}}, \dots, n_0^{\mathfrak{D}}, \dots \right\rangle$ is an $\mathcal{MS}$-model, where $\langle D, \delta \rangle$ is a metric space, the *spatial domain* of $\mathfrak{M}$, $X_i^{\mathfrak{D}} \subseteq D$, and each $n_i^{\mathfrak{D}}$ is a singleton subset of D;
- $\rightsquigarrow$ is a binary relation on $\Delta \times D$.

Let $\mathfrak{M} = \langle \mathfrak{I}, \mathfrak{D}, \rightsquigarrow \rangle$ be an $\mathcal{ALC}(\mathcal{MS})$-model. The *extensions* $C^{\mathfrak{M}}$ and $t^{\mathfrak{M}}$ of a concept C and a set term t, as well as the *truth-relation* $\mathfrak{M} \models \varphi$ for an $\mathcal{ALC}(\mathcal{MS})$-formula φ are defined inductively in the following way.

- The extension $C^{\mathfrak{M}} \subseteq \Delta$ of C:
  - $A_i^{\mathfrak{M}} = A_i^{\mathfrak{I}}$, A_i a concept name,
  - $(C_0 \sqcap C_1)^{\mathfrak{M}} = C_0^{\mathfrak{M}} \cap C_1^{\mathfrak{M}}$;

- $(\neg C_0)^{\mathfrak{M}} = \Delta - C_0^{\mathfrak{M}}$;
- $a \in (\exists R_i.C_0)^{\mathfrak{M}}$ iff there is a $b \in C_0^{\mathfrak{M}}$ such that $aR_i^{\mathfrak{I}}b$;
- $a \in (\varsigma^{-}(t))^{\mathfrak{M}}$ iff there is an $x \in t^{\mathfrak{M}}$ such that $a \rightsquigarrow x$.

- The extension $t^{\mathfrak{M}} \subseteq D$ of t:
  - $X_i^{\mathfrak{M}} = X_i^{\mathfrak{D}}$, X_i a set variable;
  - $n_i^{\mathfrak{M}} = n_i^{\mathfrak{D}}$, n_i a nominal;
  - $(t_0 \cap t_1)^{\mathfrak{M}} = t_0^{\mathfrak{M}} \cap t_1^{\mathfrak{M}}$;
  - $(\neg t_0)^{\mathfrak{M}} = D - t_0^{\mathfrak{M}}$;
  - $(\mathsf{E}_{\leq\alpha}t_0)^{\mathfrak{M}} = \{x \in D \mid \exists y \in D \ \big(\delta(x,y) \leq \alpha \ \& \ y \in t_0^{\mathfrak{M}}\big)\}$;
  - $(\mathsf{E}_{>\alpha}t_0)^{\mathfrak{M}} = \{x \in D \mid \exists y \in D \ \big(\delta(x,y) > \alpha \ \& \ y \in t_0^{\mathfrak{M}}\}\big)$;
  - $x \in (\varsigma(C))^{\mathfrak{M}}$ iff there is an $a \in C^{\mathfrak{M}}$ such that $a \rightsquigarrow x$;
  - $x \in (\varsigma(c_i))^{\mathfrak{M}}$ iff $c_i^{\mathfrak{I}} \rightsquigarrow x$.

- The truth-relation $\mathfrak{M} \models \varphi$:
  - $\mathfrak{M} \models C \sqsubseteq D$ iff $C^{\mathfrak{M}} \subseteq D^{\mathfrak{M}}$;
  - $\mathfrak{M} \models t_0 \sqsubseteq t_1$ iff $t_0^{\mathfrak{M}} \subseteq t_1^{\mathfrak{M}}$;
  - $\mathfrak{M} \models c : C$ iff $c^{\mathfrak{I}} \in C^{\mathfrak{M}}$;
  - $\mathfrak{M} \models cRd$ iff $c^{\mathfrak{I}}R^{\mathfrak{I}}d^{\mathfrak{I}}$;
  - $\mathfrak{M} \models \psi \wedge \chi$ iff $\mathfrak{M} \models \psi$ and $\mathfrak{M} \models \chi$;
  - $\mathfrak{M} \models \neg\psi$ iff not $\mathfrak{M} \models \psi$.

(Note that the 'closest relatives' of the metric logic $\mathcal{MS}$ introduced in [15] are the logics of place [12, 16, 14, 7, 10] metric (or quantitative) temporal logics [1, 11, 6].)

3 Decidability

As in pure description logic, many reasoning tasks for $\mathcal{ALC}(\mathcal{MS})$ can be reduced to the *satisfiability problem* for $\mathcal{ALC}(\mathcal{MS})$-formulas. For each of the two components of our combined system, the satisfiability problem is known to be decidable (see [13] and [15]). In [8], further decidability and complexity results for metric logics can be found, as well as an expressive completeness theorem relating metric logic to classical first-order logic with two variables, monadic predicates and atoms of the form $\boldsymbol{\delta}(x,y) < a$, $\boldsymbol{\delta}(x,y) = a$ interpreted in metric spaces.

We now come to the decidability of the satisfiability problem for $\mathcal{ALC}(\mathcal{MS})$. The proof can be viewed as a combination of the known decidability proofs for $\mathcal{ALC}$ and $\mathcal{MS}$. As the technical details are too involved and lengthy to be presented here, we confine ourselves only to a brief sketch of the proof and refer the reader to the full paper [9].

Theorem 1 *The satisfiability problem for $\mathcal{ALC}(\mathcal{MS})$-formulas is decidable. Moreover, every satisfiable $\mathcal{ALC}(\mathcal{MS})$-formula φ is satisfied in a finite model the size of which can be computed effectively from the length of φ.*

Here is a sketch of the proof. Suppose φ is satisfied in $\mathcal{M} = \langle \mathcal{I}, \mathcal{D}, \rightsquigarrow \rangle$, where $\mathcal{I} = \left\langle \Delta, \cdot^{\mathcal{I}} \right\rangle$ and $\mathcal{D} = \left\langle D, d, \cdot^{\mathcal{D}} \right\rangle$. Then we perform the following steps:

(1) Filtrate $\mathcal{D}$ through (an analogue of) the filter defined in [15] (page 50) to obtain a finite model $\mathcal{D}_1$ based on equivalence classes $D_1 = \{[x] : x \in D\}$. Define a binary relation $\rightsquigarrow_1$ on $\Delta \times D_1$ by taking $a \rightsquigarrow_1 [x]$ iff there exists $y \in [x]$ such that $a \rightsquigarrow y$.

(2) Use the duplication technique of [15] (page 52) to obtain from $\mathcal{D}_1$ a new model $\mathcal{D}^*$ based on $D^* = D_1^1 \cup (D_1^2 \times \{0, 1\})$, where D_1 is a disjoint union of $D_1^1 \cup D_1^2$. Define a binary relation $\rightsquigarrow_2$ on $\Delta \times D_2$ by taking $a \rightsquigarrow_2 x$ iff either $x \in D_1^1$ and $a \rightsquigarrow_1 x$ or $x = (y, i)$ and $a \rightsquigarrow_1 y$.

(3) Finally, filtrate the $\mathcal{ALC}$-model $\mathcal{I}$ (as, say, in [4]) to obtain a finite $\mathcal{ALC}$-model $\mathcal{I}^*$ based on a set of equivalence classes $\Delta^* = \{[a] : a \in \Delta\}$. Define $\mathfrak{M}^* = \langle \mathcal{I}^*, \mathcal{D}^*, \rightsquigarrow^* \rangle$ by taking $[a] \rightsquigarrow^* x$ iff there exists $b \in [a]$ such that $b \rightsquigarrow_2 x$. One can show that this model is as required.

Actually, Theorem 1 can be extended to a general transfer result for combinations of spatial and description logics. As was shown in [3], almost all description logics can be regarded as *abstract description systems* (ADS). In fact, both $\mathcal{ALC}$ and $\mathcal{MS}$ can be regarded as ADSs, and the definition of $\mathcal{ALC}(\mathcal{MS})$ can easily be generalized to a definition of a combination $\mathcal{L}_1(\mathcal{L}_2)$ of two ADSs $\mathcal{L}_1$ and $\mathcal{L}_2$ in such a way that the following transfer result holds (it can be proved using a technique similar to that used in [3]):

Theorem 2 *Suppose $\mathcal{L}_1$ and $\mathcal{L}_2$ are ADSs with decidable relativized satisfiability problems. Then the (relativized) satisfiability problem for $\mathcal{L}_1(\mathcal{L}_2)$ is decidable.*[2]

4 Discussion and further research

This note suggests a new way of combining formal conceptual and spatial reasoning (cf. e.g. [5]). It outlines only a basic idea; still much is to be done to convert the constructed metric description logic into a really useful KR&R

[2]Note that here, in contrast to [3], we do not require the ADSs to be local.

formalism. Apart from the obvious problems such as developing implementable algorithms and determining the computational complexity, there are a number of more fundamental issues:

1. The proposed method of combining description and metric logics with such a robust algorithmic behavior (see Theorem 2) seems too good for capturing various subtle interactions between object and spatial knowledge. In fact, the proof of Theorem 2 shows that such interactions are rather limited. For example, it is not possible to constrain the interpretation of a role R in such a way that $aR^{\mathfrak{I}}b$ iff the spatial extension of a is included in the spatial extension of b. Obviously, more expressive constructors connecting conceptual and spatial domains are required to express this and similar constraints.

2. As metric spaces induce in the usual way a topology, it seems natural to extend the metric description language with the modal logic $\mathbf{S4}_u$ or the $\mathsf{RCC\text{-}8}$ relations, thus bringing together qualitative, semi-qualitative, and conceptual spatial reasoning.

3. One more idea is to use more than one extension relation to treat space-granularity; cf. [11].

4. A different approach to combining conceptual and spatial reasoning uses concrete domains, see [2] and [5]. The combination method proposed here and that of applying concrete domains allow for rather different formalizations. However, the precise relation between the two approaches is still unclear.

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