Definitions and (Uniform) Interpolants in First-Order Modal Logic

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Abstract

We first consider two decidable fragments of quantified modal logic $\mathsf{S5}$: the one-variable fragment $\mathsf{Q}^1\mathsf{S5}$ and its extension $\mathsf{S5}_{\mathcal{ALC}^u}$ that combines $\mathsf{S5}$ and the description logic $\mathcal{ALC}$ with the universal role. As neither of them enjoys Craig interpolation or projective Beth definability, the existence of interpolants and explicit definitions of predicates—which is crucial in many knowledge engineering tasks—does not directly reduce to entailment. Our concern therefore is the computational complexity of deciding whether (uniform) interpolants and definitions exist for given input formulas, signatures and ontologies. We prove that interpolant and definition existence in $\mathsf{Q}^1\mathsf{S5}$ and $\mathsf{S5}_{\mathcal{ALC}^u}$ is decidable in coN2ExpTime, being 2ExpTime-hard, while uniform interpolant existence is undecidable. Then we show that interpolant and definition existence in the one-variable fragment $\mathsf{Q}^1\mathsf{K}$ of quantified modal logic K is nonelementary decidable, while uniform interpolant existence is undecidable.

1 Introduction

Decidable fragments of first-order modal logics have been a well-established KR formalism for many decades, e.g., in the form of epistemic, temporal, or standpoint description logics (Donini et al. 1998; Lutz, Wolter, and Zakharyaschev 2008; Artale et al. 2017; Álvarez, Rudolph, and Strass 2022), spatio-temporal logics (Kontchakov et al. 2007), and logics of knowledge and belief (Belardinelli and Lomuscio 2009; Wang 2017; Liu et al. 2022; Wang, Wei, and Seligman 2022). While significant progress has been made in understanding the computational complexity of entailment in these 'two-dimensional' logics, little is known about the algorithmic properties of logic-based support mechanisms for engineering knowledge bases or specifications in these logics. Important examples of relevant problems are:

(***definition existence***) Given a knowledge base (KB), a predicate P, and a signature σ, is it possible to give a definition of P in terms of σ-predicates modulo the KB?

(***forgetting/uniform interpolants***) Given a KB and a signature σ, is it possible to 'forget σ', i.e., find a new KB without σ-predicates that says the same about non-σ-symbols as the original KB?

(***conservative extensions***) Given a KB and a set of additional axioms, is it the case that the expanded KB does not entail new relationships between the original predicates?

These and related problems have been studied extensively for many KR formalisms (Eiter and Kern-Isberner 2019) including propositional logic (Lang and Marquis 2008), answer set programming (Gonçalves, Knorr, and Leite 2023), and description logics (Konev et al. 2009; Botoeva et al. 2016) but investigating them for first-order modal logics (FOMLs) poses particular challenges. In contrast to many other KR formalisms, FOMLs used in KR typically do not enjoy the Craig interpolation property (CIP) as $\models \varphi \to \psi$ does not necessarily entail the existence of an *interpolant* χ whose predicate symbols occur in both φ and ψ, with $\models \varphi \to \chi$ and $\models \chi \to \psi$. Nor do they enjoy the projective Beth definability property (BDP) according to which implicit definability of a predicate in a given signature (which can be reduced to entailment) implies its explicit definability as required in definition existence. Forgetting and conservative extensions in FOMLs become dependent on predicates that do not occur in the original KB. In fact, Fine (1979) showed that no FOML with constant domains (the standard assumption in KR applications) between the basic quantified modal logics K and $\mathsf{S5}$ enjoys CIP or BDP.

Example 1 (based on (Fine 1979))**.** Interpreting $\Box$ as the $\mathsf{S5}$-modality 'always', let a KB contain the axioms

$$\mathsf{rep} \to \Diamond\forall x\,\big(\mathsf{inPower}(x) \to \Box(\mathsf{rep} \to \neg\mathsf{inPower}(x))\big),$$

$$\neg\mathsf{rep} \to \Box\exists x\,\big(\mathsf{inPower}(x) \land \Box(\neg\mathsf{rep} \to \mathsf{inPower}(x))\big),$$

where rep stands for the proposition 'replaceable'. Then rep is true at a world w satisfying the KB iff there is a world w' where all those who were in power at w lose it. Thus, rep is implicitly defined via $\mathsf{inPower}$. However, there is no explicit definition of rep via $\mathsf{inPower}$ in FOML (see Example 8).

Fine's example shows that CIP/BDP fail already in typical decidable fragments of FOML lying between the one-variable fragment FOM^1 and full FOML. Because of their wide use, 'repairing' CIP/BDP has become a major research challenge. For instance, it is shown in (Fitting 2002; Areces, Blackburn, and Marx 2003) that by adding second-order quantifiers or the machinery of hybrid logic constructors to FOML, one obtains natural logics with CIP and BDP. The price, however, is that these extensions are undecidable even if applied to decidable fragments of FOML.

In this paper, we take a fundamentally different, non-uniform approach. Instead of repairing CIP/BDP by enriching the language, we stay within its original boundaries and explore if it is possible to check the existence of interpolants/definitions even though the reduction to entailment via CIP/BDP is blocked. We conjecture that, in real-world applications, interpolants and definitions often do exist, so the failure of CIP/BDP will have a limited effect on the users.

We first focus on two decidable fragments of quantified $\mathsf{S5}$: its one-variable fragment $\mathsf{Q^1S5}$ illustrated in Example 1 and $\mathsf{S5}_{\mathcal{ALC}^u}$, the FOML obtained by combining $\mathsf{S5}$ and the description logic (DL) $\mathcal{ALC}^u$ extending the basic DL $\mathcal{ALC}$ with the universal role. In $\mathsf{S5}_{\mathcal{ALC}^u}$, we admit the application of modal operators to concepts and concept inclusions but not to roles, and so consider a typical *monodic* fragment of FOML, in which modal operators are only applied to formulas with at most one free variable (Hodkinson, Wolter, and Zakharyaschev 2000; Wolter and Zakharyaschev 2001). $\mathsf{Q^1S5}$ is a fragment of $\mathsf{S5}_{\mathcal{ALC}^u}$, and satisfiability is NExpTime-complete for both languages (Gabbay et al. 2003).

We chose $\mathsf{S5}$ as our starting point as it is widely used in Knowledge Representation, underpinning fundamental modalities such as necessity and agents' knowledge (Fagin et al. 1995). Combined with DLs, it has also been proposed as a logic of change interpreting $\Box$ as 'always' (Artale, Lutz, and Toman 2007) and can naturally encode a rather expressive version of standpoint logic (Álvarez, Rudolph, and Strass 2022); see Appendix B for details.

Example 2. In $\mathsf{S5}_{\mathcal{ALC}^u}$, we can encode different standpoints s_i by concept names S_i that hold everywhere in the domain of any world conceivable by s_i. That $C \sqsubseteq D$ holds according to s_i can then be represented as $\Box(S_i \sqcap C \sqsubseteq D)$ or $\Box_{s_i}(C \sqsubseteq D)$ for short. Suppose that our KB, K, contains the axioms

$\Box(\top \sqsubseteq S_1 \sqcup S_2)$,

$\Box(\mathsf{KR} \sqcup \mathsf{Databases} \sqcup \mathsf{Verification} \equiv \mathsf{CS} \sqcap \exists \mathsf{uses.Logic})$,

$\Box(\mathsf{Databases} \sqcup \mathsf{Verification} \sqsubseteq \neg\exists \mathsf{historicAreaOf.AI})$,

$\Box_{s_1}(\mathsf{KR} \equiv \mathsf{CS} \sqcap \exists \mathsf{areaOf.AI} \sqcap \exists \mathsf{uses.Logic})$,

$\Box_{s_2}(\mathsf{KR} \sqsubseteq \exists \mathsf{historicAreaOf.AI})$,

$\Box_{s_2}(\exists \mathsf{areaOf.AI} \sqsubseteq \neg\exists \mathsf{uses.Logic})$,

The first says that s_1 and s_2 cover all standpoints: if they agree on something, then it is generally agreed. According to the next two, it is generally agreed that the areas of CS that use Logic are KR, Databases and Verification, while Verification and Databases are not historic areas of AI. The last three express the s_i's diverging views on KR. Then

$$\mathsf{KR} \equiv \mathsf{CS} \sqcap \exists \mathsf{uses.Logic} \sqcap (\exists \mathsf{areaOf.AI} \sqcup \exists \mathsf{historicAreaOf.AI})$$

is entailed by K, and so explicitly defines KR modulo K without referring to S_i, KR, Databases, and Verification. $\dashv$

Our main result is that interpolant and definition existence in $\mathsf{Q^1S5}$ and $\mathsf{S5}_{\mathcal{ALC}^u}$ is decidable in coN2ExpTime, being 2ExpTime-hard. The proof is based on novel 'componentwise' bisimulations that replace standard FOML bisimulations in our characterisation of interpolant/definition existence. For the upper bound, we show that there are bisimilar models witnessing non-existence of interpolants/definitions of double-exponential size. The proof is inspired by the recent upper bound proofs of interpolant existence in the two-variable first-order logic FO^2 (Jung and Wolter 2021) but requires a novel use of types. The lower bound proof combines the interpolation counterexample of (Marx and Areces 1998), the exponential grid generation from (Hodkinson et al. 2003; Göller, Jung, and Lohrey 2015), and the representation of exponentially space bounded ATMs from (Jung and Wolter 2021). As a corollary we obtain a 2ExpTime lower bound for FO^2 without equality, answering an open question of (Jung and Wolter 2021).

We then consider uniform interpolant existence and conservative extension and show that both problems are undecidable for $\mathsf{Q^1S5}$ and $\mathsf{S5}_{\mathcal{ALC}^u}$. The proof extends a reduction proving undecidability of conservative extensions for FO^2 (with and without equality) from (Jung et al. 2017). As a corollary of our proof, we obtain that uniform interpolant existence is undecidable for FO^2 (with and without equality), settling an open problem from (Jung et al. 2017).

Finally, we consider the one-variable fragment $\mathsf{Q^1K}$ of quantified K and prove a non-elementary upper bound for interpolant/definition existence using the fact that $\mathsf{Q^1K}$ has finitely many non-equivalent formulas of bounded modal depth. To our surprise, conservative extensions and uniform interpolant existence are still undecidable in $\mathsf{Q^1K}$, which is proved by adapting the undecidability proof for $\mathsf{Q^1S5}$.

Related Work on Interpolant Existence. Except for work on linear temporal logic *LTL* by (Henkell 1988; Henkell et al. 2010; Place and Zeitoun 2016), the non-uniform approach to Craig interpolants has only very recently been studied by (Jung and Wolter 2021) for the guarded and two-variable fragment, by (Artale et al. 2021; Jung, Mazzullo, and Wolter 2022) for classical DLs, and by (Benedikt et al. 2016; Fortin, Konev, and Wolter 2022) for Horn logics. The non-uniform investigation of uniform interpolants started with complexity results by (Lutz, Seylan, and Wolter 2012; Lutz and Wolter 2011) and upper bounds on their size in (Nikitina and Rudolph 2014). The practical computation of uniform interpolants is an active research area for many years (Konev, Walther, and Wolter 2009; Koopmann and Schmidt 2015; Zhao and Schmidt 2016), see (Zhao et al. 2018; Koopmann 2020) for recent system descriptions.

Omitted proofs, definitions, and constructions can be found in the Appendix.

2 Preliminaries

Logics. The formulas of the *one-variable fragment* FOM^1 *of first-order modal logic* are built from unary predicate symbols $\boldsymbol{p} \in \mathcal{P}$ in a countably-infinite set $\mathcal{P}$ and a single variable x using $\top$, $\neg$, $\wedge$, $\exists x$, and the possibility operator $\Diamond$ via which the other Booleans, $\forall x$, and the necessity operator $\Box$ are standardly definable. A *signature* is any finite set $\sigma \subseteq \mathcal{P}$; the signature $sig(\varphi)$ of a formula φ comprises the predicate symbols in φ. If $sig(\varphi) \subseteq \sigma$, we call φ a σ*-formula*. By $sub(\varphi)$ we denote the closure under single negation of the set of subformulas of φ, and by $|\varphi|$ the cardinality of $sub(\varphi)$.

We interpret FOM^1 in (*Kripke*) *models* with *constant domains* of the form $\mathfrak{M} = (W, R, D, I)$, where $W \neq \emptyset$ is a set of *worlds*, $R \subseteq W \times W$ an *accessibility relation* on W, $D \neq \emptyset$ an (FO-)*domain* of $\mathfrak{M}$, and $I(w)$, for each $w \in W$, is an *interpretation* of the $\boldsymbol{p} \in \mathcal{P}$ over D, that is, $\boldsymbol{p}^{I(w)} \subseteq D$.

The *truth-relation* $\mathfrak{M}, w, d \models \varphi$, for any $w \in W$, $d \in D$ and FOM^1-formula φ, is defined inductively by taking

- $\mathfrak{M}, w, d \models \boldsymbol{p}(x)$ iff $d \in \boldsymbol{p}^{I(w)}$, for $\boldsymbol{p} \in \mathcal{P}$,
- $\mathfrak{M}, w, d \models \exists x\, \varphi$ iff there is $d' \in D$ with $\mathfrak{M}, w, d' \models \varphi$,
- $\mathfrak{M}, w, d \models \Diamond\varphi$ iff there is $w' \in W$ with $R(w, w')$ and $\mathfrak{M}, w', d \models \varphi$,

and the standard clauses for $\top$, $\neg$, $\wedge$. If φ is a sentence (i.e., every occurrence of x in φ is in the scope of $\exists$), then $\mathfrak{M}, w, d \models \varphi$ iff $\mathfrak{M}, w, d' \models \varphi$, for any $d, d' \in D$, and so we can omit d and write $\mathfrak{M}, w \models \varphi$. In a similar way, we can use $\mathfrak{M}, d \models \psi$ if every $\boldsymbol{p}$ in ψ is in the scope of $\Diamond$.

The set of formulas φ with $\mathfrak{M}, w, d \models \varphi$, for all $\mathfrak{M}$, w, d, is denoted by $\mathsf{Q}^1\mathsf{K}$; it is the FOM^1-extension of the modal logic K. Those φ that are true everywhere in all models $\mathfrak{M}$ with $R = W \times W$ comprise $\mathsf{Q}^1\mathsf{S5}$, the FOM^1-extension of the modal logic $\mathsf{S5}$. Let L be one of these two logics.

A *knowledge base* (KB), K, is any finite set of sentences. We say that K (*locally*) *entails* φ *in* L and write $K \models_L \varphi$ if $\mathfrak{M}, w \models K$ implies $\mathfrak{M}, w, d \models \varphi$, for any L-model $\mathfrak{M}$ and any w and d in it. Shortening $\emptyset \models_L \varphi$ to $\models_L \varphi$ (i.e., $\varphi \in L$), we note that $K \models_L \varphi$ iff $\models_L (\bigwedge_{\psi \in K} \psi \to \varphi)$, which reduces KB-entailment in L to L-validity which is known to be coNExpTime-complete (Marx 1999).

Bisimulations. Given two models $\mathfrak{M} = (W, R, D, I)$ with w, d and $\mathfrak{M}' = (W', R', D', I')$ with w', d', we write $\mathfrak{M}, w, d \equiv_\sigma \mathfrak{M}', w', d'$, for a signature σ, if the same σ-formulas are true at w, d in $\mathfrak{M}$ and at w', d' in $\mathfrak{M}'$. We characterise $\equiv_\sigma$ using bisimulations. Namely, a relation

$$\boldsymbol{\beta} \subseteq (W \times D) \times (W' \times D')$$

is called a σ-*bisimulation* between models $\mathfrak{M}$ and $\mathfrak{M}'$ if the following conditions hold for all $((w, d), (w', d')) \in \boldsymbol{\beta}$ and $\boldsymbol{p} \in \sigma$:

(a) $\mathfrak{M}, w, d \models \boldsymbol{p}$ iff $\mathfrak{M}', w', d' \models \boldsymbol{p}$;

(w) if $(w, v) \in R$, then there is v' such that $(w', v') \in R'$ and $((v, d), (v', d')) \in \boldsymbol{\beta}$, and the other way round;

(d) for every $e \in D$, there is $e' \in D'$ such that $((w, e), (w', e')) \in \boldsymbol{\beta}$, and the other way round.

We say that $\mathfrak{M}, w, d$ and $\mathfrak{M}', w', d'$ are σ-*bisimilar* and write $\mathfrak{M}, w, d \sim_\sigma \mathfrak{M}', w', d'$ if there is a σ-bisimulation $\boldsymbol{\beta} \ni ((w, d), (w', d'))$ between $\mathfrak{M}$ and $\mathfrak{M}'$. The next characterisation is proved in a standard way using ω-saturated models (Chang and Keisler 1998; Goranko and Otto 2007):

Lemma 3. *For any signature σ and any ω-saturated models $\mathfrak{M}$ with w, d and $\mathfrak{M}'$ with w', d', we have:*

$$\mathfrak{M}, w, d \equiv_\sigma \mathfrak{M}', w', d' \quad \textit{iff} \quad \mathfrak{M}, w, d \sim_\sigma \mathfrak{M}', w', d'.$$

The direction from right to left holds for arbitrary models.

Modal products and succinct notation. As observed by (Wajsberg 1933), $\mathsf{S5}$ is a notational variant of the one-variable fragment FO^1 of FO: just drop x from $\exists x$ and $\boldsymbol{p}(x)$ in FO^1-formulas, treating $\exists$ as a possibility operator and $\boldsymbol{p}$ as a *propositional variable*. The same operation transforms FOM^1-formulas into more succinct *bimodal formulas* with $\Diamond$ interpreted over the (W, R) 'dimension' and $\exists$ over the $(D, D \times D)$ 'dimension'. This way we view the FOM^1-extensions of $\mathsf{S5}$ and K as two-dimensional *products* of modal logics: $\mathsf{S5} \times \mathsf{S5}$ and $\mathsf{K} \times \mathsf{S5}$. The former is known to be the 'equality and substitution-free' fragment of two-variable FO-logic FO^2 (Gabbay et al. 2003); the latter is embedded into FO by the *standard translation* *:

$$\boldsymbol{p}^* = \boldsymbol{q}(z, x), \;\; (\neg\varphi)^* = \neg\varphi^*, \;\; (\varphi \wedge \psi)^* = \varphi^* \wedge \psi^*,$$
$$(\exists\varphi)^* = \exists x\, \varphi^*, \;\; (\Diamond\varphi)^* = \exists y\, (R(z, y) \wedge \varphi^*\{y/z\}),$$

where y is a fresh variable not occurring in φ^* and $\{y/z\}$ means a substitution of y in place of z.

From now on, we write FOM^1-formulas as *bimodal* ones: for example, $\exists\Box\boldsymbol{p}$ instead of $\exists x\, \Box\boldsymbol{p}(x)$. By a formula we mean an FOM^1-formula unless indicated otherwise; a logic, L, is one of $\mathsf{Q}^1\mathsf{S5}$ and $\mathsf{Q}^1\mathsf{K}$, again unless stated otherwise.

3 Main Notions and Characterisations

We now introduce the main notions studied in this paper and provide their model-theoretic characterisations.

Craig interpolants. A formula χ is an *interpolant* of formulas φ and ψ *in a logic* L if $sig(\chi) \subseteq sig(\varphi) \cap sig(\psi)$, $\models_L \varphi \to \chi$ and $\models_L \chi \to \psi$. L enjoys the *Craig interpolation property* (*CIP*) if an interpolant for φ and ψ exists whenever $\models_L \varphi \to \psi$. One of our main concerns here is the *interpolant existence problem* (*IEP*) for L: decide if given φ and ψ have an interpolant in L. For logics with CIP, IEP reduces to entailment, and so is not interesting. This is the case for many logics including propositional $\mathsf{S5}$ and K, but not for FOMLs with constant domain between $\mathsf{Q}^1\mathsf{K}$ and $\mathsf{Q}^1\mathsf{S5}$ (Fine 1979; Marx and Areces 1998).

Explicit definitions. Given formulas φ, ψ and a signature σ, an *explicit σ-definition of ψ modulo φ in L* is a σ-formula χ with $\models_L \varphi \to (\psi \leftrightarrow \chi)$. The *explicit σ-definition existence problem* (*EDEP*) for L is to decide, given φ, ψ and σ, whether there exists an explicit σ-definition of ψ modulo φ in L. EDEP reduces trivially to entailment for logics enjoying the *projective Beth definability property* (*BDP*) according to which ψ is explicitly σ-definable modulo φ in L iff it is implicitly σ-definable modulo φ in the sense that $\{\varphi, \varphi'\} \models_L \psi \leftrightarrow \psi'$, where φ', ψ' result from φ, ψ by uniformly replacing all non-σ-symbols with fresh ones. Again, many logics including $\mathsf{S5}$ and K enjoy BDP while FOMLs with constant domains between $\mathsf{Q}^1\mathsf{K}$ and $\mathsf{Q}^1\mathsf{S5}$ do not.

Note that, in typical KR applications, φ in our formulation of EDEP corresponds to a KB K and ψ is a predicate $\boldsymbol{p}$. Then the problem whether there exists an explicit σ-definition of $\boldsymbol{p}$ modulo K is the problem of deciding whether there is χ with $sig(\chi) \subseteq \sigma$ and $K \models_L \forall x(\boldsymbol{p}(x) \leftrightarrow \chi(x))$. This problem trivially translates to EDEP using our discussion of KBs above. In more detail, this view of EDEP is discussed in Section 6 in the context of $\mathsf{S5}_{\mathcal{ALC}^u}$.

IEP and EDEP are closely related (Gabbay and Maksimova 2005). In this paper, we only require the following:

Theorem 4. *For any $L \in \{\mathsf{Q}^1\mathsf{S5}, \mathsf{Q}^1\mathsf{K}\}$, EDEP for L and IEP for L are polynomially reducible to each other.*

The proof, given in Appendix C, is based on a characterisation of IEP and EDEP using bisimulations.

Lemma 3 together with the fact that FOM^1 is a fragment of FO are used to obtain, again in a standard way, the following criterion of interpolant existence. We call formulas φ and ψ *σ-bisimulation consistent* in L if there exist L-models $\mathfrak{M}$ with w, d and $\mathfrak{M}'$ with w', d' such that $\mathfrak{M}, w, d \models \varphi$, $\mathfrak{M}', w', d' \models \psi$ and $\mathfrak{M}, w, d \sim_\sigma \mathfrak{M}', w', d'$.

Theorem 5. *For any φ and ψ, the following are equivalent:*

– *there does not exist an interpolant of φ and ψ in L;*

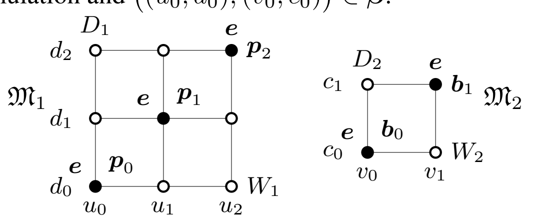
– *φ, $\neg\psi$ are $sig(\varphi) \cap sig(\psi)$-bisimulation consistent in L.*

Proof. Suppose φ and ψ do not have an interpolant in L and $\sigma = sig(\varphi) \cap sig(\psi)$. Consider the set Ξ of σ-formulas χ with $\models_L \varphi \to \chi$. By compactness, we have an ω-saturated model $\mathfrak{M}$ of L with w and d such that $\mathfrak{M}, w, d \models \chi$, for all $\chi \in \Xi$, and $\mathfrak{M}, w, d \models \neg\psi$. Take the set Ξ' of σ-formulas χ with $\mathfrak{M}, w, d \models \chi$ and an ω-saturated model $\mathfrak{M}'$ with $\mathfrak{M}', w', d' \models \Xi'$ and $\mathfrak{M}', w', d' \models \varphi$, for some w' and d'. Then $\mathfrak{M}, w, d \equiv_\sigma \mathfrak{M}', w', d'$, and so $\mathfrak{M}, w, d \sim_\sigma \mathfrak{M}', w', d'$ by Lemma 3. The converse implication is straightforward. $\dashv$

Example 6. For every $n < \omega$, Marx and Areces (1998) constructed FOM^1-formulas φ and ψ with $\models_{\mathsf{Q^1S5}} \varphi \to \psi$ and $sig(\varphi) \cap sig(\psi) = \{\boldsymbol{e}\}$ that have no interpolant in the n-variable $\mathsf{Q^nS5}$. For $n = 1$, φ and ψ look as follows:

$$\begin{aligned}\varphi \;=\;& \boldsymbol{p}_0 \wedge \Diamond\exists(\boldsymbol{p}_1 \wedge \Diamond\exists \boldsymbol{p}_2) \wedge \\ & \Box\forall\big[(\boldsymbol{e} \leftrightarrow \boldsymbol{p}_0 \vee \boldsymbol{p}_1 \vee \boldsymbol{p}_2) \wedge \textstyle\bigwedge_{i \neq j}(\boldsymbol{p}_i \to \neg \boldsymbol{p}_j) \wedge \\ & \qquad \textstyle\bigwedge_i \big(\boldsymbol{p}_i \to \Box(\boldsymbol{e} \to \boldsymbol{p}_i) \wedge \forall(\boldsymbol{e} \to \boldsymbol{p}_i)\big)\big], \\ \psi \;=\;& \Box\forall(\boldsymbol{e} \leftrightarrow \boldsymbol{b}_0 \vee \boldsymbol{b}_1) \to \\ & \Diamond\exists\big(\boldsymbol{b}_0 \wedge \Diamond(\neg\boldsymbol{e} \wedge \exists \boldsymbol{b}_0)\big) \vee \Diamond\exists\big(\boldsymbol{b}_1 \wedge \Diamond(\neg\boldsymbol{e} \wedge \exists \boldsymbol{b}_1)\big).\end{aligned}$$

To see that φ, $\neg\psi$ are $\{\boldsymbol{e}\}$-bisimulation consistent in $\mathsf{Q^1S5}$, take the models $\mathfrak{M}_1$ and $\mathfrak{M}_2$ below with $\mathfrak{M}_1, u_0, d_0 \models \varphi$ and $\mathfrak{M}_2, v_0, c_0 \models \neg\psi$. (In our pictures, the possible worlds are always shown along the horizontal axis and the domain elements along the vertical one, giving points of the form (w, d).) The relation $\boldsymbol{\beta}$ connecting each $\boldsymbol{e}$-point in $\mathfrak{M}_1$ with each $\boldsymbol{e}$-point in $\mathfrak{M}_2$, and similarly for $\neg\boldsymbol{e}$-points, is an $\{\boldsymbol{e}\}$-bisimulation and $\big((u_0, d_0), (v_0, c_0)\big) \in \boldsymbol{\beta}$.



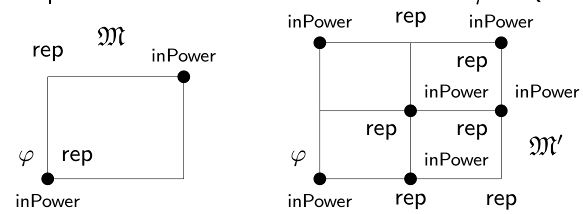
Similarly to Theorem 5 we obtain the following criterion of explicit definition existence:

Theorem 7. *For any φ, ψ, σ, the following are equivalent:*

– *there is no explicit σ-definition of ψ modulo φ in L;*

– *$\varphi \wedge \psi$ and $\varphi \wedge \neg\psi$ are σ-bisimulation consistent in L.*

Example 8. Let φ be the conjunction of the two KB axioms from Example 1, $\sigma = \{\mathsf{inPower}\}$, and let $\psi = \mathsf{rep}$.[1] Then the second condition of Theorem 7 holds for the $\mathsf{Q^1S5}$-models shown below, in which (w, d) in $\mathfrak{M}$ is bisimilar to (w', d') in $\mathfrak{M}'$ iff (w, d) and (w', d') agree on σ. It follows that rep has no definition via inPower modulo φ in $\mathsf{Q^1S5}$. $\dashv$



We next define conservative extensions, an important notion in the context of ontology modules and modularisation (Grau et al. 2008; Botoeva et al. 2016).

Conservative extensions. Given formulas φ and ψ, we call φ an *L-conservative extension of ψ* if (a) $\models_L \varphi \to \psi$ and (b) $\models_L \varphi \to \chi$ implies $\models_L \psi \to \chi$, for any χ with $sig(\chi) \subseteq sig(\psi)$. In typical KR applications, ψ is given by a KB K and φ is obtained by adding fresh axioms to K. (The translation of our results to the language of KBs is obvious.) The next example shows that this notion of conservative extension is syntax-dependent in the sense that it is not robust under the addition of fresh predicates.

Example 9. For the formulas

$\varphi = \mathsf{rep} \wedge \Diamond\forall\big(\mathsf{inPower} \to \Box(\mathsf{rep} \to \neg\mathsf{inPower})\big)$,

$\psi = \Box\forall(\Diamond\mathsf{inPower} \wedge \Diamond\neg\mathsf{inPower} \wedge \exists\mathsf{inPower} \wedge \exists\neg\mathsf{inPower})$

$\varphi \wedge \psi$ is a conservative extension of ψ in $\mathsf{Q^1S5}$ (as all models of ψ are $\{\mathsf{inPower}\}$-bisimilar to $\mathfrak{M}$ in Example 8). Now, let $\psi' = \psi \wedge (\mathsf{p} \vee \neg\mathsf{p})$, for a fresh proposition p. Then $\varphi \wedge \psi'$ is not a conservative extension of ψ' as witnessed by the formula $\chi = \neg\big(\mathsf{p} \wedge \Box\exists(\mathsf{inPower} \wedge \Box(\mathsf{p} \to \mathsf{inPower}))\big)$. $\dashv$

If in the previous definition we require (b) to hold for all χ with $sig(\chi) \cap sig(\varphi) \subseteq sig(\psi)$, then φ is called a *strong L-conservative extension of ψ*. As observed by (Jung et al. 2017), the difference between conservative and strong conservative extensions is closely related to the failure of CIP: if L enjoys CIP, then L-conservative extensions coincide with strong L-conservative extensions. The problem of deciding whether a given φ is a (strong) conservative extension of a given ψ will be referred to as (S)CEP. The study of the complexity of (S)CEP for DLs and modal logics started with (Ghilardi, Lutz, and Wolter 2006) and (Ghilardi et al. 2006); see (Botoeva et al. 2019; Jung, Lutz, and Marcinkowski 2022) for more recent work.

Uniform interpolants. Given a formula φ and a signature σ, we call a formula ψ a *σ-uniform interpolant of φ in L* if $sig(\psi) = \sigma$ and φ is a strong L-conservative extension of ψ.

A logic L has the *uniform interpolation property* (*UIP*) if, for any φ and σ, there is a σ-uniform interpolant of φ in L. UIP entails CIP but not the other way round. For example, modal logic S4 and $\mathcal{ALC}^u$ enjoy CIP but not UIP (Ghilardi and Zawadowski 1995; Lutz and Wolter 2011). This leads to the *uniform interpolant existence problem* (*UIEP*): given φ and σ, decide whether φ has a uniform σ-interpolant in L. Uniform interpolants are closely related to forgetting introduced by (Lin and Reiter 1994). In this case, one often drops the requirement that the conservative extension is strong.

[1] As our FOM^1 has no 0-ary predicates, the *proposition* rep is given as $\forall x\, \mathsf{rep}(x)$ assuming that $\models_L \forall x\, \mathsf{rep}(x) \vee \forall x\, \neg\mathsf{rep}(x)$.

4 Deciding IEP and EDEP for $\mathsf{Q}^1\mathsf{S5}$

In this section, we first give a simpler—yet equivalent—definition of bisimulation between $\mathsf{Q}^1\mathsf{S5}$-models and then use it to show that, when checking bisimulation consistency in $\mathsf{Q}^1\mathsf{S5}$, it is enough to look for bisimilar models of double-exponential size in the size of the given formulas.

As $R = W \times W$ in any $\mathsf{Q}^1\mathsf{S5}$-model $\mathfrak{M} = (W, R, D, I)$, we drop R and write simply $\mathfrak{M} = (W, D, I)$. Given a signature σ and $(w, d) \in W \times D$, the *literal σ-type* $\ell^\sigma_{\mathfrak{M}}(w, d)$ of (w, d) in $\mathfrak{M}$ is the set

$$\{\boldsymbol{p} \in \sigma \mid \mathfrak{M}, w, d \models \boldsymbol{p}\} \cup \{\neg \boldsymbol{p} \mid \boldsymbol{p} \in \sigma,\ \ \mathfrak{M}, w, d \not\models \boldsymbol{p}\}.$$

A pair $(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2)$ of relations $\boldsymbol{\beta}_1 \subseteq W \times W'$ and $\boldsymbol{\beta}_2 \subseteq D \times D'$ is called a *σ-$\mathsf{S5}$-bisimulation* between $\mathfrak{M} = (W, D, I)$ and $\mathfrak{M}' = (W', D', I')$ when the following conditions hold:

($\mathsf{s5}_1$) if $(w, w') \in \boldsymbol{\beta}_1$ then, for any $d \in D$, there is $d' \in D'$ such that $(d, d') \in \boldsymbol{\beta}_2$ and $\ell^\sigma_{\mathfrak{M}}(w, d) = \ell^\sigma_{\mathfrak{M}'}(w', d')$, and the other way round;

($\mathsf{s5}_2$) if $(d, d') \in \boldsymbol{\beta}_2$ then, for any $w \in W$, there is $w' \in W'$ such that $(w, w') \in \boldsymbol{\beta}_1$ and $\ell^\sigma_{\mathfrak{M}}(w, d) = \ell^\sigma_{\mathfrak{M}'}(w', d')$, and the other way round.

We say that $\mathfrak{M}, w, d$ and $\mathfrak{M}', w', d'$ are *σ-$\mathsf{S5}$-bisimilar* and write $\mathfrak{M}, w, d \sim^{\mathsf{S5}}_\sigma \mathfrak{M}', w', d'$ if there is a σ-$\mathsf{S5}$-bisimulation $(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2)$ with $(w, w') \in \boldsymbol{\beta}_1$, $(d, d') \in \boldsymbol{\beta}_2$ and $\ell^\sigma_{\mathfrak{M}}(w, d) = \ell^\sigma_{\mathfrak{M}'}(w', d')$. Note that in this case we have $\mathrm{dom}(\boldsymbol{\beta}_1) = W$, $\mathrm{ran}(\boldsymbol{\beta}_1) = W'$, $\mathrm{dom}(\boldsymbol{\beta}_2) = D$, and $\mathrm{ran}(\boldsymbol{\beta}_2) = D'$.

Theorem 10. *$\mathfrak{M}, w, d \sim^{\mathsf{S5}}_\sigma \mathfrak{M}', w', d'$ if and only if $\mathfrak{M}, w, d \sim_\sigma \mathfrak{M}', w', d'$.*

Proof. If $\mathfrak{M}, w, d \sim^{\mathsf{S5}}_\sigma \mathfrak{M}', w', d'$ is witnessed by $(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2)$, then $\boldsymbol{\beta}$ defined by setting $((v, e), (v', e')) \in \boldsymbol{\beta}$ iff $(v, v') \in \boldsymbol{\beta}_1$, $(e, e') \in \boldsymbol{\beta}_2$ and $\ell^\sigma_{\mathfrak{M}}(v, e) = \ell^\sigma_{\mathfrak{M}'}(v', e')$ satisfies **(a)**, **(w)** and **(d)**. Conversely, if $\mathfrak{M}, w, d \sim_\sigma \mathfrak{M}', w', d'$ is witnessed by $\boldsymbol{\beta}$, then $(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2)$ below satisfies ($\mathsf{s5}_1$), ($\mathsf{s5}_2$)

$$\boldsymbol{\beta}_1 = \{(v, v') \mid \exists e, e' \, ((v, e), (v', e')) \in S\},$$
$$\boldsymbol{\beta}_2 = \{(e, e') \mid \exists v, v' \, (v, e), (v', e) \in S\},$$

$(w, w') \in \boldsymbol{\beta}_1$, $(d, d') \in \boldsymbol{\beta}_2$, and $\ell^\sigma_{\mathfrak{M}}(w, d) = \ell^\sigma_{\mathfrak{M}'}(w', d')$. ⊣

In this section, we only deal with σ-$\mathsf{S5}$-bisimulations, and so omit explicit $\mathsf{S5}$ from the relevant notations. We write $\mathfrak{M}_1, w_1 \sim_\sigma \mathfrak{M}_2, w_2$ if there is a σ-bisimulation $(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2)$ with $(w_1, w_2) \in \boldsymbol{\beta}_1$. By ($\mathsf{s5}_1$), $\mathfrak{M}_1, w_1 \sim_\sigma \mathfrak{M}_2, w_2$ entails that the interpretations $I_1(w_1)$ in $\mathfrak{M}_1$ and $I_2(w_2)$ in $\mathfrak{M}_2$ are *globally σ-bisimilar* in the sense that, for any $d_1 \in D_1$, there exists $d_2 \in D_2$ satisfying the same $\boldsymbol{p} \in \sigma$ in $I_1(w_1)$ and $I_2(w_2)$, and the other way round. Similarly, we write $\mathfrak{M}_1, d_1 \sim_\sigma \mathfrak{M}_2, d_2$ if there is a σ-bisimulation $(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2)$ with $(d_1, d_2) \in \boldsymbol{\beta}_2$. We omit $\mathfrak{M}_1$ and $\mathfrak{M}_2$ and write simply $(w_1, d_1) \sim_\sigma (w_2, d_2)$, $w_1 \sim_\sigma w_2$, $d_1 \sim_\sigma d_2$ if understood.

Observe that σ-bisimulations between the same models are preserved under set-theoretic union: if Γ is a set of σ-bisimulations, then $(\bigcup_{(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2) \in \Gamma} \boldsymbol{\beta}_1, \bigcup_{(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2) \in \Gamma} \boldsymbol{\beta}_2)$ is a σ-bisimulation too. It follows that $(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2)$ defined by taking $(w_1, w_2) \in \boldsymbol{\beta}_1$ if $w_1 \sim_\sigma w_2$ and $(d_1, d_2) \in \boldsymbol{\beta}_2$ if $d_1 \sim_\sigma d_2$ is the maximal σ-bisimulation between the given models.

Example 11. Consider $\mathfrak{M}_1$, $\mathfrak{M}_2$, and $\sigma = \{\boldsymbol{e}\}$ from Example 6. Then $(W_1 \times W_1, D_1 \times D_1)$ is a σ-bisimulation between $\mathfrak{M}_1$ and $\mathfrak{M}_1$ witnessing $(u_i, d_i) \sim_\sigma (u_j, d_j)$ and $(u_k, d_l) \sim_\sigma (u_m, d_n)$, for $i, j, k, l, m, n \in \{0, 1, 2\}$, $k \neq l$, $m \neq n$. The pair $(W_1 \times W_2, D_1 \times D_2)$ is a σ-bisimulation between $\mathfrak{M}_1$ and $\mathfrak{M}_2$ witnessing $(u_i, d_i) \sim_\sigma (v_j, c_j)$ and $(u_k, d_l) \sim_\sigma (v_m, c_n)$, for $i, k, l \in \{0, 1, 2\}$, $k \neq l$, and $j, m, n \in \{0, 1\}$, $m \neq n$ (cf. $\boldsymbol{\beta}$ in Example 6).

We now use σ-bisimulations to develop an algorithm deciding IEP for $\mathsf{Q}^1\mathsf{S5}$ in CON2EXPTIME. Suppose we want to check whether φ and ψ have an interpolant in $\mathsf{Q}^1\mathsf{S5}$. By Theorem 5, this is not the case iff there are $\mathsf{Q}^1\mathsf{S5}$-models $\mathfrak{M}_1$ with w_1, d_1 and $\mathfrak{M}_2$ with w_2, d_2 such that $\mathfrak{M}_1, w_1, d_1 \models \varphi$, $\mathfrak{M}_2, w_2, d_2 \not\models \psi$, and $\mathfrak{M}_1, w_1, d_1 \sim_\sigma \mathfrak{M}_2, w_2, d_2$. We are going to show that if such $\mathfrak{M}_i$ do exist, they can be chosen to be of double-exponential size in $|\varphi|$ and $|\psi|$.

Fix φ, ψ and $\sigma = \mathit{sig}(\varphi) \cap \mathit{sig}(\psi)$. Denote by $\mathit{sub}_\exists(\varphi, \psi)$ the closure under single negation of the set of formulas of the form $\exists \xi$ in $\mathit{sub}(\varphi, \psi) = \mathit{sub}(\varphi) \cup \mathit{sub}(\psi)$. The *world-type* of $w \in W$ in a model $\mathfrak{M} = (W, D, I)$ is defined as

$$\mathsf{wt}_{\mathfrak{M}}(w) = \{\rho \in \mathit{sub}_\exists(\varphi, \psi) \mid \mathfrak{M}, w \models \rho\}.$$

A *world-type*, wt, in $\mathfrak{M}$ is the world-type of some $w \in W$.

Similarly, let $\mathit{sub}_\Diamond(\varphi, \psi)$ be the closure under single negation of the set of formulas of the form $\Diamond \xi$ in $\mathit{sub}(\varphi, \psi)$. The *domain-type* of $d \in D$ in $\mathfrak{M}$ is the set

$$\mathsf{dt}_{\mathfrak{M}}(d) = \{\rho \in \mathit{sub}_\Diamond(\varphi, \psi) \mid \mathfrak{M}, d \models \rho\}.$$

A *domain-type*, dt, in $\mathfrak{M}$ is the domain-type of some $d \in D$.

The *full type* of $(w, d) \in W \times D$ in $\mathfrak{M}$ is the set

$$\mathsf{ft}_{\mathfrak{M}}(w, d) = \{\rho \in \mathit{sub}(\varphi, \psi) \mid \mathfrak{M}, w, d \models \rho\}.$$

A *full type*, ft, in $\mathfrak{M}$ is the full type of some (w, d) in $\mathfrak{M}$.

The main result of this section generalises the following construction that shows how, given any $\mathsf{Q}^1\mathsf{S5}$-model $\mathfrak{M}$ satisfying a formula φ, we can construct from the world and domain types in $\mathfrak{M}$ a model $\mathfrak{M}'$ satisfying φ and having exponential size in $|\varphi|$. Intuitively, as a first approximation, we could start by taking the worlds W' (domain D') in $\mathfrak{M}'$ to comprise all the world- (domain-) types in $\mathfrak{M}$. But then we might have w, w' and d, d' with $\mathsf{wt}_{\mathfrak{M}}(w) = \mathsf{wt}_{\mathfrak{M}}(w')$, $\mathsf{dt}_{\mathfrak{M}}(d) = \mathsf{dt}_{\mathfrak{M}}(d')$ and different truth-values of some variables $\boldsymbol{p}$ at (w, d) and (w', d') in $\mathfrak{M}$. To deal with this issue, we introduce, as shown in the example below, sufficiently many copies of each world- and domain-type so that we can accommodate *all* possible truth-values in $\mathfrak{M}$ of the $\boldsymbol{p}$ in φ.

Example 12. Let $\mathfrak{M}, w, d \models \varphi$, for $\mathfrak{M} = (W, D, I)$, and let n be the number of full types in $\mathfrak{M}$ (over $\mathit{sub}(\varphi)$) and $[n] = \{1, \dots, n\}$. Define D' to be a set that contains n distinct copies of each dt in $\mathfrak{M}$ over $\mathit{sub}_\Diamond(\varphi)$, denoting the kth copy by dt^k. For any wt and dt in $\mathfrak{M}$, let $\pi_{\mathsf{wt},\mathsf{dt}}$ be a function from $[n]$ *onto* the set of full types ft in $\mathfrak{M}$ with $\mathsf{wt} = \mathsf{ft} \cap \mathit{sub}_\exists(\varphi)$ and $\mathsf{dt} = \mathsf{ft} \cap \mathit{sub}_\Diamond(\varphi)$. Let Π be a smallest set of sequences π of such $\pi_{\mathsf{wt},\mathsf{dt}}$ for which the following condition holds: for any $\mathsf{ft} = \mathsf{ft}_{\mathfrak{M}}(u, e)$ and $k \in [n]$, there exists $\pi \in \Pi$ with $\pi_{\mathsf{wt}_{\mathfrak{M}}(u), \mathsf{dt}_{\mathfrak{M}}(e)}(k) = \mathsf{ft}$. We then set $W' = \{\mathsf{wt}^\pi_{\mathfrak{M}}(u) \mid u \in W,\ \pi \in \Pi\}$, treating each $\mathsf{wt}^\pi_{\mathfrak{M}}(u)$ as a fresh π-copy of $\mathsf{wt}_{\mathfrak{M}}(u)$. As $|\Pi| \leq n^2$, both $|W'|$ and $|D'|$

are exponential in $|\varphi|$. Define a model $\mathfrak{M}' = (W', D', I')$ by taking $\mathfrak{M}', \mathsf{wt}^\pi, \mathsf{dt}^k \models \boldsymbol{p}$ iff $\boldsymbol{p} \in \pi_{\mathsf{wt},\mathsf{dt}}(k)$. One can show by induction that $\mathfrak{M}', \mathsf{wt}^\pi, \mathsf{dt}^k \models \rho$ iff $\rho \in \pi_{\mathsf{wt},\mathsf{dt}}(k)$, for any $\rho \in \mathit{sub}(\varphi)$; see Appendix E.1 for details. $\dashv$

We now introduce more complex 'data structures' that allow us to extend the construction above from satisfiability to σ-bisimulation consistency. Let $\mathfrak{M}_i = (W_i, D_i, I_i)$, for $i = 1, 2$, be two $\mathsf{Q^1S5}$-models with pairwise disjoint W_i and D_i. For any $w \in W_1 \cup W_2$ and $i \in \{1, 2\}$, we set

$$T_i(w) = \{\mathsf{wt}_{\mathfrak{M}_i}(v) \mid v \in W_i,\ v \sim_\sigma w\} \tag{1}$$

and call $\mathsf{wm}(w) = (T_1(w), T_2(w))$ the *world mosaic* of w in $\mathfrak{M}_1, \mathfrak{M}_2$. The pair $\mathsf{wp}_i(w) = (\mathsf{wt}_{\mathfrak{M}_i}(w), \mathsf{wm}(w))$, for $w \in W_i$, is called the *i-world point* of w in $\mathfrak{M}_1$, $\mathfrak{M}_2$. A *world mosaic*, wm, and an *i-world point*, wp_i, in $\mathfrak{M}_1, \mathfrak{M}_2$ are defined as the world mosaic and i-world point of some $w \in W_1 \cup W_2$ in $\mathfrak{M}_1, \mathfrak{M}_2$ (in the latter case, $w \in W_i$).

Similarly, for any $d \in D_1 \cup D_2$ and $i \in \{1, 2\}$, we set

$$S_i(d) = \{\mathsf{dt}_{\mathfrak{M}_i}(e) \mid e \in D_i,\ e \sim_\sigma d\} \tag{2}$$

and call $\mathsf{dm}(d) = (S_1(d), S_2(d))$ the *domain mosaic* of d in $\mathfrak{M}_1$, $\mathfrak{M}_2$. If $d \in D_i$, the pair $\mathsf{dp}_i(d) = (\mathsf{dt}_{\mathfrak{M}_i}(d), \mathsf{dm}(d))$ is called the *i-domain point* of d in $\mathfrak{M}_1$, $\mathfrak{M}_2$. A *domain mosaic*, dm, and an *i-domain point*, dp_i, in $\mathfrak{M}_1$, $\mathfrak{M}_2$ are defined as the domain mosaic and i-domain point of some $d \in D_1 \cup D_2$. As follows from the definitions and Lemma 3 coupled with Theorem 10,

(wm) $u \sim_\sigma v$ implies $\mathsf{wm}(u) = \mathsf{wm}(v)$,

(dm) $d \sim_\sigma e$ implies $\mathsf{dm}(d) = \mathsf{dm}(e)$.

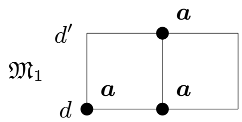
Observe that the number of distinct wp_i and dp_i is at most double-exponential in $|\varphi|$ and $|\psi|$.

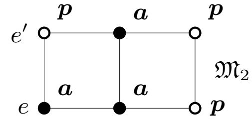
Example 13. (a) Consider models $\mathfrak{M}_1$ and $\mathfrak{M}_2$ from Example 6, $\sigma = \{\boldsymbol{e}\}$ and $\tau = \{\boldsymbol{e}, \boldsymbol{p}_0, \boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{b}_0, \boldsymbol{b}_1\}$. Then $\mathsf{wt}_{\mathfrak{M}_1}(u_i)$ and $\mathsf{dt}_{\mathfrak{M}_2}(c_i)$ contain, respectively, the sets

$$\{\exists(\boldsymbol{p}_i \wedge \boldsymbol{e})\} \cup \{\exists\neg\boldsymbol{p} \mid \boldsymbol{p} \in \tau\} \cup \{\neg\exists\boldsymbol{p}_j \mid j \neq i\},$$
$$\{\Diamond(\boldsymbol{p}_i \wedge \boldsymbol{e})\} \cup \{\Diamond\neg\boldsymbol{p} \mid \boldsymbol{p} \in \tau\} \cup \{\neg\Diamond\boldsymbol{p}_j \mid j \neq i\}.$$

From the σ-bisimulations shown in Example 11 we obtain $\mathsf{wm}(u_0) = \mathsf{wm}(u_1) = \mathsf{wm}(u_2) = \mathsf{wm}(v_0) = \mathsf{wm}(v_1)$, and so $\mathfrak{M}_1, \mathfrak{M}_2$ have only one world mosaic: $\mathsf{wm} = (\{\mathsf{wt}_{\mathfrak{M}_1}(u_i) \mid i = 0, 1, 2\}, \{\mathsf{wt}_{\mathfrak{M}_2}(v_i) \mid i = 0, 1\})$. $\mathfrak{M}_1$ has three distinct 1-world points $(\mathsf{wt}_{\mathfrak{M}_1}(u_i), \mathsf{wm})$, for $i = 0, 1, 2$; $\mathfrak{M}_2$ has two 2-world points. Similarly, $\mathfrak{M}_1, \mathfrak{M}_2$ define one domain mosaic, $\mathfrak{M}_1$ has three distinct 1-domain points and $\mathfrak{M}_2$ has two 2-domain points.

(b) It can happen that non-bisimilar domain elements give the same domain-point. Consider the models $\mathfrak{M}_1$ and $\mathfrak{M}_2$ below and suppose that $\mathit{sub}_\Diamond(\varphi, \psi)$ has no formulas with $\exists$





in the scope of $\Diamond$, $\sigma = \{\boldsymbol{a}\}$ and $\mathit{sig}(\varphi, \psi) = \{\boldsymbol{a}, \boldsymbol{p}\}$. Then $\mathsf{dt}_{\mathfrak{M}_1}(d) = \mathsf{dt}_{\mathfrak{M}_1}(d')$ but $d \not\sim_\sigma d'$ as $\Diamond(\boldsymbol{a} \wedge \exists\neg\boldsymbol{a})$ is true at d and false at d'; likewise, $\mathsf{dt}_{\mathfrak{M}_2}(e) = \mathsf{dt}_{\mathfrak{M}_2}(e')$ but $e \not\sim_\sigma e'$. Since $d \sim_\sigma e$ and $d' \sim_\sigma e'$, we have $\mathsf{dm}(d) = (\{\mathsf{dt}_{\mathfrak{M}_1}(d)\}, \{\mathsf{dt}_{\mathfrak{M}_2}(e)\}) = \mathsf{dm}(e)$, $\mathsf{dm}(d') = (\{\mathsf{dt}_{\mathfrak{M}_1}(d')\}, \{\mathsf{dt}_{\mathfrak{M}_2}(e')\}) = \mathsf{dm}(e')$, $\mathsf{dp}_1(d) = \mathsf{dp}_1(d')$, and $\mathsf{dp}_2(e) = \mathsf{dp}_2(e')$. $\dashv$

Suppose $\mathfrak{M}_1, w_1, d_1 \sim_\sigma \mathfrak{M}_2, w_2, d_2$, $\mathfrak{M}_1, w_1, d_1 \models \varphi$ and $\mathfrak{M}_2, w_2, d_2 \not\models \psi$. We construct $\mathfrak{M}'_i = (W'_i, D'_i, I'_i)$, $i = 1, 2$, witnessing σ-bisimulation consistency of φ and $\neg\psi$ and having at most double-exponential size in $|\varphi|$ and $|\psi|$. Intuitively, W'_i and D'_i consist of copies of the i-world and, respectively, i-domain points in $\mathfrak{M}_1$, $\mathfrak{M}_2$ rather than copies of the world- and domain-types as in Example 12. Then we obtain the required σ-bisimulation $(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2)$ by including in $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$ exactly those 1/2-world and, respectively, 1/2-domain points that share the same world and domain mosaic.

Let n be the number of full types over $\mathit{sub}(\varphi, \psi)$ and let $[n] = \{1, \dots, n\}$. For $i = 1, 2$, we set

$$D'_i = \{\mathsf{dp}_i^k \mid \mathsf{dp}_i \text{ an } i\text{-domain point in } \mathfrak{M}_1, \mathfrak{M}_2,\ k \in [n]\},$$

treating dp_i^k as the kth copy of dp_i and assuming all of the copies to be distinct. Next, we define W'_i, $i = 1, 2$, using *surjective* functions of the form

$$\pi_{\mathsf{wp}_i,\mathsf{dp}_i} \colon [n] \to \{\mathsf{ft}_{\mathfrak{M}_i}(w, d) \mid (w, d) \in W_i \times D_i,$$
$$\mathsf{wp}_i = \mathsf{wp}_i(w),\ \mathsf{dp}_i = \mathsf{dp}_i(d)\}.$$

Observe that, for any $\mathsf{wp}_i = (\mathsf{wt}, \mathsf{wm})$, $\mathsf{dp}_i = (\mathsf{dt}, \mathsf{dm})$, and $k \in [n]$, we have $\mathsf{wt} = \pi_{\mathsf{wp}_i,\mathsf{dp}_i}(k) \cap \mathit{sub}_\exists(\varphi, \psi)$ and $\mathsf{dt} = \pi_{\mathsf{wp}_i,\mathsf{dp}_i}(k) \cap \mathit{sub}_\Diamond(\varphi, \psi)$.

Let Π be a smallest set of sequences π of $\pi_{\mathsf{wp}_i,\mathsf{dp}_i}$ such that, for any $\mathsf{ft} = \mathsf{ft}_{\mathfrak{M}_i}(w, d)$, $\mathsf{wp}_i = \mathsf{wp}_i(w)$, $\mathsf{dp}_i = \mathsf{dp}_i(d)$ with (w, d) in $\mathfrak{M}_i$ and any $k \in [n]$, there is $\pi \in \Pi$ with $\pi_{\mathsf{wp}_i,\mathsf{dp}_i}(k) = \mathsf{ft}$. Clearly, $|\Pi| \leq n^2$. Then we set

$$W'_i = \{\mathsf{wp}_i^\pi \mid \mathsf{wp}_i \text{ an } i\text{-world point in } \mathfrak{M}_1, \mathfrak{M}_2,\ \pi \in \Pi\},$$

treating wp_i^π as a fresh π-copy of wp_i. Clearly, both $|D'_i|$ and $|W'_i|$ are double-exponential in $|\varphi|$, $|\psi|$. Finally, we set

$$\mathfrak{M}'_i, \mathsf{wp}_i^\pi, \mathsf{dp}_i^k \models \boldsymbol{p} \quad \text{iff} \quad \boldsymbol{p} \in \pi_{\mathsf{wp}_i,\mathsf{dp}_i}(k) \tag{3}$$

and define $\boldsymbol{\beta}_1 \subseteq W'_1 \times W'_2$ and $\boldsymbol{\beta}_2 \subseteq D'_1 \times D'_2$ by taking $\boldsymbol{\beta}_1(\mathsf{wp}_1^{\pi^1}, \mathsf{wp}_2^{\pi^2})$ iff $\mathsf{wm}_1 = \mathsf{wm}_2$, where $\mathsf{wp}_i = (\mathsf{wt}_i, \mathsf{wm}_i)$, for $i = 1, 2$; and $\boldsymbol{\beta}_2(\mathsf{dp}_1^{k_1}, \mathsf{dp}_2^{k_2})$ iff $\mathsf{dm}_1 = \mathsf{dm}_2$, where $\mathsf{dp}_i = (\mathsf{dt}_i, \mathsf{dm}_i)$, for $i = 1, 2$.

Lemma 14. *(i) $\mathfrak{M}'_i, \mathsf{wp}_i^\pi, \mathsf{dp}_i^k \models \rho$ iff $\rho \in \pi_{\mathsf{wp}_i,\mathsf{dp}_i}(k)$, for every $\rho \in \mathit{sub}(\varphi, \psi)$. (ii) The pair $(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2)$ is a σ-bisimulation between $\mathfrak{M}'_1$ and $\mathfrak{M}'_2$.*

The construction and lemmas above yield the following:

Theorem 15. *Any formulas φ and ψ are $\mathit{sig}(\varphi) \cap \mathit{sig}(\psi)$-bisimulation consistent in $\mathsf{Q^1S5}$ iff there are witnessing $\mathsf{Q^1S5}$-models of size double-exponential in $|\varphi|$ and $|\psi|$.*

Theorems 15, 5 and 4 give the upper bound of

Theorem 16. *(i) Both IEP and EDEP for $\mathsf{Q^1S5}$ are decidable in* coN2ExpTime.

(ii) IEP and EDEP for $\mathsf{Q^1S5}$ are both 2ExpTime*-hard.*

Note that the lower bound results hold even if we want to decide, for any FOM^1-formulas φ and ψ, whether an interpolant or an explicit definition exists not only in $\mathsf{Q}^1\mathsf{S5}$ but in any finite-variable fragment of quantified $\mathsf{S5}$.

The lower bounds are established in Appendix E.2. Here, we only comment on the intuition behind the proof. Given a 2^n-space bounded alternating Turing Machine M and an input word $\overline{a}$ of length n, we construct in polytime formulas φ and ψ such that $\models_{\mathsf{Q}^1\mathsf{S5}} \varphi \to \psi$ and M accepts $\overline{a}$ iff $\varphi, \neg\psi$ are σ-bisimulation consistent, where $\sigma = sig(\varphi) \cap sig(\psi)$.

One aspect of our construction is similar to that of (Artale et al. 2021; Jung and Wolter 2021): we also represent accepting computation-trees as binary trees whose nodes are coloured by predicates in σ. However, unlike the formalisms in the cited work, $\mathsf{Q}^1\mathsf{S5}$ cannot express the uniqueness of properties, and so the remaining ideas are novel. One part of φ 'grows' 2^n-many copies of σ-coloured binary trees, using a technique from 2D propositional modal logic (Hodkinson et al. 2003; Göller, Jung, and Lohrey 2015). Another part of φ colours the tree-nodes with non-σ-symbols to ensure that, in the mth tree, for each $m < 2^n$, the content of the mth tape-cell is properly changing during the computation. Then we use ideas from Example 6 to make sure that the generated 2^n-many trees are all σ-bisimilar, and so represent the same accepting computation-tree.

The following corollary is also proved in Appendix E.2:

Corollary 17. *IEP and EDEP for* FO^2 *without equality are both* 2ExpTime*-hard.*

5 (S)CEP and UIEP in $\mathsf{Q}^1\mathsf{S5}$: Undecidability

We now turn to the (strong) conservative extension and uniform interpolant existence problems, which, in contrast to interpolant existence, turn out to be undecidable.

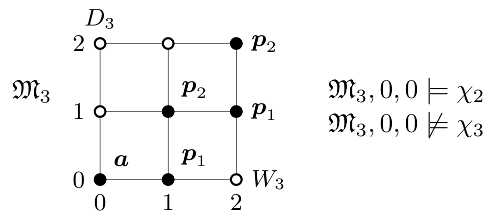
Theorem 18. (i) *(S)CEP in* $\mathsf{Q}^1\mathsf{S5}$ *is undecidable.*
(ii) *UIEP in* $\mathsf{Q}^1\mathsf{S5}$ *is undecidable.*

The undecidability proof for CEP is by adapting an undecidability proof for CEP of FO^2 in (Jung et al. 2017). The main new idea is the generation of arbitrary large binary trees within $\mathsf{Q}^1\mathsf{S5}$-models that can then be forced to be grids in case one does not have a (strong) conservative extension. The undecidability proof for UIEP merges a counterexample to UIP with the formulas constructed to prove undecidability of CEP. Here we provide the counterexample to UIP, details of the proofs are given in Appendix F.

Example 19. Let $\sigma = \{\boldsymbol{a}, \boldsymbol{p}_1, \boldsymbol{p}_2\}$ and

$$\varphi_0 = \Box\forall\big(\boldsymbol{a} \to \Diamond(\boldsymbol{p}_1 \land \boldsymbol{b})\big) \land \Box\forall\big(\boldsymbol{p}_1 \land \boldsymbol{b} \to \exists(\boldsymbol{p}_2 \land \boldsymbol{b})\big) \land \Box\forall\big(\boldsymbol{p}_2 \land \boldsymbol{b} \to \Diamond(\boldsymbol{p}_1 \land \boldsymbol{b})\big).$$

To show that $\boldsymbol{a} \land \varphi_0$ has no σ-uniform interpolant in $\mathsf{Q}^1\mathsf{S5}$, for every positive $r < \omega$, we define a formula χ_r inductively by taking $\chi_0 = \top$ and $\chi_{r+1} = \boldsymbol{p}_1 \land \exists(\boldsymbol{p}_2 \land \Diamond\chi_r)$. Then $\models_{\mathsf{Q}^1\mathsf{S5}} \boldsymbol{a} \land \varphi_0 \to \Diamond\chi_r$ for all $r > 0$. Thus, if ϱ were a σ-uniform interpolant of $\boldsymbol{a} \land \varphi_0$, then $\models_{\mathsf{Q}^1\mathsf{S5}} \varrho \to \Diamond\chi_r$ would follow for all $r > 0$. Consider a model $\mathfrak{M}_r = (W_r, D_r, I_r)$ with $W_r = D_r = \{0, \dots, r-1\}$, in which $\boldsymbol{a}$ is true at $(0,0)$, $\boldsymbol{p}_1$ at $(k, k-1)$, and $\boldsymbol{p}_2$ at (k,k), for $0 < k < r$, as illustrated in the picture below.



Then $\mathfrak{M}_r, 0, 0 \not\models \Diamond\chi_r$, for any $r > 0$, and so $\mathfrak{M}_r, 0, 0 \not\models \varrho$. On the other hand, $\mathfrak{M}_r, 0, 0 \models \Diamond\chi_{r'}$ for all $r' < r$. Now consider the ultraproduct $\prod_U \mathfrak{M}_r$ with a non-principal ultrafilter U on $\omega \setminus \{0\}$. As each $\Diamond\chi_{r'}$ is true at $(0,0)$ in almost all $\mathfrak{M}_r$, it follows from the properties of ultraproducts (Chang and Keisler 1998) that, for a suitable $\overline{0}$ and all $r > 0$, we have $\prod_U \mathfrak{M}_r, \overline{0}, \overline{0} \models \boldsymbol{a} \land \neg\varrho \land \Diamond\chi_r$. One can interpret $\boldsymbol{b}$ in $\prod_U \mathfrak{M}_r$ so that $\mathfrak{M}, \overline{0}, \overline{0} \models \varphi_0$ for the resulting model $\mathfrak{M}$. Then $\mathfrak{M} \models \boldsymbol{a} \land \varphi_0 \land \neg\varrho$, contrary to the fact that $\models_{\mathsf{Q}^1\mathsf{S5}} \boldsymbol{a} \land \varphi_0 \to \varrho$ for any uniform interpolant ϱ of $\boldsymbol{a} \land \varphi_0$.

Remark 20. Example 19 can be translated into FO^2 to prove that the latter does not have UIP. It can then be merged with the proof of undecidability of CEP in FO^2 from (Jung et al. 2017)—in the same way as we combined Example 19 with the undecidability proof for UIEP in $\mathsf{Q}^1\mathsf{S5}$—to show that UIEP is undecidable in FO^2 (with and without $=$). The latter problem has so far remained open.

6 Modal Description Logic $\mathsf{S5}_{\mathcal{ALC}^u}$

Next, we extend the results of Sections 4, 5 to the description modal logic $\mathsf{S5}_{\mathcal{ALC}^u}$, where $\mathcal{ALC}^u$ is the basic description logic $\mathcal{ALC}$ with the universal role (Baader et al. 2017), which is a notational variant of multimodal K with the universal modality (and can be regarded as a fragment of FO^2).

The *concepts* of $\mathsf{S5}_{\mathcal{ALC}^u}$ are constructed from *concept names* $A \in \mathcal{C}$, *role names* $R \in \mathcal{R}$, for some countably-infinite and disjoint sets $\mathcal{C}$ and $\mathcal{R}$, and a distinguished *universal role* $U \in \mathcal{R}$ by means of the following grammar:

$$C, D \;:=\; A \mid \top \mid C \sqcap D \mid \neg C \mid \exists R.C \mid \exists U.C \mid \Diamond C.$$

A *signature* σ is any finite set of concept and role names. The signature $sig(C)$ of a concept C comprises the concept and role names in C. We interpret $\mathsf{S5}_{\mathcal{ALC}^u}$ in models $\mathfrak{M} = (W, \Delta, I)$, where $I(w)$ is an interpretation of the concept and role names at each world $w \in W$ over domain $\Delta \neq \emptyset$: $A^{I(w)} \subseteq \Delta$, $R^{I(w)} \subseteq \Delta \times \Delta$, and $U^{I(w)} = \Delta \times \Delta$. The *truth-relation* $\mathfrak{M}, w, d \models C$ is defined by taking

- $\mathfrak{M}, w, d \models \top$, $\quad \mathfrak{M}, w, d \models A$ iff $d \in A^{I(w)}$,
- $\mathfrak{M}, w, d \models \exists R.C$ iff there is $(d, d') \in R^{I(w)}$ such that $\mathfrak{M}, w, d' \models C$,
- $\mathfrak{M}, w, d \models \Diamond C$ iff there is $w' \in W$ with $\mathfrak{M}, w', d \models C$,

and standard clauses for Boolean $\sqcap$, $\neg$. We sometimes use more conventional $C^{I(w)} = \{d \in \Delta \mid \mathfrak{M}, w, d \models C\}$, writing $\mathfrak{M}, w \models C \sqsubseteq D$ if $C^{I(w)} \subseteq D^{I(w)}$, and $\models C \sqsubseteq D$ if $\mathfrak{M}, w \models C \sqsubseteq D$ for all $\mathfrak{M}$ and w. The problem of deciding if $\models C \sqsubseteq D$, for given C and D, is coNExpTime-complete (Gabbay et al. 2003).

Typical applications of description logics use reasoning modulo *ontologies*—finite sets $\mathcal{O}$ of *concept inclusions*

(CIs) $C' \sqsubseteq D'$ regarded as axioms—by taking $\mathcal{O} \models C \sqsubseteq D$ iff whenever $\mathfrak{M}, w \models \alpha$ for all $\alpha \in \mathcal{O}$ then $\mathfrak{M}, w \models C \sqsubseteq D$. Reasoning modulo ontologies is reducible to the ontology-free case by the following equivalence: $\mathcal{O} \models C \sqsubseteq D$ iff $\models \top \sqsubseteq \bigsqcup_{C' \sqsubseteq D' \in \mathcal{O}} \exists U.(C' \sqcap \neg D') \sqcup \forall U.(\neg C \sqcup D)$.

An *interpolant for* $C \sqsubseteq D$ *in* $\mathsf{S5}_{\mathcal{ALC}^u}$ is a concept E such that $sig(E) \subseteq sig(C) \cap sig(D)$, $\models C \sqsubseteq E$, and $\models E \sqsubseteq D$. The IEP for $\mathsf{S5}_{\mathcal{ALC}^u}$ is to decide whether a given concept inclusion $C \sqsubseteq D$ has an interpolant in $\mathsf{S5}_{\mathcal{ALC}^u}$. The following related problems can easily be reduced to IEP in polytime:

(IEP modulo ontologies) Given an ontology $\mathcal{O}$, a signature σ, and a CI $C \sqsubseteq D$, does there exist a σ-concept E such that $\mathcal{O} \models C \sqsubseteq E$ and $\mathcal{O} \models E \sqsubseteq D$?

(ontology interpolant existence, OIEP) Given an ontology $\mathcal{O}$, a signature σ, and a CI $C \sqsubseteq D$, is there an ontology $\mathcal{O}'$ with $sig(\mathcal{O}') \subseteq \sigma$, $\mathcal{O} \models \mathcal{O}'$, and $\mathcal{O}' \models C \sqsubseteq D$?

(EDEP modulo ontologies) Given an ontology $\mathcal{O}$, a signature σ, and a concept name A, does there exist a concept C such that $sig(C) \subseteq \sigma$ and $\mathcal{O} \models A \equiv C$?

(See Example 2 for an illustration.) Explicit definitions have been proposed for query rewriting in ontology-based data access (Franconi, Kerhet, and Ngo 2013; Toman and Weddell 2021), developing and maintaining ontology alignments (Geleta, Payne, and Tamma 2016), and ontology engineering (ten Cate et al. 2006). IEP is fundamental for robust modularisations and decompositions of ontologies (Konev et al. 2009; Botoeva et al. 2016).

Our main result in this section is the following:

Theorem 21. *IEP, EDEP* (*modulo ontologies*)*, and OIEP are decidable in* coN2ExpTime*, being* 2ExpTime*-hard.*

The detailed proof is given in Appendix G. Here, we only formulate a model-theoretic characterisation of interpolant existence in $\mathsf{S5}_{\mathcal{ALC}^u}$ in terms of the following generalisation of σ-bisimulations for $\mathsf{Q}^1\mathsf{S5}$ from Section 4.

A σ-*bisimulation* between models $\mathfrak{M}_i = (W_i, \Delta_i, I_i)$, $i = 1, 2$, is any triple $(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \boldsymbol{\beta})$ with $\boldsymbol{\beta}_1 \subseteq W_1 \times W_2$, $\boldsymbol{\beta}_2 \subseteq \Delta_1 \times \Delta_2$, and $\boldsymbol{\beta} \subseteq (W_1 \times \Delta_1) \times (W_2 \times \Delta_2)$ if

(w) for any $(w_1, w_2) \in \boldsymbol{\beta}_1$ and $d_1 \in \Delta_1$, there is $d_2 \in \Delta_2$ with $((w_1, d_1), (w_2, d_2)) \in \boldsymbol{\beta}$ and similarly for $d_2 \in \Delta_2$,

(d) for any $(d_1, d_2) \in \boldsymbol{\beta}_2$ and $w_1 \in W_1$, there is $w_2 \in W_2$ with $((w_1, d_1), (w_2, d_2)) \in \boldsymbol{\beta}$ and similarly for $w_2 \in W_2$,

(c) $((w_1, d_1), (w_2, d_2)) \in \boldsymbol{\beta}$ implies both $(w_1, w_2) \in \boldsymbol{\beta}_1$ and $(d_1, d_2) \in \boldsymbol{\beta}_2$,

and the following hold for all $((w_1, d_1), (w_2, d_2)) \in \boldsymbol{\beta}$:

(a) $\mathfrak{M}_1, w_1, d_1 \models A$ iff $\mathfrak{M}_2, w_2, d_2 \models A$, for all $A \in \sigma$;

(r) if $(d_1, e_1) \in R^{I(w_1)}$ and $R \in \sigma$, then there is $e_2 \in \Delta_2$ with $(d_2, e_2) \in R^{I(w_2)}$ and $((w_1, e_1), (w_2, e_2)) \in \boldsymbol{\beta}$, and the other way round.

The criterion below—in which σ-bisimulation consistency is defined as in Section 3 with concepts C, D in place of formulas φ, ψ—is an $\mathsf{S5}_{\mathcal{ALC}^u}$-analogue of Theorem 5:

Theorem 22. *The following conditions are equivalent for any concept inclusion* $C \sqsubseteq D$:

– *there does not exist an interpolant for* $C \sqsubseteq D$ *in* $\mathsf{S5}_{\mathcal{ALC}^u}$;

– C *and* $\neg D$ *are* $sig(C) \cap sig(D)$*-bisimulation consistent.*

We then extend the 'filtration' construction of Section 4 from $\mathsf{Q}^1\mathsf{S5}$ to $\mathsf{S5}_{\mathcal{ALC}^u}$. In contrast to $\mathsf{Q}^1\mathsf{S5}$, we now have to deal with non-trivial σ-bisimulations between the respective $\mathcal{ALC}$-models $I(w_1)$ and $I(w_2)$ (satisfying conditions $(\mathbf{a})$ and $(\mathbf{r})$). To this end we introduce *full mosaics* (sets of full types realised in σ-bisimilar pairs (w, d)) and *full points* (full mosaics with a distinguished full type). The range of the surjections π used to construct W' and D' then consists of full points rather than full types. This provides us with the data structure to define σ-bisimilar $\mathcal{ALC}$-models $I(w)$ when required. This construction establishes an upper bound on the size of models witnessing bisimulation consistency:

Theorem 23. *Any concepts C and D do not have an interpolant in* $\mathsf{S5}_{\mathcal{ALC}^u}$ *iff there are witnessing* $\mathsf{S5}_{\mathcal{ALC}^u}$*-models of size double-exponential in* $|C|$ *and* $|D|$.

This result gives the upper bound of Theorem 21. The lower one follows from Theorem 16 (ii) as, treating FOM^1-formulas φ, ψ as role-free $\mathsf{S5}_{\mathcal{ALC}^u}$-concepts and using Theorems 5 and 22, one can readily show that φ and ψ have an interpolant in $\mathsf{Q}^1\mathsf{S5}$ iff they have an interpolant in $\mathsf{S5}_{\mathcal{ALC}^u}$.

The (strong) conservative extension problem, (S)CEP, and the uniform interpolant existence problem, UIEP, in $\mathsf{S5}_{\mathcal{ALC}^u}$ are defined in the obvious way. Using the same argument as for interpolation, the undecidability of (S)CEP and UIEP in $\mathsf{S5}_{\mathcal{ALC}^u}$ follows directly from the undecidability of both problems for $\mathsf{Q}^1\mathsf{S5}$. Note that, for the component logics—propositional $\mathsf{S5}$ and $\mathcal{ALC}^u$—CEP is coNExpTime and 2ExpTime-complete, respectively (Ghilardi et al. 2006; Jung et al. 2017).

7 Quantified Modal Logic $\mathsf{Q}^1\mathsf{K}$

Finally, we consider the one-variable quantified modal logic $\mathsf{Q}^1\mathsf{K}$. By the *modal depth* $md(\varphi)$ of a FOM^1-formula φ we mean the maximal number of nestings of $\Diamond$ in φ; if φ has no modal operators, then $md(\varphi) = 0$. Formulas of modal depth k can be characterised using a finitary version of bisimulations, called k-bisimulations, defined below.

For a signature σ and two models $\mathfrak{M} = (W, R, D, I)$ and $\mathfrak{M}' = (W', R', D', I')$, a sequence $\boldsymbol{\beta}_0, \dots, \boldsymbol{\beta}_k$ of relations $\boldsymbol{\beta}_i \subseteq (W \times D) \times (W' \times D')$ is a σ-k-*bisimulation* between $\mathfrak{M}$ and $\mathfrak{M}'$ if the following conditions hold for all $\boldsymbol{p} \in \sigma$ and $((w, d), (w', d')) \in \boldsymbol{\beta}_i$: **(a)**, **(d)** from Section 2 as well as

(w′) if $i > 0$, $(w, v) \in R$, then there is v' with $(w', v') \in R'$ and $((v, d), (v', d')) \in \boldsymbol{\beta}_{i-1}$, and the other way round.

We say that $\mathfrak{M}, w, d$ and $\mathfrak{M}', w', d'$ are σ-k-*bisimilar* and write $\mathfrak{M}, w, d \sim^k_\sigma \mathfrak{M}', w', d'$ if there is a σ-k-bisimulation $\boldsymbol{\beta}_0, \dots, \boldsymbol{\beta}_k$ with $\boldsymbol{\beta}_k \ni ((w, d), (w', d'))$ between $\mathfrak{M}$ and $\mathfrak{M}'$. We write $\mathfrak{M}, w, d \equiv^k_\sigma \mathfrak{M}', w', d'$ when $\mathfrak{M}, w, d \models \varphi$ iff $\mathfrak{M}', w', d' \models \varphi$, for every σ-formula φ with $md(\varphi) \leq k$.

We can define formulas $\tau^k_{\mathfrak{M},\sigma}$, generalising the characteristic formulas of (Goranko and Otto 2007), that describe every model $\mathfrak{M}$ up to σ-k-bisimulations in the sense that the following equivalences hold (see Appendix H for details):

Lemma 24. *For any models $\mathfrak{M}$ with w, d and $\mathfrak{N}$ with v, e, and any $k < \omega$, the following conditions are equivalent:*

(*i*) $\mathfrak{N}, v, e \equiv_\sigma^k \mathfrak{M}, w, d$;

(*ii*) $\mathfrak{N}, v, e \models \tau^k_{\mathfrak{M},\sigma}(w, d)$;

(*iii*) $\mathfrak{N}, v, e \sim_\sigma^k \mathfrak{M}, w, d$.

Intuitively, $\tau^k_{\mathfrak{M},\sigma}(w, d)$ is the strongest formula of modal depth k that is true at w, d in $\mathfrak{M}$. For any formula φ with $md(\varphi) \leq k$, we now set

$$\exists^{\sim\sigma,k}\varphi = \bigvee_{\mathfrak{M},w,d\models\varphi} \tau^k_{\mathfrak{M},\sigma}(w, d).$$

Thus, for any $\mathfrak{N}, v, e$, we have $\mathfrak{N}, v, e \models \exists^{\sim\sigma,k}\varphi$ iff there is $\mathfrak{M}, w, d$ with $\mathfrak{M}, w, d \models \varphi$ and $\mathfrak{N}, v, e \sim_\sigma^k \mathfrak{M}, w, d$, i.e., $\exists^{\sim\sigma,k}$ is an existential depth restricted bisimulation quantifier (D'Agostino and Lenzi 2006; French 2006). Clearly, $\models_{\mathsf{Q^1K}} \varphi \to \exists^{\sim\sigma,k}\varphi$.

Theorem 25. *The following conditions are equivalent, for any formula ψ with $md(\psi) = k'$ and $n = \max\{k, k'\}$:*

(a) *there is χ such that $sig(\chi) \subseteq \sigma$, $\models_{\mathsf{Q^1K}} \varphi \to \chi$, and $\models_{\mathsf{Q^1K}} \chi \to \psi$;*

(b) $\models_{\mathsf{Q^1K}} \exists^{\sim\sigma,n}\varphi \to \psi$.

Proof. (a) $\Rightarrow$ (b) If $\not\models_{\mathsf{Q^1K}} \exists^{\sim\sigma,n}\varphi \to \psi$, there is $\mathfrak{M}, w, d$ with $\mathfrak{M}, w, d \models \exists^{\sim\sigma,n}\varphi$ and $\mathfrak{M}, w, d \models \neg\psi$. By the definition of $\exists^{\sim\sigma,n}\varphi$, we then have $\mathfrak{M}, w, d \models \tau^n_{\mathfrak{M}',\sigma}(w', d')$ and $\mathfrak{M}', w', d' \models \varphi$, for some model $\mathfrak{M}'$, w', d'. By Lemma 24, $\mathfrak{M}', w', d' \sim_\sigma^n \mathfrak{M}, w, d$. Using a standard unfolding argument, we may assume that (W, R) in $\mathfrak{M}$ and (W', R') in $\mathfrak{M}'$ are tree-shaped with respective roots w, w'. As φ and ψ have modal depth $\leq n$, we may also assume that the depth of (W, R) and (W', R') is $\leq n$. But then $\mathfrak{M}', w', d' \sim_\sigma \mathfrak{M}, w, d$, contrary to (a). The implication (b) $\Rightarrow$ (a) is trivial. $\dashv$

We do not know whether $\exists^{\sim\sigma,k}\varphi$ is equivalent to a formula whose size can be bounded by an elementary function in $|\sigma|$, $|\varphi|$, k. For pure $\mathcal{ALC}$, it is indeed equivalent to an exponential-size concept (ten Cate et al. 2006).

Condition (b) in Theorem 25 gives an obvious non-elementary algorithm for checking whether given formulas have an interpolant in $\mathsf{Q^1K}$. Thus, by Theorem 4, we obtain:

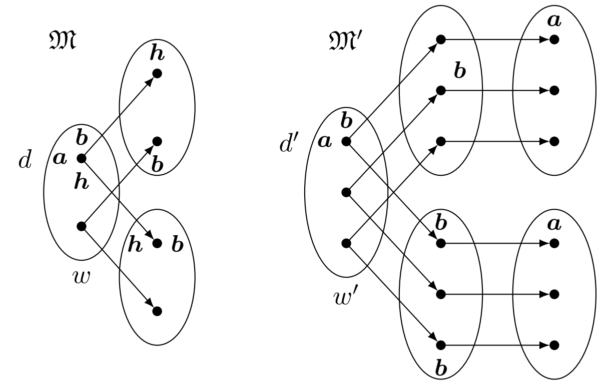
Theorem 26. *IEP and EDEP for* $\mathsf{Q^1K}$ *are decidable in non-elementary time.*

The proof above seems to give a hint that UIEP for $\mathsf{Q^1K}$ might also be decidable as (an analogue of) $\exists^{\sim\sigma,k}\varphi$ of modal depth $md(\varphi)$ is a uniform interpolant of any propositional modal formula φ in K (Visser 1996). The next example illustrates why this is not the case for 'two-dimensional' $\mathsf{Q^1K}$.

Example 27. Suppose $\sigma = \{\boldsymbol{a}, \boldsymbol{b}\}$,

$$\varphi = \forall\big((\boldsymbol{a} \leftrightarrow \boldsymbol{b} \leftrightarrow \boldsymbol{h}) \wedge (\boldsymbol{h} \leftrightarrow \Box\boldsymbol{h} \leftrightarrow \Diamond\boldsymbol{h})\big) \wedge \Diamond\forall(\boldsymbol{b} \leftrightarrow \boldsymbol{h}),$$
$$\psi = \forall(\boldsymbol{a} \leftrightarrow \Box\Box\boldsymbol{a} \leftrightarrow \Diamond\Diamond\boldsymbol{a}) \wedge \Box\Diamond\top \to \Diamond\forall(\boldsymbol{b} \leftrightarrow \Diamond\boldsymbol{a}).$$

One can check (see Appendix H) that $\models_{\mathsf{Q^1K}} \varphi \to \psi$. However, $\not\models_{\mathsf{Q^1K}} \exists^{\sim\sigma,1}\varphi \to \psi$ as, for $\mathfrak{M}$ and $\mathfrak{M}'$ below, $\mathfrak{M}, w, d \models \varphi$ and $\mathfrak{M}', w', d' \not\models \psi$ but $\mathfrak{M}, w, d \sim_\sigma^1 \mathfrak{M}', w', d'$. $\dashv$



In fact, by adapting the undecidability proof for $\mathsf{Q^1S5}$ we prove the following:

Theorem 28. (*i*) *(S)CEP for* $\mathsf{Q^1K}$ *is undecidable.*
(*ii*) *UIEP for* $\mathsf{Q^1K}$ *is undecidable.*

8 Outlook

Craig interpolation and Beth definability have been studied for essentially all logical systems, let alone those applied in KR, AI, verification and databases. In fact, one of the first questions typically asked about a logic L of interest is whether L has interpolants for *all* valid implications $\varphi \to \psi$. Some L enjoy this property, while others miss it. This paper and preceding (Jung and Wolter 2021; Artale et al. 2021) open a new, *non-uniform* perspective on interpolation/definability for the latter type of L by regarding formulas φ and ψ as input (say, coming from an application) and deciding whether they have an interpolant in L.

In the context of first-order modal logics, challenging open questions that arise from this work are: What is the tight complexity of IEP for $\mathsf{Q^1S5}$ and $\mathsf{S5}_{\mathcal{ALC}^u}$? Is the non-elementary upper bound for IEP in $\mathsf{Q^1K}$ optimal? Is IEP decidable for $\mathsf{K}_{\mathcal{ALC}^u}$? More generally, what happens if we replace S5 and K by other standard modal logics, e.g., S4, multimodal S5, or the linear temporal logic *LTL*, and/or use in place of $\mathcal{ALC}^u$ other DLs or other decidable fragments of FO such as the guarded or two-variable fragment?

A different line of research is computing interpolants. For logics with CIP, this is typically done using resolution, tableau, or sequent calculi. A more recent approach is based on type-elimination known from complexity proofs for modal and guarded logics (Benedikt, ten Cate, and Vanden Boom 2016; ten Cate 2022). While no attempt has yet been made to use traditional methods for computing interpolants in logics without CIP, type elimination has been adapted to $\mathcal{ALC}$ with role inclusions (Artale et al. 2020) that does not have CIP. Rather than eliminating types, one eliminates pairs of sets of types—i.e., mosaics in our proofs above. The question whether these proofs can be turned into an algorithm computing interpolants in, say, $\mathsf{Q^1S5}$ is non-trivial and open. More generally, one can try to develop calculi for the consequence relation '$\varphi \models \psi$ iff there are no $sig(\varphi) \cap sig(\psi)$-bisimilar models satisfying φ and $\neg\psi$' and use them to compute interpolants; see (Barwise and van Benthem 1999) for a model-theoretic account of such consequence relations for infinitary logics without CIP.

Acknowledgments

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Appendix

A Connections with FO^2

The *atoms* of FO^2 are of the form $x = y$, $\boldsymbol{p}(x,y)$, $\boldsymbol{p}(y,x)$, $\boldsymbol{p}(x,x)$, $\boldsymbol{p}(y,y)$ for some binary predicate symbol $\boldsymbol{p} \in \mathcal{P}$.[2] A *signature* is any finite set $\sigma \subseteq \mathcal{P}$. FO^2-*formulas* are built up from atoms using $\neg, \wedge, \exists x, \exists y$. We consider two proper fragments of FO^2: in the *equality-free* fragment, we do not have atoms of the form $x = y$, and in the *equality- and substitution-free* fragment, the only available atoms are of the form $\boldsymbol{p}(y,x)$. We interpret FO^2-formulas in usual FO-*models* of the form $\mathfrak{A} = (A^{\mathfrak{A}}, \boldsymbol{p}^{\mathfrak{A}})_{\boldsymbol{p}\in\mathcal{P}}$, where $A^{\mathfrak{A}}$ is a nonempty set and $\boldsymbol{p}^{\mathfrak{A}} \subseteq A^{\mathfrak{A}} \times A^{\mathfrak{A}}$, for each $\boldsymbol{p} \in \mathcal{P}$.

Now, fix some signature σ. We connect two FO-models $\mathfrak{A}, \mathfrak{B}$ with three different kinds of σ-bisimulations, depending on the chosen fragment $\mathcal{L}$ of FO^2, as follows. Given $\mathcal{L}$, let $\mathit{Lit}_{\mathcal{L}(\sigma)}$ denote the set of available *literals* for $\boldsymbol{p} \in \sigma$ (atoms and negated atoms) in $\mathcal{L}$. Given $\mathfrak{A}$ and $a, a' \in A^{\mathfrak{A}}$, we define

$$\ell_{\mathfrak{A}}^{\mathcal{L}(\sigma)}(a,a') = \big\{\ell \in \mathit{Lit}_{\mathcal{L}(\sigma)} \mid \mathfrak{A} \models \ell[a/y, a'/x]\big\}.$$

A relation $\boldsymbol{\beta} \subseteq (A^{\mathfrak{A}} \times A^{\mathfrak{A}}) \times (B^{\mathfrak{B}} \times B^{\mathfrak{B}})$ is a σ-*bisimulation between* $\mathfrak{A}$ *and* $\mathfrak{B}$ *in* $\mathcal{L}$ if the following hold, for all $\big((a,a'),(b,b')\big) \in \boldsymbol{\beta}$:

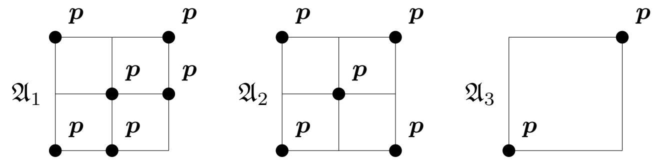
1. $\ell_{\mathfrak{A}}^{\mathcal{L}(\sigma)}(a,a') = \ell_{\mathfrak{B}}^{\mathcal{L}(\sigma)}(b,b')$;
2. for every $a'' \in A^{\mathfrak{A}}$ there is $b'' \in B^{\mathfrak{B}}$ such that $\big((a,a''),(b,b'')\big) \in \boldsymbol{\beta}$, and the other way round;
3. for every $a'' \in A^{\mathfrak{A}}$ there is $b'' \in B^{\mathfrak{B}}$ such that $\big((a'',a'),(b'',b')\big) \in \boldsymbol{\beta}$, and the other way round.

If $\big((a,a'),(b,b')\big) \in \boldsymbol{\beta}$ for some $\boldsymbol{\beta}$ as above, then we say that $\mathfrak{A}, a, a'$ *and* $\mathfrak{B}, b, b'$ *are* σ-*bisimilar in* $\mathcal{L}$.

Given FO-models $\mathfrak{A}, \mathfrak{B}$ and $a, a' \in A^{\mathfrak{A}}$, $b, b' \in B^{\mathfrak{B}}$, we write $\mathfrak{A}, a, a' \equiv_{\mathcal{L}(\sigma)} \mathfrak{B}, b, b'$ whenever $\mathfrak{A} \models \varphi[a/y, a'/x]$ iff $\mathfrak{B} \models \varphi[b/y, b'/x]$ hold for all $\mathcal{L}(\sigma)$-formulas φ. Then we have the following well-known equivalence, for any pair $\mathfrak{A}, \mathfrak{B}$ of saturated models:

$$\mathfrak{A}, a, a' \equiv_{\mathcal{L}(\sigma)} \mathfrak{B}, b, b' \quad \text{iff} \quad \mathfrak{A}, a, a' \text{ and } \mathfrak{B}, b, b' \text{ are } \sigma\text{-bisimilar in } \mathcal{L}.$$

Clearly, if $\mathfrak{A}, a, a'$ and $\mathfrak{B}, b, b'$ are σ-bisimilar in FO^2, then they are σ-bisimilar in the equality-free fragment, and if they are σ-bisimilar in the equality-free fragment, then they are σ-bisimilar in the equality- and substitution-free fragment. However, as the models below show, the converse directions do not always hold.



Here, $\mathfrak{A}_1$ and $\mathfrak{A}_3$ are $\{\boldsymbol{p}\}$-bisimilar in the equality- and substitution-free fragment of FO^2, but not in the equality-free fragment: $\forall x \forall y\big(\boldsymbol{p}(x,y) \leftrightarrow \boldsymbol{p}(y,x)\big) \wedge \forall x \boldsymbol{p}(x,x)$ is true in $\mathfrak{A}_3$, while it is not true in $\mathfrak{A}_1$. $\mathfrak{A}_2$ and $\mathfrak{A}_3$ are $\{\boldsymbol{p}\}$-bisimilar in the equality-free fragment, but not in FO^2: $\forall x \forall y\big(\boldsymbol{p}(x,y) \to x = y\big)$ is true in $\mathfrak{A}_3$, while it is not true in $\mathfrak{A}_2$.

Next, we discuss connections between $\mathsf{Q}^1\mathsf{S5}$ and the above three fragments of FO^2. With a slight abuse of notation, we consider the predicate symbols in $\mathcal{P}$ (and thus in any signature σ) as unary symbols when dealing with FOM^1 and binary ones when dealing with (fragments of) FO^2. We translate each FOM^1-formula φ to an FO^2-formula $\varphi^\dagger$ by taking

$$\begin{aligned}
\boldsymbol{p}(x)^\dagger &= \boldsymbol{p}(y,x)\\
(\neg\varphi)^\dagger &= \neg\varphi^\dagger\\
(\varphi \wedge \psi)^\dagger &= \varphi^\dagger \wedge \psi^\dagger\\
(\Diamond\varphi)^\dagger &= \exists y \varphi^\dagger\\
(\exists\varphi)^\dagger &= \exists x \varphi^\dagger
\end{aligned}$$

Observe that the image of this translation is in the equality- and substitution-free fragment of FO^2. It is easy to show the following:

Lemma 29. *For all* FOM^1*-formulas* φ, ψ *and* FOM^1*-signature* σ,

(i) $\models_{\mathsf{Q}^1\mathsf{S5}} \varphi$ *iff* $\models_{\mathsf{FO}^2} \varphi^\dagger$*;*

(ii) φ, ψ *are* σ*-bisimulation consistent in* $\mathsf{Q}^1\mathsf{S5}$ *iff* $\varphi^\dagger$, $\psi^\dagger$ *are* σ*-bisimulation consistent in the equality- and substitution-free fragment of* FO^2.

Proof. It is straightforward, using the following observations (a) and (b):

(a) FO-models for the equality- and substitution-free fragment of FO^2 and *square* $\mathsf{Q}^1\mathsf{S5}$-models (W, D, I) (where $|W| = |D|$) are in one-to-one correspondence in the following sense.

- For every FO-model $\mathfrak{A}$, take the square $\mathsf{Q}^1\mathsf{S5}$-model $\mathfrak{M}_{\mathfrak{A}} = (A^{\mathfrak{A}}, A^{\mathfrak{A}}, I)$ where, for all $a, b \in A^{\mathfrak{A}}$ and $\boldsymbol{p} \in \mathcal{P}$, $b \in \boldsymbol{p}^{I(a)}$ iff $(a,b) \in \boldsymbol{p}^{\mathfrak{A}}$. Then we have
$$\mathfrak{M}_{\mathfrak{A}}, a, b \models \varphi \quad \text{iff} \quad \mathfrak{A} \models \varphi^\dagger[a/y, b/x].$$
- For every square $\mathsf{Q}^1\mathsf{S5}$-model $\mathfrak{M} = (W, D, I)$ and every bijection $f\colon D \to W$, take the FO-model $\mathfrak{A}_{\mathfrak{M},f} = (D, \boldsymbol{p}^{\mathfrak{A}_{\mathfrak{M},f}})$ where, for all $a, b \in D$ and $\boldsymbol{p} \in \mathcal{P}$, $(a,b) \in \boldsymbol{p}^{\mathfrak{A}_{\mathfrak{M},f}}$ iff $b \in \boldsymbol{p}^{I(f(a))}$. Then we have
$$\mathfrak{M}, f(a), b \models \varphi \quad \text{iff} \quad \mathfrak{A}_{\mathfrak{M},f} \models \varphi^\dagger[a/y, b/x]. \tag{4}$$

[2] It is shown in (Grädel, Kolaitis, and Vardi 1997) that, in FO^2, one can replace relations of arbitrary arity by binary relations as far the complexity of satisfiability is concerned. This has been extended to bisimulation consistency in (Jung and Wolter 2021). We therefore consider FO^2 with binary relations only.

(b) For all $\mathsf{Q}^1\mathsf{S5}$-models $\mathfrak{M}$, w, d there exist a square $\mathsf{Q}^1\mathsf{S5}$-model $\mathfrak{M}'$, w', d' such that $\mathfrak{M}, w, d$ and $\mathfrak{M}', w', d'$ are bisimilar in $\mathsf{Q}^1\mathsf{S5}$ (see e.g. (Gabbay et al. 2003, Prop.3.12)). $\dashv$

B Encoding of Standpoints

We give a few more details regarding an encoding of a version of standpoint logic based on $\mathcal{ALC}^u$ without individual names into $\mathsf{S5}_{\mathcal{ALC}^u}$. We refer the reader to (Álvarez, Rudolph, and Strass 2022) for a detailed introduction to standpoint logic.

In standpoint logic one constructs *standpoint expressions* e from a set of *primitive standpoints* $s_1, \ldots, s_n$ and the *universal standpoint* $*$ using Boolean operators. For $\mathcal{ALC}^u$-concepts C, D and standpoint expression e the formula $\Box_e(C \sqsubseteq D)$ then states that $C \sqsubseteq D$ is true according to e and the concept $\Box_e C$ applies to all individuals that are a member of C according to standpoint e. In more detail, one interprets the above expressions in standpoint structures $\mathfrak{S} = (\Delta, \Pi, \sigma, \gamma)$, where Δ is a non-empty set, the *domain* of $\mathfrak{S}$, Π is a non-empty set of *precisifications*, σ is a function mapping every s_i to a set $\sigma(s_i) \subseteq \Pi$, and γ is a function mapping every $\pi \in \Pi$ to a DL-interpretation $\gamma(\pi)$ with domain Δ. Set $\sigma(*) = \Pi$ and interpret Boolean combinations of primitive standpoints as expected by setting $\sigma(e_1 \wedge e_2) = \sigma(e_1) \cap \sigma(e_2)$ and $\sigma(\neg e) = \Pi \setminus \sigma(e)$. Then we set for $\pi \in \Pi$ and $d \in \Delta$

- $\mathfrak{S}, \pi \models C(d)$ if $d \in C^{\gamma(\pi)}$ for C an $\mathcal{ALC}^u$-concept;
- $\mathfrak{S}, \pi \models C \sqcap D(d)$ if $\mathfrak{S}, \pi \models C(d)$ and $\mathfrak{S}, \pi \models D(d)$;
- $\mathfrak{S}, \pi \models \neg C(d)$ if $\mathfrak{S}, \pi \not\models C(d)$;
- $\mathfrak{S}, \pi \models \Box_e C(d)$ if $\mathcal{S}, \pi' \models C(d)$ for all $\pi' \in \sigma(e)$;
- $\mathfrak{S}, \pi \models \exists R.C(d)$ if there exists f with $(d, f) \in R^{\gamma(\pi)}$ and $\mathfrak{S}, \pi \models C(f)$;
- $\mathfrak{S}, \pi \models \exists U.C(d)$ if there exists $f \in \Delta$ with $\mathfrak{S}, \pi \models C(f)$.

Also set

- $\mathfrak{S}, \pi \models \Box_e(C \sqsubseteq D)$ if $\mathfrak{S}, \pi \models \Box_e \forall U.(\neg C \sqcup D)(d)$, for some (equivalently all) $d \in \Delta$.

We set $\mathfrak{S} \models \Box_e(C \sqsubseteq D)$ if $\mathfrak{S}, \pi \models \Box_e(C \sqsubseteq D)$ for some (equivalently all) $\pi \in \Pi$. Call formulas of the form $\Box_e(C \sqsubseteq D)$ *standpoint inclusions* and set, for a set $\mathcal{O}$ of standpoint inclusions and a standpoint inclusion $\Box_e(C \sqsubseteq D)$, $\mathcal{O} \models_{\mathcal{S}} \Box_e(C \sqsubseteq D)$ if for all standpoint structures $\mathfrak{S}$ the following holds: if $\mathfrak{S} \models \alpha$ for all $\alpha \in \mathcal{O}$, then $\mathfrak{S} \models \Box_e(C \sqsubseteq D)$.

We encode standpoint logic in $\mathsf{S5}_{\mathcal{ALC}^u}$ in a natural and straightforward way. First take concept names $S_1, \ldots, S_n$ representing primitive standpoints. We take the axioms

$$\alpha_i = \Box\forall U.(\exists U.S_i \leftrightarrow \forall U.S_i)$$

which entail that S_i behaves like a proposition (a 0-ary predicate). Any standpoint expression e then corresponds to a Boolean combination $e^\dagger$ of the S_i (with $*$ encoded as $\top$).

To translate any standpoint inclusion $\alpha = \Box_e(C \sqsubseteq D)$ into an $\mathsf{S5}_{\mathcal{ALC}^u}$-inclusion $\alpha^\dagger$ it then only remains to set

$$\alpha^\dagger = (\top \sqsubseteq \Box\forall U.(\neg(e^\dagger \sqcap C^\dagger) \sqcup D^\dagger)),$$

where $C^\dagger, D^\dagger$ are obtained from C, D by replacing, recursively, any occurrence of a $\Box_g F$ by $\Box\forall(\neg g^\dagger \sqcup F^\dagger)$.

The following reduction can be proved by induction:

Theorem 30. *The following conditions are equivalent for any set $\mathcal{O}$ of standpoint inclusions and standpoint inclusion α:*

1. $\mathcal{O} \models_S \alpha$;
2. $\{\delta^\dagger \mid \delta \in \mathcal{O}\} \cup \{\alpha_i \mid i \leq n\} \models_{\mathsf{S5}_{\mathcal{ALC}^u}} \alpha^\dagger$.

C Proofs for Section 3

Theorem 4. *EDEP and IEP are polynomial time reducible to each other.*

Proof. EDEP is polynomially reducible to IEP by a standard trick (Gabbay and Maksimova 2005): a formula ψ has an explicit σ-definition modulo φ in L iff the formulas $\varphi \wedge \psi$ and $\varphi^\sigma \rightarrow \psi^\sigma$ have an interpolant in L, where φ^σ, ψ^σ are obtained by replacing each variable $\boldsymbol{p} \notin \sigma$ with a fresh variable $\boldsymbol{p}^\sigma$. Indeed, any σ-formula χ with $\models_L \varphi \rightarrow (\psi \leftrightarrow \chi)$ is an interpolant of $\varphi \wedge \psi$ and $\varphi^\sigma \rightarrow \psi^\sigma$ in L. Conversely, any interpolant of $\varphi \wedge \psi$ and $\varphi^\sigma \rightarrow \psi^\sigma$ is an explicit σ-definition of ψ modulo φ in L.

For the other reduction, we observe first that the decision problem for L is polynomially reducible to EDEP because, for $\psi = \boldsymbol{p} \notin sig(\varphi)$ and $\sigma = \emptyset$, we have $\models_L \neg\varphi$ iff there is an explicit σ-definition of ψ modulo φ in L. Then we use Theorems 5 and 7 to show that formulas φ, ψ have an interpolant in L iff $\models_L \varphi \rightarrow \psi$ and there is a $sig(\varphi) \cap sig(\psi)$-definition of ψ modulo $\psi \rightarrow \varphi$ in L. Indeed, it suffices to observe that, for any L-models $\mathfrak{M}$ with w, d and $\mathfrak{M}'$ with w', d', we have $\mathfrak{M}, w, d \models \varphi$ and $\mathfrak{M}', w', d' \models \neg\psi$ iff $\mathfrak{M}, w, d \models (\psi \rightarrow \varphi) \wedge \psi$ and $\mathfrak{M}', w', d' \models (\psi \rightarrow \varphi) \wedge \neg\psi$. $\dashv$

D Proof for Example 9

For the formulas

$$\varphi = \mathsf{rep} \wedge \Diamond\forall\big(\mathsf{inPower} \rightarrow \Box(\mathsf{rep} \rightarrow \neg\mathsf{inPower})\big),$$

$$\psi = \Box\forall(\Diamond\mathsf{inPower} \wedge \Diamond\neg\mathsf{inPower} \wedge \exists\mathsf{inPower} \wedge \exists\neg\mathsf{inPower})$$

$\varphi \wedge \psi$ is a conservative extension of ψ in $\mathsf{Q}^1\mathsf{S5}$. Condition (a) of the definition of conservative extension is trivial. To show (b), suppose $\models_{\mathsf{Q}^1\mathsf{S5}} \varphi \wedge \psi \rightarrow \chi$, for some χ such that $sig(\chi) \subseteq \{\mathsf{inPower}\} = \sigma$. We need to prove $\models_{\mathsf{Q}^1\mathsf{S5}} \psi \rightarrow \chi$. Suppose ψ is true somewhere in a $\mathsf{Q}^1\mathsf{S5}$-model $\mathfrak{N}$. By the definition of ψ, it is true everywhere in $\mathfrak{N}$. Consider the relation $\boldsymbol{\beta}$ that connects each $\mathsf{inPower}$-point in $\mathfrak{N}$ with each $\mathsf{inPower}$-point in $\mathfrak{M}$ from Example 8, and each $\neg\mathsf{inPower}$-point in $\mathfrak{N}$ with each $\neg\mathsf{inPower}$-point in $\mathfrak{M}$. It follows from the definition of ψ and the structure of $\mathfrak{M}$ that $\boldsymbol{\beta}$ is a σ-bisimulation between $\mathfrak{N}$ and $\mathfrak{M}$. The reader can check that $\mathfrak{M}, w \models \varphi \wedge \psi$, and so $\mathfrak{M}, w, d \models \chi$ and $\mathfrak{M}, w, e \models \chi$. As $\boldsymbol{\beta}$ is a σ-bisimulation, we obtain that χ is true everywhere in $\mathfrak{N}$, establishing (b).

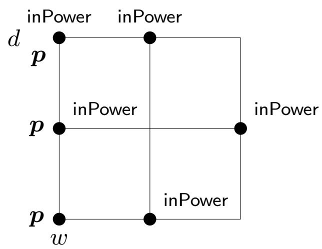
Now, let $\psi' = \psi \wedge (\mathsf{p} \vee \neg\mathsf{p})$, for a fresh proposition p. Then $\varphi \wedge \psi'$ is not a conservative extension of ψ' as witnessed by the formula χ below

$$\chi = \neg\big(\mathsf{p} \wedge \Box\exists(\mathsf{inPower} \wedge \Box(\mathsf{p} \rightarrow \mathsf{inPower}))\big).$$

Indeed, we have $\models_{\mathsf{Q^1S5}} \varphi \wedge \psi' \to \chi$. For suppose $\mathfrak{M}, w \models \varphi \wedge \psi'$, and so $w \models \mathsf{rep}$. Then, by φ, there is a world u with $u \models \forall(\mathsf{inPower} \to \Box(\mathsf{rep} \to \neg\mathsf{inPower}))$. By ψ', there is a domain element d with $u, d \models \mathsf{inPower}$, from which $w, d \models \neg\mathsf{inPower}$. Moreover, this is the case for all d with $u, d \models \mathsf{inPower}$.

Now, if $w \models \neg\mathsf{p}$, we have $w \models \chi$. So assume $w \models \mathsf{p}$. Then $u \not\models \exists(\mathsf{inPower} \wedge \Box(\mathsf{p} \to \mathsf{inPower}))$ because if we had $u, d' \models \mathsf{inPower}$ for some d', then $w, d' \models \neg\mathsf{inPower}$, which is a contradiction. Thus, we obtain $w \models \chi$, which proves $\models_{\mathsf{Q^1S5}} \varphi \wedge \psi' \to \chi$.

On the other hand, in the $\mathsf{Q^1S5}$-model shown in the picture below, ψ' is true at w while χ is false, and so $\not\models_{\mathsf{Q^1S5}} \psi' \wedge \psi' \to \chi$.



E Proofs for Section 4

E.1 Upper bounds

Theorem 10 $\mathfrak{M}, w, d \sim_\sigma^{\mathsf{S5}} \mathfrak{M}', w', d'$ *if and only if* $\mathfrak{M}, w, d \sim_\sigma \mathfrak{M}', w', d'$.

Proof. ($\Rightarrow$) Suppose $\mathfrak{M}, w, d \sim_\sigma^{\mathbf{S5}} \mathfrak{M}', w', d'$ is witnessed by $(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2)$. Define $\boldsymbol{\beta}$ by setting $((v, e), (v', e')) \in \boldsymbol{\beta}$ iff $(v, v') \in \boldsymbol{\beta}_1$, $(e, e') \in \boldsymbol{\beta}_2$ and $\ell^\sigma_{\mathfrak{M}}(v, e) = \ell^\sigma_{\mathfrak{M}'}(v', e')$. It follows that $((w, d), (w', d')) \in \boldsymbol{\beta}$. We show that $\boldsymbol{\beta}$ satisfies **(a)**, **(w)** and **(d)**. Let $((v, e), (v', e')) \in \boldsymbol{\beta}$. The first of them follows from $\ell^\sigma_{\mathfrak{M}}(v, e) = \ell^\sigma_{\mathfrak{M}'}(v', e')$. To show **(w)**, take any $v \in W$. As $(e, e') \in \boldsymbol{\beta}_2$, there is v' with $(v, v') \in \boldsymbol{\beta}_1$ and $\ell^\sigma_{\mathfrak{M}}(v, e) = \ell^\sigma_{\mathfrak{M}'}(v', e')$ by ($\mathsf{s5}_2$), from which $(v, e), (v', e') \in \boldsymbol{\beta}$. The converse implication of **(w)** is symmetric. Finally, take any $c \in D$. By ($\mathsf{s5}_1$), there is c' with $(c, c') \in \boldsymbol{\beta}_2$ and $\ell^\sigma_{\mathfrak{M}}(v, c) = \ell^\sigma_{\mathfrak{M}'}(v', c')$. This and a symmetric argument establish **(d)**.

($\Leftarrow$) Let $\mathfrak{M}, w, d \sim_\sigma \mathfrak{M}', w', d'$ be witnessed by $\boldsymbol{\beta}$. Set

$$\boldsymbol{\beta}_1 = \{(v, v') \mid \exists e, e' \, ((v, e), (v', e')) \in S\},$$
$$\boldsymbol{\beta}_2 = \{(e, e') \mid \exists v, v' \, (v, e), (v', e) \in S\}.$$

Then $(w, w') \in \boldsymbol{\beta}_1$, $(d, d') \in \boldsymbol{\beta}_2$, $\ell^\sigma_{\mathfrak{M}}(w, d) = \ell^\sigma_{\mathfrak{M}'}(w', d')$. To show ($\mathsf{s5}_1$), suppose $(v, v') \in \boldsymbol{\beta}_1$ and $c \in D$. Then there are e, e' with $((v, e), (v', e')) \in \boldsymbol{\beta}$, and so, by **(d)**, there is c' with $((v, c), (v', c')) \in \boldsymbol{\beta}$, and so $(c, c') \in \boldsymbol{\beta}_2$ and $\ell^\sigma_{\mathfrak{M}}(v, c) = \ell^\sigma_{\mathfrak{M}'}(v', c')$. Condition ($\mathsf{s5}_2$) is proved similarly using **(w)**. $\dashv$

Example 12 Let $\mathfrak{M}, w, d \models \varphi$, for $\mathfrak{M} = (W, D, I)$, and let n be the number of full types in $\mathfrak{M}$ (over $\mathit{sub}(\varphi)$) and $[n] = \{1, \dots, n\}$. Define D' to be a set that contains n distinct copies of each dt in $\mathfrak{M}$ over $\mathit{sub}_\Diamond(\varphi)$, denoting the kth copy by dt^k. For any wt and dt in $\mathfrak{M}$, let $\pi_{\mathsf{wt},\mathsf{dt}}$ be a function from $[n]$ *onto* the set of full types ft in $\mathfrak{M}$ with $\mathsf{wt} = \mathsf{ft} \cap \mathit{sub}_\exists(\varphi)$ and $\mathsf{dt} = \mathsf{ft} \cap \mathit{sub}_\Diamond(\varphi)$. Let Π be the smallest set of sequences π of such $\pi_{\mathsf{wt},\mathsf{dt}}$ satisfying the following condition: for any $\mathsf{ft} = \mathsf{ft}_{\mathfrak{M}}(u, e)$ and $k \in [n]$, there is $\pi \in \Pi$ with $\pi_{\mathsf{wt}_{\mathfrak{M}}(u),\mathsf{dt}_{\mathfrak{M}}(e)}(k) = \mathsf{ft}$. Set $W' = \{\mathsf{wt}^\pi_{\mathfrak{M}}(u) \mid u \in W,\ \pi \in \Pi\}$, treating each $\mathsf{wt}^\pi_{\mathfrak{M}}(u)$ as a fresh π-copy of $\mathsf{wt}_{\mathfrak{M}}(u)$. As $|\Pi| \leq n^2$, both $|W'|$ and $|D'|$ are exponential in $|\varphi|$. Define an $\mathsf{Q^1S5}$-model $\mathfrak{M}' = (W', D', I')$ by taking $\mathfrak{M}', \mathsf{wt}^\pi, \mathsf{dt}^k \models \boldsymbol{p}$ iff $\boldsymbol{p} \in \pi_{\mathsf{wt},\mathsf{dt}}(k)$. We show by induction that $\mathfrak{M}', \mathsf{wt}^\pi, \mathsf{dt}^k \models \rho$ iff $\rho \in \pi_{\mathsf{wt},\mathsf{dt}}(k)$, for any $\rho \in \mathit{sub}(\varphi)$. The basis and the Booleans are straightforward.

Case $\rho = \exists\xi$. If $\mathsf{wt}^\pi, \mathsf{dt}^k \models \rho$, there is $\mathsf{dt}'^{k'}$ with $\mathsf{wt}^\pi, \mathsf{dt}'^{k'} \models \xi$. By IH, $\xi \in \pi_{\mathsf{wt},\mathsf{dt}'}(k')$, so $\rho \in \pi_{\mathsf{wt},\mathsf{dt}'}(k')$ and $\rho \in \mathsf{wt}$, whence $\rho \in \pi_{\mathsf{wt},\mathsf{dt}}(k)$. Conversely, let $\rho \in \pi_{\mathsf{wt},\mathsf{dt}}(k) = \mathsf{ft}_{\mathfrak{M}}(u, e)$. Then there is e' with $\mathfrak{M}, u, e' \models \xi$, and so $\xi \in \mathsf{ft}(u, e')$. Let $\mathsf{dt}' = \mathsf{dt}_{\mathfrak{M}}(e')$. As $\pi_{\mathsf{wt},\mathsf{dt}'}$ is surjective, there is k' with $\pi_{\mathsf{wt},\mathsf{dt}'}(k') = \mathsf{ft}(u, e')$, and so $\xi \in \pi_{\mathsf{wt},\mathsf{dt}'}(k')$. By IH, $\mathsf{wt}^\pi, \mathsf{dt}'^{k'} \models \xi$, and so $\mathsf{wt}^\pi, \mathsf{dt}^k \models \rho$.

Case $\rho = \Diamond\xi$. If $\mathsf{wt}^\pi, \mathsf{dt}^k \models \rho$, there exists $\mathsf{wt}'^{\pi'}$ with $\mathsf{wt}'^{\pi'}, \mathsf{dt}^k \models \xi$. By IH, $\xi \in \pi'_{\mathsf{wt}',\mathsf{dt}}(k)$, so $\Diamond\xi \in \pi'_{\mathsf{wt}',\mathsf{dt}}(k)$ and $\rho \in \mathsf{dt}$, whence $\rho \in \pi_{\mathsf{wt},\mathsf{dt}}(k)$. Conversely, if $\rho \in \pi_{\mathsf{wt},\mathsf{dt}}(k) = \mathsf{ft}_{\mathfrak{M}}(u, e)$, there is u' with $\mathfrak{M}, u', e \models \xi$. Let $\mathsf{wt}' = \mathsf{wt}_{\mathfrak{M}}(u')$. By the choice of Π, it has π' with $\pi'_{\mathsf{wt}',\mathsf{dt}}(k) = \mathsf{ft}_{\mathfrak{M}}(u', e)$. Then $\xi \in \pi'_{\mathsf{wt}',\mathsf{dt}}(k)$, so $\mathsf{wt}'^{\pi'}, \mathsf{dt} \models \xi$ by IH and $\mathsf{wt}^\pi, \mathsf{dt} \models \Diamond\xi$. $\dashv$

Lemma 14 (i) *For every* $\rho \in \mathit{sub}(\varphi, \psi)$,

$$\mathfrak{M}'_i, \mathsf{wp}^\pi_i, \mathsf{dp}^k_i \models \rho \quad \textit{iff} \quad \rho \in \pi_{\mathsf{wp}_i,\mathsf{dp}_i}(k).$$

Proof. The proof is by induction on the construction of ρ, with the basis given by (3). For the induction step, suppose first that $\rho = \exists\xi$. If $\mathfrak{M}'_i, \mathsf{wp}^\pi_i, \mathsf{dp}^k_i \models \rho$, then there is $\mathsf{dp}'^{k'}_i$ such that $\mathfrak{M}'_i, \mathsf{wp}^\pi_i, \mathsf{dp}'^{k'}_i \models \xi$. By IH, $\xi \in \pi_{\mathsf{wp}_i,\mathsf{dp}'_i}(k')$ and $\exists\xi \in \pi_{\mathsf{wp}_i,\mathsf{dp}'_i}(k')$. Then $\exists\xi \in \mathsf{wt}$ for $\mathsf{wp}_i = (\mathsf{wt}, \mathsf{wm})$, and so $\rho \in \pi_{\mathsf{wp}_i,\mathsf{dp}_i}(k)$. Conversely, suppose $\rho \in \pi_{\mathsf{wp}_i,\mathsf{dp}_i}(k)$, where $\pi_{\mathsf{wp}_i,\mathsf{dp}_i}(k) = \mathsf{ft}_{\mathfrak{M}_i}(w, d)$ with $\mathsf{wp}_i = \mathsf{wp}_i(w)$ and $\mathsf{dp}_i = \mathsf{dp}_i(d)$. Then there is d' with $\mathfrak{M}_i, w, d' \models \xi$. Let $\mathsf{dp}'_i = \mathsf{dp}_i(d')$. As $\pi_{\mathsf{wp}_i,\mathsf{dp}_i}$ is surjective, there is k' with $\xi \in \pi_{\mathsf{wp}_i,\mathsf{dp}'_i}(k')$. By IH, $\mathfrak{M}'_i, \mathsf{wp}^\pi_i, \mathsf{dp}'^{k'}_i \models \xi$. It follows that $\mathfrak{M}'_i, \mathsf{wp}^\pi_i, \mathsf{dp}^k_i \models \rho$.

Next, let $\rho = \Diamond\xi$. Suppose $\mathfrak{M}'_i, \mathsf{wp}^\pi_i, \mathsf{dp}^k_i \models \rho$. Then there is $\mathsf{wp}'^{\pi'}_i$ with $\mathfrak{M}'_i, \mathsf{wp}'^{\pi'}_i, \mathsf{dp}^k_i \models \xi$. By IH, $\xi \in \pi'_{\mathsf{wp}'_i,\mathsf{dp}_i}(k)$ and $\Diamond\xi \in \pi'_{\mathsf{wp}'_i,\mathsf{dp}_i}(k)$. Then $\Diamond\xi \in \mathsf{dt}$ for $\mathsf{dp}_i = (\mathsf{dt}, \mathsf{dm})$. It follows that $\rho \in \pi_{\mathsf{wp}_i,\mathsf{dp}_i}(k)$. Conversely, let $\rho \in \pi_{\mathsf{wp}_i,\mathsf{dp}_i}(k) = \mathsf{ft}_{\mathfrak{M}_i}(w, d)$ with $\mathsf{wp}_i = \mathsf{wp}_i(w)$ and $\mathsf{dp}_i = \mathsf{dp}_i(d)$. Then there is w' with $\mathfrak{M}_i, w', d \models \xi$. By the choice of Π, there exists $\pi' \in \Pi$ such that $\pi'_{\mathsf{wp}'_i,\mathsf{dp}_i}(k) = \mathsf{ft}_{\mathfrak{M}_i}(w', d)$, where $\mathsf{wp}'_i = \mathsf{wp}_i(w')$. Then $\xi \in \pi'_{\mathsf{wp}'_i,\mathsf{dp}_i}(k)$, and so $\mathfrak{M}'_i, \mathsf{wp}'^{\pi'}_i, \mathsf{dp}_i \models \xi$ by IH, whence $\mathfrak{M}'_i, \mathsf{wp}^\pi_i, \mathsf{dp}_i \models \Diamond\xi$.

The induction step for the Booleans is straightforward. $\dashv$

Lemma 14 (ii) *The pair* $(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2)$ *is a* σ*-bisimulation between* $\mathfrak{M}'_1$ *and* $\mathfrak{M}'_2$.

Proof. To check ($\mathbf{s5}_1$), suppose $\boldsymbol{\beta}_1(\mathsf{wp}^{\pi^1}_1, \mathsf{wp}^{\pi^2}_2)$ with $\mathsf{wp}_i = (\mathsf{wt}_i, \mathsf{wm}_i)$, for $i = 1, 2$, so $\mathsf{wm}_1 = \mathsf{wm}_2$. Take any $\mathsf{dp}_1^{k_1} \in D_1'$ with $\mathsf{dp}_1 = (\mathsf{dt}_1, \mathsf{dm}_1)$. Our aim is to find $\mathsf{dp}_2^{k_2} \in D_2'$ with $\mathsf{dp}_2 = (\mathsf{dt}_2, \mathsf{dm}_2)$ such that $\mathsf{dm}_1 = \mathsf{dm}_2$ and $\boldsymbol{p} \in \pi^1_{\mathsf{wp}_1,\mathsf{dp}_1}(k_1)$ iff $\boldsymbol{p} \in \pi^2_{\mathsf{wp}_2,\mathsf{dp}_2}(k_2)$, for every $\boldsymbol{p} \in \sigma$.

Suppose $\pi^1_{\mathsf{wp}_1,\mathsf{dp}_1}(k_1) = \mathsf{ft}_{\mathfrak{M}_1}(u, d)$, for some $(u, d) \in W_1 \times D_1$. Then $\mathsf{wp}_1 = \mathsf{wp}_1(u) = (\mathsf{wt}_{\mathfrak{M}_1}(u), \mathsf{wm}(u))$, $\mathsf{wm}_1 = \mathsf{wm}(u) = (T_1(u), T_2(u))$, and $\mathsf{dp}_1 = \mathsf{dp}_1(d) = (\mathsf{dt}_{\mathfrak{M}_1}(d), \mathsf{dm}(d))$ and $\mathsf{dm}_1 = \mathsf{dm}(d) = (S_1(d), S_2(d))$. As $\mathsf{wm}_1 = \mathsf{wm}_2$, we also have $\mathsf{wt}_2 \in T_2(u)$, and so there is $v \in W_2$ with $u \sim_\sigma v$ and $\mathsf{wp}_2 = (\mathsf{wt}_{\mathfrak{M}_2}(v), \mathsf{wm}(v))$.

Now, by (**s5**$_1$) for $\mathfrak{M}_i$, $i = 1, 2$, there is $e \in D_2$ such that $d \sim_\sigma e$ and $\ell^\sigma_{\mathfrak{M}_1}(u, d) = \ell^\sigma_{\mathfrak{M}_2}(v, e)$. By **(dm)**, $\mathsf{dm}(d) = \mathsf{dm}(e)$. As all functions $\pi_{\mathsf{wp}_i,\mathsf{dp}_i}$ are surjective, there exists $k_2 \in [n]$ with $\pi^2_{\mathsf{wp}_2,\mathsf{dp}_2}(k_2) = \mathsf{ft}_{\mathfrak{M}_2}(v, e)$, which implies that $\boldsymbol{p} \in \pi^1_{\mathsf{wp}_1,\mathsf{dp}_1}(k_1)$ iff $\boldsymbol{p} \in \pi^2_{\mathsf{wp}_2,\mathsf{dp}_2}(k_2)$, for every $\boldsymbol{p} \in \sigma$.

$$
\begin{tikzcd}
\mathsf{wp}_1^{\pi_1} \arrow[r, dash, "\mathsf{ft}(u,d)"] \arrow[d, dashed, dash, "\sigma"'] & \mathsf{dp}_1^{k_1} \arrow[d, dashed, dash, "\sigma"'] \\
\mathsf{wp}_2^{\pi_2} \arrow[r, dash, "\mathsf{ft}(v,e)"] & \mathsf{dp}_2^{k_2}
\end{tikzcd}
\qquad
\begin{tikzcd}
u \arrow[r, dash] \arrow[d, dashed, dash, "\sigma"'] & d \arrow[d, dashed, dash, "\sigma"'] \\
v \arrow[r, dash] & e
\end{tikzcd}
$$

To check (**s5**$_2$), let $\boldsymbol{\beta}_2(\mathsf{dp}_1^{k_1}, \mathsf{dp}_2^{k_2})$ with $\mathsf{dp}_i = (\mathsf{dt}_i, \mathsf{dm}_i)$, for $i = 1, 2$, so $\mathsf{dm}_1 = \mathsf{dm}_2$. Take any $\mathsf{wp}_1^{\pi^1} \in W_1'$ with $\mathsf{wp}_1 = (\mathsf{wt}_1, \mathsf{wm}_1)$. We need to find $\mathsf{wp}_2^{\pi^2} \in W_2'$ with $\mathsf{wp}_2 = (\mathsf{wt}_2, \mathsf{wm}_2)$ such that $\mathsf{wm}_1 = \mathsf{wm}_2$ and $\boldsymbol{p} \in \pi^1_{\mathsf{wp}_1,\mathsf{dp}_1}(k_1)$ iff $\boldsymbol{p} \in \pi^2_{\mathsf{wp}_2,\mathsf{dp}_2}(k_2)$, for every $\boldsymbol{p} \in \sigma$.

Let $\pi^1_{\mathsf{wp}_1,\mathsf{dp}_1}(k_1) = \mathsf{ft}_{\mathfrak{M}_1}(u, d)$, for some $(u, d) \in W_1 \times D_1$. Then $\mathsf{wp}_1 = \mathsf{wp}_1(u) = (\mathsf{wt}_{\mathfrak{M}_1}(u), \mathsf{wm}(u))$, $\mathsf{wm}_1 = \mathsf{wm}(u) = (T_1(u), T_2(u))$, and $\mathsf{dp}_1 = \mathsf{dp}_1(d) = (\mathsf{dt}_{\mathfrak{M}_1}(d), \mathsf{dm}(d))$ and $\mathsf{dm}_1 = \mathsf{dm}(d) = (S_1(d), S_2(d))$.

There exists $e \in D_2$ such that $e \sim_\sigma d$ and $\mathsf{dp}_2 = (\mathsf{dt}_{\mathfrak{M}_2}(e), \mathsf{dm}(e))$. By (**s5**$_2$) for $\mathfrak{M}_i$, $i = 1, 2$, there is $v \in W_2$ with $u \sim_\sigma v$ and $\ell^\sigma_{\mathfrak{M}_1}(u, d) = \ell^\sigma_{\mathfrak{M}_2}(v, e)$. By **(wm)**, we have $\mathsf{wm}(u) = \mathsf{wm}(v)$. By the choice of Π, there is $\pi^2 \in \Pi$ such that $\pi^2_{\mathsf{wp}_2,\mathsf{dp}_2}(k_2) = \mathsf{ft}_{\mathfrak{M}_2}(v, e)$, which implies that $\boldsymbol{p} \in \pi^1_{\mathsf{wp}_1,\mathsf{dp}_1}(k_1)$ iff $\boldsymbol{p} \in \pi^2_{\mathsf{wp}_2,\mathsf{dp}_2}(k_2)$, for all $\boldsymbol{p} \in \sigma$. ⊣

Theorem 15 *Any formulas φ and ψ are $sig(\varphi) \cap sig(\psi)$-bisimulation consistent in* $\mathsf{Q^1S5}$ *iff there are witnessing* $\mathsf{Q^1S5}$*-models of size double-exponential in $|\varphi|$ and $|\psi|$.*

Proof. Let $\mathfrak{M}_1, w_1, d_1 \sim_\sigma \mathfrak{M}_2, w_2, d_2$, $\mathfrak{M}_1, w_1, d_1 \models \varphi$, $\mathfrak{M}_2, w_2, d_2 \not\models \psi$, and let $\sigma = sig(\varphi) \cap sig(\psi)$. Consider the models $\mathfrak{M}_1'$, $\mathfrak{M}_2'$ with σ-bisimulation $(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2)$. For $i = 1, 2$, let $\mathsf{wp}_i = \mathsf{wp}_i(w_i) = (\mathsf{wt}_{\mathfrak{M}_i}(w_i), \mathsf{wm}(w_i))$ and let $\mathsf{dp}_i = \mathsf{dp}_i(d_i) = (\mathsf{dt}_{\mathfrak{M}_i}(d_i), \mathsf{dm}(w_i))$. By the choice of Π, we have π^i with $\pi^i_{\mathsf{wp}_i,\mathsf{dp}_i}(1) = \mathsf{ft}_{\mathfrak{M}_i}(w_i, d_i)$. Then $\mathfrak{M}_1', \mathsf{wp}_1^{\pi^1}, \mathsf{dp}_1^1 \models \varphi$ and $\mathfrak{M}_2', \mathsf{wp}_2^{\pi^2}, \mathsf{dp}_2^1 \not\models \psi$ by Lemma 14 (i). Since $w_1 \sim_\sigma w_2$ and $d_1 \sim_\sigma d_2$, **(wm)** and **(dm)** imply $\mathsf{wm}(w_1) = \mathsf{wm}(w_2)$ and $\mathsf{dm}(d_1) = \mathsf{dm}(d_2)$. By Lemma 14 (ii), $(\mathsf{wp}_1^{\pi^1}, \mathsf{wp}_2^{\pi^2}) \in \boldsymbol{\beta}_1$ and $(\mathsf{dp}_1^{\pi^1}, \mathsf{dp}_2^{\pi^2}) \in \boldsymbol{\beta}_2$, and so $\mathfrak{M}_1', \mathsf{wp}_1^{\pi^1}, \mathsf{dp}_1^1 \sim_\sigma \mathfrak{M}_2', \mathsf{wp}_2^{\pi^2}, \mathsf{dp}_2^1$. ⊣

Now, Theorems 15, 4 and 5 give the upper bound of

Theorem 16 (i) Both IEP and EDEP for $\mathsf{Q^1S5}$ are decidable in coN2ExpTime.

Note that, as explained in Section A above, the upper bound result of Theorem 16 (i) for $\mathsf{Q^1S5}$ (or equivalently, for the equality- and substitution-free fragment of FO^2) does not directly follow from the upper bound result for FO^2 in (Jung and Wolter 2021) (though can also be proved by using a simplified version of the proof in (Jung and Wolter 2021)).

E.2 Lower bounds

Theorem 16 (ii) *IEP and EDEP for* $\mathsf{Q^1S5}$ *are both* 2ExpTime*-hard.*

Proof. We reduce the word problem for languages recognised by exponentially space bounded alternating Turing machines (ATMs). It is well-known that there are 2^n-space bounded ATMs for which the recognised language is 2ExpTime-hard (Chandra, Kozen, and Stockmeyer 1981).

A *2^n-space bounded ATM* is a tuple $M = (Q, q_0, \Gamma, \Delta)$, whose set Q of states is partitioned to $\forall$-states and $\exists$-states, with the initial state q_0 being a $\forall$-state; Γ is the tape alphabet containing the blank symbol $\flat$; and

$$\Delta \colon Q \times \Gamma \to \mathcal{P}\big(Q \times \Gamma \times \{L, R\}\big)$$

is the transition function such that $|\Delta(q, a)|$ is always either 0 or 2, and $\forall$-states and $\exists$-states alternate on every computation path. $\forall$- and $\exists$-*configurations* are represented by 2^n-long sequences of symbols from $\Gamma \cup (Q \times \Gamma)$, with a single symbol in the sequence being from $Q \times \Gamma$.

Similarly to (Jung and Wolter 2021), we use the following (slightly non-standard) acceptance condition. An *accepting computation-tree* is an infinite tree of configurations such that $\forall$-configurations always have 2 children, and $\exists$-configurations always have 1 child (marked by 0 or 1). We say that *M accepts an input word* $\overline{a} = (a_0, a_1, \dots, a_{n-1})$ if there is an accepting computation-tree with the configuration

$$c_{\mathit{init}} = \big((q_0, a_0), a_1, \dots, a_{n-1}, \flat, \dots, \flat\big)$$

at its root. Note that, starting from the standard ATM acceptance condition defined via accepting states, this can be achieved by assuming that the 2^n-space bounded ATM terminates on every input and then modifying it to enter an infinite loop from the accepting state.

Given a 2^n-space bounded ATM M and an input word $\overline{a}$ of length n, we will construct in polytime formulas φ and ψ such that

1. $\models_{\mathsf{Q^1S5}} \varphi \to \psi$, and
2. M accepts $\overline{a}$ iff φ, $\neg\psi$ are σ-bisimulation consistent, where $\sigma = sig(\varphi) \cap sig(\psi)$.

By Theorems 4 and 5, it follows that both IEP and EDEP are 2ExpTime-hard for $\mathsf{Q^1S5}$.

One aspect of our construction is similar to that of (Artale et al. 2021; Jung and Wolter 2021): we also represent accepting computation-trees as binary trees whose nodes are coloured by predicates in σ. However, unlike the formalisms in the cited work, $\mathsf{Q^1S5}$ cannot express the uniqueness of properties, and so the remaining ideas are novel. One part of

φ 'grows' 2^n-many copies of σ-coloured binary trees, using a technique from 2D propositional modal logic (Hodkinson et al. 2003; Göller, Jung, and Lohrey 2015). Another part of φ colours the tree-nodes with non-σ-symbols to ensure that, in the mth tree, for each $m < 2^n$, the content of the mth tape-cell is properly changing during the computation. Then we use ideas from Example 6 to make sure that the generated 2^n-many trees are all σ-bisimilar, and so represent the same accepting computation-tree.

We begin with defining the conjuncts (5)–(38) of φ. We will use three counters, B, U and V, each counting modulo 2^n and implemented using $2n$-many unary predicate symbols: $\boldsymbol{h}_0^A, \dots, \boldsymbol{h}_{n-1}^A, \boldsymbol{v}_0^A, \dots, \boldsymbol{v}_{n-1}^A$ for $A \in \{B, U, V\}$. We write equ^A for $\bigwedge_{i<n}(\boldsymbol{h}_i^A \leftrightarrow \boldsymbol{v}_i^A)$, and write $[A = m]$ for $m < 2^n$, if equ^A holds and the $\boldsymbol{h}^A$- and $\boldsymbol{v}^A$-sequences represent m in binary. We use notation $[A < m]$ and $[A \neq m]$ similarly. We write succ^A for expressing that '$\boldsymbol{h}^A$-value $=$ $\boldsymbol{v}^A$-value+1 (mod 2^n)':

$$\bigvee_{i<n} \Big(\boldsymbol{h}_i^A \land \neg \boldsymbol{v}_i^A \land \bigwedge_{j<i} (\neg \boldsymbol{h}_j^A \land \boldsymbol{v}_j^A) \land \bigwedge_{i<j<n} (\boldsymbol{h}_j^A \leftrightarrow \boldsymbol{v}_j^A) \Big) \lor \bigwedge_{i<n} (\neg \boldsymbol{h}_i^A \land \boldsymbol{v}_i^A).$$

We express, for $A \in \{B, U, V\}$, that the $\boldsymbol{h}^A$-predicates are 'modally-stable' and the $\boldsymbol{v}^A$-predicates are 'FO-stable':

$$\Box\forall \bigwedge_{i<n} \big((\boldsymbol{h}_i^A \to \Box \boldsymbol{h}_i^A) \land (\neg \boldsymbol{h}_i^A \to \Box \neg \boldsymbol{h}_i^A) \big), \tag{5}$$

$$\Box\forall \bigwedge_{i<n} \big((\boldsymbol{v}_i^A \to \forall \boldsymbol{v}_i^A) \land (\neg \boldsymbol{v}_i^A \to \forall \neg \boldsymbol{v}_i^A) \big). \tag{6}$$

We use the B-counter to generate 2^n-many 'special' equ^B-points 'coloured' by a fresh predicate $\boldsymbol{r}$ for the root-node of the trees representing the computation. The succ^B-points used in generating the $\boldsymbol{r}$-points will be marked by a fresh predicate $\boldsymbol{n}^B$ (for 'next B'):

$$[B = 0] \land \boldsymbol{r}, \tag{7}$$

$$\Box\forall \big(\boldsymbol{r} \land [B \neq 2^n - 1] \to \exists \boldsymbol{n}^B \big), \tag{8}$$

$$\Box\forall (\boldsymbol{n}^B \to \Diamond \boldsymbol{r}), \tag{9}$$

$$\Box\forall (\boldsymbol{r} \to \mathsf{equ}^B), \tag{10}$$

$$\Box\forall (\boldsymbol{n}^B \to \mathsf{succ}^B). \tag{11}$$

Then, at each $\boldsymbol{r}$-point, we 'grow' an infinite binary rooted tree that we will use to represent the accepting computation-tree of M on $\overline{a}$ as follows. The binary tree is divided into 2^n-long 'linear' levels (where each node has one child only): each linear 2^n-long subpath within such a level represents a configuration. In addition, the infinite binary tree is branching to two at the last node of the linear subpath representing each $\forall$-configuration (see more details in the proof of Lemma 32 below).

We grow this infinite binary tree with the help of the U-counter. Nodes of this infinite 'U-tree' are marked by a fresh predicate $\boldsymbol{t}$, and the succ^U-points used in generating the $\boldsymbol{t}$-points will be marked by a fresh predicate $\boldsymbol{n}^U$. First, we generate a computation-tree 'skeleton' of alternating $\forall$- and $\exists$-levels, and with appropriate branching. We use fresh predicates $\boldsymbol{q}_\forall$ and $\boldsymbol{q}_\exists^i$, $i = 0, 1$, to mark the levels, and an additional predicate $\boldsymbol{z}$ to enforce two different children at $\forall$-levels. Given any formula χ, we write $\mathsf{next}(\chi)$ for $\forall\big(\boldsymbol{n}^U \to \Box(\boldsymbol{t} \to \chi)\big)$. We add the following conjuncts, for $i = 0, 1$:

$$\Box\forall \big(\boldsymbol{r} \to [U = 0] \land \boldsymbol{q}_\forall \land \boldsymbol{t} \big), \tag{12}$$

$$\Box\forall (\boldsymbol{t} \to \exists \boldsymbol{n}^U), \tag{13}$$

$$\Box\forall (\boldsymbol{n}^U \to \Diamond \boldsymbol{t}), \tag{14}$$

$$\Box\forall (\boldsymbol{t} \to \mathsf{equ}^U), \tag{15}$$

$$\Box\forall (\boldsymbol{n}^U \to \mathsf{succ}^U), \tag{16}$$

$$\Box\forall \big(\boldsymbol{t} \land [U \neq 2^n - 1] \land \boldsymbol{q}_\forall \to \mathsf{next}(\boldsymbol{q}_\forall) \big), \tag{17}$$

$$\Box\forall \big(\boldsymbol{t} \land [U \neq 2^n - 1] \land \boldsymbol{q}_\exists^i \to \mathsf{next}(\boldsymbol{q}_\exists^i) \big), \tag{18}$$

$$\Box\forall \big(\boldsymbol{t} \land [U = 2^n - 1] \land \boldsymbol{q}_\forall \to \exists(\boldsymbol{n}^U \land \boldsymbol{z}) \land \exists(\boldsymbol{n}^U \land \neg \boldsymbol{z}) \big), \tag{19}$$

$$\Box\forall \big(\boldsymbol{t} \land [U = 2^n - 1] \land \boldsymbol{q}_\forall \to \mathsf{next}(\boldsymbol{q}_\exists^0 \lor \boldsymbol{q}_\exists^1) \big), \tag{20}$$

$$\Box\forall \big(\boldsymbol{t} \land [U = 2^n - 1] \land \boldsymbol{q}_\exists^i \to \mathsf{next}(\boldsymbol{q}_\forall) \big). \tag{21}$$

Next, for each $\gamma \in \Gamma \cup (Q \times \Gamma)$, we introduce a fresh predicate $\boldsymbol{s}_\gamma$. We initialise the computation on input $\overline{a} = (a_0, a_1, \dots, a_{n-1})$, where $a_i \neq \flat$ for $i < n$:

$$\Box\forall \Big(\boldsymbol{r} \to \boldsymbol{s}_{(q_0, a_0)} \land \mathsf{next}(\boldsymbol{s}_{a_1} \land \dots \mathsf{next}(\boldsymbol{s}_{a_{n-1}} \land \mathsf{next}(\boldsymbol{s}_\flat)) \dots) \Big), \tag{22}$$

$$\Box\forall \big(\boldsymbol{t} \land \boldsymbol{s}_\flat \land [U \neq 2^n - 1] \to \mathsf{next}(\boldsymbol{s}_\flat) \big). \tag{23}$$

Next, we ensure that the subsequent configurations are properly represented. Using the V-counter, we ensure that, for each $m < 2^n$, the U-tree that is grown at the mth $\boldsymbol{r}$-point properly describes the 'evolution' of the mth tape-cell's content during the accepting computation. We begin with ensuring that the V-counter increases along the U-counter, and with initialising it as $2^n - 1 - m$ of the value m of the B-counter:

$$\Box\forall (\boldsymbol{t} \to \mathsf{equ}^V), \tag{24}$$

$$\Box\forall (\boldsymbol{n}^U \to \mathsf{succ}^V), \tag{25}$$

$$\Box\forall \Big[\boldsymbol{r} \to \bigwedge_{i<n} \big((\boldsymbol{h}_i^B \leftrightarrow \neg \boldsymbol{h}_i^V) \land (\boldsymbol{v}_i^B \leftrightarrow \neg \boldsymbol{v}_i^V) \big) \Big]. \tag{26}$$

Below we enforce the proper evolution of the 'middle' section of the 2^n-long tape (when $0 < m < 2^n - 1$), the two missing cells at the beginning and the end of the tape can be handled similarly.

In order to do this, we represent the transition function Δ of M by two partial functions

$$f_i \colon \big(\Gamma \cup (Q \times \Gamma) \big)^3 \to \big(\Gamma \cup (Q \times \Gamma) \big), \quad \text{for } i = 0, 1,$$

giving the next content of the middle-cell for each triple of cells. We ensure that the domain of the f_i is proper by taking, for all (q, a) with $|\Delta(q, a)| = 0$, the conjunct

$$\Box\forall \neg \boldsymbol{s}_{(q,a)}. \tag{27}$$

For each $\overline{\gamma} = (\gamma_0, \gamma_1, \gamma_2) \in \big(\Gamma \cup (Q \times \Gamma)\big)^3$ in the domain of any of the f_i, we write $\mathsf{cells}_{\overline{\gamma}}$ for

$$\boldsymbol{s}_{\gamma_0} \wedge \exists\Big[\boldsymbol{n}^U \wedge \Diamond\Big(\boldsymbol{t} \wedge \big(\boldsymbol{s}_{\gamma_1} \wedge \exists(\boldsymbol{n}^U \wedge \Diamond \boldsymbol{s}_{\gamma_2})\big)\Big)\Big].$$

In addition to the $\boldsymbol{s}_\gamma$ variables, for some $\gamma \in \Gamma \cup (Q \times \Gamma)$, we will use additional variables $\boldsymbol{s}^0_\gamma$, $\boldsymbol{s}^1_\gamma$, and $\boldsymbol{s}^+_\gamma$, and have the conjuncts, for $i = 0, 1$ and $\overline{\gamma}$ in the domain of any of the f_i:

$$\Box\forall\big(\boldsymbol{t} \wedge [V = 2^n - 1] \wedge [U < 2^n - 2] \wedge \mathsf{cells}_{\overline{\gamma}} \wedge \boldsymbol{q}_\forall \rightarrow \mathsf{next}(\boldsymbol{s}^0_{f_0(\overline{\gamma})} \wedge \boldsymbol{s}^1_{f_1(\overline{\gamma})})\big), \quad (28)$$

$$\Box\forall\big(\boldsymbol{t} \wedge [V = 2^n - 1] \wedge [U < 2^n - 2] \wedge \mathsf{cells}_{\overline{\gamma}} \wedge \boldsymbol{q}^i_\exists \rightarrow \mathsf{next}(\boldsymbol{s}^i_{f_i(\overline{\gamma})})\big), \quad (29)$$

$$\Box\forall\big(\boldsymbol{t} \wedge [U \neq 2^n - 1] \wedge \boldsymbol{s}^i_\gamma \rightarrow \mathsf{next}(\boldsymbol{s}^i_\gamma)\big), \quad (30)$$

$$\Box\forall\Big(\boldsymbol{t} \wedge [U = 2^n - 1] \wedge \boldsymbol{q}_\forall \wedge \boldsymbol{s}^0_\gamma \rightarrow \forall\big(\boldsymbol{n}^U \wedge \boldsymbol{z} \rightarrow \Box(\boldsymbol{t} \rightarrow \boldsymbol{s}^+_\gamma)\big)\Big), \quad (31)$$

$$\Box\forall\Big(\boldsymbol{t} \wedge [U = 2^n - 1] \wedge \boldsymbol{q}_\forall \wedge \boldsymbol{s}^1_\gamma \rightarrow \forall\big(\boldsymbol{n}^U \wedge \neg\boldsymbol{z} \rightarrow \Box(\boldsymbol{t} \rightarrow \boldsymbol{s}^+_\gamma)\big)\Big), \quad (32)$$

$$\Box\forall\big(\boldsymbol{t} \wedge [U = 2^n - 1] \wedge \boldsymbol{q}^i_\exists \wedge \boldsymbol{s}^i_\gamma \rightarrow \mathsf{next}(\boldsymbol{s}^+_\gamma)\big), \quad (33)$$

$$\Box\forall\big(\boldsymbol{t} \wedge [V \neq 2^n - 1] \wedge \boldsymbol{s}^+_\gamma \rightarrow \mathsf{next}(\boldsymbol{s}^+_\gamma)\big), \quad (34)$$

$$\Box\forall\big(\boldsymbol{t} \wedge [V = 2^n - 1] \wedge \boldsymbol{s}^+_\gamma \rightarrow \mathsf{next}(\boldsymbol{s}_\gamma)\big). \quad (35)$$

Finally, we introduce a fresh predicate $\boldsymbol{e}$ that will 'interact' with the formula ψ. We add conjuncts to φ ensuring that each of the generated U-trees stays within the 'B-domain' of its root $\boldsymbol{r}$-point (meaning every node of these trees is an $\boldsymbol{e}$-point having the same B-value):

$$\Box\forall(\boldsymbol{t} \vee \boldsymbol{n}^U \rightarrow \boldsymbol{e}), \quad (36)$$

$$\Box\forall(\boldsymbol{e} \rightarrow \mathsf{equ}^B), \quad (37)$$

$$\Box\forall \bigwedge_{i<n} \big(\boldsymbol{e} \wedge \boldsymbol{v}^B_i \rightarrow \Box(\boldsymbol{e} \rightarrow \boldsymbol{v}^B_i)\big). \quad (38)$$

By this, we have completed the definition of φ.

Next, using the second formula of Example 6, we define the formula ψ such that $sig(\varphi) \cap sig(\psi)$ is the set

$$\sigma = \big\{\boldsymbol{e}, \boldsymbol{r}, \boldsymbol{n}^U, \boldsymbol{z}, \boldsymbol{t}, \boldsymbol{q}_\forall, \boldsymbol{q}^0_\exists, \boldsymbol{q}^1_\exists\big\} \cup \big\{\boldsymbol{s}_\gamma \mid \gamma \in \Gamma \cup (Q \times \Gamma)\big\}.$$

We let

$$\psi \;=\; \chi \wedge \Box\forall(\boldsymbol{e} \leftrightarrow \boldsymbol{b}_0 \vee \boldsymbol{b}_1) \rightarrow \Diamond\exists\big(\boldsymbol{b}_0 \wedge \Diamond(\neg\boldsymbol{e} \wedge \exists\boldsymbol{b}_0)\big) \vee \Diamond\exists\big(\boldsymbol{b}_1 \wedge \Diamond(\neg\boldsymbol{e} \wedge \exists\boldsymbol{b}_1)\big),$$

where $\chi = \bigwedge_{\boldsymbol{p} \in \sigma \setminus \{\boldsymbol{e}\}}(\boldsymbol{p} \rightarrow \boldsymbol{p})$ and $\boldsymbol{b}_0$, $\boldsymbol{b}_1$ are fresh predicates.

Lemma 31. *If $n > 1$ then $\models_{\mathsf{Q^1S5}} \varphi \rightarrow \psi$.*

Proof. Suppose $\mathfrak{M}, w_0, d_0 \models \varphi \wedge \Box\forall(\boldsymbol{e} \leftrightarrow \boldsymbol{b}_0 \vee \boldsymbol{b}_1)$ for some model $\mathfrak{M} = (W, D, I)$. Then, by (7)–(11), we have at least $2^n > 3$ different $\boldsymbol{r}$-points (w_0, d_0), $\dots, (w_{2^n-1}, d_{2^n-1})$ in $W \times D$, with the respective B-values $0, \dots, 2^n - 1$. As $\boldsymbol{r}$-points are also $\boldsymbol{e}$-points by (12) and (36), the pigeonhole principle implies that there are $i \neq j < 2^{n-1}$, $k \in \{0, 1\}$ such that $\mathfrak{M}, w_i, d_i \models \boldsymbol{b}_k$ and $\mathfrak{M}, w_j, d_j \models \boldsymbol{b}_k$. Then $\mathfrak{M}, w_j, d_i \models \neg\boldsymbol{e}$ by (38), and so $\mathfrak{M}, w_0, d_0 \models \psi$. ⊣

Lemma 32. *If M accepts $\overline{a}$ then φ, $\neg\psi$ are σ-bisimulation consistent.*

Proof. Let $\mathfrak{T} = (T, S_0, S_1, \boldsymbol{q}_\forall, \boldsymbol{q}^0_\exists, \boldsymbol{q}^1_\exists, \boldsymbol{s}_\gamma)_{\gamma \in \Gamma \cup (Q \times \Gamma)}$ be the infinite binary tree-shaped FO-structure with root $r \in T$ and binary predicates S_0, S_1, that represents the accepting computation-tree of M on $\overline{a}$ as discussed after formula (11) above, that is, configurations are represented by subpaths of 2^n-many nodes linked by S_0. Every node of the 2^n-long subpath representing a $\forall$-configuration is coloured by $\boldsymbol{q}_\forall$. The last node representing a $\forall$-configuration has one S_i-child, for each of $i = 0, 1$, where the representations of the two subsequent $\exists$-configurations start. For $i = 0, 1$, if it is the i-child of an $\exists$-configuration c that is present in the computation-tree, then every node of the 2^n-long subpath representing c is coloured by $\boldsymbol{q}^i_\exists$ (see Fig. 1 for an example). The last node representing an $\exists$-configuration has one S_0-child, where the representation of the next configuration starts. Nodes representing a configuration are also coloured with $\boldsymbol{s}_\gamma$ for the corresponding symbol γ from $\Gamma \cup (Q \times \Gamma)$.

We begin by defining a model $\mathfrak{M} = (W, D, I)$ making φ true. Take $2 \cdot 2^n$ many disjoint copies W_m and D_m, $m < 2^n$, of T and let $W = \bigcup_{m<2^n} W_m$ and $D = \bigcup_{m<2^n} D_m$. For each $m < 2^n$ and $t \in T$, let w^t_m and d^t_m denote the copy of t in W_m and D_m, respectively. We define I first for the symbols in σ. For all $m < 2^n$, $t \in T$ and $\boldsymbol{p} \in \{\boldsymbol{q}_\forall, \boldsymbol{q}^0_\exists, \boldsymbol{q}^1_\exists\} \cup \{\boldsymbol{s}_\gamma \mid \gamma \in \Gamma \cup (Q \times \Gamma)\}$, we let

$$\boldsymbol{e}^{I(w^t_m)} = \{d^{t'}_m \mid t' \in T\}, \quad (39)$$

$$\boldsymbol{r}^{I(w^t_m)} = \begin{cases} \{d^t_m\}, & \text{if } t = r, \\ \emptyset, & \text{otherwise,} \end{cases} \quad (40)$$

$$(\boldsymbol{n}^U)^{I(w^t_m)} = \{d^{t'}_m \mid S_0(t, t') \text{ or } S_1(t, t')\}, \quad (41)$$

$$\boldsymbol{z}^{I(w^t_m)} = \{d^{t'}_m \mid S_0(t, t')\}, \quad (42)$$

$$\boldsymbol{t}^{I(w^t_m)} = \{d^t_m\}, \quad (43)$$

$$\boldsymbol{p}^{I(w^t_m)} = \begin{cases} \{d^t_m\}, & \text{if } \boldsymbol{p}(t) \text{ holds in } \mathfrak{T}, \\ \emptyset, & \text{otherwise.} \end{cases} \quad (44)$$

Next, we define I for the symbols not in σ. The $\boldsymbol{h}^B_i$- and $\boldsymbol{v}^B_i$-predicates, for $i < n$, set up a binary counter counting from 0 to $2^n - 1$ on pairs (w^r_0, d^r_0), $\dots$, $(w^r_{2^n-1}, d^r_{2^n-1})$ in such a way that they are

- stable within each $W_m \times D_m$, $m < 2^n$: if $\mathfrak{M}, w^r_m, d^r_m \models \boldsymbol{h}^B_i$ then $\mathfrak{M}, w, d \models \boldsymbol{h}^B_i$ for all $w \in W_m$, $d \in D_m$; if $\mathfrak{M}, w^r_m, d^r_m \models \boldsymbol{v}^B_i$ then $\mathfrak{M}, w, d \models \boldsymbol{v}^B_i$ for all $w \in W_m$, $d \in D_m$;
- the $\boldsymbol{h}^B_i$-predicates are modally-stable: if $\mathfrak{M}, w, d \models \boldsymbol{h}^B_i$ for some $w \in W$ and $d \in D$ then $\mathfrak{M}, w', d \models \boldsymbol{h}^B_i$ for all $w' \in W$;
- the $\boldsymbol{v}^B_i$-predicates are FO-stable: if $\mathfrak{M}, w, d \models \boldsymbol{v}^B_i$ for some $w \in W$ and $d \in D$ then $\mathfrak{M}, w, d' \models \boldsymbol{v}^B_i$ for all $d' \in D$.

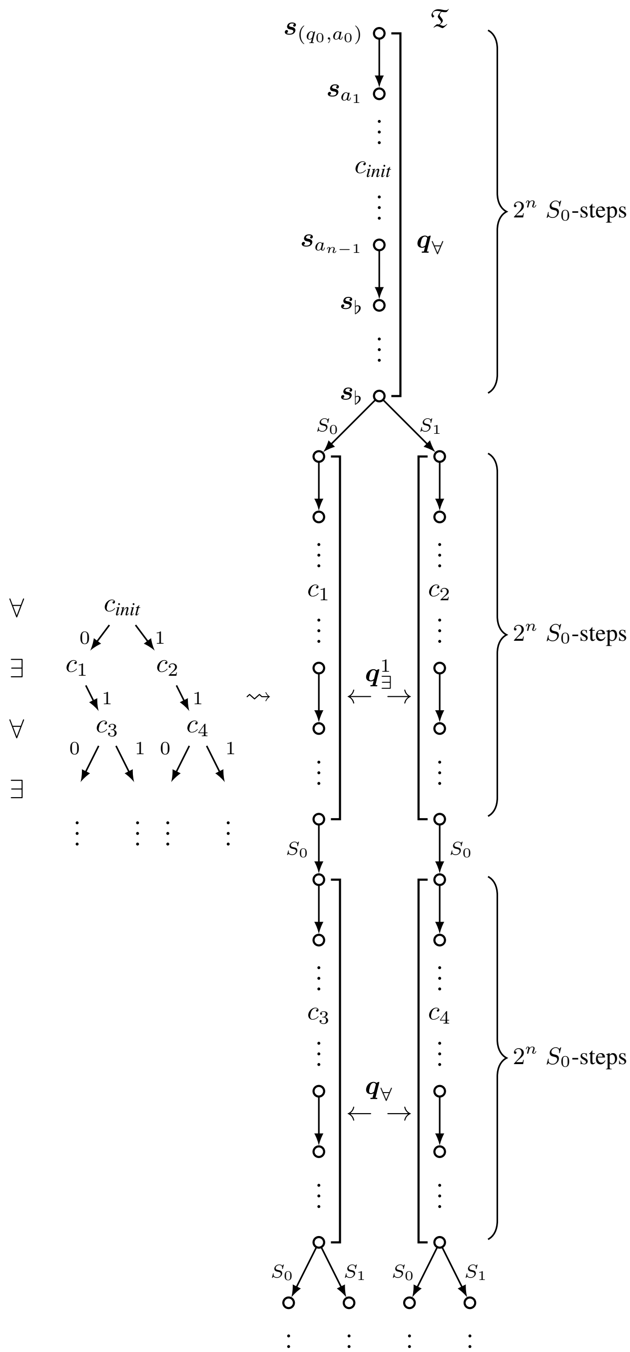


Figure 1: Representing accepting computation-trees.

We let, for all $m < 2^n$ and $t \in T$,

$$(\boldsymbol{n}^B)^{I(w^t_m)} = \begin{cases} \{d^t_{m+1}\}, & \text{if } m < 2^n - 1 \text{ and } t = r, \\ \emptyset, & \text{otherwise.} \end{cases}$$

For each $m < 2^n$, the $\boldsymbol{h}^U_i$- and $\boldsymbol{v}^U_i$-predicates, for $i < n$, set up a binary counter counting from 0 modulo 2^n infinitely along the levels of the tree $\mathfrak{T}$, on pairs of the form (w^t_m, d^t_m), for $t \in T$. The $\boldsymbol{h}^U_i$-predicates are modally-stable, while the $\boldsymbol{v}^U_i$-predicates are FO-stable, in the above sense.

Then, for each $m < 2^n$, the modally-stable $\boldsymbol{h}^V_i$- and the FO-stable $\boldsymbol{v}^V_i$-predicates set up a binary counter counting from $2^n - 1 - m$ modulo 2^n infinitely along the levels of the tree $\mathfrak{T}$, on pairs of the form (w^t_m, d^t_m), for $t \in T$. Also, we extend the FO-structure $\mathfrak{T}$ to $\mathfrak{T}^+_m$ by adding unary predicates $\boldsymbol{s}^0_\gamma$, $\boldsymbol{s}^1_\gamma$, $\boldsymbol{s}^+_\gamma$, for $\gamma \in \Gamma \cup (Q \times \Gamma)$, see Fig. 2. For all $m < 2^n$, $t \in T$, $\boldsymbol{p} \in \{\boldsymbol{s}^0_\gamma, \boldsymbol{s}^1_\gamma, \boldsymbol{s}^+_\gamma \mid \gamma \in \Gamma \cup (Q \times \Gamma)\}$, we let

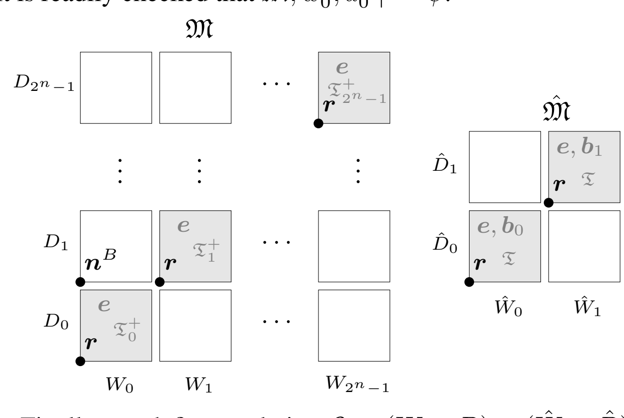
$$\boldsymbol{p}^{I(w^t_m)} = \begin{cases} \{d^t_m\}, & \text{if } \boldsymbol{p}(t) \text{ holds in } \mathfrak{T}^+_m, \\ \emptyset, & \text{otherwise.} \end{cases}$$

It is readily checked that $\mathfrak{M}, w^r_0, d^r_0 \models \varphi$.

Next, we define a model $\hat{\mathfrak{M}} = (\hat{W}, \hat{D}, \hat{I})$ making $\neg\psi$ true. We take $2 \cdot 2$ disjoint copies $\hat{W}_0, \hat{W}_1$ and $\hat{D}_0, \hat{D}_1$ of T and let $\hat{W} = \hat{W}_0 \cup \hat{W}_1$ and $\hat{D} = \hat{D}_0 \cup \hat{D}_1$. For each $k < 2$ and $t \in T$, let $\hat{w}^t_m$ and $\hat{d}^t_m$ denote the copy of t in $\hat{W}_k$ and $\hat{D}_k$, respectively. Now, for symbols in σ we define $\hat{I}$ similarly to I in (39)–(44) above. For symbols not in σ the only ones with non-empty $\hat{I}$-extensions are the $\boldsymbol{b}_i$, for $i < 2$: For all $i, k < 2$, $t \in \mathfrak{T}$, we let

$$\boldsymbol{b}_i^{\hat{I}(\hat{w}^t_k)} = \begin{cases} \{\hat{d}^{t'}_i \mid t' \in T\}, & \text{if } k = i, \\ \emptyset, & \text{otherwise.} \end{cases}$$

It is readily checked that $\hat{\mathfrak{M}}, \hat{w}^r_0, \hat{d}^r_0 \models \neg\psi$.



Finally, we define a relation $\boldsymbol{\beta} \subseteq (W \times D) \times (\hat{W} \times \hat{D})$ by taking, for any $w, d, \hat{w}, \hat{d}$, $\big((w,d),(\hat{w},\hat{d})\big) \in \boldsymbol{\beta}$ iff there exist $t, t' \in T$, $m, m' < 2^n$, $k, k' < 2$ such that $w = w^t_m$, $d = d^{t'}_{m'}$, $\hat{w} = \hat{w}^t_k$, $\hat{d} = \hat{d}^{t'}_{k'}$, and $m = m'$ iff $k = k'$. It is not hard to show that $\boldsymbol{\beta}$ is a σ-bisimulation between $\mathfrak{M}$ and $\hat{\mathfrak{M}}$ with $\big((w^r_0, d^r_0), (\hat{w}^r_0, \hat{d}^r_0)\big) \in \boldsymbol{\beta}$. $\dashv$

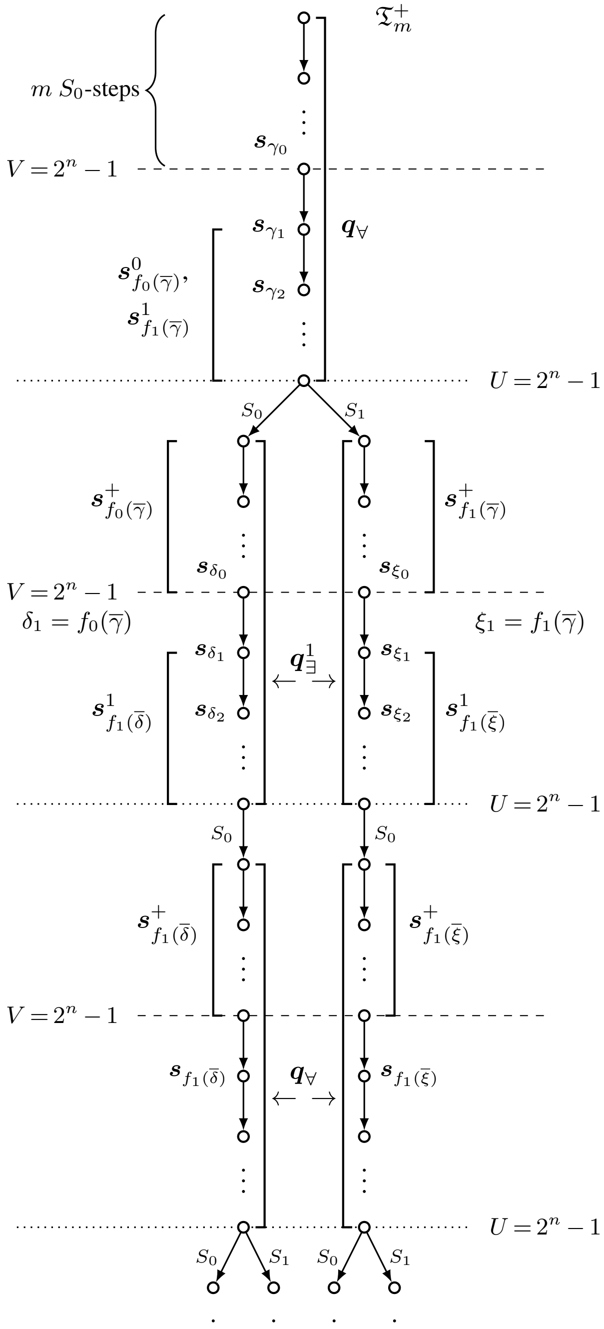


Figure 2: Passing information from one configuration to the next.

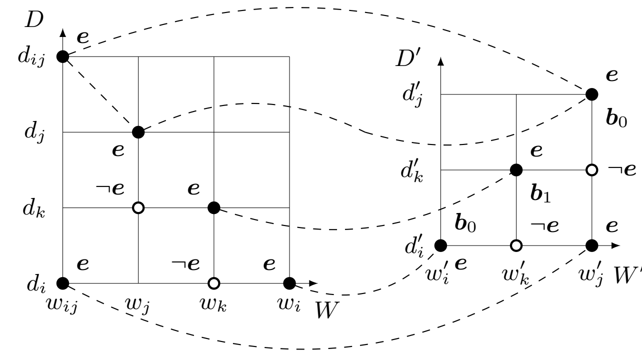


Figure 3: Enforcing 2^n σ-bisimilar trees.

Lemma 33. *If $n > 1$ and φ, $\neg\psi$ are σ-bisimulation consistent, then M accepts $\overline{a}$.*

Proof. Let $\mathfrak{M}, w_0, d_0$ and $\mathfrak{M}', w_0', d_0'$ be models such that $\mathfrak{M}, w_0, d_0 \models \varphi$, $\mathfrak{M}', w_0', d_0' \models \neg\psi$, and $\mathfrak{M}, w_0, d_0 \sim_\sigma \mathfrak{M}', w_0', d_0'$. Let $(w_0, d_0), \dots, (w_{2^n-1}, d_{2^n-1})$ be subsequent $\boldsymbol{r}$-points in $\mathfrak{M}$ generated by (7)–(11), with the respective B-values $0, \dots, 2^n - 1$. By (12) and (36), we have $\mathfrak{M}, w_i, d_i \models \boldsymbol{e}$, for $i < 2^n$. We claim that for all $i, j < 2^n$ there exist w_{ij}, d_{ij} such that

$$\mathfrak{M}, w_{ij}, d_{ij} \models [B = i], \tag{45}$$

$$\mathfrak{M}, w_{ij}, d_{ij} \sim_\sigma \mathfrak{M}, w_j, d_j. \tag{46}$$

Indeed, to begin with, there exist w_i', d_i' such that $\mathfrak{M}, w_i, d_i \sim_\sigma \mathfrak{M}', w_i', d_i'$, and so $\mathfrak{M}', w_i', d_i' \models \boldsymbol{e}$. Let $k < 2^n$ and $k \neq i, j$. As the B-values of (w_i, d_i) and (w_k, d_k) are different, $\mathfrak{M}, w_k, d_i \models \neg\boldsymbol{e}$ follows by (37). Thus, there exist w_k', d_k' such that $\mathfrak{M}', w_k', d_i' \models \neg\boldsymbol{e}$, $\mathfrak{M}', w_k', d_k' \models \boldsymbol{e}$, and $\mathfrak{M}, w_k, d_k \sim_\sigma \mathfrak{M}', w_k', d_k'$. Similarly, there exist w_j', d_j' such that $\mathfrak{M}', w_j', d_k' \models \neg\boldsymbol{e}$, $\mathfrak{M}', w_j', d_j' \models \boldsymbol{e}$, and $\mathfrak{M}, w_j, d_j \sim_\sigma \mathfrak{M}', w_j', d_j'$, see Fig. 3.

As $\mathfrak{M}', w_0', d_0' \models \neg\psi$, we have $\mathfrak{M}', w_i', d_i' \models \boldsymbol{b}_s$ for $s = 0$ or $s = 1$. Suppose $s = 0$ (the other case is similar). It follows from $\neg\psi$ that $\mathfrak{M}', w_k', d_k' \models \boldsymbol{b}_1$ and $\mathfrak{M}', w_j', d_j' \models \boldsymbol{b}_0$. Then $\mathfrak{M}', w_j', d_i' \models \boldsymbol{e}$ again follows from $\neg\psi$. As $\mathfrak{M}, w_i, d_i \sim_\sigma \mathfrak{M}', w_i', d_i'$, there are w_{ij}, d_{ij} such that $\mathfrak{M}, w_{ij}, d_i \models \boldsymbol{e}$, $\mathfrak{M}, w_{ij}, d_{ij} \models \boldsymbol{e}$, and $\mathfrak{M}, w_{ij}, d_{ij} \sim_\sigma \mathfrak{M}', w_j', d_j'$ Therefore, $\mathfrak{M}, w_{ij}, d_{ij} \sim_\sigma \mathfrak{M}, w_j, d_j$, and so (46) holds. We also have $\mathfrak{M}, w_{ij}, d_{ij} \models \mathsf{equ}^B$ by (37), and so (45) follows from (38) and (6).

In particular, (45) and (46) imply that, for each $m < 2^n$, there exist w_m^+, d_m^+ such that $\mathfrak{M}, w_m^+, d_m^+ \models [B = m]$ and $\mathfrak{M}, w_m^+, d_m^+ \sim_\sigma \mathfrak{M}, w_0, d_0$. Take the U-tree $\mathfrak{T}_0$ grown from (w_0, d_0) by (12)–(21). Then it has a σ-bisimilar copy $\mathfrak{T}_m'$ grown from each (w_m^+, d_m^+). Choose a computation-tree 'skeleton' from $\mathfrak{T}_0$ determined by its $\boldsymbol{q}_\exists^i$ labels. As $\boldsymbol{e}, \boldsymbol{r}, \boldsymbol{z}, \boldsymbol{n}^U, \boldsymbol{t}, \boldsymbol{q}_\forall, \boldsymbol{q}_\exists^i \in \sigma$, the formulas (22)–(35) imply that the $\boldsymbol{s}_\gamma$ labels in $\mathfrak{T}_m'$ properly describe the 'evolution' of the mth tape-cell's content via the chosen 'skeleton'. As all $\boldsymbol{s}_\gamma$ are in σ, using its $\boldsymbol{s}_\gamma$ labels and (27), we can extract an accepting computation-tree from $\mathfrak{T}_0$ (with all tape-cell contents evolving properly). $\dashv$

This completes the proof of Theorem 16. $\dashv$

Next, we show that the lower bound results of Theorem 16 (ii) hold even if we want to decide, for any FOM^1-formulas φ, ψ, whether an interpolant (or an explicit definition) exists not only in $\mathsf{Q}^1\mathsf{S5}$, but in any finite-variable fragment of quantified $\mathsf{S5}$. More precisely, we claim that, for any $n, \ell < \omega$ with $2^n > \ell + 1$, given a 2^n-space bounded ATM M and an input word $\overline{a}$ of length n, there exist polytime FOM^1-formulas φ and ψ_ℓ such that

1. $\models_{\mathsf{Q}^1\mathsf{S5}} \varphi \to \psi_\ell$, and
2. M accepts $\overline{a}$ iff φ, $\neg\psi_\ell$ are σ-bisimulation consistent in the ℓ-variable fragment of quantified $\mathsf{S5}$, where $\sigma = sig(\varphi) \cap sig(\psi_\ell)$.

Indeed, φ is like above and ψ_ℓ is similar to $\psi = \psi_1$ above: we just divide $\boldsymbol{e}$ not to two but $\ell + 1$ parts, using fresh variables $\boldsymbol{b}_0, \dots, \boldsymbol{b}_\ell$. Then the proof that these work is similar to the proof above.

Corollary 17. *IEP and EDEP for* FO^2 *without equality are both* 2EXPTIME*-hard.*

Proof. For the equality- and substitution-free fragment of FO^2, this is now a straightforward consequence of Lemma 29: We can simply use the †-translations of the formulas used in the proof of Theorem 16 (ii). In order to prove Corollary 17 for the equality-free fragment, we need an additional step: We need to show that there are suitable bijections between the FO- and modal domains of each of the two (square) $\mathsf{Q}^1\mathsf{S5}$-models constructed in the proof of Lemma 32 such that the resulting FO-models are not only σ-bisimilar in the equality- and substitution-free fragment of FO^2, but they are also σ-bisimilar in the equality-free fragment. In fact, we claim that they are σ-bisimilar in full FO^2, and so Corollary 17 can be considered as a generalisation of the lower bound result in (Jung and Wolter 2021).

To this end, take the σ-bisimulation $\boldsymbol{\beta}$ in $\mathsf{Q}^1\mathsf{S5}$ between the $\mathsf{Q}^1\mathsf{S5}$-models $\mathfrak{M} = (W, D, I)$ and $\hat{\mathfrak{M}} = (\hat{W}, \hat{D}, \hat{I})$ defined in the proof of Lemma 32. Now define a bijection $f\colon D \to W$ by taking $f(d^t_m) = w^t_m$ for all $m < 2^n$, $t \in T$, and a bijection $\hat{f}\colon \hat{D} \to \hat{W}$ by taking $\hat{f}(\hat{d}^t_k) = \hat{w}^t_k$ for all $k < 2$, $t \in T$. Using that (i) the respective restrictions of the FO-models $\mathfrak{A}_{\mathfrak{M},f}$ to D_m and $\mathfrak{A}_{\hat{\mathfrak{M}},\hat{f}}$ to $\hat{D}_k$ are σ-isomorphic for any $m < 2^n$, $k < 2$, and (ii) for all $\boldsymbol{p} \in \sigma$ we have $\mathfrak{A}_{\mathfrak{M},f} \not\models \boldsymbol{p}[y/d^t_m, x/d^{t'}_{m'}]$ if $m \neq m'$, and $\mathfrak{A}_{\hat{\mathfrak{M}},\hat{f}} \not\models \boldsymbol{p}[y/\hat{d}^t_k, x/\hat{d}^{t'}_{k'}]$ if $k \neq k'$, it is straightforward to see that the relation

$$\boldsymbol{\beta}^{f,\hat{f}} = \big\{\big((a,b),(\hat{a},\hat{b})\big) \in (D \times D) \times (\hat{D} \times \hat{D}) \mid \big((f(a),b),\hat{f}(\hat{a}),\hat{b})\big) \in \boldsymbol{\beta}\big\}$$

is a σ-bisimulation between $\mathfrak{A}_{\mathfrak{M},f}$ and $\mathfrak{A}_{\hat{\mathfrak{M}},\hat{f}}$ in FO^2. $\dashv$

F Proofs for Section 5

Theorem 18 (i) *(S)CEP in* $\mathsf{Q}^1\mathsf{S5}$ *is undecidable.*

Proof. The proof is by reduction of the following undecidable tiling problem. By a *tiling system* we mean a tuple $\mathfrak{T} = (T, H, V, \boldsymbol{o}, \boldsymbol{z}^\uparrow, \boldsymbol{z}^\rightarrow)$, where T is a finite set of *tiles* with $\boldsymbol{o}, \boldsymbol{z}^\uparrow, \boldsymbol{z}^\rightarrow \in T$, and $H, V \subseteq T \times T$ are horizontal and vertical *matching relations*. We say that $\mathfrak{T}$ *has a solution* if there exists a triple (n, m, τ), where $0 < n, m < \omega$ and $\tau\colon \{0, \dots, n-1\} \times \{0, \dots, m-1\} \to T$, such that the following hold, for all $i < n$ and $j < m$:

(t1) if $i < n-1$ then $(\tau(i,j), \tau(i+1,j)) \in H$;

(t2) if $j < m-1$ then $(\tau(i,j), \tau(i,j+1)) \in V$;

(t3) $\tau(i,j) = \boldsymbol{o}$ iff $i = j = 0$;

(t4) $\tau(i,j) = \boldsymbol{z}^\rightarrow$ iff $i = n-1$, and $\tau(i,j) = \boldsymbol{z}^\uparrow$ iff $j = m-1$.

The reader can easily show by reduction of the halting problem for Turing machines that it is undecidable whether a given tiling system has a solution; cf. (van Emde Boas 1997).

For any tiling system $\mathfrak{T} = (T, H, V, \boldsymbol{o}, \boldsymbol{z}^\uparrow, \boldsymbol{z}^\rightarrow)$, we show how to construct in polytime formulas φ and ψ such that $\mathfrak{T}$ has a solution iff $\varphi \wedge \psi$ is not a (strong) conservative extension of φ. For any model $\mathfrak{M} = (W, D, I)$, we mark the points on the finite grid to be tiled by a predicate $\boldsymbol{g}$, that is, we let

$$\boldsymbol{g}^{\mathfrak{M}} = \{(w,d) \in W \times D \mid d \in \boldsymbol{g}^{I(w)}\}.$$

Then we define the intended 'horizontal' and 'vertical' neighbour relations $R_h^{\mathfrak{M}}$ and $R_v^{\mathfrak{M}}$ on the grid by setting

$$R_h^{\mathfrak{M}} = \big\{\big((w,d),(w',d')\big) \in \boldsymbol{g}^{\mathfrak{M}} \times \boldsymbol{g}^{\mathfrak{M}} \mid (w',d) \models \boldsymbol{x},\ (w,d) \models \neg\boldsymbol{z}^\rightarrow\big\}, \tag{47}$$

$$R_v^{\mathfrak{M}} = \big\{\big((w,d),(w',d')\big) \in \boldsymbol{g}^{\mathfrak{M}} \times \boldsymbol{g}^{\mathfrak{M}} \mid (w,d') \models \boldsymbol{y},\ (w,d) \models \neg\boldsymbol{z}^\uparrow\big\}. \tag{48}$$

We set, for any formula χ,

$$\Diamond_h\chi = \boldsymbol{g} \wedge \neg\boldsymbol{z}^\rightarrow \wedge \Diamond\big(\boldsymbol{x} \wedge \exists(\boldsymbol{g} \wedge \chi)\big),$$
$$\Diamond_v\chi = \boldsymbol{g} \wedge \neg\boldsymbol{z}^\uparrow \wedge \exists\big(\boldsymbol{y} \wedge \Diamond(\boldsymbol{g} \wedge \chi)\big),$$

and $\Box_h\chi = \neg\Diamond_h\neg\chi$, $\Box_v\chi = \neg\Diamond_v\neg\chi$. Now φ uses the following conjuncts to generate the grid:

$$\boldsymbol{o} \wedge \boldsymbol{g}$$
$$\Box\forall\big(\boldsymbol{g} \wedge \neg(\boldsymbol{z}^\uparrow \wedge \boldsymbol{z}^\rightarrow) \to \Diamond\boldsymbol{x}\big)$$
$$\Box\forall(\boldsymbol{x} \to \exists\boldsymbol{g})$$
$$\Box\forall(\boldsymbol{g} \wedge \neg\boldsymbol{z}^\uparrow \to \exists\boldsymbol{y}) \tag{49}$$
$$\Box\forall(\boldsymbol{y} \to \Diamond\boldsymbol{g}). \tag{50}$$

Next, we regard each tile $\boldsymbol{t} \in T$ as a fresh predicate, and we add the following conjuncts to φ, expressing the constraints for the tiles:

$$\Box\forall(\boldsymbol{g} \wedge \neg\Diamond_h \top \to \boldsymbol{z}^{\to}) \tag{51}$$

$$\Box\forall(\boldsymbol{g} \leftrightarrow \bigvee_{\boldsymbol{t} \in T} \boldsymbol{t}) \tag{52}$$

$$\Box\forall \bigwedge_{\boldsymbol{t} \neq \boldsymbol{t}'} (\boldsymbol{t} \to \neg\boldsymbol{t}') \tag{53}$$

$$\Box\forall\big(\boldsymbol{t} \to \Box_h \bigvee_{(\boldsymbol{t},\boldsymbol{t}') \in H} \boldsymbol{t}'\big) \tag{54}$$

$$\Box\forall\big(\boldsymbol{t} \to \Box_v \bigvee_{(\boldsymbol{t},\boldsymbol{t}') \in V} \boldsymbol{t}'\big) \tag{55}$$

$$\Box\forall(\boldsymbol{g} \to \neg\Diamond_h \boldsymbol{o} \wedge \neg\Diamond_v \boldsymbol{o}) \tag{56}$$

$$\Box\forall\big((\boldsymbol{z}^{\to} \to \Box_v \boldsymbol{z}^{\to}) \wedge (\Diamond_v \boldsymbol{z}^{\to} \to \boldsymbol{z}^{\to})\big) \tag{57}$$

$$\Box\forall\big((\boldsymbol{z}^{\uparrow} \to \Box_h \boldsymbol{z}^{\uparrow}) \wedge (\Diamond_h \boldsymbol{z}^{\uparrow} \to \boldsymbol{z}^{\uparrow})\big) \tag{58}$$

Let $\sigma = sig(\varphi) = \{\boldsymbol{g}, \boldsymbol{x}, \boldsymbol{y}\} \cup \{\boldsymbol{t} \mid \boldsymbol{t} \in T\}$.

Note that we have not yet forced $R_h^{\mathfrak{M}}, R_v^{\mathfrak{M}}$ to form a grid-like structure on $\boldsymbol{g}^{\mathfrak{M}}$-points. We say that a $\boldsymbol{g}^{\mathfrak{M}}$-point (w, d) is *confluent* if, for every $R_h^{\mathfrak{M}}$-successor (w_h, d_h) and every $R_v^{\mathfrak{M}}$-successor (w_v, d_v) of (w, d), there is (w', d') that is both an $R_v^{\mathfrak{M}}$-successor of (w_h, d_h) and an $R_h^{\mathfrak{M}}$-successor of (w_v, d_v). Forcing the grid to be finite and confluence of all grid-points are achieved using the formula ψ, which contains two additional predicates, $\boldsymbol{q}$ and $\boldsymbol{s}$, behaving like second-order variables over grid-points. We set

$$\psi = \boldsymbol{q} \wedge \Box\forall\big(\boldsymbol{q} \to \Diamond_h \boldsymbol{q} \vee \Diamond_v \boldsymbol{q} \vee (\Diamond_v \Box_h \boldsymbol{s} \wedge \Diamond_h \Box_v \neg\boldsymbol{s})\big).$$

It is readily seen that whenever $\mathfrak{M}, w_0, d_0 \models \varphi$, for some model $\mathfrak{M}$, then the following are equivalent:

(c1) $\mathfrak{M}', w_0, d_0 \models \psi$, for some model $\mathfrak{M}' = (W, D, I')$ with I' the same as I on all predicates save possibly $\boldsymbol{q}$ and $\boldsymbol{s}$ (we call such an $\mathfrak{M}'$ a *variant* of $\mathfrak{M}$);

(c2) $\mathfrak{M}$ contains an infinite $R_h^{\mathfrak{M}} \cup R_v^{\mathfrak{M}}$-path starting at (w_0, d_0) or a non-confluent $\boldsymbol{g}^{\mathfrak{M}}$-point accessible from (w_0, d_0) via an $R_h^{\mathfrak{M}} \cup R_v^{\mathfrak{M}}$-path.

Lemma 34. *If $\mathfrak{T}$ has a solution, then $\varphi \wedge \psi$ is not a conservative extension of φ.*

Proof. Suppose (n, m, τ) is a solution to $\mathfrak{T}$. We enumerate the points of the $n \times m$-grid starting from the first horizontal row $(0, 0), \dots (n-1, 0)$, then continuing with the second row $(0, 1), \dots, (n-1, 1)$, and so on. We define a model $\mathfrak{N} = (W, D, J)$ with $W = D = \{0, \dots, nm-1\}$ that represents this enumeration as follows (remember that $\boldsymbol{o}, \boldsymbol{z}^{\uparrow}, \boldsymbol{z}^{\to} \in T$, $\boldsymbol{z}^{\uparrow}$ marks the tiles of last row, and $\boldsymbol{z}^{\to}$ marks the tiles of the last column). For all $k < nm$ and $\boldsymbol{t} \in T$,

$$\boldsymbol{g}^{J(k)} = \{k\} \tag{59}$$

$$\boldsymbol{x}^{J(k)} = \begin{cases} \{k-1\}, & \text{if } k > 0, \\ \emptyset, & \text{otherwise}, \end{cases} \tag{60}$$

$$\boldsymbol{y}^{J(k)} = \begin{cases} \{k+n\}, & \text{if } k < nm - n, \\ \emptyset, & \text{otherwise}, \end{cases} \tag{61}$$

$$\boldsymbol{t}^{J(k)} = \begin{cases} \{k\}, & \text{if } k = jn + i \text{ and } \tau(i, j) = \boldsymbol{t}, \\ \emptyset, & \text{otherwise}. \end{cases} \tag{62}$$

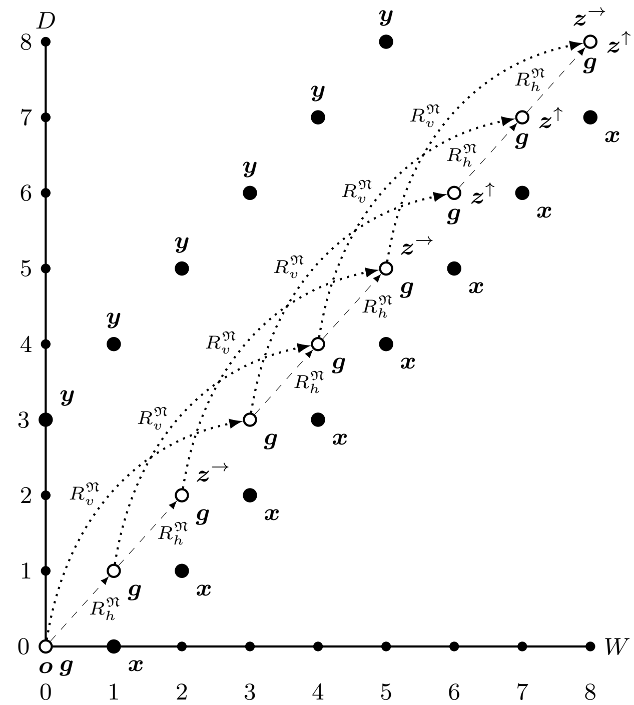


Figure 4: The model $\mathfrak{N}$.

For $n = m = 3$, the model $\mathfrak{N}$ (without tiles other than $\boldsymbol{o}, \boldsymbol{z}^{\to}, \boldsymbol{z}^{\uparrow}$) and the relations $R_h^{\mathfrak{N}}, R_v^{\mathfrak{N}}$ are illustrated in Fig. 4.

It is easy to check that $\mathfrak{N}, 0, 0 \models \varphi$ and $\mathfrak{N}', 0, 0 \models \neg\psi$, for any variant $\mathfrak{N}'$ of $\mathfrak{N}$.

We construct a formula χ with $sig(\chi) = \sigma$ such that $\models_{\mathsf{Q^1S5}} \psi \wedge \varphi \to \neg\chi$ but $\not\models_{\mathsf{Q^1S5}} \varphi \to \neg\chi$, which means that $\varphi \wedge \psi$ is not a (strong) conservative extension of φ. Intuitively, the formula χ characterises the model $\mathfrak{N}$ at $(0, 0)$. First, for every $(i, j) \in W \times D$, we construct a formula $\varphi_{i,j}$ such that, for all $(i', j') \in W \times D$,

$$\mathfrak{N}, i', j' \models \varphi_{i,j} \quad \text{iff} \quad (i', j') = (i, j). \tag{63}$$

For instance, we can set inductively

$$\begin{aligned} \varphi_{0,0} &= \boldsymbol{o}, \\ \varphi_{i+1,i+1} &= \boldsymbol{g} \wedge \exists(\boldsymbol{x} \wedge \Diamond\varphi_{i,i}), \\ \varphi_{i,j} &= \exists\varphi_{i,i} \wedge \Diamond\varphi_{j,j}, \text{ for } i \neq j. \end{aligned}$$

Now let

$$\chi_{i,j} = \varphi_{i,j} \wedge \bigwedge_{\boldsymbol{p} \in \sigma,\ \mathfrak{N}, i, j \models \boldsymbol{p}} \boldsymbol{p} \wedge \bigwedge_{\boldsymbol{p} \in \sigma,\ \mathfrak{N}, i, j \models \neg\boldsymbol{p}} \neg\boldsymbol{p}, \tag{64}$$

and let χ be the conjunction of

$$\chi_{0,0}, \tag{65}$$

$$\Box\forall(\chi_{i,i} \to \Box_h \chi_{i+1,i+1}), \text{ for } i < nm - 1, \tag{66}$$

$$\Box\forall(\chi_{i,i} \to \Box_v \chi_{i+n,i+n}), \text{ for } i < nm - n, \tag{67}$$

$$\Box\forall(\chi_{i,i} \to \Diamond\chi_{j,i}), \text{ for } i, j < nm, \tag{68}$$

$$\Box\forall(\chi_{i,i} \to \exists\chi_{i,j}), \text{ for } i, j < nm, \tag{69}$$

$$\Box\forall(\Diamond\chi_{l,i} \wedge \exists\chi_{i,j} \to \chi_{i,i}), \text{ for } i, j, l < nm. \tag{70}$$

Using (73), it is easy to see that $\mathfrak{N}, 0, 0 \models \chi$, and so $\varphi \wedge \chi$ is satisfiable, i.e., $\not\models_{\mathsf{Q^1S5}} \varphi \rightarrow \neg\chi$. Now suppose that $\mathfrak{M}$ is any model such that $\mathfrak{M}, w_0, d_0 \models \varphi \wedge \chi$ for some w_0, d_0. Using the equivalence **(c1)** $\Leftrightarrow$ **(c2)** above, we show that $\mathfrak{M}, w_0, d_0 \models \neg\psi$, which implies $\models_{\mathsf{Q^1S5}} \psi \wedge \varphi \rightarrow \neg\chi$.

To begin with, by (47), (48), the definition of $\mathfrak{N}$, and (64)–(67), there cannot exist an infinite $R_h^{\mathfrak{M}} \cup R_v^{\mathfrak{M}}$-chain. Now suppose there is an $R_h^{\mathfrak{M}} \cup R_v^{\mathfrak{M}}$-chain from (w_0, d_0) to some node (w, d) with an $R_h^{\mathfrak{M}}$-successor (w_1, d_1) and $R_v^{\mathfrak{M}}$-successor (w_2, d_2). Then $\mathfrak{M}, w, d \models \chi_{i,i}$ for some i, $\mathfrak{M}, w_1, d_1 \models \chi_{i+1,i+1}$ and $\mathfrak{M}, w_2, d_2 \models \chi_{i+n,i+n}$, by (65)–(67). By (68) and (69), there exist d_1' with $\mathfrak{M}, w_1, d_1' \models \chi_{i+1,i+n+1}$, and w_2' with $\mathfrak{M}, w_2', d_2 \models \chi_{i+n+1,i+n}$. By (70), $\mathfrak{M}, w_2', d_1' \models \chi_{i+n+1,i+n+1}$. Moreover, as by (64), $\boldsymbol{z}^{\uparrow}$ is not a conjunct of $\chi_{i+1,i+1}$, and $\boldsymbol{z}^{\rightarrow}$ is not a conjunct of $\chi_{i+n,i+n}$, we have that $\boldsymbol{y}$ is a conjunct of $\chi_{i+1,i+n+1}$, and $\boldsymbol{x}$ is a conjunct of $\chi_{i+n+1,i+n}$. Thus, by (47) and (48), $((w_1, d_1), (w_2', d_1')) \in R_v^{\mathfrak{M}}$ and $((w_2, d_2)(w_2', d_1')) \in R_h^{\mathfrak{M}}$, and so (w, d) is confluent. ⊣

We say that a formula α is a *model conservative extension* of a formula β if $\models_L \alpha \rightarrow \beta$ and, for any model $\mathfrak{M}, w, d$ with $\mathfrak{M}, w, d \models \beta$, there exists a model $\mathfrak{M}'$ with $\mathfrak{M}', w, d \models \alpha$, which coincides with $\mathfrak{M}$ except for the interpretation of the predicates in $sig(\alpha) \setminus sig(\beta)$. Clearly, if α is a model conservative extension of β, then α is also a strong conservative extension of β. Thus, if $\varphi \wedge \psi$ in our proof is not a conservative extension of φ, then $\varphi \wedge \psi$ is not a model conservative extension of φ.

Lemma 35. *If $\varphi \wedge \psi$ is not a conservative extension of φ, then $\mathfrak{T}$ has a solution.*

Proof. Consider a model $\mathfrak{M} = (W, D, I)$ such that $\mathfrak{M}, w, d \models \varphi$ but $\mathfrak{M}', w, d \models \neg\psi$ in any variant $\mathfrak{M}'$ of $\mathfrak{M}$. Using the equivalence **(c1)** $\Leftrightarrow$ **(c2)**, one can easily find within $\mathfrak{M}$ a finite grid-shaped (with respect to $R_h^{\mathfrak{M}}$ and $R_v^{\mathfrak{M}}$) submodel, which gives a solution to $\mathfrak{T}$.

For instance, we can start by taking an $R_h^{\mathfrak{M}}$-path of $\boldsymbol{g}^{\mathfrak{M}}$-points

$$(w, d) = (w_0^0, d_0^0) R_h^{\mathfrak{M}} (w_1^0, d_1^0) R_h^{\mathfrak{M}} \dots R_h^{\mathfrak{M}} (w_{n-1}^0, d_{n-1}^0),$$

for some $n > 0$, that ends with the first point (w_{n-1}^0, d_{n-1}^0) such that $\mathfrak{M}, w_{n-1}^0, d_{n-1}^0 \models \boldsymbol{z}^{\rightarrow}$. Such a path must exist by $\neg$**(c2)** and (51). The chosen points (w_i^0, d_i^0) form the first row of the required grid.

Next, observe that $\Box\forall(\boldsymbol{g} \wedge \neg\Diamond_v\top \rightarrow \boldsymbol{z}^{\uparrow})$ follows from (49) and (50). So, similarly to the above, by $\neg$**(c2)** we can take an $R_v^{\mathfrak{M}}$-path

$$(w_0^0, d_0^0) R_v^{\mathfrak{M}} (w_0^1, d_0^1) R_v^{\mathfrak{M}} \dots R_h^{\mathfrak{M}} (w_0^{m-1}, d_0^{m-1}),$$

for some $m > 0$, that ends with the first point (w_0^{m-1}, d_0^{m-1}) such that $\mathfrak{M}, w_0^{m-1}, d_0^{m-1} \models \boldsymbol{z}^{\uparrow}$. It forms the first column of the grid. By $\neg$**(c2)** again, the point (w_0^0, d_0^0) is confluent, and so we find (w_1^1, d_1^1) with

$$(w_1^0, d_1^0) R_v^{\mathfrak{M}} (w_1^1, d_1^1) \quad \text{and} \quad (w_0^1, d_0^1) R_h^{\mathfrak{M}} (w_1^1, d_1^1).$$

Similarly, we find the remaining $\boldsymbol{g}^{\mathfrak{M}}$-points (w_i^j, d_i^j) for the whole $n \times m$-grid. By (52) and (53), each (w_i^j, d_i^j) makes exactly one tile $\boldsymbol{t} \in T$ true. By (54) and (55), the matching conditions of **(t1)** and **(t2)** are satisfied. By (56), we have **(t3)**. Finally, **(t4)** is satisfied by (57) and (58). ⊣

This completes the proof of Theorem 18 (i). ⊣

Theorem 18 (ii) *UIEP in* $\mathsf{Q^1S5}$ *is undecidable.*

Proof. We prove the undecidability using a combination of the proof of Theorem 18 (i) and Example 19. Take $\mathfrak{T}$, φ, and ψ from the proof of Theorem 18 (i). Using fresh predicates $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{p}_1, \boldsymbol{p}_2$, and φ_0 from Example 19, set

$$\varphi' = \varphi \wedge (\boldsymbol{p}_1 \rightarrow \boldsymbol{p}_1) \wedge (\boldsymbol{p}_2 \rightarrow \boldsymbol{p}_2),$$
$$\psi' = (\psi \vee \boldsymbol{a}) \wedge \varphi_0.$$

Let $\sigma = sig(\varphi')$. We show that it is undecidable whether there exists a σ-uniform interpolant of $\varphi' \wedge \psi'$ in $\mathsf{Q^1S5}$.

Lemma 36. *If there is no σ-uniform interpolant of $\varphi' \wedge \psi'$ in* $\mathsf{Q^1S5}$*, then $\mathfrak{T}$ has a solution.*

Proof. As the assumption implies that $\varphi' \wedge \psi'$ is not a model conservative extension of φ', the proof is a straightforward variant of the proof of Lemma 35. Consider a model $\mathfrak{M} = (W, D, I)$ with $\mathfrak{M}, w, d \models \varphi'$ but $\mathfrak{M}', w, d \models \neg\psi'$ in any variant $\mathfrak{M}'$ of $\mathfrak{M}$ obtained by interpreting the predicates $\boldsymbol{q}, \boldsymbol{s}, \boldsymbol{a}, \boldsymbol{b}$. In particular, if $\boldsymbol{a}$ and $\boldsymbol{b}$ are both interpreted as $\emptyset$ in all worlds, then $\mathfrak{M}', w, d \models \varphi_0$, and so $\mathfrak{M}', w, d \models \neg\psi$ follows. But this case is considered in the proof of Lemma 35. ⊣

Lemma 37. *If $\mathfrak{T}$ has a solution, then there is no σ-uniform interpolant of $\varphi' \wedge \psi'$ in* $\mathsf{Q^1S5}$.

Proof. Assume that (n, m, τ) is a solution to $\mathfrak{T}$. Take the formulas χ and χ_s constructed in the proof of Lemma 34 and in Example 19, respectively. As it was shown, we have $\models_{\mathsf{Q^1S5}} \varphi \wedge \psi \rightarrow \neg\chi$ and $\models_{\mathsf{Q^1S5}} \boldsymbol{a} \wedge \varphi_0 \rightarrow \chi_s$ for all $s > 0$. Thus, $\models_{\mathsf{Q^1S5}} \varphi' \wedge \psi' \rightarrow (\chi \rightarrow \chi_s)$ for all $s > 0$. Therefore, if ϱ were a uniform interpolant of $\varphi' \wedge \psi'$, then

$$\models_{\mathsf{Q^1S5}} \varrho \rightarrow (\chi \rightarrow \Diamond\chi_s) \text{ for all } s > 0 \qquad (71)$$

would follow.

On the other hand, we combine $\mathfrak{N} = (W, D, J)$ from the proof of Lemma 34 with the models $\mathfrak{M}_s = (W_s, D_s, I_s)$ constructed in Example 19. For every $s \geq nm$, we define a model $\mathfrak{N}_s = (W_s, D_s, J_s)$ as follows. For every $k < s$, we let

$$\boldsymbol{p}^{J_s(k)} = \begin{cases} \boldsymbol{p}^{J(k)}, & \text{if } k < nm \text{ and } \boldsymbol{p} \in sig(\varphi), \\ \boldsymbol{p}^{I_s(k)}, & \text{if } \boldsymbol{p} \in \{\boldsymbol{a}, \boldsymbol{p}_1, \boldsymbol{p}_2\}, \\ \emptyset, & \text{otherwise.} \end{cases}$$

As shown in the proof of Lemma 34, $\mathfrak{N}, 0, 0 \models \varphi \wedge \chi$, so we have $\mathfrak{N}_s, 0, 0 \models \varphi' \wedge \chi$. As it was shown in Example 19 that $\mathfrak{M}_s, 0, 0 \not\models \chi_s$ for all $s > 0$, we have $\mathfrak{N}_s, 0, 0 \not\models \chi_s$ for all $s \geq nm$. So it follows from (71) that $\mathfrak{N}_s, 0, 0 \models \neg\varrho$. Note that also $\mathfrak{N}_s, 0, 0 \models \chi_{s'}$ for all $s > s' \geq nm$. Now consider the ultraproduct $\prod_U \mathfrak{N}_s$ with a non-principal ultrafilter U on $\omega \setminus \{0, \dots, nm-1\}$. As each $\chi_{s'}$ is true in almost all $\mathfrak{N}_s, 0, 0$, it follows from the properties of ultraproducts (Chang and Keisler 1998) that

$\prod_U \mathfrak{N}_s, \bar{0}, \bar{0} \models \boldsymbol{a} \wedge \neg\varrho \wedge \chi_{s'}$ for all $s' > 0$, for a suitable $\bar{0}$. But then one can interpret $\boldsymbol{b}$ in $\prod_U \mathfrak{N}_s$ so that $\mathfrak{N}, \bar{0}, \bar{0} \models \varphi_0$ for the resulting model $\mathfrak{N}$. Then $\mathfrak{N}, \bar{0}, \bar{0} \models \boldsymbol{a} \wedge \varphi_0 \wedge \varphi' \wedge \neg\varrho$ and so $\mathfrak{N}, \bar{0}, \bar{0} \models \varphi' \wedge \psi' \wedge \neg\varrho$. As $\models_{\mathsf{Q^1S5}} \varphi' \wedge \psi' \to \varrho$ should hold for a uniform interpolant ϱ of $\varphi' \wedge \psi'$, we have derived a contradiction. $\dashv$

This completes the proof of Theorem 18 (ii). $\dashv$

We give the example announced in Remark 20 showing that FO^2 does not have the UIP. Let $\sigma = \{A, R, P\}$ and

$$\begin{aligned}\varphi_0 = \forall x(A(x) \to \exists y(R(x,y) \wedge B(y) \wedge P(y)) \wedge \\ \forall x(B(x) \wedge P(x) \to \exists y(R(x,y) \wedge B(y) \wedge P(y))).\end{aligned}$$

Then $A(x) \wedge \varphi_0$ has no σ-uniform interpolant in FO^2, which can be shown similarly to the proof in Example 19.

G Proofs for Section 6

We start by providing polytime reductions of interpolant existence problems relative to ontologies to IEP. For an ontology $\mathcal{O}$, define a concept $\mathcal{O}^c$ by setting

$$\mathcal{O}^c = \bigwedge_{C \sqsubseteq D \in \mathcal{O}} \forall U.(\neg C \sqcup D)$$

Recall that IEP modulo ontologies is defined as follows. Given an ontology $\mathcal{O}$, a signature σ, and a CI $C \sqsubseteq D$, does there exist a σ-concept E such that $\mathcal{O} \models C \sqsubseteq E$ and $\mathcal{O} \models E \sqsubseteq D$? For the reduction, assume $\mathcal{O}$, a signature σ, and a CI $C \sqsubseteq D$ are given. Then the following conditions are equivalent:

– there exists a σ-interpolant for $\mathcal{O}^c \sqcap C \sqsubseteq \neg\mathcal{O}^c \sqcup D$;

– there exists a σ-concept E such that $\mathcal{O} \models C \sqsubseteq E$ and $\mathcal{O} \models E \sqsubseteq D$.

Next recall that ontology interpolant existence is defined as follows. Given an ontology $\mathcal{O}$, a signature σ, and a CI $C \sqsubseteq D$, is there an ontology $\mathcal{O}'$ with $sig(\mathcal{O}') \subseteq \sigma$, $\mathcal{O} \models \mathcal{O}'$, and $\mathcal{O}' \models C \sqsubseteq D$? For the reduction, assume $\mathcal{O}$, a signature σ, and a CI $C \sqsubseteq D$ are given. Then the following conditions are equivalent:

– there exists a σ-interpolant for $\mathcal{O}^c \sqsubseteq \neg C \sqcup D$;

– there is an ontology $\mathcal{O}'$ with $sig(\mathcal{O}') \subseteq \sigma$, $\mathcal{O} \models \mathcal{O}'$, and $\mathcal{O}' \models C \sqsubseteq D$.

Finally recall that DEP modulo ontologies is defined as follows. Given an ontology $\mathcal{O}$, a signature σ, and a concept name A, does there exist a σ-concept C such that $\mathcal{O} \models A \equiv C$? We reduce this problem to IEP modulo ontologies. Assume an ontology $\mathcal{O}$, a signature σ, and a concept name A are given. Then let $\mathcal{O}$ be the resulting ontology if all symbols X not in σ are replaced by fresh symbols X'. Then the following conditions are equivalent:

– there exists a σ-concept C such that $\mathcal{O} \cup \mathcal{O}' \models A \sqsubseteq C$ and $\mathcal{O} \cup \mathcal{O}' \models C \sqsubseteq A'$;

– there exists a σ-concept C such that $\mathcal{O} \models A \equiv C$.

We next analyse bisimulations for $\mathsf{S5}_{\mathcal{ALC}^u}$. We write $\mathfrak{M}_1, w_1, d_2 \sim_\sigma \mathfrak{M}_2, w_2, d_2$ to say that there is a σ-bisimulation $(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \boldsymbol{\beta})$ between $\mathfrak{M}_1$ and $\mathfrak{M}_2$ for which $((w_1, d_1), (w_2, d_2)) \in \boldsymbol{\beta}$; $\mathfrak{M}_1, w_1 \sim_\sigma \mathfrak{M}_2, w_2$ says that there is a σ-bisimulation $(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \boldsymbol{\beta})$ with $(w_1, w_2) \in \boldsymbol{\beta}_1$; and $\mathfrak{M}_1, d_1 \sim_\sigma \mathfrak{M}_2, d_2$ that there is $(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \boldsymbol{\beta})$ with $(d_1, d_2) \in \boldsymbol{\beta}_2$. The usage of $\mathfrak{M}_1, w_1, d_1 \equiv_\sigma \mathfrak{M}_2, w_2, d_2$, $\mathfrak{M}_1, w_1 \equiv_\sigma \mathfrak{M}_2, w_2$, and $\mathfrak{M}_1, w_1 \equiv_\sigma \mathfrak{M}_2, w_2$ is as in Section 2 but for any σ-concepts and those of the form $\exists U.C$ and $\Diamond C$, respectively.

Lemma 38. *For any saturated $\mathsf{S5}_{\mathcal{ALC}^u}$-models $\mathfrak{M}_1$ with w_1, d_1 and $\mathfrak{M}_2$ with w_2, d_2, we have*

– $\mathfrak{M}_1, w_1, d_1 \equiv_\sigma \mathfrak{M}_2, w_2, d_2$ *iff* $\mathfrak{M}_1, w_1, d_1 \sim_\sigma \mathfrak{M}_2, w_2, d_2$,

– $\mathfrak{M}_1, w_1 \equiv_\sigma \mathfrak{M}_2, w_2$ *iff* $\mathfrak{M}_1, w_1 \sim_\sigma \mathfrak{M}_2, w_2$,

– $\mathfrak{M}_1, d_1 \equiv_\sigma \mathfrak{M}_2, d_2$ *iff* $\mathfrak{M}_1, d_1 \sim_\sigma \mathfrak{M}_2 d_2$.

The implication $(\Rightarrow)$ holds for arbitrary models.

Proof. The proof of $(\Leftarrow)$ is by a straightforward induction on the construction of concepts C, using **(a)** for concept names, $(\mathbf{r})$ for $\exists R.C$, **(c)** and **(w)** for $\exists U.C$, and **(c)** and **(d)** for $\Diamond C$.

For $(\Rightarrow)$, we define $\boldsymbol{\beta}_1$, $\boldsymbol{\beta}_2$, and $\boldsymbol{\beta}$ via $\equiv_\sigma$ in the obvious way. Then we observe that $\mathfrak{M}_1, w_1, d_1 \equiv_\sigma \mathfrak{M}_2, w_2, d_2$ implies $\mathfrak{M}_1, w_1 \sim_\sigma \mathfrak{M}_2, w_2$ and $\mathfrak{M}_1, d_1 \equiv_\sigma \mathfrak{M}_2, d_2$. Hence we obtain **(c)**. We obtain **(w)** and **(d)** using saturatedness; **(a)** is trivial and **(r)** follows again from saturatedness. $\dashv$

We now extend the 'filtration' construction of Section 4 from $\mathsf{Q^1S5}$ to $\mathsf{S5}_{\mathcal{ALC}^u}$ to establish an upper bound on the size of models witnessing bisimulation consistency.

Theorem 23 *Any concepts C and D do not have an interpolant in $\mathsf{S5}_{\mathcal{ALC}^u}$ iff there are witnessing $\mathsf{S5}_{\mathcal{ALC}^u}$-models of size double-exponential in $|C|$ and $|D|$.*

Proof. Given concepts C and D, we define the sets $sub(C, D)$, $sub_\Diamond(C, D)$, and $sub_\exists(C, D)$ as in Section 4 regarding $\exists U$ as the $\mathsf{S5}_{\mathcal{ALC}^u}$-counterpart of $\exists$ in $\mathsf{Q^1S5}$. The *world-type* $\mathsf{wt}_{\mathfrak{M}}(w)$ of $w \in W$ in $\mathfrak{M} = (W, \Delta, I)$, the *domain-type* $\mathsf{dt}_{\mathfrak{M}}(d)$ of $d \in \Delta$, and the *full type* $\mathsf{ft}_{\mathfrak{M}}(w, d)$ of (w, d) in $\mathfrak{M}$ are also defined as in Section 4. Observe that we have $\mathsf{wt}_{\mathfrak{M}}(w) = \mathsf{ft}_{\mathfrak{M}}(w, d)^{\mathsf{wt}}$ and $\mathsf{dt}_{\mathfrak{M}}(d) = \mathsf{ft}_{\mathfrak{M}}(w, d)^{\mathsf{dt}}$, where

$$\begin{aligned}\mathsf{ft}_{\mathfrak{M}}(w, d)^{\mathsf{wt}} &= sub_\exists(C, D) \cap \mathsf{ft}_{\mathfrak{M}}(w, d),\\ \mathsf{ft}_{\mathfrak{M}}(w, d)^{\mathsf{dt}} &= sub_\Diamond(C, D) \cap \mathsf{ft}_{\mathfrak{M}}(w, d).\end{aligned}$$

Now, suppose that $\mathfrak{M}_i = (W_i, \Delta_i, I_i)$, for $i = 1, 2$, have pairwise disjoint W_i and Δ_i, $\mathfrak{M}_1, w_1, d_1 \sim_\sigma \mathfrak{M}_2, w_2, d_2$, for $\sigma = sig(C) \cap sig(D)$, with $\mathfrak{M}_1, w_1, d_1 \models C$, and $\mathfrak{M}_2, w_2, d_2 \not\models D$. For $w \in W_1 \cup W_2$, we define the *world mosaic* $\mathsf{wm}(w) = (T_1(w), T_2(w))$ by (1) and the *i-world point* $\mathsf{wp}_i(w) = (\mathsf{wt}_{\mathfrak{M}_i}(w), \mathsf{wm}(w))$ of w in $\mathfrak{M}_1$, $\mathfrak{M}_2$. Using (2) with Δ_i in place of D_i, we define the *domain mosaic* $\mathsf{dm}(d) = (S_1(d), S_2(d))$ and *i-domain point* $\mathsf{dp}_i(d) = (\mathsf{dt}_{\mathfrak{M}_i}(d), \mathsf{dm}(d))$ of $d \in \Delta_1 \cup \Delta_2$ in $\mathfrak{M}_1$, $\mathfrak{M}_2$. Then, for $(w, d) \in (W_1 \times \Delta_1) \cup (W_2 \times \Delta_2)$, we set

$$F_i(w, d) = \{\mathsf{ft}_{\mathfrak{M}_i}(v, e) \mid (v, e) \in W_i \times \Delta_i, (v, e) \sim_\sigma (w, d)\}$$

calling $\mathsf{fm}(w,d) = (F_1(w,d), F_2(w,d))$ the *full mosaic* and $\mathsf{fp}_i(w,d) = (\mathsf{ft}_{\mathfrak{M}_i}(w,d), \mathsf{fm}(w,d))$ the *i-full point* of (w,d) in $\mathfrak{M}_1, \mathfrak{M}_2$. Given $\mathsf{fm} = (F_1, F_2)$, we set

$$\mathsf{fm}^{\mathsf{wt}} = (\{\mathsf{ft}^{\mathsf{wt}} \mid \mathsf{ft} \in F_1\}, \{\mathsf{ft}^{\mathsf{wt}} \mid \mathsf{ft} \in F_2\}),$$
$$\mathsf{fm}^{\mathsf{dt}} = (\{\mathsf{ft}^{\mathsf{dt}} \mid \mathsf{ft} \in F_1\}, \{\mathsf{ft}^{\mathsf{dt}} \mid \mathsf{ft} \in F_2\}).$$

Lemma 39. *Suppose* $\mathsf{fm} = \mathsf{fm}(w,d)$*,* $\mathsf{wm} = \mathsf{wm}(w)$*, and* $\mathsf{dm} = \mathsf{dm}(d)$*. Then* $\mathsf{fm}^{\mathsf{wt}} = \mathsf{wm}$ *and* $\mathsf{fm}^{\mathsf{dt}} = \mathsf{dm}$.

Proof. The inclusions $\mathsf{wm} \subseteq \mathsf{fm}^{\mathsf{wt}}$ and $\mathsf{dm} \subseteq \mathsf{fm}^{\mathsf{dt}}$ follow from **(w)** and **(d)**. Indeed, suppose $\mathsf{wm} = (T_1(w), T_2(w))$, $\mathsf{wt} \in T_i(w)$, and $\mathsf{fm} = (F_1(w,d), F_2(w,d))$. By definition, there exists $v \in W_i$ with $v \sim_\sigma w$ such that $\mathsf{wt} = \mathsf{wt}_{\mathfrak{M}_i}(v)$. By **(w)**, there exists $e \in \Delta_i$ with $(w,d) \sim_\sigma (v,e)$. But then $\mathsf{ft}_{\mathfrak{M}_i}(v,e) \in F_i(w,d)$ and $\mathsf{ft}_{\mathfrak{M}_i}(v,e)^{\mathsf{wt}} = \mathsf{wt}$, as required. The second claim is proved in the same way using **(d)**.

The converse inclusions $\mathsf{fm}^{\mathsf{wt}} \subseteq \mathsf{wm}$ and $\mathsf{fm}^{\mathsf{dt}} \subseteq \mathsf{dm}$ follow from **(c)**. To see this, let $\mathsf{fm} = (F_1(w,d), F_2(w,d))$, $\mathsf{wm} = (T_1(w), T_2(w))$, and $\mathsf{dm} = (S_1(d), S_2(d))$. Let $\mathsf{ft} \in F_i(w,d)$. Then there are $(v,e) \in W_i \times \Delta_i$ with $(v,e) \sim_\sigma (w,d)$. Therefore, by **(c)**, $v \sim_\sigma w$ and $e \sim_\sigma d$, and so $\mathsf{wt} \in T_i(w)$ and $\mathsf{dt} \in S_i(d)$, as required. ⊣

As in Section 4, we construct models $\mathfrak{M}_1' = (W_1', \Delta_1', I_1')$ and $\mathfrak{M}_2' = (W_2', \Delta_2', I_2')$ from copies of i-world points wp_i and i-domain points dp_i in $\mathfrak{M}_1, \mathfrak{M}_2$. When defining them, we use the following notation:

$$C^{\mathfrak{M}_i'} = \{(w,d) \mid d \in C^{I_i'(w)}\},$$
$$R^{\mathfrak{M}_i'} = \{((w,d),(w,d')) \mid (d,d') \models R^{I_i'(w)}\}.$$

Let $n = m_1 \times m_2$, where m_1 and m_2 are the number of full types and, respectively, full mosaics over $sub(C,D)$ in $\mathfrak{M}_1$, $\mathfrak{M}_2$. For $i = 1,2$, we set

$$\Delta_i' = \{\mathsf{dp}_i^k \mid \mathsf{dp}_i \text{ an } i\text{-domain point in } \mathfrak{M}_1, \mathfrak{M}_2, k \in [n]\}.$$

For an i-world point wp_i and an i-domain point dp_i, let

$$L^i_{\mathsf{wp}_i,\mathsf{dp}_i} = \{\mathsf{fp}_i(w,d) \mid \mathsf{wp}_i = \mathsf{wp}_i(w),\ \mathsf{dp}_i = \mathsf{dp}_i(d), \\ (w,d) \in W_i \times \Delta_i\}.$$

We define W_i' using surjective functions

$$\pi_{\mathsf{wp}_i,\mathsf{dp}_i} \colon [n] \to L^i_{\mathsf{wp}_i,\mathsf{dp}_i}.$$

Let Π be a smallest set of sequences $\pi = (\pi_{\mathsf{wp}_i,\mathsf{dp}_i})$ such that, for any $\mathsf{fp}_i = \mathsf{fp}_i(w,d) = (\mathsf{ft}_{\mathfrak{M}_i}(w,d), \mathsf{fm}(w,d))$, $\mathsf{wp}_i = \mathsf{wp}_i(w)$, and $\mathsf{dp}_i = \mathsf{dp}_i(d)$ with $(w,d) \in \mathfrak{M}_i$ and any $k \in [n]$, there is $\pi \in \Pi$ with $\pi_{\mathsf{wp}_i,\mathsf{dp}_i}(k) = \mathsf{fp}_i$. Now let

$$W_i' = \{\mathsf{wp}_i^\pi \mid \mathsf{wp}_i \text{ an } i\text{-world point in } \mathfrak{M}_1, \mathfrak{M}_2 \text{ and } \pi \in \Pi\}.$$

Note that $|\Delta_i'|$ and $|W_i'|$ are double-exponential in $|C|$, $|D|$. It remains to define the extensions of concept and role names in $\mathfrak{M}_1'$ and $\mathfrak{M}_2'$. For the former, we set

$$(\mathsf{wp}_i^\pi, \mathsf{dp}_i^k) \in A^{\mathfrak{M}_i'} \text{ iff } A \in \mathsf{ft} \text{ for } \pi_{\mathsf{wp}_i,\mathsf{dp}_i}(k) = (\mathsf{ft}, \mathsf{fm}).$$

The definition of $R^{\mathfrak{M}_i'}$ is more involved. Call a pair $\mathsf{ft}_1, \mathsf{ft}_2$ *R-coherent* if $\exists R.C \in \mathsf{ft}_1$ whenever $C \in \mathsf{ft}_2$, for all $\exists R.E \in sub(C,D)$, and call $\mathsf{ft}_1, \mathsf{ft}_2$ *R-witnessing* if they are R-coherent and $\mathsf{ft}_1^{\mathsf{wt}} = \mathsf{ft}_2^{\mathsf{wt}}$. For full mosaics $\mathsf{fm} = (F_1, F_2)$ and $\mathsf{fm}' = (F_1', F_2')$ and a role $R \in \sigma$, we set $\mathsf{fm} \preceq_R \mathsf{fm}'$ if there exist functions $f_i \colon F_i \to F_i'$, $i = 1,2$, such that, for all $\mathsf{ft} \in F_i$, the pair $\mathsf{ft}, f_i(\mathsf{ft})$ is R-witnessing. Now suppose that $\pi_{\mathsf{wp}_i,\mathsf{dp}_i}(k) = (\mathsf{ft}, \mathsf{fm})$ and $\pi_{\mathsf{wp}_i,\mathsf{dp}_i'}(k') = (\mathsf{ft}', \mathsf{fm}')$. For $R \in \sigma$, we set

$$((\mathsf{wp}_i^\pi, \mathsf{dp}_i^k), (\mathsf{wp}_i^{pi}, \mathsf{dp}_i^{k'})) \in R^{\mathfrak{M}_i'} \text{ iff} \\ \mathsf{fm} \preceq_R \mathsf{fm}' \text{ and } \mathsf{ft}, \mathsf{ft}' \text{ is } R\text{-witnessing}. \quad (72)$$

For $R \notin \sigma$, we omit the condition $\mathsf{fm} \preceq_R \mathsf{fm}'$ from (72).

Lemma 40. *Let* $E \in sub(C,D)$*. Then* $(\mathsf{wp}_i^\pi, \mathsf{dp}_i^k) \in E^{\mathfrak{M}_i'}$ *iff* $E \in \mathsf{ft}$*, for* $\pi_{\mathsf{wp}_i,\mathsf{dp}_i}(k) = (\mathsf{ft}, \mathsf{fm})$.

Proof. The proof is by induction on the construction of E. The basis of induction follows from the definition, and the inductive step for the Booleans is trivial.

Let $E = \exists U.E'$. Suppose first $(\mathsf{wp}_i^\pi, \mathsf{dp}_i^k) \in E^{\mathfrak{M}_i'}$ and $\pi_{\mathsf{wp}_i,\mathsf{dp}_i}(k) = (\mathsf{ft}, \mathsf{fm})$. By definition, there exists ${\mathsf{dp}_i'}^{k'}$ with $(\mathsf{wp}_i^\pi, {\mathsf{dp}_i'}^{k'}) \in E'^{\mathfrak{M}_i'}$. By IH, $E' \in \mathsf{ft}'$ for $\pi_{\mathsf{wp}_i,\mathsf{dp}_i'}(k') = (\mathsf{ft}', \mathsf{fm}')$. Then $\exists U.E' \in \mathsf{ft}'$, and so $\exists U.E' \in \mathsf{ft}'^{\mathsf{wt}}$. As $\mathsf{ft}^{\mathsf{wt}} = \mathsf{ft}'^{\mathsf{wt}}$, we obtain $E \in \mathsf{ft}$, as required. Conversely, let $E = \exists U.E' \in \mathsf{ft}$, for $\pi_{\mathsf{wp}_i,\mathsf{dp}_i}(k) = (\mathsf{ft}, \mathsf{fm})$. Take (w,d) with $\mathsf{ft} = \mathsf{ft}(w,d)$ and $\mathsf{fm} = \mathsf{fm}(w,d)$. By definition, there is d' with $E' \in \mathsf{ft}'$ for $\mathsf{ft}' = \mathsf{ft}(w,d')$. Let $\mathsf{dp}_i' = \mathsf{dp}_i(d')$ and $\mathsf{fm}' = \mathsf{fm}(w,d')$. As $\pi_{\mathsf{wp}_i,\mathsf{dp}_i'}$ is surjective, there is k' with $\pi_{\mathsf{wp}_i,\mathsf{dp}_i'}(k') = (\mathsf{ft}', \mathsf{fm}')$. Then, by IH, we obtain $(\mathsf{wp}_i^\pi, \mathsf{dp}_i'^{k'}) \in E'^{\mathfrak{M}_i'}$, and so $(\mathsf{wp}_i^\pi, \mathsf{dp}_i^k) \in E^{\mathfrak{M}_i'}$.

Let $E = \Diamond E'$. Suppose first $(\mathsf{wp}_i^\pi, \mathsf{dp}_i^k) \in E^{\mathfrak{M}_i'}$. Let $\pi_{\mathsf{wp}_i,\mathsf{dp}_i}(k) = (\mathsf{ft}, \mathsf{fm})$ and $\mathsf{dp}_i = (\mathsf{dt}, \mathsf{dm})$. By definition, there is $\mathsf{wp}_i'^{\pi'}$ with $(\mathsf{wp}_i'^{\pi'}, \mathsf{dp}_i^k) \in E'^{\mathfrak{M}_i'}$. By IH, $E' \in \mathsf{ft}'$ for $\pi'_{\mathsf{wp}_i',\mathsf{dp}_i}(k) = (\mathsf{ft}', \mathsf{fm}')$. By definition, $\Diamond E' \in \mathsf{ft}'$, and so $\Diamond E' \in \mathsf{ft}'^{\mathsf{dt}} = \mathsf{dt}$. But then, again by definition, $E \in \mathsf{ft}$, as required.

Conversely, let $E = \Diamond E' \in \mathsf{ft}$ for $\pi_{\mathsf{wp}_i,\mathsf{dp}_i}(k) = (\mathsf{ft}, \mathsf{fm})$. Take (w,d) with $\mathsf{ft} = \mathsf{ft}(w,d)$ and $\mathsf{fm} = \mathsf{fm}(w,d)$. By definition, there is w' with $E' \in \mathsf{ft}'$ for $\mathsf{ft}' = \mathsf{ft}(w',d)$. Let $\mathsf{wp}_i' = \mathsf{wp}_i(w')$ and $\mathsf{fm}' = \mathsf{fm}(w',d)$. Then there is π' with $\pi'_{\mathsf{wp}_i',\mathsf{dp}_i}(k) = (\mathsf{ft}', \mathsf{fm}')$. By IH, $(\mathsf{wp}_i'^{\pi'}, \mathsf{dp}_i^k) \in E'^{\mathfrak{M}_i'}$, and so $(\mathsf{wp}_i^\pi, \mathsf{dp}_i^k) \in E^{\mathfrak{M}_i'}$.

Let $E = \exists R.E'$. Suppose that $(\mathsf{wp}_i^\pi, \mathsf{dp}_i^k) \in E^{\mathfrak{M}_i'}$ and $R \in \sigma$. Let $\pi_{\mathsf{wp}_i,\mathsf{dp}_i}(k) = (\mathsf{ft}, \mathsf{fm})$. By definition, there exists $\mathsf{dp}_i'^{k'}$ with $((\mathsf{wp}_i^\pi, \mathsf{dp}_i^k)(\mathsf{wp}_i^\pi, \mathsf{dp}_i'^{k'})) \in R^{\mathfrak{M}_i'}$ and $(\mathsf{wp}_i^\pi, \mathsf{dp}_i'^{k'}) \in E'^{\mathfrak{M}_i'}$. By IH, $E' \in \mathsf{ft}'$ for $\pi_{\mathsf{wp}_i,\mathsf{dp}_i'}(k') = (\mathsf{ft}', \mathsf{fm}')$. By the definition of $R^{\mathfrak{M}_i'}$, we obtain that $\mathsf{ft}, \mathsf{ft}'$ are R-coherent. But then $E \in \mathsf{ft}$, as required. The case $(\mathsf{wp}_i^\pi, \mathsf{dp}_i^k) \in E^{\mathfrak{M}_i'}$ and $R \notin \sigma$ is similar.

Conversely, let $E = \exists R.E' \in \mathsf{ft}$ for $\pi_{\mathsf{wp}_i,\mathsf{dp}_i}(k) = (\mathsf{ft}, \mathsf{fm})$. Take (w,d) with $\mathsf{ft} = \mathsf{ft}(w,d)$ and $\mathsf{fm} = \mathsf{fm}(w,d)$. By definition, there exists e with $E' \in \mathsf{ft}'$ for $\mathsf{ft}' = \mathsf{ft}(w,e)$ and $((w,d),(w,e)) \in R^{\mathfrak{M}_i}$. Let $\mathsf{dp}_i' = \mathsf{dp}_i(e)$ and $\mathsf{fm}' = \mathsf{fm}(w,e)$. Assume first that $R \in \sigma$. We define functions f_j, $j = 1,2$ witnessing $\mathsf{fm} \preceq_R \mathsf{fm}'$.

Suppose $\mathsf{ft} \in F_j$, where $\mathsf{fm} = (F_1, F_2)$. Then we find $(w', d') \sim_\sigma (w, d)$ with $\mathsf{ft} = \mathsf{ft}(w', d')$. By **(r)**, $(w, d), (w, e) \in R^{\mathfrak{M}_i'}$ and $(w', d') \sim_\sigma (w, d)$ give us e' with $(w', d'), (w', e') \in R^{\mathfrak{M}_i'}$ and $(w, e) \sim_\sigma (w', e')$. Let $\mathsf{ft}' = \mathsf{ft}(w', e')$. We define the required f_j by taking $f_j(\mathsf{ft}) = (\mathsf{ft}')$.

Since $\pi_{\mathsf{wp}_i, \mathsf{dp}_i'}$ is surjective, there exists k' such that $\pi_{\mathsf{wp}_i, \mathsf{dp}_i'}(k') = (\mathsf{ft}', \mathsf{fm}')$. By IH, $(\mathsf{wp}_i^\pi, \mathsf{dp}_i'^{k'}) \in E'^{\mathfrak{M}_i'}$. By the definition of $R^{\mathfrak{M}_i'}$, $((\mathsf{wp}_i^\pi, \mathsf{dp}_i^k), (\mathsf{wp}_i^\pi, \mathsf{dp}_i'^{k'})) \in R^{\mathfrak{M}_i'}$, so $(\mathsf{wp}_i^\pi, \mathsf{dp}_i^k) \in E^{\mathfrak{M}_i'}$. The case $R \notin \sigma$ is similar. $\dashv$

We define relations $\boldsymbol{\beta}_1 \subseteq W_1' \times W_2'$, $\boldsymbol{\beta}_2 \subseteq \Delta_1' \times \Delta_2'$, and $\boldsymbol{\beta} \subseteq (W_1' \times \Delta_1') \times (W_2' \times \Delta_2')$ by taking

– $((\mathsf{wt}, \mathsf{wm})^\pi, (\mathsf{wt}', \mathsf{wm}')^{\pi'}) \in \boldsymbol{\beta}_1$ iff $\mathsf{wm} = \mathsf{wm}'$;

– $((\mathsf{dt}, \mathsf{dm})^k, (\mathsf{dt}', \mathsf{dm}')^{k'}) \in \boldsymbol{\beta}_2$ iff $\mathsf{dm} = \mathsf{dm}'$;

– $((\mathsf{wp}_i^\pi, \mathsf{dp}_i^k), (\mathsf{wp}_i'^{\pi'}, \mathsf{dp}_i'^{k'})) \in \boldsymbol{\beta}$ iff $\mathsf{wp}_i = (\mathsf{wt}, \mathsf{wm})$, $\mathsf{wp}_i' = (\mathsf{wt}', \mathsf{wm}')$, $\mathsf{dp}_i = (\mathsf{dt}, \mathsf{dm})$, and $\mathsf{dp}_i' = (\mathsf{dt}', \mathsf{dm}')$ with $\mathsf{wm} = \mathsf{wm}'$, $\mathsf{dm} = \mathsf{dm}'$, $\mathsf{fm} = \mathsf{fm}'$, and $\pi_{\mathsf{wp}_i, \mathsf{dp}_i}(k) = (\mathsf{ft}, \mathsf{fm})$ and $\pi'_{\mathsf{wp}_i', \mathsf{dp}_i'}(k') = (\mathsf{ft}', \mathsf{fm}')$.

Lemma 41. *The triple $(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \boldsymbol{\beta})$ is a σ-bisimulation between $\mathfrak{M}_1'$ and $\mathfrak{M}_2'$.*

Proof. We show that $(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \boldsymbol{\beta})$ satisfies conditions **(w)**, **(d)**, **(c**, **(r)**, and **(r)**.

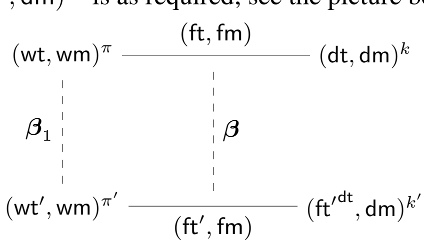
(w) Suppose $((\mathsf{wt}, \mathsf{wm})^\pi, (\mathsf{wt}', \mathsf{wm}')^{\pi'}) \in \boldsymbol{\beta}_1$ and $(\mathsf{dt}, \mathsf{dm})^k \in \Delta_i'$. We need to find $(\mathsf{dt}', \mathsf{dm}')^{k'}$ such that

$$((\mathsf{wt}, \mathsf{wm})^\pi, (\mathsf{dt}, \mathsf{dm})^k), ((\mathsf{wt}', \mathsf{wm}')^{\pi'}, (\mathsf{dt}', \mathsf{dm}')^{k'})) \in \boldsymbol{\beta}.$$

We have $\mathsf{wm} = \mathsf{wm}'$ and set $(\mathsf{ft}, \mathsf{fm}) = \pi_{\mathsf{wt}, \mathsf{wm}, \mathsf{dt}, \mathsf{dm}}(k)$. Assume $\mathsf{fm} = (F_1, F_2)$. By Lemma 39, there exists $\mathsf{ft}' \in F_i$ with $\mathsf{ft}'^{\mathsf{wt}} = \mathsf{wt}'$. As the component $\pi'_{\mathsf{wt}', \mathsf{wm}, \mathsf{ft}'^{\mathsf{dt}}, \mathsf{dm}}$ of π' is surjective, there exists $k' \in [n]$ such that

$$\pi'_{\mathsf{wt}', \mathsf{wm}, \mathsf{ft}'^{\mathsf{dt}}, \mathsf{dm}}(k') = (\mathsf{ft}', \mathsf{fm}).$$

Then $(\mathsf{ft}'^{\mathsf{dt}}, \mathsf{dm})^{k'}$ is as required; see the picture below.



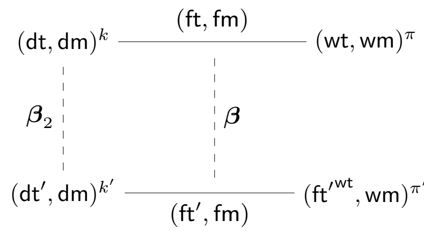
(d) Suppose that $((\mathsf{dt}, \mathsf{dm})^k, (\mathsf{dt}', \mathsf{dm}')^{k'}) \in \boldsymbol{\beta}_2$ and $(\mathsf{wt}, \mathsf{wm})^\pi \in W_i'$. We need to construct $(\mathsf{wt}', \mathsf{wm}')^{\pi'}$ with

$$((\mathsf{wt}, \mathsf{wm})^\pi, (\mathsf{dt}, \mathsf{dm})^k), ((\mathsf{wt}', \mathsf{wm}')^{\pi'}, (\mathsf{dt}', \mathsf{dm}')^{k'})) \in \boldsymbol{\beta}.$$

We have $\mathsf{dm} = \mathsf{dm}'$ and set $(\mathsf{ft}, \mathsf{fm}) = \pi_{\mathsf{wt}, \mathsf{wm}, \mathsf{dt}, \mathsf{dm}}(k)$. Assume $\mathsf{fm} = (F_1, F_2)$. By Lemma 39, there exists $\mathsf{ft}' \in F_i$ with $\mathsf{ft}'^{\mathsf{dt}} = \mathsf{dt}'$. By the construction of Π, there exists π' such that, for the component $\pi'_{\mathsf{wt}'^{\mathsf{wt}}, \mathsf{wm}, \mathsf{ft}', \mathsf{dm}}$ of π', we have

$$\pi'_{\mathsf{wt}'^{\mathsf{wt}}, \mathsf{wm}, \mathsf{ft}', \mathsf{dm}}(k') = (\mathsf{ft}', \mathsf{fm}).$$

Then $(\mathsf{ft}'^{\mathsf{wt}}, \mathsf{wm})^{\pi'}$ is as required; see the picture below.



Condition **(c)** follows from the definition of $\boldsymbol{\beta}_1$, $\boldsymbol{\beta}_2$, $\boldsymbol{\beta}$; and condition **(a)** follows from the definition of $\boldsymbol{\beta}$.

(r) Suppose $R \in \sigma$, $\boldsymbol{\beta}$ contains

$$(((\mathsf{wt}_0, \mathsf{wm})^{\pi^0}, (\mathsf{dt}_0, \mathsf{dm})^{k_0}), ((\mathsf{wt}_1, \mathsf{wm})^{\pi^1}, (\mathsf{dt}_1, \mathsf{dm})^{k_1})))$$

and $R^{\mathfrak{M}_i}$ contains

$$((\mathsf{wt}_0, \mathsf{wm})^{\pi^0}, (\mathsf{dt}_0, \mathsf{dm})^{k_0}), ((\mathsf{wt}_0, \mathsf{wm})^{\pi^0}, (\mathsf{dt}_2, \mathsf{dm}_2)^{k_2})).$$

Let $(\mathsf{ft}, \mathsf{fm}) = \pi^0_{\mathsf{wt}_0, \mathsf{wm}, \mathsf{dt}_0, \mathsf{dm}}(k_0)$. Then there exists ft_1 such that $(\mathsf{ft}_1, \mathsf{fm}) = \pi^1_{\mathsf{wt}_1, \mathsf{wm}, \mathsf{dt}_1, \mathsf{dm}}(k_1)$. Moreover, for $(\mathsf{ft}_2, \mathsf{fm}_2) = \pi^0_{\mathsf{wt}_0, \mathsf{wm}, \mathsf{dt}_2, \mathsf{dm}_2}(k_2)$, we have

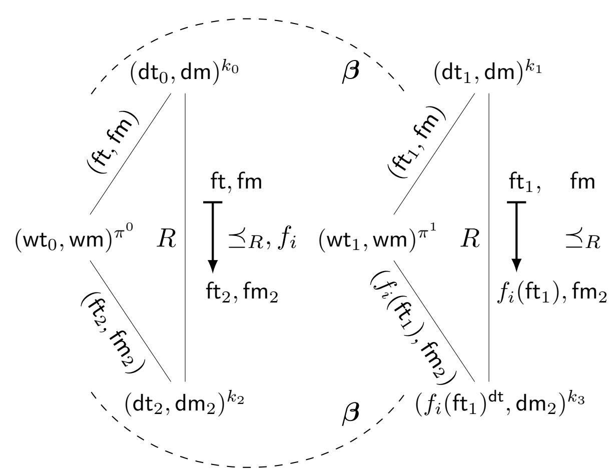
– $\mathsf{fm} \preceq_R \mathsf{fm}_2$ and

– we may assume that $f_i(\mathsf{ft}) = \mathsf{ft}_2$, for the function f_i witnessing $\mathsf{fm} \preceq_R \mathsf{fm}_2$.

Then $(f_i(\mathsf{ft}_1)^{\mathsf{dt}}, \mathsf{dm}_2)^{k_3}$ with $k_3 \in [n]$ such that

$$\pi^1_{\mathsf{wt}_1, \mathsf{wm}, f_i(\mathsf{ft}_1)^{\mathsf{dt}}, \mathsf{dm}_2}(k_3) = (f_i(\mathsf{ft}_1), \mathsf{fm}_2)$$

is as required; see the picture below.



This completes the proof of the lemma. $\dashv$

Theorem 23 follows. $\dashv$

H Proofs for Section 7

We now define inductively formulas that describe models up to σ-k-bisimulations. For $\mathfrak{M} = (W, R, D, I)$ and σ, let

$$t^0_{\mathfrak{M},\sigma}(w,d) = \bigwedge\{\boldsymbol{p} \in \sigma \mid \mathfrak{M}, w, d \models \boldsymbol{p}\} \wedge \bigwedge\{\neg\boldsymbol{p} \mid \boldsymbol{p} \in \sigma,\ \mathfrak{M}, w, d \not\models \boldsymbol{p}\},$$

$$\tau^0_{\mathfrak{M},\sigma}(w,d) = t^0_{\mathfrak{M},\sigma}(w,d) \wedge \bigwedge_{e\in D} \exists t^0_{\mathfrak{M},\sigma}(w,e) \wedge \forall \bigvee_{e\in D} t^0_{\mathfrak{M},\sigma}(w,e),$$

and let $\tau^{k+1}_{\mathfrak{M},\sigma}(w,d)$ be a conjunction of the formulas below:

$$t^0_{\mathfrak{M},\sigma}(w,d) \wedge \bigwedge_{(w,v)\in R} \Diamond\tau^k_{\mathfrak{M},\sigma}(v,d) \wedge \Box \bigvee_{(w,v)\in R} \tau^k_{\mathfrak{M},\sigma}(v,d),$$

$$\bigwedge_{e\in D} \exists\big(t^0_{\mathfrak{M}}(w,e) \wedge \bigwedge_{(w,v)\in R} \Diamond\tau^k_{\mathfrak{M}}(v,e) \wedge \Box \bigvee_{(w,v)\in R} \tau^k_{\mathfrak{M}}(v,e)\big),$$

$$\forall \bigvee_{e\in D} \big(t^0_{\mathfrak{M}}(w,e) \wedge \bigwedge_{(w,v)\in R} \Diamond\tau^k_{\mathfrak{M}}(v,e) \wedge \Box \bigvee_{(w,v)\in R} \tau^k_{\mathfrak{M}}(v,e)\big).$$

The following lemma says that $\tau^k_{\mathfrak{M},\sigma}(w,d)$ is the strongest formula of modal depth k that is true at w, d in $\mathfrak{M}$:

Lemma 24. *Suppose models $\mathfrak{M}$ with w, d and $\mathfrak{N}$ with v, e and $k < \omega$ are given. Then the following conditions are equivalent:*

1. $\mathfrak{N}, v, e \equiv^k_\sigma \mathfrak{M}, w, d$;
2. $\mathfrak{N}, v, e \models \tau^k_{\mathfrak{M},\sigma}(w,d)$;
3. $\mathfrak{N}, v, e \sim^k_\sigma \mathfrak{M}, w, d$.

Proof. The proof is by induction over k. For $k = 0$ the equivalences hold by definition. Assume the equivalences have been shown for k. For "1. $\Rightarrow$ 2." assume that $\mathfrak{N}, v, e \equiv^{k+1}_\sigma \mathfrak{M}, w, d$. Obviously $\mathfrak{M}, w, d \models \tau^{k+1}_{\mathfrak{M},\sigma}(w,d)$ and $\tau^{k+1}_{\mathfrak{M},\sigma}(w,d)$ has modal depth $k+1$. Hence $\mathfrak{N}, v, e \models \tau^{k+1}_{\mathfrak{M},\sigma}(w,d)$, as required. "2. $\Rightarrow$ 3." Assume $\mathfrak{N}, v, e \models \tau^{k+1}_{\mathfrak{M},\sigma}(w,d)$. Then define $\boldsymbol{\beta}_i$ for $i \le k+1$ by taking

$$\boldsymbol{\beta}_i = \big\{\big((v',e'),(w',d')\big) \mid \mathfrak{N}, v', e' \models \tau^i_{\mathfrak{M},\sigma}(w',d')\big\}.$$

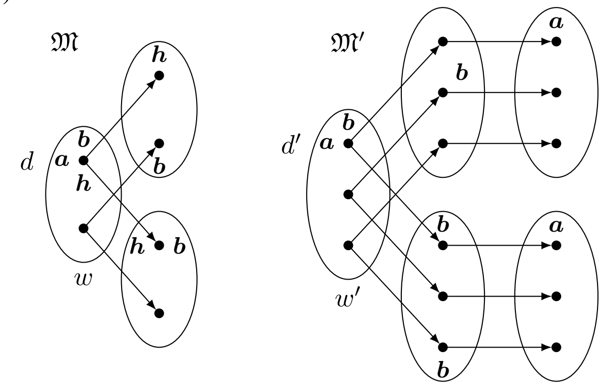
It is readily seen that $\boldsymbol{\beta}_0, \dots, \boldsymbol{\beta}_{k+1}$ is a σ-$(k+1)$-bisimulation. Hence $\mathfrak{N}, v, e \sim^{k+1}_\sigma \mathfrak{M}, w, d$. "3. $\Rightarrow$ 1." holds by definition of σ-k-bisimulations. $\dashv$

Example 27 Suppose $\sigma = \{\boldsymbol{a}, \boldsymbol{b}\} = \mathrm{sig}(\psi)$ and

$$\varphi = \forall(\boldsymbol{a} \leftrightarrow \boldsymbol{b} \leftrightarrow \boldsymbol{h}) \wedge \forall(\boldsymbol{h} \leftrightarrow \Box\boldsymbol{h} \leftrightarrow \Diamond\boldsymbol{h}) \wedge \Diamond\forall(\boldsymbol{b} \leftrightarrow \boldsymbol{h}),$$
$$\psi = \forall(\boldsymbol{a} \leftrightarrow \Box\Box\boldsymbol{a} \leftrightarrow \Diamond\Diamond\boldsymbol{a}) \wedge \Box\Diamond\top \to \Diamond\forall(\boldsymbol{b} \leftrightarrow \Diamond\boldsymbol{a}).$$

Intuitively, φ at a world w says (using the 'help' predicate $\boldsymbol{h}$) that there is an R-successor v such that, for every e, we have $v, e \models \boldsymbol{b}$ iff $w, e \models \boldsymbol{b}$. The premise of ψ at w says two things: first, at distance 2 R-successors u of w, for every e, we have $u, e \models \boldsymbol{a}$ iff $w, e \models \boldsymbol{a}$; and second, every R-successor of w, in particular v, also has an R-successor u. These conditions imply that, for every e, we have $v, e \models \boldsymbol{b}$ iff $u, e \models \boldsymbol{a}$, and so we obtain $\models_{\mathsf{Q^1K}} \varphi \to \psi$.

On the other hand, $\not\models_{\mathsf{Q^1K}} \exists^{\sim\sigma,1}\varphi \to \psi$ because, for the models $\mathfrak{M}$ and $\mathfrak{M}'$ below, we have $\mathfrak{M}, w, d \models \varphi$ and $\mathfrak{M}', w', d' \not\models \psi$ but $\mathfrak{M}, w, d \sim^1_\sigma \mathfrak{M}', w', d'$ (a σ-1-bisimulation connects all points in the roots w and w' that agree on σ, and all points in the depth 1 worlds that agree on σ).



Theorem 28 (i) *(S)CEP in* $\mathsf{Q^1K}$ *is undecidable.*

Proof. We modify the undecidability proof for $\mathsf{Q^1S5}$: for any tiling system $\mathfrak{T}$, we show how to construct in polytime formulas φ and ψ such that $\mathfrak{T}$ has a solution iff $\varphi \wedge \psi$ is not a (strong) conservative extension of φ.

Let $\mathfrak{T} = (T, H, V, \boldsymbol{o}, \boldsymbol{z}^\uparrow, \boldsymbol{z}^\rightarrow)$ be a tiling system. To prove undecidability of strong conservative extensions we work with models $\mathfrak{M} = (W, R, D, I)$ of modal depth 1 having a root $r \in W$ and R-successors $W' = W \setminus \{r\}$ of r. We encode the finite grid to be tiled on $W' \times D$ in essentially the same way as previously on the whole $W \times D$. In particular, $\boldsymbol{g}^{\mathfrak{M}} \subseteq W' \times D$ and $R^{\mathfrak{M}}_h, R^{\mathfrak{M}}_v \subseteq \boldsymbol{g}^{\mathfrak{M}} \times \boldsymbol{g}^{\mathfrak{M}}$ are defined as in (47) and (48) before. We cannot, however, define the modalities $\Diamond_h\chi$ and $\Diamond_v\chi$ using $\mathsf{FOM^1}$-formulas, as we cannot directly refer from (w,d) to (w',d) in our model $\mathfrak{M}$. Instead, we have to 'speak about' $R^{\mathfrak{M}}_h$ and $R^{\mathfrak{M}}_v$ from the viewpoint of points of the form (r,d), for the root r of (W,R). For instance, $\Box\forall(\chi_1 \to \Diamond_h\chi_2)$ is expressed using

$$\forall\big[\Diamond(\boldsymbol{g} \wedge \chi_1 \wedge \neg\boldsymbol{z}^\rightarrow) \to \Diamond\big(\boldsymbol{x} \wedge \exists(\boldsymbol{g} \wedge \chi_2)\big)\big]$$

and $\Box\forall(\chi_1 \to \Diamond_v\chi_2)$ using

$$\forall\big[\Diamond\big(\boldsymbol{y} \wedge \exists(\boldsymbol{g} \wedge \chi_1 \wedge \neg\boldsymbol{z}^\uparrow)\big) \to \Diamond(\boldsymbol{g} \wedge \chi_2)\big].$$

We now define the new formula φ in detail. The following conjuncts generate the grid:

$$\begin{aligned}
&\Diamond(\boldsymbol{o} \wedge \boldsymbol{g}),\\
&\forall\big[\Diamond\big(\boldsymbol{g} \wedge \neg(\boldsymbol{z}^\uparrow \wedge \boldsymbol{z}^\rightarrow)\big) \to \Diamond\boldsymbol{x}\big],\\
&\Box\forall(\boldsymbol{x} \to \exists\boldsymbol{g}),\\
&\Box\forall(\boldsymbol{g} \wedge \neg\boldsymbol{z}^\uparrow \to \exists\boldsymbol{y}),\\
&\forall(\Diamond\boldsymbol{y} \to \Diamond\boldsymbol{g}).
\end{aligned}$$

The constraints on the tiles are expressed by the following conjuncts:

$$\forall\big(\Diamond\boldsymbol{g} \wedge \neg\Diamond\boldsymbol{x} \rightarrow \Box(\boldsymbol{g} \rightarrow \boldsymbol{z}^{\rightarrow})\big),$$
$$\Box\forall(\boldsymbol{g} \leftrightarrow \bigvee_{\boldsymbol{t}\in T} \boldsymbol{t}),$$
$$\Box\forall \bigwedge_{\boldsymbol{t}\neq\boldsymbol{t}'} (\boldsymbol{t} \rightarrow \neg\boldsymbol{t}'),$$
$$\forall\big[\Diamond(\boldsymbol{t} \wedge \neg\boldsymbol{z}^{\rightarrow}) \rightarrow \Box\big(\boldsymbol{x} \rightarrow \forall(\boldsymbol{g} \rightarrow \bigvee_{(\boldsymbol{t},\boldsymbol{t}')\in H} \boldsymbol{t}')\big)\big],$$
$$\forall\big[\Diamond\big(\boldsymbol{y} \wedge \exists(\boldsymbol{t} \wedge \neg\boldsymbol{z}^{\uparrow})\big) \rightarrow \Box(\boldsymbol{g} \rightarrow \bigvee_{(\boldsymbol{t},\boldsymbol{t}')\in V} \boldsymbol{t}')\big],$$
$$\forall\big(\Diamond(\boldsymbol{y} \wedge \exists\boldsymbol{z}^{\rightarrow}) \rightarrow \Box(\boldsymbol{g} \rightarrow \boldsymbol{z}^{\rightarrow})\big),$$
$$\forall\big[\Diamond\boldsymbol{z}^{\rightarrow} \rightarrow \Box\big(\boldsymbol{y} \rightarrow \forall(\boldsymbol{g} \rightarrow \boldsymbol{z}^{\rightarrow})\big)\big],$$
$$\forall\big[\Diamond\boldsymbol{z}^{\uparrow} \rightarrow \Box\big(\boldsymbol{x} \rightarrow \forall(\boldsymbol{g} \rightarrow \boldsymbol{z}^{\uparrow})\big)\big],$$
$$\forall\big(\Diamond(\boldsymbol{x} \wedge \exists\boldsymbol{z}^{\uparrow}) \rightarrow \Box(\boldsymbol{g} \rightarrow \boldsymbol{z}^{\uparrow})\big).$$

Finally, we take a fresh predicate $\boldsymbol{p}_0$ and add the conjunct $\boldsymbol{p}_0 \rightarrow \boldsymbol{p}_0$ to φ.

We now aim to construct a formula ψ for which, as previously, the equivalence **(c1)** $\Leftrightarrow$ **(c2)** holds. This is slightly more involved, as we cannot directly express case distinctions using disjunction and nested 'modalities'. We require auxiliary predicates to achieve this: we use $\boldsymbol{a}_h$ to encode that $\boldsymbol{q}$ is true in an R_h-successor of a $\boldsymbol{q}$-node, $\boldsymbol{a}_v$ to encode that $\boldsymbol{q}$ is true in an R_v-successor of a $\boldsymbol{q}$-node, and $\boldsymbol{b}', \boldsymbol{b}''$ are used to encode that a $\boldsymbol{q}$-node is not confluent. In detail, ψ starts with the conjunct

$$\Diamond(\boldsymbol{g} \wedge \boldsymbol{q}).$$

Next we add a conjunct making a case distinction between $\boldsymbol{a}_h$, $\boldsymbol{a}_v$, and $\boldsymbol{b}'$:

$$\Box\forall\big(\boldsymbol{q} \rightarrow (\neg\boldsymbol{z}^{\rightarrow} \wedge \boldsymbol{a}_h) \vee (\neg\boldsymbol{z}^{\uparrow} \wedge \exists(\boldsymbol{y} \wedge \boldsymbol{a}_v)) \vee (\neg\boldsymbol{z}^{\rightarrow} \wedge \neg\boldsymbol{z}^{\uparrow} \wedge \exists(\boldsymbol{y} \wedge \boldsymbol{b}'))\big).$$

The next two conjuncts state the consequences of $\boldsymbol{a}_h$ and $\boldsymbol{a}_v$, respectively:

$$\forall\big[\Diamond(\boldsymbol{q} \wedge \boldsymbol{a}_h) \rightarrow \Diamond\big(\boldsymbol{x} \wedge \exists(\boldsymbol{g} \wedge \boldsymbol{q})\big)\big],$$
$$\forall\big[\Diamond\big(\boldsymbol{y} \wedge \exists(\boldsymbol{q} \wedge \boldsymbol{a}_v)\big) \rightarrow \Diamond(\boldsymbol{g} \wedge \boldsymbol{q})\big].$$

The next conjunct forces $\boldsymbol{s}$ to be true in the horizontal successors of a vertical successor:

$$\forall\big[\Diamond(\boldsymbol{y} \wedge \exists\boldsymbol{b}') \rightarrow \Box\big(\boldsymbol{x} \rightarrow \forall(\boldsymbol{g} \rightarrow \boldsymbol{s})\big)\big].$$

Finally, the following formulas force $\neg\boldsymbol{s}$ in the horizontal successors of a vertical successor:

$$\forall\big[\Diamond(\boldsymbol{q} \wedge \boldsymbol{b}') \rightarrow \Diamond\big(\boldsymbol{x} \wedge \forall(\boldsymbol{y} \rightarrow \boldsymbol{b}'')\big)\big],$$
$$\forall\big(\Diamond(\boldsymbol{y} \wedge \boldsymbol{b}'') \rightarrow \Box(\boldsymbol{g} \rightarrow \neg\boldsymbol{s})\big).$$

It is not difficult to show that the equivalence **(c1)** $\Leftrightarrow$ **(c2)** holds for φ and ψ.

Lemma 42. *If $\mathfrak{T}$ has a solution, then $\varphi \wedge \psi$ is not a conservative extension of φ.*

Proof. The proof is obtained by modifying the proof of Lemma 34. To begin with, we modify the model $\mathfrak{N} = (W, D, J)$ defined in that proof by adding a root world to W from where every other world is accessible in one R-step. More precisely, we define a new model $\mathfrak{N} = (W, R, D, J)$ as follows. We let $D = W' = \{0, \dots, nm-1\}$, $W = W' \cup \{r\}$, $R = \{(r, w) \mid w \in W'\}$, and J is defined by (59)–(62), plus having $\boldsymbol{p}^{J(r)} = \emptyset$ for $\boldsymbol{p} \in \{\boldsymbol{g}, \boldsymbol{x}, \boldsymbol{y}\} \cup T$, and $\boldsymbol{p}_0^{J(w)} = \emptyset$ for all $w \in W$.

It is straightforward to check that $\mathfrak{N}, r, 0 \models \varphi$ and $\mathfrak{N}, r, 0 \models \neg\psi$. First, we show that $\varphi \wedge \psi$ is not a *strong* conservative extension of φ. We construct a formula χ with $sig(\chi) \cap sig(\psi) \subseteq sig(\varphi)$ such that $\varphi \wedge \chi$ is satisfiable but $\models_{\mathsf{Q^1S5}} \varphi \wedge \chi \rightarrow \neg\psi$. It then follows that $\varphi \wedge \psi$ is not a strong conservative extension of φ. The formula χ provides a description of the model $\mathfrak{N}$ at $(r, 0)$. We take, for every $(i, j) \in W \times D$, a fresh predicate $\boldsymbol{p}_{i,j}$ and extend $\mathfrak{N}$ to $\mathfrak{N}'$ by setting, for all $(i', j') \in W \times D$,

$$\mathfrak{N}', i', j' \models \boldsymbol{p}_{i,j} \quad \text{iff} \quad (i', j') = (i, j). \tag{73}$$

Now let $\sigma' = sig(\varphi) \cup \{\boldsymbol{p}_{i,j} \mid (i, j) \in W \times D\}$, and set

$$\chi_{i,j} = \bigwedge_{\boldsymbol{p}\in\sigma',\ \mathfrak{N},i,j\models\boldsymbol{p}} \boldsymbol{p} \wedge \bigwedge_{\boldsymbol{p}\in\sigma',\ \mathfrak{N},i,j\models\neg\boldsymbol{p}} \neg\boldsymbol{p}. \tag{74}$$

Let χ be the conjunction of of the following formulas:

$$\chi_{r,0} \wedge \Box(\boldsymbol{o} \wedge \boldsymbol{g} \rightarrow \chi_{0,0}), \tag{75}$$
$$\forall\big[\Diamond\chi_{i,i} \rightarrow \Box\big(\boldsymbol{x} \rightarrow (\chi_{i+1,i} \wedge \forall(\boldsymbol{g} \rightarrow \chi_{i+1,i+1}))\big)\big], \quad \text{for } i < nm-1, \tag{76}$$
$$\forall\big[\Diamond(\boldsymbol{y} \wedge \exists\chi_{i,i}) \rightarrow \Box(\boldsymbol{y} \rightarrow \chi_{i,i+n}) \wedge \Box(\boldsymbol{g} \rightarrow \chi_{i+n,i+n})\big], \quad \text{for } i < nm-n, \tag{77}$$
$$\Box\forall(\chi_{i,i} \rightarrow \exists\chi_{i,j}), \text{ for } i, j < nm, \tag{78}$$
$$\forall(\Diamond\chi_{i,i} \rightarrow \Diamond\chi_{j,i}), \text{ for } i, j < nm, \tag{79}$$
$$\forall(\Diamond\chi_{i,j} \rightarrow \chi_{r,j}), \text{ for } i, j < nm, \tag{80}$$
$$\forall\big(\chi_{r,j} \rightarrow \Box(\exists\chi_{l,k} \rightarrow \chi_{l,j})\big), \text{ for } j, k, l < nm. \tag{81}$$

It is easy to see that $\mathfrak{N}', r, 0 \models \chi$, and so $\varphi \wedge \chi$ is satisfiable. Now suppose that $\mathfrak{M}$ is any model such that $\mathfrak{M}, w_0, d_0 \models \varphi \wedge \chi$ for some w_0, d_0. We show that $\mathfrak{M}, w_0, d_0 \models \neg\psi$. Observe that if $\big((w, d), (w', d')\big) \in R_h^{\mathfrak{M}}$ and $\mathfrak{M}, w, d \models \chi_{i,i}$, then $\mathfrak{M}, w', d' \models \chi_{i+1,i+1}$, by (76), and if $\big((w, d), (w', d')\big) \in R_v^{\mathfrak{M}}$ and $\mathfrak{M}, w, d \models \chi_{i,i}$, then $\mathfrak{M}, w', d' \models \chi_{i+n,i+n}$, by (77). Hence, there cannot be an infinite $R_h^{\mathfrak{M}} \cup R_v^{\mathfrak{M}}$-chain.

Now suppoae there is an $R_h^{\mathfrak{M}} \cup R_v^{\mathfrak{M}}$-chain from (w_0, d_0) to some node (w, d) which has an $R_h^{\mathfrak{M}}$-successor (w_1, d_1) and $R_v^{\mathfrak{M}}$-successor (w_2, d_2). Then $\mathfrak{M}, w, d \models \chi_{i,i}$ for some i, $\mathfrak{M}, w_1, d_1 \models \chi_{i+1,i+1}$ and $\mathfrak{M}, w_2, d_2 \models \chi_{i+n,i+n}$, by (75)–(77).

There exist d_1' with $\mathfrak{M}, w_1, d_1' \models \chi_{i+1,i+n+1}$, and w_2' with $\mathfrak{M}, w_2', d_2 \models \chi_{i+n+1,i+n}$, by (78) and (79). Then $\mathfrak{M}, w_0, d' \models \chi_{r,i+n+1}$ by (80). By (81) for $l = j = i+n+1$ and $k = i+n$, we obtain $\mathfrak{M}, w_2', d_1' \models \chi_{i+n+1,i+n+1}$. Moreover, as $\boldsymbol{z}^{\uparrow}$ is not a conjunct of $\chi_{i+1,i+1}$, and $\boldsymbol{z}^{\rightarrow}$ is not a conjunct of $\chi_{i+n,i+n}$, we have that $\boldsymbol{y}$ is a conjunct of $\chi_{i+1,i+n+1}$, and $\boldsymbol{x}$ is a conjunct of $\chi_{i+n+1,i+n}$. Thus, $\big((w_1, d_1), (w_2', d_1')\big) \in R_v^{\mathfrak{M}}$ and $\big((w_2, d_2)(w_2', d_1')\big) \in R_h^{\mathfrak{M}}$, and so (w, d) is confluent.

We next aim to prove that $\varphi \wedge \psi$ is not a (necessarily strong) conservative extension of φ. In this case, we are not allowed to use the fresh predicates $\boldsymbol{p}_{i,j}$ in the formula χ to achieve (73). We instead will use the predicate $\boldsymbol{p}_0 \in sig(\varphi)$ to uniquely characterise the points of $\mathfrak{N}$. Take a bijection f from $\{0, \dots, n-1\} \times \{0, \dots, m-1\}$ to $\{0, \dots, nm-1\}$ and set, for all $(i, j) \in W \times D$:

$$\varphi_{i,j} = \Diamond^{f(i,j)+1} \boldsymbol{p}_0, \quad \varphi_{r,j} = \neg\exists \boldsymbol{g} \wedge \Diamond\varphi_{0,j}, \text{ for } i, j < nm.$$

Modify the model $\mathfrak{N} = (W, R, D, J)$ constructed above to $\mathfrak{N}^+ = (W^+, R^+, D, J^+)$ by adding an nm-long R-chain to each leaf $i < nm$ of (W, R), and define $\boldsymbol{p}_0^{J^+(w)}$ for each $w \in W^+ \setminus W$ such that we still have, for all 'old' points $(i', j') \in W \times D$, the analogue of (73):

$$\mathfrak{N}^+, i', j' \models \varphi_{i,j} \quad \text{iff} \quad (i', j') = (i, j).$$

Now let $\sigma^- = sig(\varphi) \setminus \{\boldsymbol{p}_0\}$ and set, for all $(i, j) \in W \times D$,

$$\chi'_{i,j} = \varphi_{i,j} \wedge \bigwedge_{\boldsymbol{p} \in \sigma^-,\ \mathfrak{N}, i, j \models \boldsymbol{p}} \boldsymbol{p} \wedge \bigwedge_{\boldsymbol{p} \in \sigma^-,\ \mathfrak{N}, i, j \models \neg\boldsymbol{p}} \neg\boldsymbol{p}.$$

Finaly, define χ' with $sig(\chi') \subseteq sig(\varphi)$ by replacing $\chi_{i,j}$ in the conjuncts (75)–(81) by $\chi'_{i,j}$. $\dashv$

Lemma 43. *If $\varphi \wedge \psi$ is not a model conservative extension of φ, then $\mathfrak{T}$ has a solution.*

Proof. Consider a model $\mathfrak{M} = (W, R, D, I)$ such that $\mathfrak{M}, w, d \models \varphi$ but $\mathfrak{M}', w, d \models \neg\psi$ in any extension $\mathfrak{M}'$ of $\mathfrak{M}$ obtained by interpreting the predicates $\boldsymbol{q}, \boldsymbol{s}$. Similarly, to the proof of Lemma 35, by using the equivalence **(c1)** $\Leftrightarrow$ **(c2)**, one can easily find within $\mathfrak{M}$ a finite grid-shaped (with respect to $R_h^{\mathfrak{M}}$ and $R_v^{\mathfrak{M}}$) submodel, which gives a solution to $\mathfrak{T}$. $\dashv$

This complete the proof of Theorem 28 (i). $\dashv$

We next consider uniform interpolants. We modify Example 19 so that it can be used for $\mathsf{Q}^1\mathsf{K}$.

Example 44. Let φ_0 be the conjunction of

$$\forall\big(\Diamond\boldsymbol{a} \to \Diamond(\boldsymbol{p}_1 \wedge \boldsymbol{b})\big),$$
$$\Box\forall\big(\boldsymbol{p}_1 \wedge \boldsymbol{b} \to \exists(\boldsymbol{p}_2 \wedge \boldsymbol{b})\big),$$
$$\forall\big(\Diamond(\boldsymbol{p}_2 \wedge \boldsymbol{b}) \to \Diamond(\boldsymbol{p}_1 \wedge \boldsymbol{b})\big),$$

and let $\sigma = \{\boldsymbol{a}, \boldsymbol{p}_1, \boldsymbol{p}_2\}$. We show that there is no σ-uniform interpolant of $\Diamond\boldsymbol{a} \wedge \varphi_0$ in $\mathsf{Q}^1\mathsf{K}$.

For every $s > 0$, we define a formula χ_s as follows. Take fresh predicates $\boldsymbol{a}_1^h, \boldsymbol{a}_2^h, \dots$ and $\boldsymbol{a}_1^v, \boldsymbol{a}_2^v, \dots$. Intuitively, χ_s says that s many steps of the $\boldsymbol{p}_1, \boldsymbol{p}_2$-ladder has been constructed by forcing $\boldsymbol{a}_1^h, \dots \boldsymbol{a}_s^h$ to be true 'horizontally' and $\boldsymbol{a}_1^v, \dots \boldsymbol{a}_s^v$ to be true vertically in the corresponding steps. In detail, we let

$$\chi'_1 = \forall(\Diamond\boldsymbol{a} \to \Box\boldsymbol{a}_1^h) \wedge \Box\forall(\boldsymbol{a}_1^h \wedge \boldsymbol{p}_1 \to \forall\boldsymbol{a}_1^v),$$
$$\chi_1 = \chi'_1 \to \exists\Diamond(\boldsymbol{p}_2 \wedge \boldsymbol{a}_1^v),$$

and for $s > 0$,

$$\chi'_{s+1} = \chi'_s \wedge \forall\big(\Diamond(\boldsymbol{a}_s^v \wedge \boldsymbol{p}_2) \to \Box\boldsymbol{a}_{s+1}^h\big) \wedge \Box\forall(\boldsymbol{a}_{s+1}^h \wedge \boldsymbol{p}_1 \to \forall\boldsymbol{a}_{s+1}^v),$$
$$\chi_{s+1} = \chi'_{s+1} \to \exists\Diamond(\boldsymbol{p}_2 \wedge \boldsymbol{a}_{s+1}^v).$$

Then $\models_{\mathsf{Q}^1\mathsf{K}} \Diamond\boldsymbol{a} \wedge \varphi_0 \to \chi_s$ for all $s > 0$. Thus, if ϱ were a σ-uniform interpolant of $\Diamond\boldsymbol{a} \wedge \varphi_0$, then $\models_{\mathsf{Q}^1\mathsf{K}} \varrho \to \chi_s$ would follow, for all $s > 0$.

On the other hand, for $s > 0$, we modify the model $\mathfrak{M}_s = (W_s, D_s, I_s)$ defined in Example 19 by adding a root world to W_s from where every other world is accessible in one R_s-step. More precisely, we define a new model $\mathfrak{M}_s = (W_s, R_s, D_s, I_s)$ as follows. We let $W'_s = D_s = \{0, \dots, s-1\}$, $W_s = W'_s \cup \{r\}$, $R_s = \{(r, w) \mid w \in W'_s\}$, and I_s is defined by

$$\boldsymbol{a}^{I_s(k)} = \begin{cases} \{0\}, & \text{if } k = 0, \\ \emptyset, & \text{otherwise;} \end{cases} \tag{82}$$

$$\boldsymbol{p}_1^{I_s(k)} = \begin{cases} \{k-1\}, & \text{if } k > 0, \\ \emptyset, & \text{otherwise;} \end{cases} \tag{83}$$

$$\boldsymbol{p}_2^{I_s(k)} = \begin{cases} \{k\}, & \text{if } k > 0, \\ \emptyset, & \text{otherwise;} \end{cases} \tag{84}$$

plus having $\boldsymbol{p}^{I_s(r)} = \emptyset$ for

$$\boldsymbol{p} \in \{\boldsymbol{a}, \boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{a}_1^h, \dots, \boldsymbol{a}_s^h, \boldsymbol{a}_1^v, \dots, \boldsymbol{a}_s^v\}$$

and, for all $0 < i \leq s$ and all $k < s$,

$$(\boldsymbol{a}_i^h)^{I_s(k)} = \{i-1\},$$
$$(\boldsymbol{a}_i^v)^{I_s(k)} = \begin{cases} \{0, \dots, s-1\}, & \text{if } 0 < k = i < s, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Then, for every $s > 0$, $\mathfrak{M}_s, r, 0 \not\models \chi_s$ and so $\mathfrak{M}_s, r, 0 \models \neg\varrho$. Also, $\mathfrak{M}_s, r, 0 \models \chi_{s'}$ for all $s' < s$. Now consider the ultraproduct $\prod_U \mathfrak{M}_s$ with U a non-principal ultrafilter on $\omega \setminus \{0\}$. As each $\chi_{s'}$ is true in almost all $\mathfrak{M}_s, r, 0$, it follows from the properties of ultraproducts (Chang and Keisler 1998) that $\prod_U \mathfrak{M}_s, \overline{r}, \overline{0} \models \Diamond\boldsymbol{a} \wedge \neg\varrho \wedge \chi_{s'}$ for all $s' > 0$, for some suitable $\overline{r}, \overline{0}$. But then one can interpret $\boldsymbol{b}$ in $\prod_U \mathfrak{M}_s$ such that $\mathfrak{M}, \overline{r}, \overline{0} \models \varphi_0$ for the resulting model $\mathfrak{M}$. Then $\mathfrak{M} \models \Diamond\boldsymbol{a} \wedge \varphi_0 \wedge \neg\varrho$ and as $\models_{\mathsf{Q}^1\mathsf{K}} \Diamond\boldsymbol{a} \wedge \varphi_0 \to \varrho$ should hold for a uniform interpolant ϱ of $\Diamond\boldsymbol{a} \wedge \varphi_0$, we have derived a contradiction.

Theorem 28 (ii) *UIEP for $\mathsf{Q}^1\mathsf{K}$ is undecidable.*

Proof. The proof is by combining the construction of Theorem 28 and Example 44 in exactly the same way as the construction of Theorem 18 and Example 19 were combined in the proof of Theorem 18 (ii). $\dashv$

References

Álvarez, L. G.; Rudolph, S.; and Strass, H. 2022. How to agree to disagree - managing ontological perspectives using standpoint logic. In Sattler, U.; Hogan, A.; Keet, C. M.; Presutti, V.; Almeida, J. P. A.; Takeda, H.; Monnin, P.; Pirrò, G.; and d'Amato, C., eds., *The Semantic Web - ISWC 2022 - 21st International Semantic Web Conference, Virtual Event, October 23-27, 2022, Proceedings*, volume 13489 of *Lecture Notes in Computer Science*, 125–141. Springer.

Areces, C.; Blackburn, P.; and Marx, M. 2003. Repairing the interpolation theorem in quantified modal logic. *Ann. Pure Appl. Log.* 124(1-3):287–299.

Artale, A.; Kontchakov, R.; Kovtunova, A.; Ryzhikov, V.; Wolter, F.; and Zakharyaschev, M. 2017. Ontology-mediated query answering over temporal data: A survey (invited talk). In Schewe, S.; Schneider, T.; and Wijsen, J., eds., *24th International Symposium on Temporal Representation and Reasoning, TIME 2017, October 16-18, 2017, Mons, Belgium*, volume 90 of *LIPIcs*, 1:1–1:37. Schloss Dagstuhl - Leibniz-Zentrum für Informatik.

Artale, A.; Jung, J. C.; Mazzullo, A.; Ozaki, A.; and Wolter, F. 2020. Living without bethn and craig: Definitions and interpolants in description and modal logics with nominals and role inclusions. *CoRR* abs/2007.02736.

Artale, A.; Jung, J. C.; Mazzullo, A.; Ozaki, A.; and Wolter, F. 2021. Living without Beth and Craig: Definitions and interpolants in description logics with nominals and role inclusions. In *Proceedings of the 35th AAAI Conference on Artificial Intelligence, AAAI 2021*, 6193–6201. AAAI Press.

Artale, A.; Lutz, C.; and Toman, D. 2007. A description logic of change. In Veloso, M. M., ed., *IJCAI 2007, Proceedings of the 20th International Joint Conference on Artificial Intelligence, Hyderabad, India, January 6-12, 2007*, 218–223.

Baader, F.; Horrocks, I.; Lutz, C.; and Sattler, U. 2017. *An Introduction to Description Logic*. Cambridge University Press.

Barwise, J., and van Benthem, J. 1999. Interpolation, preservation, and pebble games. *J. Symb. Log.* 64(2):881–903.

Belardinelli, F., and Lomuscio, A. 2009. Quantified epistemic logics for reasoning about knowledge in multi-agent systems. *Artif. Intell.* 173(9-10):982–1013.

Benedikt, M.; Leblay, J.; ten Cate, B.; and Tsamoura, E. 2016. *Generating Plans from Proofs: The Interpolation-based Approach to Query Reformulation*. Synthesis Lectures on Data Management. Morgan & Claypool Publishers.

Benedikt, M.; ten Cate, B.; and Vanden Boom, M. 2016. Effective interpolation and preservation in guarded logics. *ACM Trans. Comput. Log.* 17(2):8.

Botoeva, E.; Konev, B.; Lutz, C.; Ryzhikov, V.; Wolter, F.; and Zakharyaschev, M. 2016. Inseparability and conservative extensions of description logic ontologies: A survey. In Pan, J. Z.; Calvanese, D.; Eiter, T.; Horrocks, I.; Kifer, M.; Lin, F.; and Zhao, Y., eds., *Reasoning Web: Logical Foundation of Knowledge Graph Construction and Query Answering - 12th International Summer School 2016, Aberdeen, UK, September 5-9, 2016, Tutorial Lectures*, volume 9885 of *Lecture Notes in Computer Science*, 27–89. Springer.

Botoeva, E.; Lutz, C.; Ryzhikov, V.; Wolter, F.; and Zakharyaschev, M. 2019. Query inseparability for ALC ontologies. *Artif. Intell.* 272:1–51.

Chandra, A. K.; Kozen, D. C.; and Stockmeyer, L. J. 1981. Alternation. *J. ACM* 28:114–133.

Chang, C., and Keisler, H. J. 1998. *Model Theory*. Elsevier.

D'Agostino, G., and Lenzi, G. 2006. On modal mu-calculus with explicit interpolants. *J. Appl. Log.* 4(3):256–278.

Donini, F. M.; Lenzerini, M.; Nardi, D.; Nutt, W.; and Schaerf, A. 1998. An epistemic operator for description logics. *Artif. Intell.* 100(1-2):225–274.

Eiter, T., and Kern-Isberner, G. 2019. A brief survey on forgetting from a knowledge representation and reasoning perspective. *Künstliche Intell.* 33(1):9–33.

Fagin, R.; Halpern, J. Y.; Moses, Y.; and Vardi, M. Y. 1995. *Reasoning About Knowledge*. MIT Press.

Fine, K. 1979. Failures of the interpolation lemma in quantified modal logic. *J. Symb. Log.* 44(2):201–206.

Fitting, M. 2002. Interpolation for first order S5. *J. Symb. Log.* 67(2):621–634.

Fortin, M.; Konev, B.; and Wolter, F. 2022. Interpolants and explicit definitions in extensions of the description logic $\mathcal{EL}$. In *Proceedings of the 19th International Conference on Principles of Knowledge Representation and Reasoning, KR 2022*.

Franconi, E.; Kerhet, V.; and Ngo, N. 2013. Exact query reformulation over databases with first-order and description logics ontologies. *J. Artif. Intell. Res.* 48:885–922.

French, T. 2006. Bisimulation quantifiers for modal logics.

Gabbay, D. M., and Maksimova, L. 2005. *Interpolation and Definability: Modal and Intuitionistic Logics*. Oxford, England: Oxford University Press UK.

Gabbay, D. M.; Kurucz, A.; Wolter, F.; and Zakharyaschev, M. 2003. *Many-dimensional Modal Logics: Theory and Applications*. North Holland Publishing Company.

Geleta, D.; Payne, T. R.; and Tamma, V. A. M. 2016. An investigation of definability in ontology alignment. In Blomqvist, E.; Ciancarini, P.; Poggi, F.; and Vitali, F., eds., *Knowledge Engineering and Knowledge Management - 20th International Conference, EKAW 2016, Bologna, Italy, November 19-23, 2016, Proceedings*, volume 10024 of *Lecture Notes in Computer Science*, 255–271.

Ghilardi, S., and Zawadowski, M. W. 1995. Undefinability of propositional quantifiers in the modal system S4. *Stud Logica* 55(2):259–271.

Ghilardi, S.; Lutz, C.; Wolter, F.; and Zakharyaschev, M. 2006. Conservative extensions in modal logic. In Governatori, G.; Hodkinson, I. M.; and Venema, Y., eds., *Advances in Modal Logic 6, papers from the sixth conference on "Advances in Modal Logic," held in Noosa, Queensland, Australia, on 25-28 September 2006*, 187–207. College Publications.

Ghilardi, S.; Lutz, C.; and Wolter, F. 2006. Did I damage my ontology? A case for conservative extensions in description logics. In Doherty, P.; Mylopoulos, J.; and Welty, C. A., eds., *Proceedings, Tenth International Conference on Principles of Knowledge Representation and Reasoning, Lake District of the United Kingdom, June 2-5, 2006*, 187–197. AAAI Press.

Göller, S.; Jung, J. C.; and Lohrey, M. 2015. The complexity of decomposing modal and first-order theories. *ACM Trans. Comput. Log.* 16(1):9:1–9:43.

Gonçalves, R.; Knorr, M.; and Leite, J. 2023. Forgetting in answer set programming - A survey. *Theory Pract. Log. Program.* 23(1):111–156.

Goranko, V., and Otto, M. 2007. Model theory of modal logic. In *Handbook of Modal Logic*. Elsevier. 249–329.

Grädel, E.; Kolaitis, P. G.; and Vardi, M. Y. 1997. On the decision problem for two-variable first-order logic. *Bulletin of Symbolic Logic* 3(1):53–69.

Grau, B. C.; Horrocks, I.; Kazakov, Y.; and Sattler, U. 2008. Modular reuse of ontologies: Theory and practice. *J. of Artifical Intelligence Research* 31:273–318.

Henkell, K.; Rhodes, J.; ; and Steinberg, B. 2010. Aperiodic pointlikes and beyond. *Internat. J. Algebra Comput.* 20(2):287–305.

Henkell, K. 1988. Pointlike sets: the finest aperiodic cover of a finite semigroup. *J. Pure Appl. Algebra* 55(1-2):85–126.

Hodkinson, I. M.; Kontchakov, R.; Kurucz, A.; Wolter, F.; and Zakharyaschev, M. 2003. On the computational complexity of decidable fragments of first-order linear temporal logics. In *10th International Symposium on Temporal Representation and Reasoning / 4th International Conference on Temporal Logic (TIME-ICTL 2003), 8-10 July 2003, Cairns, Queensland, Australia*, 91–98. IEEE Computer Society.

Hodkinson, I. M.; Wolter, F.; and Zakharyaschev, M. 2000. Decidable fragment of first-order temporal logics. *Ann. Pure Appl. Log.* 106(1-3):85–134.

Jung, J. C., and Wolter, F. 2021. Living without Beth and Craig: Definitions and interpolants in the guarded and two-variable fragments. In *Proceedings of the 36th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2021*, 1–14. IEEE.

Jung, J. C.; Lutz, C.; Martel, M.; Schneider, T.; and Wolter, F. 2017. Conservative extensions in guarded and two-variable fragments. In Chatzigiannakis, I.; Indyk, P.; Kuhn, F.; and Muscholl, A., eds., *44th International Colloquium on Automata, Languages, and Programming, ICALP 2017, July 10-14, 2017, Warsaw, Poland*, volume 80 of *LIPIcs*, 108:1–108:14. Schloss Dagstuhl - Leibniz-Zentrum für Informatik.

Jung, J. C.; Lutz, C.; and Marcinkowski, J. 2022. Conservative extensions for existential rules. In Kern-Isberner, G.; Lakemeyer, G.; and Meyer, T., eds., *Proceedings of the 19th International Conference on Principles of Knowledge Representation and Reasoning, KR 2022, Haifa, Israel. July 31 - August 5, 2022*.

Jung, J. C.; Mazzullo, A.; and Wolter, F. 2022. More on interpolants and explicit definitions for description logics with nominals and/or role inclusions. In *Proceedings of the 35th International Workshop on Description Logics, DL 2022*.

Konev, B.; Lutz, C.; Walther, D.; and Wolter, F. 2009. Formal properties of modularisation. In Stuckenschmidt, H.; Parent, C.; and Spaccapietra, S., eds., *Modular Ontologies: Concepts, Theories and Techniques for Knowledge Modularization*, volume 5445 of *Lecture Notes in Computer Science*. Springer. 25–66.

Konev, B.; Walther, D.; and Wolter, F. 2009. Forgetting and uniform interpolation in large-scale description logic terminologies. In *Proceedings of the 21st International Joint Conference on Artificial Intelligence, IJCAI 2009*, 830–835.

Kontchakov, R.; Kurucz, A.; Wolter, F.; and Zakharyaschev, M. 2007. Spatial logic + temporal logic = ? In Aiello, M.; Pratt-Hartmann, I.; and van Benthem, J., eds., *Handbook of Spatial Logics*. Springer. 497–564.

Koopmann, P., and Schmidt, R. A. 2015. Uniform interpolation and forgetting for $\mathcal{ALC}$ ontologies with aboxes. In *Proceedings of the 29th AAAI Conference on Artificial Intelligence, AAAI 2015*, 175–181. AAAI Press.

Koopmann, P. 2020. LETHE: forgetting and uniform interpolation for expressive description logics. *Künstliche Intell.* 34(3):381–387.

Lang, J., and Marquis, P. 2008. On propositional definability. *Artif. Intell.* 172(8-9):991–1017.

Lin, F., and Reiter, R. 1994. Forget it! In *Proc. of AAAI Fall Symposium on Relevance*.

Liu, M.; Padmanabha, A.; Ramanujam, R.; and Wang, Y. 2022. Generalized Bundled Fragments for First-Order Modal Logic. In Szeider, S.; Ganian, R.; and Silva, A., eds., *47th International Symposium on Mathematical Foundations of Computer Science (MFCS 2022)*, volume 241 of *Leibniz International Proceedings in Informatics (LIPIcs)*, 70:1–70:14. Dagstuhl, Germany: Schloss Dagstuhl – Leibniz-Zentrum für Informatik.

Lutz, C., and Wolter, F. 2011. Foundations for uniform interpolation and forgetting in expressive description logics. In *Proceedings of the 22nd International Joint Conference on Artificial Intelligence, IJCAI 2011*, 989–995. IJCAI/AAAI.

Lutz, C.; Seylan, I.; and Wolter, F. 2012. An automata-theoretic approach to uniform interpolation and approximation in the description logic EL. In *Proceedings of the 13th International Conference on Principles of Knowledge Representation and Reasoning, KR 2012*. AAAI Press.

Lutz, C.; Wolter, F.; and Zakharyaschev, M. 2008. Temporal description logics: A survey. In Demri, S., and Jensen, C. S., eds., *15th International Symposium on Temporal Representation and Reasoning, TIME 2008, Université du Québec à Montréal, Canada, 16-18 June 2008*, 3–14. IEEE Computer Society.

Marx, M., and Areces, C. 1998. Failure of interpolation in combined modal logics. *Notre Dame J. Formal Log.* 39(2):253–273.

Marx, M. 1999. Complexity of products of modal logics. *J. Log. Comput.* 9(2):197–214.

Nikitina, N., and Rudolph, S. 2014. (Non-)succinctness of uniform interpolants of general terminologies in the description logic $\mathcal{EL}$. *Artif. Intell.* 215:120–140.

Place, T., and Zeitoun, M. 2016. Separating regular languages with first-order logic. *Log. Methods Comput. Sci.* 12(1).

ten Cate, B.; Conradie, W.; Marx, M.; and Venema, Y. 2006. Definitorially complete description logics. In *Proc. of KR*, 79–89.

ten Cate, B. 2022. Lyndon interpolation for modal logic via type elimination sequences. Technical report, ILLC, Amsterdam.

Toman, D., and Weddell, G. E. 2021. FO Rewritability for OMQ using Beth Definability and Interpolation. In *Proceedings of the 34th International Workshop on Description Logics, DL 2021*. CEUR-WS.org.

van Emde Boas, P. 1997. The convenience of tilings. In *Complexity, Logic, and Recursion Theory*. 331–363.

Visser, A. 1996. Uniform interpolation and layered bisimulation. In *Gödel'96: Logical foundations of mathematics, computer science and physics–Kurt Gödel's legacy*, 139–164. Association for Symbolic Logic.

Wajsberg, M. 1933. Ein erweiterter Klassenkalkül. *Monatsh Math. Phys.* 40:113–126.

Wang, Y.; Wei, Y.; and Seligman, J. 2022. Quantifier-free epistemic term-modal logic with assignment operator. *Ann. Pure Appl. Log.* 173(3):103071.

Wang, Y. 2017. A new modal framework for epistemic logic. In Lang, J., ed., *Proceedings Sixteenth Conference on Theoretical Aspects of Rationality and Knowledge, TARK 2017, Liverpool, UK, 24-26 July 2017*, volume 251 of *EPTCS*, 515–534.

Wolter, F., and Zakharyaschev, M. 2001. Decidable fragments of first-order modal logics. *J. Symb. Log.* 66(3):1415–1438.

Zhao, Y., and Schmidt, R. A. 2016. Forgetting concept and role symbols in $\mathcal{ALCOIH}$-ontologies. In *Proceedings of the 25th International Joint Conference on Artificial Intelligence, IJCAI 2016*, 1345–1353. IJCAI/AAAI Press.

Zhao, Y.; Feng, H.; Alassaf, R.; Del-Pinto, W.; and Schmidt, R. A. 2018. The FAME family: A family of reasoning tools for forgetting in expressive description logics. In Ortiz, M., and Schneider, T., eds., *Proceedings of the 31st International Workshop on Description Logics co-located with 16th International Conference on Principles of Knowledge Representation and Reasoning (KR 2018), Tempe, Arizona, US, October 27th - to - 29th, 2018*, volume 2211 of *CEUR Workshop Proceedings*. CEUR-WS.org.