Logics of Metric Spaces

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Abstract

We investigate the expressive power and computational properties of two different types of languages for speaking about distances. First, we consider a first-order language $\mathcal{FM}$, the two-variable fragment of which turns out to be undecidable in the class of distance spaces validating the triangular inequality as well as in the class of all metric spaces. Yet, this two-variable fragment is still decidable in various weaker classes. Second, we introduce a variable-free 'modal' language $\mathcal{MS}$ which, when interpreted in metric spaces, has the same expressive power as the two-variable fragment of $\mathcal{FM}$. We determine natural and expressive fragments of $\mathcal{MS}$ which are decidable in various classes of distance spaces validating the triangular inequality, among them, the class of all metric spaces.

1 Introduction

This paper investigates the expressive power and computational properties of languages designed for speaking about distances. 'Distances' can be induced by different measures. We may be interested in the physical distance between two cities a and b, i.e., in the length of the straight (or geodesic) line between a and b. More pragmatic would be to bother about the length of the railroad connecting a and b, or even better, the time it takes to go from a to b by train (plane, ship, etc.). But we can also define the distance as the number of cities (stations, friends to visit, etc.) on the way from a to b, as the difference in altitude between a and b, and so forth.

The standard mathematical models, capturing common features of various notions of distance, are known as metric spaces. A *metric space* is a pair $\langle W, d\rangle$, where W is a set (of points) and d a function from $W \times W$ into the set $\mathbb{R}^+$ (of non-negative real numbers) satisfying the following axioms

$$d(x,y) = 0 \text{ iff } x = y, \tag{1}$$
$$d(x,z) \leq d(x,y) + d(y,z), \tag{2}$$
$$d(x,y) = d(y,x) \tag{3}$$

for all $x, y, z \in W$. The value $d(x,y)$ is called the *distance* from the point x to the point y. Perhaps the most 'popular' metric spaces are n-dimensional Euclidean spaces $\langle \mathbb{R}^n, d_n \rangle$ with the metric

$$d_n(\vec{x}, \vec{y}) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}.$$

Although acceptable in many cases, the thus defined concept of metric space is not universally applicable to all interesting measures of distance between points, especially those used in everyday life. Consider, for instance, the following two examples:

(i) If $d(x,y)$ is the flight-time from x to y then, as we know it too well, d is not necessarily symmetric, even approximately (just go from London to Tokyo and back).

(ii) Often we do not measure distances by means of real numbers but rather using more fuzzy notions such as 'short', 'medium' and 'long'. To represent these measures we can, of course, take functions d from $W \times W$ into the subset $\{1, 2, 3\}$ of $\mathbb{R}^+$ and define $short := 1$, $medium := 2$, and $long := 3$. So we can still regard these distances as real numbers. However, for measures of this type the triangular inequality (2) does not make sense (short plus short can still be short, but it can also be medium or long).

Metric spaces as well as more general *distance spaces* $\langle W, d \rangle$ satisfying only axiom (1) are the intended models of the languages we construct in this paper.

We begin by considering the first-order languages $\mathcal{FM}[M]$, for $M \subseteq \mathbb{R}^+$, with monadic predicates (for subsets of W), individual constants (for points in W), and the binary predicates $\boldsymbol{\delta}(x,y) < a$ and $\boldsymbol{\delta}(x,y) = a$, $a \in M$, saying that the distance between x and y is smaller than a or equal to a, respectively. Typical sets M of possible distances will be $\mathbb{Q}^+$ (the non-negative rational numbers) and $\mathbb{N}$ (the natural numbers including 0).

The following example will be used to illustrate the expressive power of our languages.

Example 1. Imagine (only imagine) that you are going to buy a house in London. You then inform your estate agent about your intention and provide her with a number of constraints:

(A) The house should not be too far from your college, say, not more than 10 miles.

(B) The house should be close to shops and restaurants; they should be reachable, say, within 1 mile.

(C) There should be a 'green zone' around the house, at least within 2 miles in each direction.

(D) Factories and motorways must be far from the house, not closer than 5 miles.

(E) There must be a sports center around, and moreover, all sports centers of the district should be reachable on foot, i.e., they should be within, say, 3 miles.

(F) Public transport should be easily accessible: whenever you are not more than 8 miles away from home, there should be a bus stop or a tube station within a distance of 2 miles.

(G) And, of course, there must be a tube station around, not too close, but not too far either—somewhere between 0.5 and 1 mile.

The constraints in Example 1 can be formalized in $\mathcal{FM}[\mathbb{Q}^+]$ by the following formulas:

(A′) $\boldsymbol{\delta}(college, house) \leq 10$, where *college* and *house* are constants.

(B′) $\exists x(\boldsymbol{\delta}(house, x) \leq 1 \wedge \mathsf{shop}(x))$ and $\exists x(\boldsymbol{\delta}(house, x) \leq 1 \wedge \mathsf{restaurant}(x))$, where shop and $\mathsf{restaurant}$ are unary predicates.

(C′) $\forall x(\boldsymbol{\delta}(house, x) \leq 2 \rightarrow \mathsf{green\_zone}(x))$, where $\mathsf{green\_zone}$ is a unary predicate.

(D′) $\forall x(\mathsf{factory}(x) \vee \mathsf{motorway}(x) \rightarrow \boldsymbol{\delta}(house, x) > 5)$, where $\mathsf{factory}$ and $\mathsf{motorway}$ are unary predicates.

(E′) $\exists x(\boldsymbol{\delta}(house, x) \leq 3 \wedge \mathsf{district\_sports\_center}(x)) \wedge \forall x(\boldsymbol{\delta}(house, x) > 3 \rightarrow \neg\mathsf{district\_sports\_center}(x))$, where $\mathsf{district\_sports\_center}$ is a unary predicate.

(F′) $\forall x(\boldsymbol{\delta}(house, x) \leq 8 \rightarrow \exists y(\boldsymbol{\delta}(x,y) \leq 2 \wedge \mathsf{public\_transport}(y)))$, where $\mathsf{public\_transport}$ is a unary predicate, and

(G′) $\exists x\big(\boldsymbol{\delta}(house,x) > 0.5 \wedge \boldsymbol{\delta}(house,x) \le 1 \wedge \mathsf{tube\_station}(x)\big)$, where $\mathsf{tube\_station}$ is a unary predicate.

As one might expect, the satisfiability problem for $\mathcal{FM}[\mathbb{Q}^+]$- and $\mathcal{FM}[\mathbb{N}]$-formulas in any class of distance spaces containing the class $\mathcal{M}$ of all metric spaces is undecidable (see Theorem 2 below). Trying to find decidable but still reasonably expressive fragments of $\mathcal{FM}[\mathbb{Q}^+]$, we then consider its two-variable fragment $\mathcal{FM}^2[\mathbb{Q}^+]$ consisting of all $\mathcal{FM}[\mathbb{Q}^+]$-formulas with the variables x and y only. (All formulas in the example above belong to this fragment.) The two-variable fragment of classical first-order logic is known to be decidable [26, 22, 14] and **NExpTime**-complete [8, 4]. We use this result to show that the satisfiability problem for $\mathcal{FM}^2[\mathbb{Q}^+]$-formulas is decidable

- in the class $\mathcal{D}$ of arbitrary distance spaces, and
- in the class $\mathcal{D}_{sym}$ of all distance spaces satisfying (3).

Unfortunately, this does not hold any more as soon as we add the triangular inequality (2): we show that the satisfiability problem for $\mathcal{FM}^2[\mathbb{Q}^+]$-formulas is undecidable both in

- the class $\mathcal{M}$ of all metric spaces and
- the class $\mathcal{D}_{tr}$ of distance spaces satisfying the triangular inequality.

We then introduce variable-free languages $\mathcal{MS}[M]$, $M \subseteq \mathbb{R}^+$, which instead of first-order quantifiers use *distance operators* 'somewhere in the circle of radius a', 'somewhere outside the circle of radius b,' etc., where $a, b \in M$. This brings us close to the field of temporal, modal, and description logics, which also avoid the use of first-order quantification by replacing it with various kinds of 'modal' operators like 'sometime in the future', 'it is possible', etc. The constraints in Example 1 can be formulated in $\mathcal{MS}$ as follows. As before, we treat '*house*' and '*college*' as constants representing certain points in the space; however, 'shop', 'restaurant' and other unary predicates are now understood as *set variables* interpreted as subsets of the domain of the distance space.

(A″) $\boldsymbol{\delta}(house, college) \le 10$.

(B″) $house \sqsubseteq (\mathsf{E}^{\le 1}\mathsf{shop} \sqcap \mathsf{E}^{\le 1}\mathsf{restaurant})$.

(C″) $house \sqsubseteq \mathsf{A}^{\le 2}\mathsf{green\_zone}$.

(D″) $house \sqsubseteq \neg\mathsf{E}^{\le 5}(\mathsf{factories} \sqcup \mathsf{motorways})$.

(E″) $house \sqsubseteq (\mathsf{E}^{\le 3}\mathsf{district\_sports\_center} \sqcap \mathsf{A}^{>3}\neg\mathsf{district\_sports\_center})$.

(F″) $house \sqsubseteq \mathsf{A}^{\le 8}\mathsf{E}^{\le 2}\mathsf{public\_transport}$.

(G″) $house \sqsubseteq \mathsf{E}^{>0.5}_{\le 1}\mathsf{tube\_station}$.

The intended meaning of the set term constructors above is as follows. The set $\mathsf{E}^{\le 1}\mathsf{shop}$ contains all points in the domain from which at least one shop is reachable within 1 mile. Likewise, for every point x in $\mathsf{A}^{\le 2}\mathsf{green\_zone}$, the whole circle of radius 2 around x belongs to the green zone, whereas $\mathsf{E}^{>0.5}_{\le 1}\mathsf{tube\_station}$ denotes the set of all points located in a distance between 0.5 and 1 mile (excluding 0.5) from at least one tube station.[1]

By replacing quantifiers with distance operators, we do not loose expressive power as compared with $\mathcal{FM}^2[M]$. In fact, we show that $\mathcal{MS}[M]$ is *expressively complete* for $\mathcal{FM}^2[M]$ in the class $\mathcal{M}$ of all metric spaces, for any $M \subseteq \mathbb{R}^+$. This theorem (the proof of which is similar to proofs in [7] and [20]) has two interesting consequences. First, any decidable

[1] By the way, the end of the imaginary story about buying a house in London was not satisfactory. Having checked her knowledge base, the estate agent said: "Unfortunately, your constraints (A)–(G) are not satisfiable in London, where we have

$$\mathsf{tube\_station} \sqsubseteq \mathsf{E}^{\le 3.5}(\mathsf{factory} \sqcup \mathsf{motorway}).$$

In view of the triangular inequality, this contradicts constraints (D) and (G)."

fragment of $\mathcal{FM}^2[M]$ can be obtained as a decidable fragment of $\mathcal{MS}[M]$. And second, since the translation from $\mathcal{FM}^2[M]$ into $\mathcal{MS}[M]$ is effective, decidable fragments of $\mathcal{MS}[M]$ have to be proper, in particular, $\mathcal{MS}[\mathbb{Q}^+]$ itself is undecidable when interpreted in spaces satisfying the triangular inequality.

We prove two results concerning fragments of $\mathcal{MS}[M]$. The first one identifies a rather expressive and natural fragment $\mathcal{MS}^\#[M]$, which has the finite model property (even for parameters from $\mathbb{R}^+$) and is decidable (if parameters are taken from $\mathbb{Q}^+$). All the constraints in Example 1 save (G) can be formulated in $\mathcal{MS}^\#[\mathbb{Q}^+]$. The second result shows that seemingly weak fragments of $\mathcal{MS}[\mathbb{N}]$ are already undecidable. Roughly speaking, we loose decidability as soon as we are able to speak about 'rings', as in constraint (G).

Table 1 summarizes the main decidability results of this paper: $+$ ($-$) means that the satisfiability problem for the corresponding language in the corresponding class of structures is decidable (undecidable). The results do not depend on whether the parameters are from $\mathbb{N}$ or $\mathbb{Q}^+$. For various fragments we will also obtain **NExpTime** upper bounds for the computational complexity. The fragments $\mathcal{MS}_i[\mathbb{Q}^+/\mathbb{N}]$ are defined in Section 3.

	$\mathcal{D}$	$\mathcal{D}_{sym}$	$\mathcal{D}_{tr}$	$\mathcal{M}$
$\mathcal{FM}[\mathbb{Q}^+/\mathbb{N}]$	$-$	$-$	$-$	$-$
$\mathcal{FM}^2[\mathbb{Q}^+/\mathbb{N}]$	$+$	$+$	$-$	$-$
$\mathcal{MS}[\mathbb{Q}^+/\mathbb{N}]$	$+$	$+$	$-$	$-$
$\mathcal{MS}_i[\mathbb{Q}^+/\mathbb{N}]$	$+$	$+$	$-$	$-$
$\mathcal{MS}^\#[\mathbb{Q}^+/\mathbb{N}]$	$+$	$+$	$+$	$+$

Table 1: The satisfiability problem for metric logics.

The structure of the paper is as follows: Section 2 introduces the syntax and semantics of both, the first-order and the 'modal' languages of metric spaces. Here, we also establish the expressive completeness result for $\mathcal{FM}^2[M]$. In Section 3, we prove the undecidability of $\mathcal{MS}_i[\mathbb{Q}^+/\mathbb{N}]$ by means of a reduction to the undecidable $\mathbb{N}\times\mathbb{N}$-tiling problem. Finally, in Sections 4 and 5, we prove our decidability results for metric and weaker distance spaces.

The idea of constructing logical formalisms capable of speaking about distances is not new. For example, somewhat weaker spatial 'modal logics of distance' were introduced in [25, 33, 27, 17, 19]. However, their computational behavior has remained unexplored. More attention has recently been devoted to metric (or quantitative) temporal logics (see e.g. [1, 21, 16]), which clearly reflects the fact that temporal logic in general is more developed than spatial logic. For example, starting with Kamp's [18] classical result on the expressive completeness of temporal logic with respect to monadic first-order logic, a beautiful theory comparing the expressive power of first-order, second-order and temporal languages for trees and linear orderings has been developed [10, 24]. Nothing like this has been done for spatial logics. We hope this paper, which has grown up from [29, 28], will help to fill the gap.

2 The logics

2.1 First-order metric logic $\mathcal{FM}[M]$

Let $M \subseteq \mathbb{R}^+$ and $0 \in M$. The language $\mathcal{FM}[M]$ (of first-order metric logic) contains a countably infinite set $c_1, c_2, \dots$ of constant symbols, a countably infinite set $x_1, x_2, \dots$ of individual variables, a countably infinite set $P_1, P_2, \dots$ of unary predicate symbols, the equality symbol $\doteq$, two (possibly infinite) sets of binary predicates

$$\boldsymbol{\delta}(\_,\_) < a \quad \text{and} \quad \boldsymbol{\delta}(\_,\_) = b \quad (a, b \in M),$$

the Booleans (including the propositional constants $\top$ for *verum* and $\bot$ for *falsum*), and the quantifier $\exists x_i$ for every variable x_i. Thus, the atomic formulas of $\mathcal{FM}[M]$ are of the form

$$\top, \quad \bot, \quad \boldsymbol{\delta}(t,t') < a, \quad \boldsymbol{\delta}(t,t') = a, \quad t \doteq t', \quad \text{and} \quad P_i(t),$$

where t and t' are *terms*, i.e. variables or constants, and $a \in M$. Compound $\mathcal{FM}[M]$-formulas are obtained from atomic ones by applying the Booleans and quantifiers in the usual way.

We use $\boldsymbol{\delta}(t_1,t_2) > a$ as an abbreviation for $\neg\boldsymbol{\delta}(t_1,t_2) < a \wedge \neg\boldsymbol{\delta}(t_1,t_2) = a$. Moreover, if $M = \mathbb{Q}^+$ we usually write $\mathcal{FM}$ instead of $\mathcal{FM}[\mathbb{Q}^+]$. The same abbreviation applies to the languages $\mathcal{MS}[M]$ introduced below. $\mathcal{FM}^2[M]$ denotes the two-variable fragment of $\mathcal{FM}[M]$, that is, the set of all $\mathcal{FM}[M]$-formulas containing occurrences of at most two variables, say, x and y.

$\mathcal{FM}[M]$-formulas are interpreted in structures of the form

$$\mathfrak{A} = \left\langle W, d, P_1^{\mathfrak{A}}, \ldots, c_1^{\mathfrak{A}}, \ldots \right\rangle,$$

where $\langle W, d\rangle$ is a distance space, the $P_i^{\mathfrak{A}}$ are subsets of W interpreting the unary predicates P_i, and the $c_i^{\mathfrak{A}}$ elements of W interpreting the constants c_i. An *assignment* $\mathfrak{a}$ in $\mathfrak{A}$ is a function assigning elements of W to variables. The pair $\mathfrak{M} = \langle \mathfrak{A}, \mathfrak{a}\rangle$ will be called an $\mathcal{FM}[M]$-*model.* For a term t, let $t^{\mathfrak{M}}$ denote $c_i^{\mathfrak{A}}$ if t is the constant c_i, and $\mathfrak{a}(x)$ if t is the variable x. Now, the *truth-relation* $\mathfrak{M} \models \varphi$, for an $\mathcal{FM}[M]$-fomula φ, is defined inductively as follows:

- $\mathfrak{M} \models \top$ and $\mathfrak{M} \not\models \bot$;
- $\mathfrak{M} \models \boldsymbol{\delta}(t_1,t_2) < a$ iff $d(t_1^{\mathfrak{M}}, t_2^{\mathfrak{M}}) < a$;
- $\mathfrak{M} \models \boldsymbol{\delta}(t_1,t_2) = a$ iff $d(t_1^{\mathfrak{M}}, t_2^{\mathfrak{M}}) = a$;
- $\mathfrak{M} \models t_1 \doteq t_2$ iff $t_1^{\mathfrak{M}} = t_2^{\mathfrak{M}}$;
- $\mathfrak{M} \models P_i(t)$ iff $t^{\mathfrak{M}} \in P_i^{\mathfrak{A}}$;
- $\mathfrak{M} \models \exists x_i\, \varphi$ iff $\langle \mathfrak{A}, \mathfrak{b}\rangle \models \varphi$ for some assignment $\mathfrak{b}$ in $\mathfrak{A}$ that may differ from $\mathfrak{a}$ only on x_i;
- $\mathfrak{M} \models \neg\varphi$ iff $\mathfrak{M} \not\models \varphi$;
- $\mathfrak{M} \models \varphi \wedge \psi$ iff $\mathfrak{M} \models \varphi$ and $\mathfrak{M} \models \psi$.

Unfortunately, from the computational point of view, the constructed logic turns out to be too expressive. We have the following undecidability result, where $\mathcal{M}$ is the class of all metric spaces and $\mathcal{D}_{tr}$ the class of all distance spaces satisfying the triangular inequality, (2).

Theorem 2. (i) *Let $\mathcal{K}$ be any class of distance spaces containing $\mathcal{M}$. Then the satisfiability problem for $\mathcal{FM}[\mathbb{N}]$-formulas in $\mathcal{K}$ is undecidable.*

(ii) *The satisfiability problem for $\mathcal{FM}^2[\mathbb{N}]$-formulas in any class $\mathcal{A}$ of distance spaces such that $\mathcal{A} \subseteq \mathcal{D}_{tr}$ and $\langle \mathbb{R}^2, d_2\rangle \in \mathcal{A}$ is undecidable as well.*

Proof. To prove the former claim, it suffices to observe that $\mathcal{FM}[\mathbb{N}]$ is powerful enough to interpret the theory of graphs (i.e., the theory of structures $\langle W, R\rangle$, where R is a symmetric and reflexive binary relation on W), which is known to be hereditarily undecidable [23]. Indeed, let $\varphi(x,y)$ be the formula

$$\boldsymbol{\delta}(x,y) = 1 \vee \boldsymbol{\delta}(x,y) = 0.$$

Given a graph $\langle W, R\rangle$, we can define a metric space $\langle W, d\rangle$ by taking, for all $a, b \in W$,

$$d(a,b) = \begin{cases} 0, & \text{if } a = b, \\ 1, & \text{if } a \neq b \text{ and } aRb, \\ 2, & \text{if not } aRb. \end{cases}$$

We then clearly have $\langle W, d\rangle \models \varphi[a,b]$ iff aRb.

(ii) follows from Theorem 3 (i) and Theorem 5 to be proved below. ❑

2.2 'Modal' metric logic $\mathcal{MS}[M]$

As an alternative to the first-order language $\mathcal{FM}[M]$, we now introduce a purely propositional language $\mathcal{MS}[M]$, whose 'distance operators' are similar to various operators considered in modal logic. The alphabet of $\mathcal{MS}[M]$ contains the following symbols:

- an infinite list of *set* (or *region*) *variables* $X_1, X_2, \dots$;
- an infinite list of *location constants* $c_1, c_2, \dots$;
- a *set constant* $\{c\}$ for every location constant c;
- the set constants $\top$ and $\bot$;
- the Boolean operators for set terms ($\sqcap$ and $\neg$) and formulas ($\wedge$ and $\neg$);
- the equality symbol $\doteq$ for set terms as well as the symbol $\sqsubseteq$ for elementship;
- the set term constructors $\mathsf{E}^{<a}$, $\mathsf{E}^{>a}$, $\mathsf{E}^{=a}$ and $\mathsf{E}^{>a}_{<b}$ (and their duals $\mathsf{A}^{<a}$, $\mathsf{A}^{>a}$, $\mathsf{A}^{=a}$, $\mathsf{A}^{>a}_{<b}$), where $a, b \in M$ and $a < b$.

Each set variable and set constant is an (*atomic*) *set term.* If s and t are set terms, so are

$$s \sqcap t, \quad \neg s, \quad \mathsf{E}^{<a}s, \quad \mathsf{E}^{>a}s, \quad \mathsf{E}^{=a}s, \quad \text{and} \quad \mathsf{E}^{>a}_{<b},$$

where $a \in M$. *Atomic formulas* of $\mathcal{MS}[M]$ are of the form:

- $c \sqsubseteq s$, where c is a location constant and s a set term,
- $s \doteq t$, where s and t are set terms,
- $\boldsymbol{\delta}(c_1, c_2) = a$ and $\boldsymbol{\delta}(c_1, c_2) < a$, where c_1, c_2 are location constants and $a \in M$.

Finally, an $\mathcal{MS}[M]$-*formula* is simply a Boolean combination of atomic ones.

As we have already mentioned, the language $\mathcal{MS}[M]$ contains a number of constructors known from modal and description logic. First, we have an analogue of the difference operator [6]: $\mathsf{A}^{>0}t$ (i.e., $\neg\mathsf{E}^{>0}\neg t$) says that t holds 'everywhere but here.' The universal modalities of [13], denoted here by $\Box$ ('everywhere') and $\Diamond$ ('somewhere'), can be defined by putting

$$\Box t := t \sqcap \mathsf{A}^{>0}t \qquad \text{and} \qquad \Diamond t := t \sqcup \mathsf{E}^{>0}t.$$

Furthermore, the set constants $\{c\}$ play the role of nominals, [2]. Using these we can state, for example, that

$$(\mathsf{E}^{\leq 1100}\{Leipzig\} \sqcap \mathsf{E}^{\leq 1100}\{Malaga\}) \sqsubseteq France.$$

I.e., 'if you are not more than 1100 km away from Leipzig and not more than 1100 km away from Malaga, then you are in France.' Here, $s \sqsubseteq t$ stands for $s \sqcap t \doteq s$.

An $\mathcal{MS}[M]$-*model* is a structure of the form:

$$\mathfrak{B} = \left\langle W, d, X_1^{\mathfrak{B}}, X_2^{\mathfrak{B}}, \dots, c_1^{\mathfrak{B}}, c_2^{\mathfrak{B}} \dots \right\rangle,$$

where $\langle W, d\rangle$ is a distance space, the $X_i^{\mathfrak{B}}$ are subsets of W, and the $c_i^{\mathfrak{B}}$ are elements of W. Thus, $\mathfrak{B}$ defines explicitly the values of set variables and location constants. The *value* $s^{\mathfrak{B}}$ of an arbitrary $\mathcal{MS}[M]$-term in $\mathfrak{B}$ is computed inductively as follows:

- $\{c_i\}^{\mathfrak{B}} = \{c_i^{\mathfrak{B}}\}$, where $\{c_i\}$ is a set constant;
- $(\top)^{\mathfrak{B}} = W$ and $(\bot)^{\mathfrak{B}} = \emptyset$;

- $(s_1 \sqcap s_2)^{\mathfrak{B}} = s_1^{\mathfrak{B}} \cap s_2^{\mathfrak{B}}$, where s_1 and s_2 are set terms;
- $(\neg s)^{\mathfrak{B}} = W - s^{\mathfrak{B}}$;
- $(\mathsf{E}^{=a} s)^{\mathfrak{B}} = \{x \in W : \exists y \in W \left(d(x,y) = a \;\&\; y \in s^{\mathfrak{B}}\right)\}$;
- $(\mathsf{E}^{<a} s)^{\mathfrak{B}} = \{x \in W : \exists y \in W \left(d(x,y) < a \;\&\; y \in s^{\mathfrak{B}}\right)\}$;
- $(\mathsf{E}^{>a} s)^{\mathfrak{B}} = \{x \in W : \exists y \in W \left(d(x,y) > a \;\&\; y \in s^{\mathfrak{B}}\right)\}$;
- $(\mathsf{E}^{>a}_{<b} s)^{\mathfrak{B}} = \{x \in W : \exists y \in W \left(a < d(x,y) < b \;\&\; y \in s^{\mathfrak{B}}\right)\}$.

The truth-relation $\mathfrak{B} \models \varphi$, φ an $\mathcal{MS}[M]$-formula, is defined in the expected way:

- $\mathfrak{B} \models c \sqsubseteq s$ iff $c^{\mathfrak{B}} \in s^{\mathfrak{B}}$,
- $\mathfrak{B} \models s_1 \doteq s_2$ iff $s_1^{\mathfrak{B}} = s_2^{\mathfrak{B}}$,
- $\mathfrak{B} \models \boldsymbol{\delta}(c_1, c_2) = a$ iff $d(c_1^{\mathfrak{B}}, c_2^{\mathfrak{B}}) = a$,
- $\mathfrak{B} \models \boldsymbol{\delta}(c_1, c_2) < a$ iff $d(c_1^{\mathfrak{B}}, c_2^{\mathfrak{B}}) < a$,

plus the standard definitions for the Boolean connectives.

In what follows we will be using abbreviations like $\mathsf{E}^{\leq a} s$, $\mathsf{E}^{\geq a}$, $\mathsf{E}^{\geq a}_{\leq b} s$, $\mathsf{E}^{\geq a}_{<b}$ and $\mathsf{E}^{>a}_{\leq b}$, the meaning of which should be clear. For instance, $\mathsf{E}^{\geq a}_{\leq b} s$ stands for

$$\mathsf{E}^{>a}_{<b} s \sqcup \mathsf{E}^{=a} s \sqcup \mathsf{E}^{=b} s.$$

Every $\mathcal{FM}$-structure $\mathfrak{A} = \left\langle W, d, P_1^{\mathfrak{A}}, \dots, c_1^{\mathfrak{A}}, \dots \right\rangle$ gives rise to its $\mathcal{MS}$-counterpart

$$\mathfrak{A}_* = \left\langle W, d, X_1^{\mathfrak{A}_*}, \dots, c_i^{\mathfrak{A}_*}, \dots \right\rangle,$$

where $X_i^{\mathfrak{A}_*} = P_i^{\mathfrak{A}}$ and $c_i^{\mathfrak{A}_*} = c_i^{\mathfrak{A}}$ for all i. This correspondence is clearly bijective. If an $\mathcal{FM}[M]$-structure (or an $\mathcal{MS}[M]$-model) is based on a metric space, we call it a *metric* $\mathcal{FM}[M]$-structure ($\mathcal{MS}[M]$-model).

The theorem we are about to prove shows that $\mathcal{MS}[M]$ is expressively complete for (i.e., has the same expressive power as) the two-variable fragment of $\mathcal{FM}[M]$.

Theorem 3. *For every $\mathcal{MS}[M]$-formula φ there exists an $\mathcal{FM}^2[M]$-sentence $\varphi^\dagger$ such that its length is linear in the length of φ and, for any metric $\mathcal{FM}$-structure $\mathfrak{A}$, we have*

$$\mathfrak{A} \models \varphi^\dagger \quad \textit{iff} \quad \mathfrak{A}_* \models \varphi.$$

Conversely, for every $\mathcal{FM}^2[M]$-sentence φ there is an $\mathcal{MS}[M]$-formula $\varphi^\ddagger$ such that its length is exponential in the length of φ and, for any metric $\mathcal{FM}$-structure $\mathfrak{A}$, we have

$$\mathfrak{A} \models \varphi \quad \textit{iff} \quad \mathfrak{A}_* \models \varphi^\ddagger.$$

Proof. Assume, for simplicity, that $M = \mathbb{Q}^+$. The proof of the former claim is pretty standard; cf. [9]. First we translate set terms occurring in φ into $\mathcal{FM}^2$-formulas with at most one free variable and then extend the translation to subformulas of φ using only two variables, x and y.

Let z and z' be metavariables ranging over $\{x, y\}$. The translation $\cdot^\dagger$ of set terms is defined inductively as follows:

$$(X_i)^\dagger = P_i(x);$$
$$(\top)^\dagger = \top \text{ and } (\bot)^\dagger = \bot;$$
$$(\{c_i\})^\dagger = (c_i \doteq x);$$
$$(s_1 \sqcap s_2)^\dagger = s_1^\dagger[x/z] \wedge s_2^\dagger[x/z']), \text{ where } z, z' \text{ are free in } s_1^\dagger \text{ and } s_2^\dagger, \text{ respectively;}$$
$$(\neg s)^\dagger = \neg s^\dagger;$$
$$(\mathsf{E}^{<a} s)^\dagger = \exists z\, (\boldsymbol{\delta}(z', z) < a \wedge s^\dagger(z)), \text{ where } z \neq z' \text{ is free in } s^\dagger;$$
$$(\mathsf{E}^{>a} s)^\dagger = \exists z\, (\boldsymbol{\delta}(z', z) > a \wedge s^\dagger(z)), \text{ where } z \neq z' \text{ is free in } s^\dagger;$$
$$(\mathsf{E}^{=a} s)^\dagger = \exists z\, (\boldsymbol{\delta}(z', z) = a \wedge s^\dagger(z)), \text{ where } z \neq z' \text{ is free in } s^\dagger;$$
$$(\mathsf{E}^{>a}_{<b} s)^\dagger = \exists z\, (a < \boldsymbol{\delta}(z', z) < b \wedge s^\dagger(z)), \text{ where } z \neq z' \text{ is free in } s^\dagger.$$

Now, turning to $\mathcal{MS}$-formulas, we define $\cdot^\dagger$ as commuting with the Booleans and by taking

$$(c \sqsubseteq s)^\dagger = s^\dagger[c/z], \text{ where } z \text{ is free in } s^\dagger;$$
$$(s_1 \doteq s_2)^\dagger = \forall x\, (s_1^\dagger[x/z_1] \leftrightarrow s_2^\dagger[x/z_2]), \; z_1, z_2 \text{ free in } s_1^\dagger, s_2^\dagger, \text{ respectively;}$$
$$(\boldsymbol{\delta}(c_1, c_2) = a)^\dagger = (\boldsymbol{\delta}(c_1, c_2) = a);$$
$$(\boldsymbol{\delta}(c_1, c_2) < a)^\dagger = (\boldsymbol{\delta}(c_1, c_2) < a).$$

By a straightforward induction one can easily check that $\mathfrak{A} \models \varphi^\dagger$ iff $\mathfrak{A}_* \models \varphi$.

To define the converse translation, we first observe that the following transformations of an $\mathcal{FM}^2$-formula $\varphi(x, y)$ result in an equivalent formula with respect to metric spaces: every occurrence of equality $t_1 \doteq t_2$ can be replaced by $\boldsymbol{\delta}(t_1, t_2) = 0$; $\boldsymbol{\delta}(t, t) = 0$ by $\top$; $\boldsymbol{\delta}(t, t) < 0$ by $\bot$; $\boldsymbol{\delta}(t, t) = a$ by $\bot$ if $a > 0$; $\boldsymbol{\delta}(t, t) < a$ by $\top$ if $a > 0$; $\boldsymbol{\delta}(y, t) = a$ by $\boldsymbol{\delta}(t, y) = a$; $\boldsymbol{\delta}(y, t) < a$ by $\boldsymbol{\delta}(t, y) < a$; $\boldsymbol{\delta}(t, x) = a$ by $\boldsymbol{\delta}(x, t) = a$, and $\boldsymbol{\delta}(t, x) < a$ by $\boldsymbol{\delta}(x, t) < a$. In what follows, we assume that these transformations have been applied to all our formulas, in particular, to φ.

We distinguish between three types of atomic formulas in $\mathcal{FM}^2$: *binary atoms* are of the form $\boldsymbol{\delta}(x, y) < a$ or $\boldsymbol{\delta}(x, y) = a$ (they have two free variables); *unary atoms* are of the form $\boldsymbol{\delta}(x, c) < a$, $\boldsymbol{\delta}(x, c) = a$, $P_i(x)$, $P_i(y)$, $\boldsymbol{\delta}(c, y) < a$ or $\boldsymbol{\delta}(c, y) = a$ (having only one free variable); atoms without free variables can be called *nullary*.

Given an $\mathcal{FM}^2$-sentence φ, we first translate it into a set term $\varphi^\star$ by inductively defining a map $\cdot^\star$ from subformulas of φ with ≤ 1 free variable into $\mathcal{MS}$-set terms (using the 'universal modalities' $\Box$ and $\Diamond$ defined on page 6).

1. If $\psi = \top$ then $\psi^\star = \top$, and if $\psi = \bot$ then $\psi^\star = \bot$.
2. If $\psi \in \{P_i(x), P_i(y)\}$ then $\psi^\star = X_i$.
3. If $\psi = P_i(c)$ then $\psi^\star = \Box(\{c\} \to X_i)$.
4. If $\psi \in \{\boldsymbol{\delta}(x, c) = a,\; \boldsymbol{\delta}(c, y) = a\}$ then $\psi^\star = \mathsf{E}^{=a}\{c\}$.
5. If $\psi = \boldsymbol{\delta}(c_1, c_2) = a$ then $\psi^\star = \Box(\{c_1\} \to \mathsf{E}^{=a}\{c_2\})$.
6. If $\psi = \boldsymbol{\delta}(c_1, c_2) < a$ then $\psi^\star = \Box(\{c_1\} \to \mathsf{E}^{<a}\{c_2\})$.
7. If $\psi \in \{\boldsymbol{\delta}(x, c) < a,\; \boldsymbol{\delta}(c, y) < a\}$ then $\psi^\star = \mathsf{E}^{<a}\{c\}$.
8. If $\psi = \chi_1 \wedge \chi_2$ then $\psi^\star = \chi_1^\star \sqcap \chi_2^\star$.
9. If $\psi = \neg\chi$ then $\psi^\star = \neg(\chi^\star)$.

The remaining cases of $\psi = \exists y \chi(x, y)$ and $\psi = \exists x \chi(x, y)$ are more sophisticated. We consider only the former. The formula $\chi(x, y)$ can be regarded as a Boolean combination of binary atoms β_i and formulas $\upsilon_i(x)$ and $\xi_i(y)$ with at most one free variable. Denote this Boolean combination by κ, i.e.,

$$\chi(x, y) = \kappa(\beta_1, \dots, \beta_r, \upsilon_1(x), \dots, \upsilon_l(x), \xi_1(y), \dots, \xi_s(y)).$$

Let us first move all components in κ without free y out of the scope of the outmost $\exists y$ in ψ. Then ψ can be equivalently rewritten as

$$\bigvee_{\langle \nu_1, \dots, \nu_l \rangle \in \{\top, \bot\}^l} \Big(\exists y\, \kappa(\beta_1, \dots, \beta_r, \nu_1, \dots, \nu_l, \xi_1, \dots, \xi_s) \wedge \bigwedge_{1 \le i \le l} (\upsilon_i \leftrightarrow \nu_i) \Big).$$

Now let $0 = a_0 < a_1 < \dots < a_n$ be the list of all rational numbers occurring in ψ together with 0. So this list is non-empty. Consider the set $\mathcal{R}_\psi$ containing the following formulas:

- $\boldsymbol{\delta}(x, y) = a_i$, for $i \le n$;
- $a_i < \boldsymbol{\delta}(x, y) < a_{i+1}$, for $i < n$;
- $\boldsymbol{\delta}(x, y) > a_n$.

For every $\beta \in \mathcal{R}_\psi$ and every binary atom β_i in ψ, we have either $\beta \models \beta_i$ or $\beta \models \neg\beta_i$. In other words, by assigning a truth-value to some β in $\mathcal{R}_\psi$, we fix the truth values of all binary atoms in ψ. Let $\beta_i^\beta = \top$ if $\beta \models \beta_i$, and $\beta_i^\beta = \bot$, otherwise. Then ψ is equivalent to the formula

$$\bigvee_{\langle \nu_1, \dots, \nu_l \rangle \in \{\top, \bot\}^l} \Big(\bigvee_{\beta \in \mathcal{R}_\varphi} \exists y \left(\beta \wedge \kappa(\beta_1^\beta, \dots, \beta_r^\beta, \nu_1, \dots, \nu_l, \xi_1, \dots, \xi_s) \right) \wedge \bigwedge_{1 \le i \le l} (\upsilon_i \leftrightarrow \nu_i) \Big).$$

Next, we replace each $\beta \in \mathcal{R}_\varphi$ by the distance operator α^* defined by taking

- $(\boldsymbol{\delta}(x, y) = a_i)^* = \mathsf{E}^{=a_i}$, for $i \le n$;
- $(a_i < \boldsymbol{\delta}(x, y) < a_{i+1})^* = \mathsf{E}^{>a_i}_{<a_{i+1}}$, for $i < n$;
- $(\boldsymbol{\delta}(x, y) > a_n)^* = \mathsf{E}^{>a_n}$,

and recursively compute the values of $\upsilon_i^\star$ and $\xi_i^\star$. This yields

$$\psi^\star = \bigsqcup_{\langle \nu_1, \dots, \nu_l \rangle \in \{\top, \bot\}^l} \Big(\bigsqcup_{\beta \in \mathcal{R}_\varphi} \beta^* \left(\kappa(\beta_1^\beta, \dots, \beta_r^\beta, \nu_1, \dots, \nu_l, \xi_1^\star, \dots, \xi_s^\star) \right) \sqcap \bigsqcap_{1 \le i \le l} (\upsilon_i^\star \leftrightarrow \nu_i) \Big).$$

Finally, we put $\varphi^\ddagger = (\varphi^\star \doteq \top)$. It should be clear from the construction that

$$\mathfrak{A} \models \varphi \quad \text{iff} \quad \mathfrak{A}_* \models \varphi^\ddagger.$$

The reader can restore details of the proof using the example below. ❑

Example 4. Consider the $\mathcal{FM}^2$-formula

$$\varphi = \exists y \Big(\exists x \big(\boldsymbol{\delta}(x, y) > 0 \wedge P_i(x) \big) \wedge \neg P_i(y) \Big).$$

Let $\xi_1(y) = \exists x\big(\boldsymbol{\delta}(x,y) > 0 \wedge P_i(x)\big)$ and $\xi_2(y) = \neg P_i(y)$. Then, we represent φ as

$$\exists y\,(\xi_1(y) \wedge \xi_2(y))$$

which is equivalent to

$$\exists y\big(\boldsymbol{\delta}(x,y) = 0 \wedge \xi_1(y) \wedge \xi_2(y)\big) \vee \exists y\big(\boldsymbol{\delta}(x,y) > 0 \wedge \xi_1(y) \wedge \xi_2(y)\big).$$

Thus, we obtain the $\mathcal{MS}$-set term

$$\varphi^\star = \mathsf{E}^{=0}(\xi_1^\star \sqcap \xi_2^\star) \sqcup \mathsf{E}^{>0}(\xi_1^\star \sqcap \xi_2^\star),$$

where $\xi_1^\star = \mathsf{E}^{=0}(\bot \sqcap X_i) \sqcup \mathsf{E}^{>0}(\top \sqcap X_i)$ or, equivalently, $\xi_1^\star = \mathsf{E}^{>0}X_i$, and $\xi_2^\star = \neg X_i$. So the resulting translation is

$$\varphi^\star = \mathsf{E}^{=0}(\mathsf{E}^{>0}X_i \sqcap \neg X_i) \sqcup \mathsf{E}^{>0}(\mathsf{E}^{>0}X_i \sqcap \neg X_i).$$

Using the universal $\Diamond$, we finally obtain

$$\varphi^\ddagger = \big(\Diamond(\mathsf{E}^{>0}X_i \sqcap \neg X_i) = \top\big).$$

The reader can easily check that φ and $\varphi^\dagger$ indeed say the same.

3 Undecidability

In this section we show that the satisfiability problem for fragments of $\mathcal{MS}$ containing distance operators like $\mathsf{E}^{>0}_{\leq a}$ is undecidable in natural classes of spaces satisfying the triangular inequality. Consider the following languages:

- $\mathcal{MS}_1$ is the fragment of $\mathcal{MS}$ whose set terms are constructed from set variables, the operators $\sqcap$, $\neg$, and $\mathsf{E}^{>0}_{\leq b}$ for $b \in \mathbb{N}$, and whose formulas are Boolean combinations of atoms of the form $s \sqsubseteq t$.
- $\mathcal{MS}_2$ results from $\mathcal{MS}_1$ by replacing $\mathsf{E}^{>0}_{\leq b}$ with $\mathsf{E}^{>0}_{<b}$.
- $\mathcal{MS}_3$ results from $\mathcal{MS}_1$ by replacing $\mathsf{E}^{>0}_{\leq b}$ with $\mathsf{E}^{\geq 1}_{\leq b}$.
- $\mathcal{MS}_4$ results from $\mathcal{MS}_1$ by replacing $\mathsf{E}^{>0}_{\leq b}$ with $\mathsf{E}^{\geq 1}_{<b}$.

Theorem 5. *Let $\mathcal{K} \subseteq \mathcal{D}_{tr}$ contain $\langle \mathbb{R}^2, d_2\rangle$. Then the satisfiability problem for $\mathcal{MS}_i$-formulas in $\mathcal{K}$ is undecidable, for any $1 \leq i \leq 4$.*

Proof. We consider only $\mathcal{MS}_1$; the other languages are treated analogously. The proof is by reduction of the undecidable $\mathbb{N} \times \mathbb{N}$-tiling problem (see [32, 4] and references therein). We remind the reader that the tiling problem for $\mathbb{N} \times \mathbb{N}$ is formulated as follows: given a finite set $\mathcal{T} = \{T_1, \dots, T_l\}$ of tiles (i.e., squares T_i with colors $left(T_i)$, $right(T_i)$, $up(T_i)$, and $down(T_i)$ on their edges), determine whether tiles in $\mathcal{T}$ can cover the grid $\mathbb{N} \times \mathbb{N}$ in such a way that the colors of adjacent edges on adjacent tiles match, or, more precisely, whether there exists a function $\tau : \mathbb{N} \times \mathbb{N} \mapsto \mathcal{T}$ such that, for all $n, m \in \mathbb{N}$, we have

$$right(\tau(n,m)) = left(\tau(n+1,m)),$$
$$up(\tau(n,m)) = down(\tau(n,m+1)).$$

So, suppose a set of tiles $\mathcal{T} = \{T_1, \dots, T_l\}$ is given. Our aim is to construct an $\mathcal{MS}_1$-formula that is satisfiable in $\mathcal{K}$ iff $\mathcal{T}$ can tile $\mathbb{N} \times \mathbb{N}$.

Note that $\mathsf{E}^{\leq b}t$ is definable in $\mathcal{MS}_1$: $\mathsf{E}^{\leq b}t := t \sqcup \mathsf{E}^{\geq 0}_{\leq b}t$. Hence, $\mathsf{A}^{\leq b}$ is definable as well. Take set variables $Z_1, \dots, Z_l$, $X_0, \dots, X_4, Y_0, \dots, Y_4$. Let $\chi_{i,j} = \mathsf{A}^{\leq 9}(X_i \sqcap Y_j)$, for $i, j \leq 4$, and let Γ be the set of the following formulas, where $i, j \leq 4$ and $k \leq l$:

$$X_i \sqcap Y_j \sqsubseteq \mathsf{E}^{\leq 9}\chi_{i,j}, \quad \chi_{i,j} \sqsubseteq \mathsf{A}^{\geq 0}_{\leq 80}\neg\chi_{i,j}, \quad \chi_{i,j} \sqsubseteq \neg\chi_{m,n} \ ((i,j) \neq (m,n)), \tag{4}$$

$$\chi_{i,j} \sqsubseteq \bigsqcup_{k \leq l} \mathsf{A}^{\leq 9} Z_k, \quad Z_m \sqsubseteq \neg Z_n \ (n \neq m), \tag{5}$$

$$\chi_{i,j} \sqcap Z_k \sqsubseteq \mathsf{E}^{\leq 20}(\mathsf{E}^{\leq 20}\chi_{i,j} \sqcap \chi_{i+_5 1,j} \sqcap \bigsqcup_{right(T_k)=left(T_m)} Z_m), \tag{6}$$

$$\chi_{i,j} \sqcap Z_k \sqsubseteq \mathsf{E}^{\leq 20}(\mathsf{E}^{\leq 20}\chi_{i,j} \sqcap \chi_{i,j+_5 1} \sqcap \bigsqcup_{up(T_k)=down(T_m)} Z_m), \tag{7}$$

where $+_5$ denotes addition modulo 5.[2] The first formula in (4) is satisfied in a model $\mathfrak{B}$ iff $X_i^{\mathfrak{B}} \cap Y_j^{\mathfrak{B}}$ is the union of a set of spheres of radius 9. The second one is satisfied in $\mathfrak{B}$ iff the distance between any two distinct centers of spheres, all points in which belong to $X_i^{\mathfrak{B}} \cap Y_j^{\mathfrak{B}}$, is more than 80, while the third formula guarantees that the sets $\chi_{i,j}^{\mathfrak{B}}$ are pairwise disjoint. We think of $\chi_{i,j}^{\mathfrak{B}}$, for $i, j \leq 4$, as a finite family of infinite sets making up the grid for the tiling. The formulas in (5) ensure that every point of the grid is covered by some tile and that different tiles never cover the same point. Finally, the formulas in (6) and (7) give sufficient tiling-conditions in the vertical and horizontal direction, respectively.

Note that if $x \in \chi_{i,j}^{\mathfrak{B}}$ then, in view of (6), it follows that there exist $y \in \chi_{i+_5 1,j}^{\mathfrak{B}}$ and $z \in \chi_{i,j}^{\mathfrak{B}}$ for which $d(x,y) \leq 20$ and $d(y,z) \leq 20$. But then, by the triangular inequality, $d(x,z) \leq 40$, and so, by the second formula in (4), $x = z$. The situation in the vertical direction is similar.

We are going to show that the conjunction of formulas in $\{\neg(\chi_{0,0} \sqsubseteq \bot)\} \cup \Gamma$ is satisfiable in $\mathcal{K}$ iff $\mathcal{T}$ can tile $\mathbb{N} \times \mathbb{N}$. This will be done in two steps.

Lemma 6. *If $\mathcal{T}$ tiles $\mathbb{N} \times \mathbb{N}$, then $\{\neg(\chi_{0,0} \sqsubseteq \bot)\} \cup \Gamma$ is satisfiable in the 2-dimensional Euclidean space $\langle \mathbb{R}^2, d_2 \rangle$.*

Proof. Suppose $\tau : \mathbb{N} \times \mathbb{N} \mapsto \mathcal{T}$ is a tiling. For $r \in \mathbb{R}^2$, put

$$S(r) = \{y \in \mathbb{R}^2 : d_2(r,y) \leq 9\}.$$

Define a model $\mathfrak{B}$ on $\mathbb{R}^2$ by taking, for $i, j \leq 4$ and $k \leq l$,

$$X_i^{\mathfrak{B}} = \bigcup_{m,n \in \mathbb{N}} S(20(5n+i), 20m),$$

$$Y_j^{\mathfrak{B}} = \bigcup_{m,n \in \mathbb{N}} S(20n, 20(5m+j)),$$

$$Z_k^{\mathfrak{B}} = \bigcup_{\tau(n,m)=T_k} S(20n, 20m).$$

It is not difficult to see that this model satisfies $\{\neg(\chi_{0,0} \sqsubseteq \bot)\} \cup \Gamma$ (see Fig. 1). ❑

Lemma 7. *Suppose a model $\mathfrak{B}$ based on a space $\langle W, d \rangle \in \mathcal{D}_{tr}$ satisfies the conjunction of $\{\neg(\chi_{0,0} \sqsubseteq \bot)\} \cup \Gamma$. Then there exists a function $f : \mathbb{N} \times \mathbb{N} \mapsto W$ such that, for all $i, j \leq 4$ and $k_1, k_2 \in \mathbb{N}$,*

(a) $f(5k_1 + i, 5k_2 + j) \in \chi_{i,j}^{\mathfrak{B}}$;

[2] The first conjunct in the right hand sides of (6) and (7) is redundant if $\mathcal{K}$ consists of only symmetric spaces.

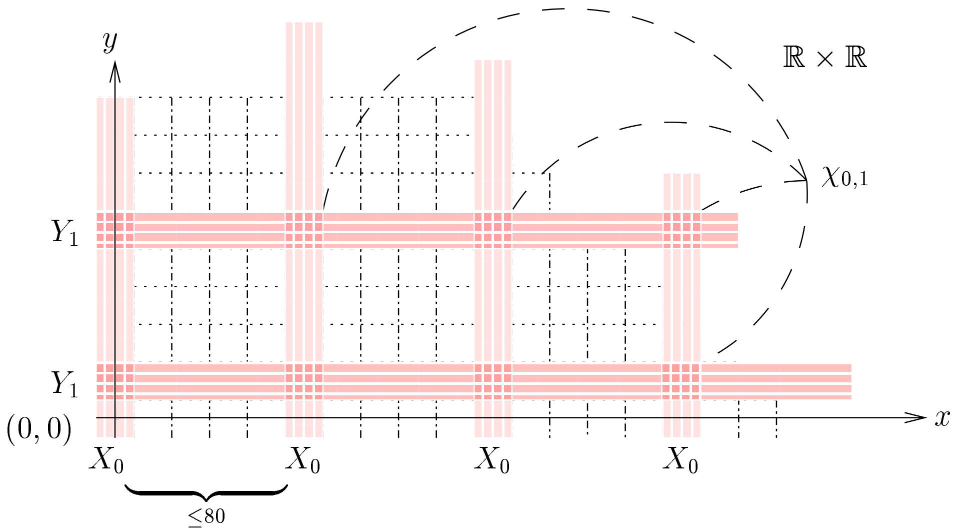

Figure 1: Building the grid.

(b) $d(f(k_1, k_2), f(k_1 + 1, k_2)) \leq 20$ *and* $d(f(k_1 + 1, k_2), f(k_1, k_2)) \leq 20$*;*

(c) $d(f(k_1, k_2), f(k_1, k_2 + 1)) \leq 20$ *and* $d(f(k_1, k_2 + 1), f(k_1, k_2)) \leq 20$.

The map $\tau : \mathbb{N} \times \mathbb{N} \mapsto \mathfrak{T}$ *defined by taking* $\tau(n, m) = T_k$ *iff* $f(n, m) \in Z_k^{\mathfrak{B}}$*, for all* $k \leq l$ *and all* $n, m \in \mathbb{N}$*, is a tiling.*

Proof. We define f inductively. Pick $f(0, 0) \in \chi_{0,0}^{\mathfrak{B}}$. By (6), we then find a sequence $w_n \in W$, for $n \in \mathbb{N}$, such that

- $w_0 = f(0, 0)$,
- $w_{5k+i} \in (\chi_{i,0})^{\mathfrak{B}}$ for all $i \leq 4$ and $k \in \mathbb{N}$,
- $d(w_n, w_{n+1}) \leq 20$ and $d(w_{n+1}, w_n) \leq 20$.

We put $f(n, 0) = w_n$ for all $n \in \mathbb{N}$. Similarly, by (7), we find a sequence v_n, for $n \in \mathbb{N}$, such that

- $v_0 = f(0, 0)$,
- $v_{5k+j} \in (\chi_{0,j})^{\mathfrak{B}}$ for all $j \leq 4$ and $k \in \mathbb{N}$,
- $d(v_n, v_{n+1}) \leq 20$ and $d(v_{n+1}, v_n) \leq 20$.

Put $f(0, m) = v_m$ for all $m \in \mathbb{N}$. Suppose now that f has been defined for all (m', n') with $m' + n' < m + n$ so that it satisfies conditions (a)–(c). Without loss of generality we can assume that $n = 5k_1$, $m = 5k_2 + 1$, for some $k_1, k_2 \in \mathbb{N}$. Then $f(n, m - 1) \in (\chi_{0,0})^{\mathfrak{B}}$, and hence $f(n, m - 1) \in (\mathsf{E}^{\leq 20}\chi_{0,1})^{\mathfrak{B}}$. So we can find a $w' \in (\chi_{0,1})^{\mathfrak{B}}$ with $d(f(n, m - 1), w') \leq 20$ and $d(w', f(n, m - 1)) \leq 20$. We then put $f(n, m) = w'$. It remains to prove that f still satisfies (a)–(c). To this end it suffices to show that $d(f(n - 1, m), w') \leq 20$ and $d(w', f(n - 1, m)) \leq 20$. We have $f(n - 1, m) \in (\chi_{4,1})^{\mathfrak{B}}$, and so there exists a $w'' \in (\chi_{0,1})^{\mathfrak{B}}$ such that $d(f(n - 1, m), w'') \leq 20$ and $d(w'', f(n - 1, m)) \leq 20$. Thus it is enough to show that $w' = w''$. Suppose otherwise. Then we have

- $d(w'', f(n - 1, m)) \leq 20$;
- $d(f(n - 1), m), f(n - 1, m - 1)) \leq 20$;

- $d(f(n-1,m-1), f(n,m-1)) \leq 20$;
- $d(f(n,m-1), w') \leq 20$.

By the triangular inequality, it follows that $d(w'', w') \leq 80$, contrary to the second formula in (4). It is readily seen now that τ is a tiling. ❑

This completes the proof of Theorem 5. ❑

4 Decidable logics of metric spaces

Consider the language $\mathcal{MS}^{\#}[M]$ whose set term constructors are the Booleans, $\mathsf{E}^{>a}$ and $\mathsf{E}^{\leq a}$, for all $a \in M$, their duals $\mathsf{A}^{>a}$ and $\mathsf{A}^{\leq a}$, as well as the nominal constructor which gives the set term $\{c\}$ for any location constant c. The atomic formulas of $\mathcal{MS}^{\#}[M]$ are $\boldsymbol{\delta}(c,d) < a$ and $\boldsymbol{\delta}(c,d) = a$, for $a \in M$, $c \sqsubseteq s$ and $s_1 \doteq s_2$, where c, d are constants and s_1, s_2 set terms. Complex $\mathcal{MS}^{\#}[M]$-formulas are Boolean combinations of atoms.

Note that in $\mathcal{MS}^{\#}[\mathbb{Q}^+]$ we can express all constraints from Example 1, save (G) (the formula $\mathit{house} \sqsubseteq \mathsf{E}^{>0.5}\mathsf{tube\_station} \sqcap \mathsf{E}^{\leq 1}\mathsf{tube\_station}$ is clearly not equivalent to $\mathit{house} \sqsubseteq \mathsf{E}^{\geq 0.5}_{\leq 1}\mathsf{tube\_station}$). Note also that the difference operator and the universal modality are still definable in $\mathcal{MS}^{\#}[M]$.

The satisfiability problem for arbitrary $\mathcal{MS}^{\#}[M]$-formulas can be reduced to the satisfiability problem for $\mathcal{MS}^{\#}[M]$-formulas without the nominal constructor. Indeed, suppose that $c_1, \dots, c_n$ are all location constants occurring in a $\mathcal{MS}^{\#}[M]$-formula φ as set terms $\{c_i\}$. Take fresh set variables $X_1, \dots, X_n$ and let φ' be the result of replacing all $\{c_i\}$ in φ with X_i. It is readily checked that φ is satisfiable in a model based on a metric space $\mathfrak{D}$ iff the formula

$$\varphi' \wedge \bigwedge_{i \leq n} \neg(X_i \sqcap \neg \mathsf{E}^{>0} X_i \doteq \bot)$$

is satisfiable in $\mathfrak{D}$.[3] Since our main concern is the finite model property and decidability, from now on we will assume that no nominals $\{c\}$ occur in $\mathcal{MS}^{\#}[M]$-formulas.

Our aim in the remaining part of this section is to prove that $\mathcal{MS}^{\#}[\mathbb{R}^+]$ interpreted in metric spaces has the finite model property and that $\mathcal{MS}^{\#}[\mathbb{Q}^+]$ interpreted in metric spaces is decidable. But before turning to the details of the proof, we introduce a relational semantics that enables us to use tools and techniques from standard modal logic.

4.1 Relational semantics

As we have two kinds of 'modal operators' in $\mathcal{MS}^{\#}[\mathbb{R}^+]$, namely, $\mathsf{E}^{\leq a}$ and $\mathsf{E}^{>a}$, *relational metric M-models* of $\mathcal{MS}^{\#}[\mathbb{R}^+]$ should be quadruples of the form

$$\mathfrak{S} = \langle W, (R_a)_{a \in M}, (R_{\overline{a}})_{a \in M}, \mathfrak{a} \rangle, \tag{8}$$

where W is a non-empty set, $(R_a)_{a \in M}$ and $(R_{\overline{a}})_{a \in M}$ are two families of binary relations on W, $M \subseteq \mathbb{R}_+$, and $\mathfrak{a}$ is an assignment in W associating with every set variable X_i a subset $\mathfrak{a}(X_i)$ of W and with every location constant c_i an element $\mathfrak{a}(c_i)$ of W. We understand relations $uR_a v$ and $uR_{\overline{a}}v$ as 'v is at most a (units) far from u' and 'v is more than a (units) far from u,' respectively.

The *value* $t^{\mathfrak{S}}$ of a set term t in $\mathfrak{S}$ is now inductively defined in the standard Kripkean way:

$$(\mathsf{E}^{\leq a} t)^{\mathfrak{S}} = \{w \in W : \exists v \in t^{\mathfrak{S}}\ wR_a v\},$$
$$(\mathsf{E}^{>a} t)^{\mathfrak{S}} = \{w \in W : \exists v \in t^{\mathfrak{S}}\ wR_{\overline{a}} v\}.$$

[3] Recall that we always have $0 \in M$.

The values of $\mathsf{A}^{\leq a}t$ and $\mathsf{A}^{>a}t$ are defined dually. (As we have no explicit metric in the relational model, there is no straightforward way to interpret atoms of the form $\boldsymbol{\delta}(c,d) < a$ or $\boldsymbol{\delta}(c,d) = a$. Satisfiability of these formulas will be simulated by other constructs introduced in the next section.)

Aiming to represent metric models by means of relational models, we have to impose a numbers of restrictions on the accessibility relations. Namely, we say that a model $\mathfrak{S}$ of the form (8) is *M-standard* if the following conditions are satisfied for all $a, b \in M$ and $w, u, v \in W$:

(i) $R_a \cup R_{\overline{a}} = W \times W$,

(ii) $R_a \cap R_{\overline{a}} = \emptyset$,

(iii) if uR_av and $a \leq b$, then uR_bv,

(iv) if $uR_{\overline{a}}v$ and $a \geq b$, then $uR_{\overline{b}}v$,

(v) uR_0v iff $u = v$,

(vi) if uR_av and vR_bw, then $uR_{a+b}w$ whenever $a + b \in M$,

(vii) uR_av iff vR_au and $uR_{\overline{a}}v$ iff $vR_{\overline{a}}u$.

Properties (v), (vi) and (vii) reflect axioms (1)–(3) of metric spaces. Note that as a consequence of (i), (ii) and (vi) we have:

(viii) if uR_av and $uR_{\overline{a+b}}w$ then $vR_{\overline{b}}w$.

With every metric space model $\mathfrak{B} = \langle W, d, X_1^{\mathfrak{B}}, \ldots, c_1^{\mathfrak{B}}, \ldots \rangle$ we can associate the relational metric M-model

$$\mathfrak{S}(\mathfrak{B}) = \langle W, (R_a)_{a\in M}, (R_{\overline{a}})_{a\in M}, \mathfrak{a} \rangle$$

in which the relations R_a and $R_{\overline{a}}$ are defined by taking

$$\forall w, v \in W \ (wR_av \leftrightarrow d(w, v) \leq a),$$
$$\forall w, v \in W \ (wR_{\overline{a}}v \leftrightarrow d(w, v) > a),$$

$\mathfrak{a}(X_i) = X_i^{\mathfrak{B}}$ and $\mathfrak{a}(c_i) = c_i^{\mathfrak{B}}$. Clearly, $\mathfrak{S}(\mathfrak{B})$ is M-standard. Moreover, the following obvious lemma shows that $\mathfrak{S}(\mathfrak{B})$ can be regarded as a relational representation of $\mathfrak{B}$.

Lemma 8. *For every metric space model $\mathfrak{B}$ and every set term $t \in \mathcal{MS}^{\#}[M]$, the value of t in $\mathfrak{B}$ coincides with the value of t in $\mathfrak{S}(\mathfrak{B})$.*

4.2 The finite model property

In this section we prove the following

Theorem 9. *An $\mathcal{MS}^{\#}[\mathbb{R}^+]$-formula φ is satisfiable in metric space model iff it is satisfiable in a finite metric space model.*

Proof. First we outline the idea of the proof. Suppose $\mathfrak{B} \models \varphi$ for some metric $\mathcal{MS}$-model $\mathfrak{B} = \langle W, d, X_1^{\mathfrak{B}}, \ldots, c_1^{\mathfrak{B}}, \ldots \rangle$. Depending on $\mathfrak{B}$, we transform φ into a set Φ, containing only formulas of the form $c \sqsubseteq t$, $s \doteq t$, $s \not\doteq t$, and $\boldsymbol{\delta}(c,d) = a$, in such a way that φ is satisfiable in a finite model whenever Φ is finitely satisfiable. Starting from Φ, we compute a finite set $M[\Phi]$ of real numbers containing, in particular, all the numbers occurring in Φ. Then we replace the metric d by a new metric d' with a (finite) range $M[\Phi]$. The new model $\mathfrak{B}_1$ still satisfies Φ. The next step is to filtrate (as in modal logic; see e.g. [5]) the relational metric model $\mathfrak{S} = \mathfrak{S}(\mathfrak{B}_1)$ through some suitable set of terms $cl(\Phi)$. To define $cl(\Phi)$, we first transform Φ into a set Φ' which, roughly speaking, is obtained from Φ by replacing every formula of the form $\boldsymbol{\delta}(c,d) = a$ with two formulas $d \sqsubseteq X^d$ and $c \sqsubseteq \mathsf{E}^{\leq a} X^d$, where the X^d are fresh set variables. The set $cl(\Phi)$ is the closure of the terms in Φ' under syntactical rules that are similar to the rules of the Fischer–Ladner closure for **PDL**-formulas (cf. [15]).

As a result of the filtration we get a *finite* relational metric model $\mathfrak{S}^f$. But unlike $\mathfrak{S}$, in general $\mathfrak{S}^f$ is not $M[\Phi]$-standard, which means that we cannot directly transform it into a finite metric space model. In fact, $\mathfrak{S}^f$ satisfies all the properties (i)–(vii) save (ii): there may exist $v \in W^f$ such that $wR_a v$ and $wR_{\overline{a}} v$, for some $w \in W^f$, and $a \in M[\Phi]$. To 'cure' these defects, we make copies of such 'bad' points v and modify the relations R_a and $R_{\overline{a}}$ in $\mathfrak{S}^f$ obtaining a finite standard relational metric model $\mathfrak{S}^*$. (The 'copying-method' was developed by the Bulgarian school of modal logic; see [11, 31]. Our technique follows [12]). The final step is to transform $\mathfrak{S}^*$ into a metric $\mathcal{MS}$-model $\mathfrak{B}^*$.

Let us now turn to technical details. Suppose $\mathfrak{B} \models \varphi$. Denote by $term(\varphi)$ the set of all set terms occurring in φ including all subterms; $sub(\varphi)$ stands for the set of all subformulas of φ. Define a set $\Phi = \Phi_1 \cup \Phi_2 \cup \Phi_3$ by taking:

$$
\begin{array}{rcl}
\Phi_1 & = & \{c \sqsubseteq t : (c \sqsubseteq t) \in sub(\varphi),\ \mathfrak{B} \models c \sqsubseteq t\} \cup \\
 & & \{c \sqsubseteq \neg t : (c \sqsubseteq t) \in sub(\varphi),\ \mathfrak{B} \not\models c \sqsubseteq t\}, \\
\Phi_2 & = & \{s \doteq t : (s \doteq t) \in sub(\varphi),\ \mathfrak{B} \models s \doteq t\} \cup \\
 & & \{s \not\doteq t : (s \doteq t) \in sub(\varphi),\ \mathfrak{B} \models s \not\doteq t\}, \\
\Phi_3 & = & \{\boldsymbol{\delta}(c,d) = a : \boldsymbol{\delta}(c,d) \text{ occurs in } \varphi\ ,\ a = d(\mathfrak{a}(c), \mathfrak{a}(d))\}.
\end{array}
$$

Note that the set of parameters from $\mathbb{R}$ that occur in Φ_3 depends on the model $\mathfrak{B}$ and not just on the initial formula φ. It should be clear from the definition that we have

Lemma 10. (1) $\mathfrak{B} \models \Phi$.

(2) *For every metric $\mathcal{MS}$-model $\mathfrak{B}'$, if $\mathfrak{B}' \models \Phi$ then $\mathfrak{B}' \models \varphi$.*

Next we construct $M[\Phi]$ and Φ'. Let

$$M(\Phi) = \{a \in \mathbb{R} : a \text{ occurs in } \Phi\} \cup \{0, 1\}.$$

So $M(\Phi)$ depends on $\mathfrak{B}$, whereas the cardinality of $M(\Phi)$ only depends on φ. Denote by γ the smallest natural number that is greater than all numbers in $M(\Phi)$ and define $M[\Phi]$ as

$$M[\Phi] = \{a_1 + \cdots + a_n < \gamma : a_1, \ldots, a_n \in M(\Phi),\ n < \omega\} \cup \{\gamma\} \cup \{0\}.$$

Let $\mu = \min\{M(\Phi) - \{0\}\}$ and let χ be the least natural number such that $\chi \geq \gamma/\mu$.

Lemma 11. $|M[\Phi]| \leq |M(\Phi)|^{\chi}$.

Proof. For any $a_1, \ldots, a_n \in M(\Phi) - \{0\}$ with $a_1 + \cdots + a_n < \gamma$ we have $n \leq \chi$. The claim follows immediately. ❑

For each location constant d occurring in Φ_3 we pick a new set variable X^d and define Φ_3', Φ' and $t(\Phi)$ by taking

$$
\begin{array}{rcl}
\Phi_3' & = & \{d \sqsubseteq X^d : d \text{ occurs in } \Phi_3\} \cup \\
 & & \{c \sqsubseteq \mathsf{A}^{\leq b} \neg X^d : \boldsymbol{\delta}(c,d) = a \in \Phi_3 \text{ and } b \in M[\Phi] \text{ maximal with } b < a\} \\
\Phi' & = & \Phi_1 \cup \Phi_2 \cup \Phi_3', \\
t(\Phi) & = & \{\, t : t \in term(\Phi')\}.
\end{array}
$$

So in particular, $t(\Phi)$ is closed under subterms. The *closure* $cl(\Phi)$ of $t(\Phi)$ is the smallest set T of terms such that $t(\Phi) \subseteq T$ and

1. T is closed under subterms;

2. if $t \in T$, then $\mathsf{A}^{\leq 0} t \in T$ whenever t is not of the form $\mathsf{A}^{\leq 0} s$;

3. if $\mathsf{A}^{\leq a} t \in T$ and $a \geq a_1 + \cdots + a_n$, for $a_i \in M[\Phi] - \{0\}$, then $\mathsf{A}^{\leq a_1} \ldots \mathsf{A}^{\leq a_n} t \in T$;

4. if $\mathsf{A}^{>a} t \in T$ and $b \in M[\Phi]$, then $\neg \mathsf{A}^{\leq b} \neg \mathsf{A}^{>a} t \in T$;

5. if $\mathsf{A}^{>a} t \in T$ and $b > a$ ($b \in M[\Phi]$), then $\mathsf{A}^{>b} t \in T$ and if $c + a \in M[\Phi]$ ($c \in M[\Phi]$), then $\neg \mathsf{A}^{>a+c} \neg \mathsf{A}^{>a} t \in T$.

Lemma 12. $|cl(\Phi)| \leq S(\Phi) = 2^{\chi+4} \cdot |t(\Phi)| \cdot |M[\Phi]|^{2\chi+3}$.

Proof. Observe that $cl(\Phi)$ can be obtained from $t(\Phi)$ step-by-step as follows:

- First, take the closure of $t(\Phi)$ under subterms and (5.) and denote the result by $cl_1(\Phi)$.
- Second, take the closure of $cl_1(\Phi)$ under subterms and (4.) and denote the result by $cl_2(\Phi)$, which is still closed under (5.).
- Third, take the closure of $cl_2(\Phi)$ under subterms and (3.), denote the result by $cl_3(\Phi)$ and notice that $cl_3(\Phi)$ is closed under (4.) and (5.).
- Finally, take the closure of $cl_3(\Phi)$ under subterms and (2.). This is closed under (1)–(5).

The following is now readily checked:

- $|cl_1(\Phi)|$ is bounded by $|t(\Phi)| \cdot 2^{\chi+1} \cdot |M[\Phi]|^{\chi+1}$, because the introduced terms are of the form $(\neg)\mathsf{A}^{>a_1}(\neg)\mathsf{A}^{>a_2}(\neg)\ldots(\neg)\mathsf{A}^{>a_k} t$, with $a_i - a_{i+1} \geq \mu$ and $(\neg)$ marking a possible occurence of $\neg$. The length k of such sequences of parameters a_i is bounded by $\chi + 1$, because $a_1 \leq \gamma$.
- $|cl_2(\Phi)|$ is bounded by $4 \cdot |cl_1(\Phi)| \cdot |M[\Phi]|$.
- $|cl_3(\Phi)|$ is bounded by $|cl_2(\Phi)| \cdot |M[\Phi]|^{\chi+1}$, because no chain $\mathsf{A}^{\leq a_1} \cdots \mathsf{A}^{\leq a_n}$ of length $> \chi$ is introduced when taking the closure under (2.).
- $|cl(\Phi)|$ is bounded by $2 \cdot |cl_3(\Phi)|$.

So we obtain that $|cl(\Phi)|$ is bounded by $S(\Phi) = 2^{\chi+4} \cdot |t(\Phi)| \cdot |M[\Phi]|^{2\chi+3}$. ❑

We are in a position now to prove the following:

Theorem 13. *Φ is satisfied in a metric $\mathcal{MS}$-model*

$$\mathfrak{B}^* = \langle W^*, d^*, X_1^*, \ldots, c_1^*, \ldots \rangle$$

such that $|W^*| \leq 2 \cdot 2^{S(\Phi)}$ and the range of d^* is a subset of $M[\Phi]$.

Proof. The proof proceeds via a sequence of lemmata. Recall that $\mathfrak{B} \models \Phi$.

We first show that Φ is satisfied in a metric $\mathcal{MS}$-model

$$\mathfrak{B}_1 = \left\langle W, d', X_1^{\mathfrak{B}_1}, \ldots, c_1^{\mathfrak{B}_1}, \ldots \right\rangle$$

such that

- the range of d' is a subset of $M = M[\Phi]$;
- $(X^d)^{\mathfrak{B}_1} = \{d^{\mathfrak{B}_1}\}$, for all constants d in Φ_3;
- $\mathfrak{B}_1 \models \Phi'$.

Indeed, define d' by taking

$$d'(w,v) = \min\{\gamma, a \in M : d(w,v) \leq a\}$$

for all $w, v \in W$, and let

$$\mathfrak{B}_1 = \left\langle W, d', X_1^{\mathfrak{B}_1}, \dots, c_1^{\mathfrak{B}_1}, \dots \right\rangle,$$

where $X_i^{\mathfrak{B}_1} = X_i^{\mathfrak{B}}$ for all X_i different from X^d, $(X^d)^{\mathfrak{B}_1} = \{d^{\mathfrak{B}}\}$ for all constants d in Φ_3, and $c_i^{\mathfrak{B}_1} = c_i^{\mathfrak{B}}$.

Clearly, the range of d' is a subset of M. Let us check that d' is a metric. It satisfies (1) because $0 \in M$. That d' is symmetric follows from the symmetry of d. To show (2), suppose $d'(w,v) + d'(v,u) \leq a$, for $a \in M$. By the definition of d', we then have $d(w,v) + d(v,u) \leq a$, and so $d(w,u) \leq a$. Hence $d'(w,u) \leq a$. Thus we have shown that

$$\{a \in M : d'(w,v) + d'(v,u) \leq a\} \subseteq \{a \in M : d'(w,u) \leq a\},$$

from which one can easily conclude that $d'(w,u) \leq d'(w,v) + d'(v,u)$.

Lemma 14. *The set Φ is satisfied in $\mathfrak{B}_1$.*

Proof. Clearly, for each $(\boldsymbol{\delta}(c,d) = a) \in \Phi_3$, $d(\mathfrak{a}(c), \mathfrak{a}(d)) = d'(\mathfrak{a}(c), \mathfrak{a}(d)) = a$. So $\mathfrak{B}_1 \models \Phi_3$. To show $\mathfrak{B}_1 \models \Phi_1 \cup \Phi_2$, it suffices to prove that

$$\forall w \in W \forall t \in t(\Phi)\ (w \in t^{\mathfrak{B}} \leftrightarrow w \in t^{\mathfrak{B}_1}).$$

This can be done by a straightforward induction on the construction of t. The basis of induction and the case of Booleans are trivial. So suppose that t is $\mathsf{A}^{\leq a} s$ (whence $a \in M$). Then we have:

$$\begin{aligned} w \in t^{\mathfrak{B}} \quad &\Leftrightarrow_1 \quad \forall v \in W\ (d(w,v) \leq a \rightarrow v \in s^{\mathfrak{B}}) \\ &\Leftrightarrow_2 \quad \forall v \in W\ (d'(w,v) \leq a \rightarrow v \in s^{\mathfrak{B}_1}) \\ &\Leftrightarrow_3 \quad w \in t^{\mathfrak{B}_1}. \end{aligned}$$

The equivalences $\Leftrightarrow_1$ and $\Leftrightarrow_3$ are obvious; $\Leftrightarrow_2$ holds by the induction hypothesis and the fact that, for all $w, v \in W$ and $a \in M$, $d(x,y) \leq a$ iff $d'(x,y) \leq a$. The case of $\mathsf{A}^{>a} s$ is considered in a similar way. ❑

Consider the relational counterpart of $\mathfrak{B}_1$, i.e., the model

$$\mathfrak{S}(\mathfrak{B}_1) = \langle W, (R_a)_{a \in M}, (R_{\overline{a}})_{a \in M}, \mathfrak{b} \rangle$$

which, for brevity, will be denoted by $\mathfrak{S}$. We are going to filtrate $\mathfrak{B}_1$ through $\Theta = cl(\Phi)$. Define an equivalence relation $\equiv$ on W by taking $u \equiv v$ when $u \in t^{\mathfrak{S}}$ iff $v \in t^{\mathfrak{S}}$, for all $t \in \Theta$. Let $[u] = \{v \in W : u \equiv v\}$. Note that if $(d \sqsubseteq X^d) \in \Phi_3'$ then $[\mathfrak{b}(d)] = \{\mathfrak{b}(d)\}$, since $X^d \in \Theta$.

Construct a filtration $\mathfrak{S}^f = \left\langle W^f, (R_a^f)_{a \in M}, (R_{\overline{a}}^f)_{a \in M}, \mathfrak{b}^f \right\rangle$ of $\mathfrak{S}$ through Θ by taking

- $W^f = \{[u] : u \in W\}$;
- $\mathfrak{b}^f(c) = [\mathfrak{b}(c)]$;
- $\mathfrak{b}^f(X) = \{[u] : u \in \mathfrak{b}(X)\}$;
- $[u] R_a^f [v]$ iff for all terms $\mathsf{A}^{\leq a} t \in \Theta$,
  - $u \in (\mathsf{A}^{\leq a} t)^{\mathfrak{S}}$ implies $v \in t^{\mathfrak{S}}$ and
  - $v \in (\mathsf{A}^{\leq a} t)^{\mathfrak{S}}$ implies $u \in t^{\mathfrak{S}}$;

- $[u]R^f_{\overline{a}}[v]$ iff for all terms $\mathsf{A}^{>a}t \in \Theta$,
  - $u \in (\mathsf{A}^{>a}t)^{\mathfrak{S}}$ implies $v \in t^{\mathfrak{S}}$ and
  - $v \in (\mathsf{A}^{>a}t)^{\mathfrak{S}}$ implies $u \in t^{\mathfrak{S}}$.

Since Θ is finite, W^f is finite as well. Note also that $\mathfrak{b}^f(X^d) = \{\mathfrak{b}^f(d)\}$ whenever $(d \sqsubseteq X^d) \in \Phi'_3$.

Lemma 15. (1) *For every $t \in \Theta$ and every $u \in W$, $u \in t^{\mathfrak{S}}$ iff $[u] \in t^{\mathfrak{S}^f}$.*

(2) *For every $(\boldsymbol{\delta}(c,d) = a) \in \Phi_3$, $a = \min\{b \in M : \mathfrak{b}^f(c) R^f_b \mathfrak{b}^f(d)\}$.*

(3) *$\mathfrak{S}^f$ satisfies* (i) *and* (iii)–(viii) *from Section 4.1.*

Proof. (1) is proved by an easy induction on the construction of t. To prove (2), take $(\boldsymbol{\delta}(c,d) = a) \in \Phi_3$. We must show that $\mathfrak{b}^f(c)R^f_a\mathfrak{b}^f(d)$ and $\neg\mathfrak{b}^f(c)R^f_b\mathfrak{b}^f(d)$, for all $a > b \in M$. Notice first that uR_av implies $[u]R^f_a[v]$ and $uR_{\overline{a}}v$ implies $[u]R^f_{\overline{a}}[v]$. Since $\mathfrak{B}_1 \models \Phi$, we have $\mathfrak{B}_1 \models \boldsymbol{\delta}(c,d) = a$, and so $d'(\mathfrak{b}(c), \mathfrak{b}(d)) = a$. Hence $\mathfrak{b}(c)R_a\mathfrak{b}(d)$ and $\mathfrak{b}^f(c)R^f_a\mathfrak{b}^f(d)$. Suppose now that $b' \in M$ is maximal with $b' < a$ and consider $\mathsf{A}^{\leq b'}\neg X^d$. By definition, $\mathfrak{b}(X^d) = \{\mathfrak{b}(d)\}$. Hence $\mathfrak{b}(d) \notin (\neg X^d)^{\mathfrak{S}}$. On the other hand, we have $b' < d'(\mathfrak{b}(c), \mathfrak{b}(d))$, from which $\mathfrak{b}(c) \in (\mathsf{A}^{\leq b'}\neg X^d)^{\mathfrak{S}}$. Since $(\mathsf{A}^{\leq b'}\neg X^d) \in \Theta$, we then obtain $\neg\mathfrak{b}^f(c)R^f_{b'}\mathfrak{b}^f(d)$. Then, for $b < a$ arbitrary, it follows by (3)(iii) that $\neg\mathfrak{b}^f(c)R^f_b\mathfrak{b}^f(d)$.

To prove (3), let us check conditions (i) and (iii)–(viii).

(i) We have to show that $R^f_a \cup R^f_{\overline{a}} = W^f \times W^f$. Let $\neg[u]R^f_a[v]$. Then $\neg uR_av$, and so $uR_{\overline{a}}v$, since $\mathfrak{S}$ satisfies (i). Thus $[u]R^f_{\overline{a}}[v]$.

(iii): if $[u]R^f_a[v]$ and $a \leq b$ then $[u]R^f_b[v]$. Let $[u]R^f_a[v]$ and $a < b$, for $b \in M$. Suppose $u \in (\mathsf{A}^{\leq b}t)^{\mathfrak{S}}$. By the definition of $\Theta = cl(\Phi)$, $\mathsf{A}^{\leq a}t \in \Theta$, and so $u \in (\mathsf{A}^{\leq a}t)^{\mathfrak{S}}$. Hence $v \in t^{\mathfrak{S}}$. The other direction, $v \in (\mathsf{A}^{\leq b}t)^{\mathfrak{S}}$, is considered in the same way.

(iv): if $[u]R^f_{\overline{a}}[v]$ and $a \geq b$ then $[u]R^f_{\overline{b}}[v]$. Let $[u]R^f_{\overline{a}}[v]$ and $a > b$. Suppose that $u \in (\mathsf{A}^{>b}t)^{\mathfrak{S}}$. Then $\mathsf{A}^{>a}t \in \Theta$, $u \in (\mathsf{A}^{>a}t)^{\mathfrak{S}}$, and so $v \in t^{\mathfrak{S}}$. Again, the other direction is treated analogously.

(v): $[u]R^f_0[v]$ iff $[u] = [v]$. The implication $(\Leftarrow)$ is obvious. So suppose $[u]R^f_0[v]$. Take some $t \in \Theta$ with $u \in t^{\mathfrak{S}}$. Without loss of generality we may assume that t is not of the form $\mathsf{A}^{\leq 0}s$. Then, by the definition of Θ, $u \in (\mathsf{A}^{\leq 0}t)^{\mathfrak{S}}$ and $\mathsf{A}^{\leq 0}t \in \Theta$. Hence $v \in t^{\mathfrak{S}}$. In precisely the same way one can show that for all $t \in \Theta$, $v \in t^{\mathfrak{S}}$ implies $u \in t^{\mathfrak{S}}$. Therefore, $[u] = [v]$.

(vi): if $[u]R^f_a[v]$ and $[v]R^f_b[w]$, then $[u]R^f_{a+b}[w]$, for $(a+b) \in M$. Suppose $u \in (\mathsf{A}^{\leq a+b}t)^{\mathfrak{S}}$. Then $\mathsf{A}^{\leq a}\mathsf{A}^{\leq b}t \in \Theta$ and $u \in (\mathsf{A}^{\leq a}\mathsf{A}^{\leq b}t)^{\mathfrak{S}}$. So $v \in (\mathsf{A}^{\leq b}t)^{\mathfrak{S}}$, whence $w \in t^{\mathfrak{S}}$. For the other direction, assume $w \in (\mathsf{A}^{\leq a+b}t)^{\mathfrak{S}}$. Again, we have $\mathsf{A}^{\leq b}\mathsf{A}^{\leq a}t \in \Theta$ and $w \in (\mathsf{A}^{\leq b}\mathsf{A}^{\leq a}t)^{\mathfrak{S}}$. Then $v \in (\mathsf{A}^{\leq a}t)^{\mathfrak{S}}$, whence $u \in t^{\mathfrak{S}}$.

(vii): $[w]R^f_a[u]$ iff $[u]R^f_a[w]$ and $[w]R^f_{\overline{a}}[u]$ iff $[u]R^f_{\overline{a}}[w]$ hold by definition.

(viii): if $[u]R^f_a[v]$ and $[u]R^f_{\overline{a+b}}[w]$ then $[v]R^f_{\overline{b}}[w]$, for $(a+b) \in M$. Suppose that $v \in (\mathsf{A}^{>b}t)^{\mathfrak{S}}$. Then $\neg\mathsf{A}^{\leq a}\neg\mathsf{A}^{>b}t \in \Theta$ and $u \in (\neg\mathsf{A}^{\leq a}\neg\mathsf{A}^{>b}t)^{\mathfrak{S}}$. Hence $u \in (\mathsf{A}^{>(a+b)}t)^{\mathfrak{S}}$ and so $w \in t^{\mathfrak{S}}$. For the other direction, suppose $w \in (\mathsf{A}^{>b}t)^{\mathfrak{S}}$. Then $u \in (\neg\mathsf{A}^{>(a+b)}\neg\mathsf{A}^{>b}t)^{\mathfrak{S}}$ and $\neg\mathsf{A}^{>(a+b)}\neg\mathsf{A}^{>b}t \in \Theta$. Hence $u \in (\mathsf{A}^{\leq a}t)^{\mathfrak{S}}$ and so $v \in t^{\mathfrak{S}}$. ❑

Unfortunately, $\mathfrak{S}^f$ does not necessarily satisfy (ii) which is required to construct the model $\mathfrak{B}^*$ we need: it may happen that for some points $[u]$, $[v]$ in W^f and $a \in M$, we have both $[u]R^f_a[v]$ and $[u]R^f_{\overline{a}}[v]$. To 'cure' these defects, we have to perform some surgery. The defects form the set

$$D(W^f) = \{v \in W^f : \exists a \in M \exists u \in W^f \ (uR^f_av \ \&\ uR^f_{\overline{a}}v)\}.$$

Let
$$W^* = \{\langle v, i\rangle : v \in D(W^f), i \in \{0,1\}\} \cup \{\langle u, 0\rangle : u \in W^f - D(W^f)\}.$$
So for each $v \in D(W^f)$ we have now two copies $\langle v, 0\rangle$ and $\langle v, 1\rangle$. Define an assignment $\mathfrak{b}^*$ in W^* by taking

- $\mathfrak{b}^*(c) = \left\langle \mathfrak{b}^f(c), 0\right\rangle$ and
- $\mathfrak{b}^*(X) = \{\langle u, i\rangle \in W^* : u \in \mathfrak{b}^f(X)\}$.

Finally, we define accessibility relations R^*_a and $R^*_{\overline{a}}$ as follows:

- if $a > 0$ then $\langle u, i\rangle\, R^*_a\, \langle v, j\rangle$ iff either
  - $uR^f_a v$ and $\neg uR^f_{\overline{a}} v$, or
  - $uR^f_a v$ and $i = j$;
- if $a = 0$ then $\langle u, i\rangle\, R^*_a\, \langle v, j\rangle$ iff $\langle u, i\rangle = \langle v, j\rangle$;
- $R^*_{\overline{a}}$ is defined as the complement of R^*_a, i.e., $\langle u, i\rangle\, R^*_{\overline{a}}\, \langle v, j\rangle$ iff $\neg \langle u, i\rangle\, R^*_a\, \langle v, j\rangle$.

Lemma 16. *$\mathfrak{S}^* = \langle W^*, (R^*_a)_{a\in M}, (R^*_{\overline{a}})_{a\in M}, \mathfrak{b}^*\rangle$ is an M-standard relational metric model.*

Proof. That $\mathfrak{S}^*$ satisfies (i), (ii), and (v) follows immediately from the definition. Let us check the remaining conditions.

(iii) Suppose that $\langle u, i\rangle\, R^*_a\, \langle v, j\rangle$ and $a < b \in M$. If $i = j$ then clearly $\langle u, i\rangle\, R^*_b\, \langle v, j\rangle$. So assume $i \neq j$. Then, by definition, $uR^f_a v$ and $\neg uR^f_{\overline{a}} v$. Since $\mathfrak{S}^f$ satisfies (iii) and (iv), we obtain $uR^f_b v$ and $\neg uR^f_{\overline{b}} v$. Thus $\langle u, i\rangle\, R^*_b\, \langle v, j\rangle$.

(iv) Suppose that $\langle u, i\rangle\, R^*_{\overline{a}}\, \langle v, j\rangle$ and $a \geq b \in M$, but $\neg \langle u, i\rangle\, R^*_{\overline{b}}\, \langle v, j\rangle$. By (i), $\langle u, i\rangle\, R^*_b\, \langle v, j\rangle$. And by (iii), $\langle u, i\rangle\, R^*_a\, \langle v, j\rangle$. Finally, (ii) yields $\neg \langle u, i\rangle\, R^*_{\overline{a}}\, \langle v, j\rangle$, which is a contradiction.

(vi) Suppose $\langle u, i\rangle\, R^*_a\, \langle v, j\rangle$, $\langle v, j\rangle\, R^*_b\, \langle w, k\rangle$ and $a + b \in M$. Then $uR^f_a v$ and $vR^f_b w$. As $\mathfrak{S}^f$ satisfies (vi), we have $uR^f_{a+b} w$. If $i = k$ then clearly $\langle u, i\rangle\, R^*_{a+b}\, \langle w, k\rangle$. So assume $i \neq k$. If $i = j \neq k$ then, using (viii) for $\mathfrak{S}^f$, $\neg uR^f_{\overline{a+b}} w$, since $uR^f_a v$ and $\neg vR^f_{\overline{b}} w$. The case $i \neq j = k$ is considered analogously using the fact that the relations in $\mathfrak{S}^f$ are symmetric.

(vii) follows from the symmetry of R^f_a and $R^f_{\overline{a}}$. Namely, suppose $\langle u, i\rangle\, R^*_a\, \langle v, j\rangle$. If $a = 0$, then $\langle v, j\rangle\, R^*_a\, \langle u, i\rangle$ follows trivially, since $\langle u, i\rangle = \langle v, j\rangle$. Suppose that $a > 0$, $uR^f_a v$ and $i = j$. Then $\langle v, j\rangle\, R^*_a\, \langle u, i\rangle$ follows from the symmetry of R^f_a. If $a > 0$, $uR^f_a v$ and $\neg uR^f_{\overline{a}} v$, then the symmetry of $R^f_{\overline{a}}$ is needed as well.

Now, the symmetry of $R^*_{\overline{a}}$ follows from the symmetry of R^*_a and conditions (i) and (ii).
❑

Lemma 17. *For all $\langle v, i\rangle \in W^*$ and $t \in \Theta$, we have $\langle v, i\rangle \in t^{\mathfrak{S}^*}$ iff $v \in t^{\mathfrak{S}^f}$.*

Proof. The proof is by induction on t. The basis of induction and the case of Booleans are trivial. The cases $t = (\mathsf{A}^{\leq a} s)$ and $t = (\mathsf{A}^{> a} s)$ are consequences of the following claims:

Claim 1: If $uR^f_a v$ and $\langle u, i\rangle \in W^*$ ($i \in \{0,1\}$), then there exists a j such that $\langle u, i\rangle\, R^*_a\, \langle v, j\rangle$. Indeed, this is clear for $i = 0$. Suppose $i = 1$. If v was duplicated, then $\langle v, 1\rangle$ is as required. If v was not duplicated, then $\neg uR^f_{\overline{a}} v$, and so $\langle v, 0\rangle$ is as required.

Claim 2: If $\langle u, i\rangle\, R^*_a\, \langle v, j\rangle$ then $uR^f_a v$. This should be obvious.

Claim 3: If $uR^f_{\overline{a}} v$ and $\langle u, i\rangle \in W^*$ ($i \in \{0,1\}$), then there exists a j such that $\langle u, i\rangle\, R^*_{\overline{a}}\, \langle v, j\rangle$. Suppose $i = 0$. If v was not duplicated, then $\neg uR^f_a v$. Hence $\neg \langle u, 0\rangle\, R^*_a\, \langle v, 0\rangle$. If v was duplicated, then $\neg \langle u, 0\rangle\, R^*_a\, \langle v, 1\rangle$. In the case of $i = 1$ we have $\neg \langle u, 1\rangle\, R^*_a\, \langle v, 0\rangle$, i.e., $\langle u, 1\rangle\, R^*_{\overline{a}}\, \langle v, 0\rangle$.

Claim 4: If $\langle u,i\rangle\, R^*_{\overline{a}}\, \langle v,j\rangle$ then $uR^f_{\overline{a}}v$. Indeed, if $i=j$ then $\neg uR^f_a v$ and so $uR^f_{\overline{a}}v$. And if $i \neq j$ then $uR^f_{\overline{a}}v$. ❑

To complete the proof of Theorem 13, we transform $\mathfrak{S}^*$ into a finite metric $\mathcal{MS}$-model and show that this model satisfies Φ. Let

$$\mathfrak{B}^* = \langle W^*, d^*, X_1^*, \dots, c_1^*, \dots \rangle,$$

where for all $w, v \in W^*$, set variables X_i, and constants c_i,

$$d^*(w,v) = \min\{\gamma, a \in M : wR^*_a v\}, \quad X_i^* = \mathfrak{b}^*(X_i), \quad c_i^* = \mathfrak{b}^*(c_i).$$

As M is finite, d^* is well-defined. Using (v)–(vii), it is easy to see that d^* is a metric. So $\mathfrak{B}^*$ is a finite metric space model. It remains to show that $\mathfrak{B}^*$ satisfies Φ. Observe first that we have

(✠) for all $w \in W^*$ and $t \in t(\Phi)$, we have $w \in t^{\mathfrak{S}^*}$ iff $w \in t^{\mathfrak{B}^*}$.

This is proved by induction on t. The basis of induction and the case of Booleans are clear. So let $t = (\mathsf{A}^{\leq a} s)$ for some $a \in M$. Then

$$\begin{aligned} w \in \mathsf{A}^{\leq a} s)^{\mathfrak{S}^*} &\Leftrightarrow_1 \quad \forall v\ (wR^*_a v \to v \in s^{\mathfrak{S}^*}) \\ &\Leftrightarrow_2 \quad \forall v\ (wR^*_a v \to v \in s^{\mathfrak{B}^*}) \\ &\Leftrightarrow_3 \quad \forall v\ (d^*(w,v) \leq a \to v \in s^{\mathfrak{B}^*}) \\ &\Leftrightarrow_4 \quad w \in (\mathsf{A}^{\leq a} s)^{\mathfrak{B}^*}. \end{aligned}$$

Equivalences $\Leftrightarrow_1$ and $\Leftrightarrow_4$ are obvious; $\Leftrightarrow_2$ holds by the induction hypothesis; $\Leftarrow_3$ is an immediate consequence of the definition of d^*, and $\Rightarrow_3$ follows from (iii). The case $t = (\mathsf{A}^{>a} s)$ is considered analogously.

We can now show that $\mathfrak{B}^* \models \Phi$. Let $(c \sqsubseteq t) \in \Phi_1$. Then we have:

$$\begin{aligned} &\mathfrak{B}^* \models c \sqsubseteq t \Leftrightarrow_1 c^* \in t^{\mathfrak{B}^*} \Leftrightarrow_2 \mathfrak{b}^*(c) \in t^{\mathfrak{S}^*} \Leftrightarrow_3 \left\langle \mathfrak{b}^f(c), 0 \right\rangle \in t^{\mathfrak{S}^*} \Leftrightarrow_4 \\ &\mathfrak{b}^f(c) \in t^{\mathfrak{S}^f} \Leftrightarrow_5 [\mathfrak{b}(c)] \in t^{\mathfrak{S}^f} \Leftrightarrow_6 \mathfrak{b}(c) \in t^{\mathfrak{S}} \Leftrightarrow_7 c^{\mathfrak{B}_1} \in t^{\mathfrak{B}_1} \Leftrightarrow_8 \mathfrak{B}_1 \models c \sqsubseteq t. \end{aligned}$$

Equivalences $\Leftrightarrow_1$ and $\Leftrightarrow_8$ are obvious; $\Leftrightarrow_2$ follows from (✠); $\Leftrightarrow_3$ and $\Leftrightarrow_5$ hold by definition; $\Leftrightarrow_4$ follows from Lemma 17, $\Leftrightarrow_6$ from Lemma 15, and $\Leftrightarrow_7$ from Lemma 8.

Since $\mathfrak{B}_1 \models \Phi$, we have $\mathfrak{B}^* \models \Phi_1$. That $\mathfrak{B}^* \models \Phi_2$ is proved analogously using (✠).

It remains to show that $\mathfrak{B}^* \models \Phi_3$. Take any $\delta(c,d) = a$ from Φ_3. We must prove that $d^*(c^*, d^*) = a$. By Lemma 15 (2),

$$a = \min\{b \in M : \mathfrak{b}^f(c) R^f_b \mathfrak{b}^f(d)\}.$$

So $a = \min\{b \in M : \left\langle \mathfrak{b}^f(c), 0 \right\rangle R^*_b \left\langle \mathfrak{b}^f(d), 0 \right\rangle\}$. By the definition of $\mathfrak{b}^*$, we have $a = \min\{b \in M : \mathfrak{b}^*(c) R^*_b \mathfrak{b}^*(d)\}$, which means that $d^*(\mathfrak{b}^*(c), \mathfrak{b}^*(d)) = a$. This completes the proof of Theorem 13. ❑

Thus, by Theorem 13 and Lemma 10 (2), φ is satisfied in the finite model $\mathfrak{B}^*$, which completes the proof of Theorem 9. ❑

4.3 Decidability

The main result of this section is the following:

Theorem 18. *(i) The problem whether $\varphi \in \mathcal{MS}^{\#}[\mathbb{Q}^+]$ is satisfiable in a member of $\mathcal{M}$ is decidable.*

(ii) Let $q \in \mathbb{N}$. The problem whether $\varphi \in \mathcal{MS}^{\#}[\{0, \dots, q\}]$ is satisfiable in a member of $\mathcal{M}$ is in **NExpTime**.

We will first concentrate on (i). Note that the finite model property of $\mathcal{MS}^{\#}[\mathbb{R}^+]$ proved above is not enough to prove the decidability of $\mathcal{MS}^{\#}[\mathbb{Q}^+]$: we still do not know an effectively computable upper bound for the size of a finite model satisfying φ. Indeed, the set $M(\Phi)$ depends not only on φ, but also on the initial model $\mathfrak{B}$ satisfying φ, because of the possible introduction of new parameters $a \in \mathbb{R}$ by strings of the form $\boldsymbol{\delta}(c,d)$, occuring in φ. Note however that by Lemmas 11 and 12, an upper bound for the size of $\mathfrak{B}^*$ can be computed effectively from the maximum of $M(\Phi)$, the minimum of $M(\Phi) - \{0\}$, and φ. Hence, to obtain an effective upper bound, it suffices to start the construction with a model satisfying φ for which both the maximum of $M(\Phi)$ and the minimum of $M(\Phi) - \{0\}$ are bounded. The next lemma shows how to obtain such a model. Let n_φ and m_φ be the minimal positive number and the maximal number occurring in φ, respectively (if no such number exists, then put $m_\varphi = n_\varphi = 1$).

Lemma 19. *Suppose a formula $\varphi \in \mathcal{MS}^{\#}[\mathbb{Q}^+]$ is satisfied in a metric $\mathcal{MS}$-model $\mathfrak{B} = \langle W, d, X_1^{\mathfrak{B}}, \dots, c_1^{\mathfrak{B}}, \dots \rangle$. Denote by D the set of all $\boldsymbol{\delta}(c,d)$ occurring in φ and assume that $D \neq \emptyset$. Then there is a metric d' on W such that φ is satisfied in $\langle W, d', X_1^{\mathfrak{B}}, \dots, c_1^{\mathfrak{B}}, \dots \rangle$ and*

$$\min\{d'(c^{\mathfrak{B}}, d^{\mathfrak{B}}) > 0 : \boldsymbol{\delta}(c,d) \in D\} \geq n_\varphi/2,$$
$$\max\{d'(c^{\mathfrak{B}}, d^{\mathfrak{B}}) : \boldsymbol{\delta}(c,d) \in D\} \leq 2m_\varphi.$$

Proof. Let $a = n_\varphi$ and $b = m_\varphi$. Set

$$a' = \min\{d(c^{\mathfrak{B}}, d^{\mathfrak{B}}) > 0 : \boldsymbol{\delta}(c,d) \in D\},$$
$$b' = \max\{d(c^{\mathfrak{B}}, d^{\mathfrak{B}}) : \boldsymbol{\delta}(c,d) \in D\}.$$

We consider the case when $a' < a/2$ and $2b < b'$. The other cases are even easier and left to the reader. Define d' by putting for all $v, w \in W$

$$d'(v,w) := \begin{cases} d(v,w) & \text{if } a \leq d(v,w) \leq b \text{ or } d(v,w) = 0, \\ b + (b/(b'-b)) \cdot (d(v,w) - b) & \text{if } d(v,w) > b, \\ a + (a/2(a-a')) \cdot (d(v,w) - a) & \text{if } 0 < d(v,w) < a. \end{cases}$$

One can now readily show that d' is a metric and that $\langle W, d', X_1^{\mathfrak{B}}, \dots, c_1^{\mathfrak{B}}, \dots \rangle$ satisfies φ. $\square$

It follows that we can start the filtration with a model $\mathfrak{B}$ for which we obtain for $cl(\Phi)$ the following upper bound (by Lemmas 11 and 12):

- $|cl(\Phi)|$ is bounded by $l(\varphi)^{p(m_\varphi/n_\varphi)}$, where p is a polynomial function of degree 2 not depending on φ and $l(\varphi)$ is the length of φ.[4]

Summarizing the results obtained so far, we have

Theorem 20. *There exists a quadratic polynomial p such that every $\varphi \in \mathcal{MS}^{\#}[\mathbb{Q}^+]$ which is satisfiable in a metric space model is satisfiable in a metric space the domain of which is bounded by*

$$f(\varphi) = 2 \cdot 2^{l(\varphi)^{p(\frac{m_\varphi}{n_\varphi})}}.$$

In contrast to many standard satisfiability problems even this result does not directly imply the decidability of the satisfiability problem for $\mathcal{MS}^{\#}[\mathbb{Q}^+]$–formulas, because there are infinitely many (even uncountably many) different metric spaces based on a finite set. We now address this problem.

[4]This is done as follows: First, by Lemma 19, we obtain that $\chi \leq 4 \cdot \frac{m_\varphi}{n_\varphi}$. Further, we clearly have $|M(\Phi)| \leq l(\varphi)$ and $|t(\Phi)| \leq 4 \cdot l(\varphi)$, whence, by Lemma 11, $|M[\Phi]| \leq l(\varphi)^\chi$. Hence, by Lemma 12, we obtain $|cl(\Phi)| \leq 2^{\chi+4} \cdot 2^2 \cdot l(\varphi) \cdot l(\varphi)^{\chi \cdot (2\chi+3)}$. It follows that $cl(\Phi)$ is bounded by $l(\varphi)^{2\chi^2+4\chi+7}$.

Fix a formula φ and $n \leq f(\varphi)$. Put $W = \{1, \ldots, n\}$. Suppose φ contains constants $C(\varphi) = \{c_1, \ldots, c_k\}$, set variables $V(\varphi) = \{X_1, \ldots, X_l\}$ and parameters $P(\varphi) = \{0 = a_0, a_1, \ldots, a_p\}$, where 0 belongs to $P(\varphi)$ even if it does not occur in φ. Assume that $0 < a_1 < a_2 < \ldots < a_p$. Further, take variables x_{ij}, for every $i, j \in W$. Those variables are intended to "simulate" the distance $d(i,j)$ between i and j.

There are only finitely many pairs $\mathfrak{E} = (\mathfrak{T}, \mathfrak{C})$ consisting of structures

$$\mathfrak{T} = \left\langle W, (X^{\mathfrak{T}} \,|\, X \in V(\varphi)), (c^{\mathfrak{T}} \,|\, c \in C(\varphi)) \right\rangle$$

and sets of the form

$$\begin{array}{rcl} \mathfrak{C} & = & \{x_{ij} = a_{k(ij)} \;|\; (i,j) \in I_1\} \\ & \cup & \{x_{ij} > a_p \;|\; (i,j) \in I_2\} \\ & \cup & \{a_{k(ij)} < x_{ij} < a_{k(ij)+1} \;|\; (i,j) \in I_3\}, \end{array}$$

where $X^{\mathfrak{T}} \subseteq \{1, \ldots, n\}$ for every set variable X of φ, $c^{\mathfrak{T}} \in \{1, \ldots, n\}$ for every constant c of φ, and $\mathfrak{C}$ is a set of constraints specifying for every ordered pair of elements i, j from W whether the distance between i and j is equal to some $a_{k(ij)}$, greater than a_p or strictly between some $a_{k(ij)}$ and $a_{k(ij)+1}$. Here, I_1, I_2 and I_3 partition $W \times W$ and k is a function $W \times W \longrightarrow P(\varphi)$. Pairs $\mathfrak{E}$ of this type are called *(n-)constraint systems for φ*. Constraint systems specify a class of models based on the domain W in such a way that it is possible to determine from the system the value of all those terms which contain parameters from $P(\varphi)$ only: define the extension $s^{\mathfrak{E}}$ of a term s containing parameters from $P(\varphi)$ only by putting inductively (we list only the interesting clauses):

- $\{c_i\}^{\mathfrak{E}} = \{c_i^{\mathfrak{T}}\}$, where $\{c_i\}$ is a set constant;
- $(\mathsf{E}^{=a} s)^{\mathfrak{E}} = \{i \in W : \exists j \in W\, (x_{ij} = a \in \mathfrak{C} \;\&\; j \in s^{\mathfrak{E}})\}$;
- $(\mathsf{E}^{<a} s)^{\mathfrak{E}} = \{i \in W : \exists j \in W\, (x_{ij} < a \in \mathfrak{C} \;\&\; j \in s^{\mathfrak{E}})\}$.

The truth-relation $\mathfrak{E} \models \varphi$, φ an $\mathcal{MS}[M]$-formula with parameters from $P(\varphi)$, is defined as expected (again we list only the intersting clauses):

- $\mathfrak{E} \models \boldsymbol{\delta}(c_1, c_2) = a$ iff for $i = c^{\mathfrak{T}}$ and $j = c_2^{\mathfrak{T}}$ and $x_{ij} = a \in \mathfrak{C}$;
- $\mathfrak{E} \models \boldsymbol{\delta}(c_1, c_2) < a$ iff for $i = c^{\mathfrak{T}}$ and $j = c_2^{\mathfrak{T}}$ and $x_{ij} < a \in \mathfrak{C}$.

Say that $\mathfrak{E} = (\mathfrak{T}, \mathfrak{C})$ *satisfies* φ iff $\mathfrak{E} \models \varphi$. Of course, if φ is satisfiable in a model of size n, then φ is satisfied in an n-constraint system for φ. The converse does not hold, because it could be that there does not exist a metric d on W which conforms to $\mathfrak{C}$, where a metric d *conforms to* $\mathfrak{C}$ if by setting $x_{ij} = d(i,j)$, for all $i, j \in W$, all constraints from $\mathfrak{C}$ are satisfied.

So, say that $\mathfrak{E} = (\mathfrak{T}, \mathfrak{C})$ is *satisfiable* if the constraints in $\mathfrak{C}$ together with the following set of equalities and inequalities has a solution in $\mathbb{R}^+$:

- $x_{ii} = 0$, for all $i \in W$;
- $x_{ij} = x_{ji}$, for all $i, j \in W$ (Symmetry);
- $x_{ik} + x_{kj} \geq x_{ij}$, for all $i, j, k \in W$ (Triangular Inequality).

The following is now easily checked:

Lemma 21. *A formula $\varphi \in \mathcal{MS}^{\#}[\mathbb{Q}^+]$ is satisfiable in a metric space model of size n iff there exists a satisfiable n-constraint system for φ which satisfies φ.*

Lemma 22. *It is decidable in polynomial time $\rho(n)$ whether an n-constraint system $\mathfrak{E}$ for φ is satisfiable and also satisfies φ.*

Proof. Clearly, given a satisfiable n-constraint system $\mathfrak{C}$ for φ, it is decidable in polynomial time whether φ is indeed satisfied in $\mathfrak{C}$.

Hence, it remains to be shown that checking the satisfiability of $\mathfrak{C}$ can be done in polynomial time. First, notice that the decidability of this problem follows from Tarskis result on the decidability of the theory of real closed fields, [30]. On the other hand, the problem can be understood as a standard problem of *linear programming*, where we can choose some arbitrary *objective function* to be maximized. In fact, we are only interested in the question whether this system of equalities and inequalities has a common solution, i.e in the *linear programming feasibility problem.* Furthermore, since all of the parameters in the constraints are from $\mathbb{Q}$, a solution exists in $\mathbb{R}$ iff a solution exists in $\mathbb{Q}$, because the set of solutions can be represented as a (possibly unbounded) convex polyhydron. Hence we can restrict ourselves to searching for rational solutions. This problem has been shown, e.g. in [3], to be solvable in polynomial time measured in the number of variables, i.e., in n. ❑

Theorem 18 (i) follows from Theorem 20 and Lemmas 21 and 22. Theorem 18 (ii) follows from Theorem 20 and Lemmas 21 and 22, because for $\varphi \in \mathcal{MS}^{\#}[\{1,\dots,q\}]$ the number q is an upper bound for m_φ/n_φ. Now the decision procedure is as follows: given $\varphi \in \mathcal{MS}^{\#}[\{1,\dots,q\}]$ guess a n-constraint system $\mathfrak{C}$ with $n \leq f(\varphi)$ and check in polynomial time (in n) whether $\mathfrak{C}$ is both satisfiable and satisfies φ.

We note that it is open whether the complexity is **NExpTime**-hard.

5 Satisfiability in weaker distance spaces

Let us now consider the satisfiability problem in the class $\mathcal{D}$ of arbitrary distance spaces and its subclasses $\mathcal{D}_{sym}$ and $\mathcal{D}_{tr}$.

Theorem 23. *The satisfiability problem for $\mathcal{FM}^2[\mathbb{Q}^+]$-formulas in $\mathcal{D}$ and $\mathcal{D}_{sym}$ is decidable. Moreover, both problems are in* **NExpTime** *and in both cases any satisfiable formula is satisfiable in a finite model.*

Proof. The proof is based on a simple reduction to the satisfiability problem for the two-variable fragment of first-order logic. Recall that atomic formulas $\boldsymbol{\delta}(x,y) < a$ and $\boldsymbol{\delta}(x,y) = a$ can be regarded as binary predicates $P_{<a}(x,y)$ and $P_{=a}(x,y)$. Denote by φ^+ the result of replacing all subformulas in φ of the form $\boldsymbol{\delta}(x,y) < a$ and $\boldsymbol{\delta}(x,y) = a$ by $P_{<a}(x,y)$ and $P_{=a}(x,y)$, respectively. Let

$$0 = a_0 < a_1 < \cdots < a_n$$

be the list of rational numbers that occur in φ together with 0 and let Γ be the set of the following formulas, for $i \leq n$:

$$\begin{aligned}
&\forall x,y\big(P_{=a_i}(x,y) \to \bigwedge_{0\leq j<i} \neg P_{<a_j}(x,y) \wedge \bigwedge_{i\neq j} \neg P_{=a_j}(x,y) \wedge \bigwedge_{n\geq j>i} P_{<a_j}(x,y)\big),\\
&\forall x,y\big(P_{<a_i}(x,y) \to \bigwedge_{i<j\leq n} P_{<a_j}(x,y)\big),\\
&\forall x,y\neg P_{<0}(x,y),\\
&\forall x,y\big(P_{=0}(x,y) \leftrightarrow x = y\big).
\end{aligned}$$

We claim that the set $\Gamma \cup \{\varphi^+\}$ is satisfiable in a first-order structure

$$\mathfrak{A} = \big\langle W, P^{\mathfrak{A}}_{=a_0}, \dots, P^{\mathfrak{A}}_{<a_0}, \dots, P^{\mathfrak{A}}_1, \dots, c^{\mathfrak{A}}_1, \dots\big\rangle$$

iff φ is satisfiable in a distance space model.

The direction ($\Leftarrow$) is clear. So suppose that $\mathfrak{A}$ satisfies $\Gamma \cup \{\varphi^+\}$. Define a distance space structure

$$\mathfrak{B} = \left\langle W, d, P_1^{\mathfrak{A}}, \dots, c_1^{\mathfrak{A}}, \dots \right\rangle$$

by taking, for $a, b \in W$,

$$d(a,b) = a_i \text{ iff } \mathfrak{A} \models P_{=a_i}(a,b),$$
$$d(a,b) = \frac{a_i + a_{i+1}}{2} \text{ iff } \mathfrak{A} \models \neg P_{<a_i}(a,b) \wedge P_{<a_{i+1}}(a,b) \wedge \neg P_{=a_i}(a,b),$$
$$d(a,b) = 2 \cdot a_n \text{ iff } \mathfrak{A} \models \neg P_{<a_n}(a,b) \wedge \neg P_{=a_n}(a,b).$$

It is not difficult to see that $\mathfrak{B}$ satisfies φ. Hence, to decide whether φ is satisfiable in a distance space model, it suffices to check whether $\Gamma \cup \{\varphi^+\}$ is satisfiable in a first-order structure. This proves the decidability of satisfiability in $\mathcal{D}$.

For $\mathcal{D}_{sym}$, we take the set Γ_{sym} which is

$$\Gamma \cup \{\forall x, y \big(P_{<a_i}(x,y) \leftrightarrow P_{<a_i}(y,x)\big),\ \forall x, y \big(P_{=a_i}(x,y) \leftrightarrow P_{=a_i}(y,x)\big) : i \leq n\}.$$

It is readily checked that φ is satisfiable in $\mathcal{D}_{sym}$ iff $\Gamma_{sym} \cup \{\varphi^+\}$ is satisfiable.

The remaining claims follow immediately from the **NExpTime**-completeness of the two-variable fragment of first-order logic and its finite model property [8, 4]. ❑

Let us now consider the satisfiability problem in $\mathcal{D}_{tr}$.

Theorem 24. *(i) The satisfiability problem for $\mathcal{MS}^\#[\mathbb{Q}^+]$-formulas in $\mathcal{D}_{tr}$ is decidable. (ii) Any $\mathcal{MS}^\#[\mathbb{Q}^+]$-formula which is satisfiable in $\mathcal{D}_{tr}$ is satisfiable in a finite member of $\mathcal{D}_{tr}$. (iii) The satisfiability problem for $\mathcal{MS}^\#[\{0, \dots, q\}]$-formulas in $\mathcal{D}_{tr}$ is in* **NExpTime**.

Proof. In principle, the proof is similar to that of Theorem 18. However, two important modifications are required: one concerns the filtration, another the copying technique.

We use the same terminology as in Section 4.2 and again start with a formula φ that is satisfied in a distance space model $\mathfrak{B} \in \mathcal{D}_{tr}$. Further, we can again define the model $\mathfrak{B}_1 = \left\langle W, d', X_1^{\mathfrak{B}_1}, \dots, c_1^{\mathfrak{B}_1}, \dots \right\rangle$ such that $\mathfrak{B}_1 \models \Phi$. However, now d' is not necessarily symmetric.

The closure $cl(\Phi)$ of $t(\Phi)$ is defined in almost the same way as on page 15; the only difference is that the last condition is replaced with the following one:

5′. if $\mathsf{A}^{>a}t \in T$ and $b > a$, for $b \in M[\Phi]$, then $\mathsf{A}^{>b}t \in T$.

The relational counterpart of $\mathfrak{B}_1$, i.e., the model

$$\mathfrak{S}(\mathfrak{B}_1) = \langle W, (R_a)_{a \in M}, (R_{\overline{a}})_{a \in M}, \mathfrak{b} \rangle,$$

will again be denoted by $\mathfrak{S}$. The filtration of $\mathfrak{B}_1$ through $\Theta = cl(\Phi)$ is modified in the following way.

Define an equivalence relation $\equiv$ on W by taking $u \equiv v$ if for all $t \in \Theta$ we have $u \in t^{\mathfrak{S}}$ iff $v \in t^{\mathfrak{S}}$. Let $[u] = \{v \in W : u \equiv v\}$. Note again that if $(d \sqsubseteq X^d) \in \Phi_3'$ then $[\mathfrak{b}(d)] = \{\mathfrak{b}(d)\}$, since $X^d \in \Theta$.

Construct a filtration $\mathfrak{S}^f = \left\langle W^f, (R_a^f)_{a \in M}, (R_{\overline{a}}^f)_{a \in M}, \mathfrak{b}^f \right\rangle$ of $\mathfrak{S}$ through Θ by taking

- $W^f = \{[u] : u \in W\}$;
- $\mathfrak{b}^f(c) = [\mathfrak{b}(c)]$;
- $\mathfrak{b}^f(X) = \{[u] : u \in \mathfrak{b}(X)\}$;
- $[u] R_a^f [v]$ for $a > 0$ iff for all terms $\mathsf{A}^{\leq a}t \in \Theta$, $u \in (\mathsf{A}^{\leq a}t)^{\mathfrak{S}}$ implies $v \in t^{\mathfrak{S}}$;

- $[u]R^f_a[v]$ for $a = 0$ iff $[u] = [v]$;
- $[u]R^f_{\overline{a}}[v]$ iff for all terms $\mathsf{A}^{>a}t \in \Theta$, $u \in (\mathsf{A}^{>a}t)^{\mathfrak{S}}$ implies $v \in t^{\mathfrak{S}}$.

Since Θ is finite, W^f is finite as well. Note also that $\mathfrak{b}^f(X^d) = \{\mathfrak{b}^f(d)\}$ whenever $(d \sqsubseteq X^d) \in \Phi_3'$ and that $[u]R_a[v]$ implies $[u]R^f_a[v]$, and $[u]R_{\overline{a}}[v]$ implies $[u]R^f_{\overline{a}}[v]$.

Lemma 25. (1) *For every $t \in \Theta$ and every $u \in W$, $u \in t^{\mathfrak{S}}$ iff $[u] \in t^{\mathfrak{S}^f}$.*

(2) *For every $(\boldsymbol{\delta}(c,d) = a) \in \Phi_3$, $a = \min\{b \in M : \mathfrak{b}^f(c)R^f_b\mathfrak{b}^f(d)\}$.*

(3) *$\mathfrak{S}^f$ satisfies* (i), (iii)–(vi) *and* (viii) *from Section 4.1.*

Proof. (1) is proved by an easy induction; the proof of (2) is the same as in Lemma 15.

To prove (3), we have to check conditions (i), (iii)–(vi) and (viii). The first one, i.e., $R^f_a \cup R^f_{\overline{a}} = W^f \times W^f$, is proved as in Lemma 15.

(iii): if $[u]R^f_a[v]$ and $a \le b$ then $[u]R^f_b[v]$. Let $[u]R^f_a[v]$ and $a < b$ for some $b \in M$. Suppose $u \in (\mathsf{A}^{\le b}t)^{\mathfrak{S}}$. By the definition of Θ, $\mathsf{A}^{\le a}t \in \Theta$. Thus, since $a < b$, $u \in (\mathsf{A}^{\le a}t)^{\mathfrak{S}}$. Then $[u]R^f_a[v]$ implies $v \in t^{\mathfrak{S}}$, and $[u]R^f_b[v]$ follows.

(iv): if $[u]R^f_{\overline{a}}[v]$ and $a \ge b$ then $[u]R^f_{\overline{b}}[v]$. The proof is similar to that of (iii).

(v): $[u]R^f_0[v]$ iff $[u] = [v]$ holds by the definition of R^f_0.

(vi): if $[u]R^f_a[v]$ and $[v]R^f_b[w]$, then $[u]R^f_{a+b}[w]$, for $(a+b) \in M$. Suppose $u \in (\mathsf{A}^{\le a+b}t)^{\mathfrak{S}}$. Then $\mathsf{A}^{\le a}\mathsf{A}^{\le b}t \in \Theta$ and $u \in (\mathsf{A}^{\le a}\mathsf{A}^{\le b}t)^{\mathfrak{S}}$. So $v \in (\mathsf{A}^{\le b}t)^{\mathfrak{S}}$, whence $w \in t^{\mathfrak{S}}$.

(viii): if $[u]R^f_a[v]$ and $[u]R^f_{\overline{a+b}}[w]$ then $[v]R^f_{\overline{b}}[w]$, for $(a+b) \in M$. Let $v \in (\mathsf{A}^{>b}t)^{\mathfrak{S}}$ and $\mathsf{A}^{>b}t \in \Theta$. Then we have $\neg\mathsf{A}^{\le a}\neg\mathsf{A}^{>b}t \in \Theta$ and $u \in (\neg\mathsf{A}^{\le a}\neg\mathsf{A}^{>b}t)^{\mathfrak{S}}$, for otherwise (since Θ is closed under subterms) $u \in (\mathsf{A}^{\le a}\neg\mathsf{A}^{>b}t)^{\mathfrak{S}}$ together with $[u]R^f_a[v]$ would imply $v \in (\neg\mathsf{A}^{>b}t)^{\mathfrak{S}}$, which is a contradiction. Suppose that $uR_{\overline{a+b}}x$ for some point $x \in W$. Since $u \in (\neg\mathsf{A}^{\le a}\neg\mathsf{A}^{>b}t)^{\mathfrak{S}}$, there is a point $y \in W$ such that uR_ay and $y \in (\mathsf{A}^{>b}t)^{\mathfrak{S}}$. As $\mathfrak{S}$ satisfies (viii), it follows that $yR_{\overline{b}}x$, and so $x \in t^{\mathfrak{S}}$. Hence $u \in (\mathsf{A}^{>(a+b)}t)^{\mathfrak{S}}$, which implies $w \in t^{\mathfrak{S}}$. ❑

Now, we are again facing the problem that $\mathfrak{S}^f$ may not satisfy condition (ii) which is required for the construction of the model $\mathfrak{B}^*$. To avoid this problematic case—the situation where for some points $[u]$, $[v]$ in W^f and $a \in M$ both $[u]R^f_a[v]$ and $[u]R^f_{\overline{a}}[v]$ hold—we modify the copying technique in the following way. The problematic points form the set

$$D(W^f) = \{v \in W^f : \exists a \in M \exists u \in W^f \ (uR^f_a v \ \& \ uR^f_{\overline{a}}v)\}.$$

Let

$$W^* = \{\langle v,i\rangle : v \in D(W^f),\ i \in \{0,1,2\}\} \cup \{\langle u,0\rangle : u \in W^f - D(W^f)\}.$$

So for each $v \in D(W^f)$ we have now three copies $\langle v,0\rangle$, $\langle v,1\rangle$ and $\langle v,2\rangle$. Define an assignment $\mathfrak{b}^*$ in W^* by taking

$$\mathfrak{b}^*(c) = \left\langle \mathfrak{b}^f(c), 0\right\rangle,$$
$$\mathfrak{b}^*(X) = \{\langle u,i\rangle \in W^* : u \in \mathfrak{b}^f(X)\}.$$

Finally, we define accessibility relations R^*_a and $R^*_{\overline{a}}$ as follows:

- If $a > 0$, then $\langle u,i\rangle\, R^*_a\, \langle v,j\rangle$ iff either
  - $uR^f_a v$ and $\neg uR^f_{\overline{a}}v$, or
  - $uR^f_a v$ and $j = 0$, or
  - $\langle u,i\rangle = \langle v,j\rangle$ (then also $uR^f_a v$).

- If $a = 0$, then $\langle u,i\rangle\, R^*_a\, \langle v,j\rangle$ iff $\langle u,i\rangle = \langle v,j\rangle$.
- $R^*_{\overline{a}}$ is defined as the complement of R^*_a, i.e.,

$$\langle u,i\rangle\, R^*_{\overline{a}}\, \langle v,j\rangle \quad \text{iff} \quad \neg\, \langle u,i\rangle\, R^*_a\, \langle v,j\rangle\, .$$

Lemma 26. *The relational model* $\mathfrak{S}^* = \langle W^*, (R^*_a)_{a\in M}, (R^*_{\overline{a}})_{a\in M}, \mathfrak{b}^*\rangle$ *satisfies conditions* (i)–(vi) *of M-standard models.*

Proof. That $\mathfrak{S}^*$ satisfies (i), (ii), and (v) follows immediately from the definitions of R^*_a and $R^*_{\overline{a}}$. Let us check the remaining conditions.

(iii) Suppose $\langle u,i\rangle\, R^*_a\, \langle v,j\rangle$ and $a < b$, for $b \in M$. If $\langle u,i\rangle = \langle v,j\rangle$, then $\langle u,i\rangle\, R^*_b\, \langle v,j\rangle$ follows immediately from the definition. So assume $\langle u,i\rangle \neq \langle v,j\rangle$. By definition we have $uR^f_a v$, and since $\mathfrak{S}^f$ satisfies (iii), $uR^f_b v$ holds as well. If $\neg uR^f_{\overline{b}} v$, then clearly $\langle u,i\rangle\, R^*_b\, \langle v,j\rangle$. So suppose $uR^f_{\overline{b}} v$. Since $\mathfrak{S}^f$ satisfies (iv), we then have $uR^f_{\overline{a}} v$, whence $j = 0$ and so $\langle u,i\rangle\, R^*_b\, \langle v,j\rangle$.

(iv) Suppose $\langle u,i\rangle\, R^*_{\overline{a}}\, \langle v,j\rangle$ and $a > b$, for $b \in M$. Assume $\neg\, \langle u,i\rangle\, R^*_{\overline{b}}\, \langle v,j\rangle$. By (i), we have $\langle u,i\rangle\, R^*_b\, \langle v,j\rangle$, whence by (iv), $\langle u,i\rangle\, R^*_a\, \langle v,j\rangle$. Now (ii) implies $\neg\, \langle u,i\rangle\, R^*_{\overline{a}}\, \langle v,j\rangle$, which is a contradiction. Hence $\langle u,i\rangle\, R^*_{\overline{b}}\, \langle v,j\rangle$.

(vi) Suppose $\langle u,i\rangle\, R^*_a\, \langle v,j\rangle$ and $\langle v,j\rangle\, R^*_b\, \langle w,k\rangle$, for $a, b, a+b \in M$. We have to show that $\langle u,i\rangle\, R^*_{a+b}\, \langle w,k\rangle$. First, if $\langle u,i\rangle = \langle v,j\rangle$ or $\langle v,j\rangle = \langle w,k\rangle$, then $\langle u,i\rangle\, R^*_{a+b}\, \langle w,k\rangle$ follows immediately from (iii), since $a, b \leq a+b$. So we may assume that $\langle u,i\rangle \neq \langle v,j\rangle$ and $\langle v,j\rangle \neq \langle w,k\rangle$. Then by definition, $uR^f_a v$ and $vR^f_b w$, whence $uR^f_{a+b} w$, because $\mathfrak{S}^f$ satisfies (vi). If $\neg uR^f_{\overline{a+b}} w$, then $\langle u,i\rangle\, R^*_{a+b}\, \langle w,k\rangle$ follows from the definition. So assume $uR^f_{\overline{a+b}} w$ holds in $\mathfrak{S}^f$ as well. From $uR^f_a v$ and (viii) we obtain $vR^f_{\overline{b}} w$, and so $k = 0$. But then again, $\langle u,i\rangle\, R^*_{a+b}\, \langle w,k\rangle$ follows from the definition. ❑

Lemma 27. *For all* $\langle u,i\rangle \in W^*$, $i \in \{0,1,2\}$ *and all* $t \in \Theta$, *we have*

$$\langle u,i\rangle \in t^{\mathfrak{S}^*} \quad \textit{iff} \quad u \in t^{\mathfrak{S}^f}.$$

Proof. The proof is by induction on t. The basis of induction follows from the definition and the case of Booleans is trivial. The cases of $t = (\mathsf{A}^{\leq a} s)$ and $t = (\mathsf{A}^{>a} s)$ are consequences of the following claims.

Claim 1: If $uR^f_a v$ and $\langle u,i\rangle \in W^*$, then there is j such that $\langle u,i\rangle\, R^*_a\, \langle v,j\rangle$. Indeed, if $a > 0$, we put $j = 0$, and $\langle u,i\rangle\, R^*_a\, \langle v,j\rangle$ follows from the definition. If $a = 0$, then $u = v$; so we can take $i = j$.

Claim 2: If $\langle u,i\rangle\, R^*_a\, \langle v,j\rangle$ then $uR^f_a v$. This follows immediately from the definition of R^*_a.

Claim 3: If $uR^f_{\overline{a}} v$ and $\langle u,i\rangle \in W^*$ then there exists j such that $\langle u,i\rangle\, R^*_{\overline{a}}\, \langle v,j\rangle$. Fix some $uR^f_{\overline{a}} v$ and $\langle u,i\rangle \in W^*$. Suppose first that $a = 0$. If $\neg uR^f_0 v$ we then have $u \neq v$, since R^f_0 satisfies (v), and so we can choose $j = i$. If $uR^f_0 v$ then v has been copied, so we can choose $j = i + 1 (mod\ 2)$ and $\langle u,i\rangle \neq \langle v,j\rangle$, from which $\langle u,i\rangle\, R^*_{\overline{a}}\, \langle v,j\rangle$.

Suppose now that $a > 0$. Consider two cases.

Case 1: $uR^f_a v$. Then v has been copied, i.e., W^* contains $\langle v,0\rangle$, $\langle v,1\rangle$ and $\langle v,2\rangle$. Then put $j \neq 0, i$ which is always possible, because we have three copies of v. But then all three defining properties of $\langle u,i\rangle\, R^*_a\, \langle v,j\rangle$ are violated, which means $\langle u,i\rangle\, R^*_{\overline{a}}\, \langle v,j\rangle$.

Case 2: $\neg uR^f_a v$. Then $u \neq v$. So we can put $j = 0$, and again all three defining properties are violated.

Claim 4: If $\langle u,i\rangle\, R^*_{\overline{a}}\, \langle v,j\rangle$ then $uR^f_{\overline{a}} v$. There are again two cases.

Case 1: $a > 0$. Suppose $\neg uR^f_{\overline{a}} v$. Then, since the first defining property of $uR^*_a v$ is violated, we have $\neg uR^f_a v$, contrary to (i). Therefore $uR^f_{\overline{a}} v$.

Case 2: $a = 0$. Then $\langle u, i\rangle \neq \langle v, j\rangle$. If $u \neq v$, then $\neg u R^f_0 v$ and hence $u R^f_{\overline{0}} v$ as required. If $u = v$ and $i \neq j$, then u has been copied. So there are $w \in W^f$ and $b \in M$ such that $w R^f_b u$ and $w R^f_{\overline{b}} u$. Since the latter can be written as $w R^f_{\overline{b+0}} u$, condition (viii) yields $u R^f_{\overline{0}} u$, as required.

Now, consider the induction step for $t = (\mathsf{A}^{\leq a} s)$. Suppose $\langle u, i\rangle \in (\mathsf{A}^{\leq a} s)^{\mathfrak{S}^*}$ and pick some $u R^f_a v$. By Claim 1, there exists $j \in \{0, 1, 2\}$ such that $\langle u, i\rangle\, R^*_a\, \langle v, j\rangle$. Then $\langle v, j\rangle \in s^{\mathfrak{S}^*}$ and, by the induction hypotheses, it follows that $v \in s^{\mathfrak{S}^f}$. Hence $u \in (\mathsf{A}^{\leq a} s)^{\mathfrak{S}^f}$. Conversely, if $u \in (\mathsf{A}^{\leq a} s)^{\mathfrak{S}^f}$ and $\langle u, i\rangle\, R^*_a\, \langle v, j\rangle$, then by Claim 2, $u R^f_a v$ and $v \in s^{\mathfrak{S}^f}$, and so by the induction hypotheses, $\langle v, j\rangle \in s^{\mathfrak{S}^*}$, i.e., $\langle u, i\rangle \in (\mathsf{A}^{\leq a} s)^{\mathfrak{S}^*}$.

The case of $t = (A^{>a} s)$ follows analogously from Claims 3 and 4. ❑

To complete the proof, we can first—in the same way as in Theorem 13—transform $\mathfrak{S}^*$ into a finite distance space model, which is possibly non-symmetric, and prove that this model satisfies Φ. This shows that $\mathcal{MS}^{\#}[\mathbb{R}^+]$ has the finite model property with respect to $\mathcal{D}_{tr}$. Then, following the lines of Section 4, we can establish both the decidability and complexity claim for $\mathcal{MS}^{\#}$-formulas in non-symmetric distance spaces. ❑

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