Decidable fragments of first-order modal logics

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Abstract

The paper considers the set $\mathcal{ML}_1$ of first-order polymodal formulas the modal operators in which can be applied to subformulas of at most one free variable. Using the mosaic technique, we prove a general satisfiability criterion for formulas in $\mathcal{ML}_1$ which reduces the modal satisfiability to the classical one. The criterion is then used to single out a number of new, in a sense optimal, decidable fragments of various predicate modal logics.

1 Introduction

The classical decision problem—to single out expressive and decidable fragments of first-order logic—has a long history and hardly needs any justification: after all, classical first-order logic was and still remains in the very center of logical studies, both in mathematics and applications. Here are only three examples (out of dozens) of such fragments (the choice is not accidental—we shall use these results later on):

- the fragment containing only monadic predicate symbols [5];
- the fragment with only two individual variables [26, 22];[1]
- the guarded fragment containing formulas of the form

$$\exists \overline{y}\ (G(\overline{x}, \overline{y}) \wedge \varphi(\overline{x}, \overline{y})),$$

  where the guard $G(\overline{x}, \overline{y})$ is atomic[2] [1].

[1] The fragment with binary predicates and three variables is undecidable [30].
[2] For a precise definition see Section 5.

(The current state of art in this field is presented in the recent monograph [6].)

For modal logicians the decision problem in first-order *modal* logics seemed almost hopeless. The following list covers almost all known results and leaves not so much space for a maneuver:

- the monadic fragment (even with a single unary predicate symbol) of practically all modal predicate logics is undecidable [18] (see also [17]);
- the two-variable fragment of practically all modal predicate logics with constant domains (without equality and function symbols) is undecidable [11]; for example, such is the two-variable fragment of the predicate logics based on **K**, **K4**, **S4**, etc.;
- only the one-variable fragment of various standard modal predicate logics turns out to be decidable (see [10, 27, 28, 2]). [3]

The expressive capability of the one-variable fragment is rather weak: we can describe only properties of unary predicates in it. Two variables used without any restrictions immediately lead to undecidability (at least for logics with constant domains). But what if we somehow restrict their use?

All undecidability proofs for fragments of modal predicate logics exploit formulas of the form $\Box\psi(x,y)$ in which the necessity operator applies to subformulas of more than one free variable; in fact, such formulas play an essential role in the reduction of undecidable problems to those fragments. So it seems natural to assume that it is the possibility of quantifying into modal contexts with two or more free variables that is "responsible" for the high complexity of modal predicate logics.

The main aim of this paper is to show that this is indeed the case. We consider the fragment $\mathcal{ML}_1$ of the polymodal first-order language (without function symbols and equality) consisting of all those formulas that do not contain subformulas of the form $\Box\psi(x_1,\dots,x_n)$ for $n \geq 2$. (In other words, we restrict modalities *de re* to formulas of one free variable and impose no restrictions whatsoever on modalities *de dicto*.) The fragment itself is certainly undecidable: it still contains full classical predicate logic. But any restriction of the pure predicate (i.e., non-modal) part of $\mathcal{ML}_1$ to a decidable fragment provides us with a decidable fragment of the modal predicate logic! For instance, we obtain the decidability of the one-variable fragment of various modal predicate logics simply by using the fact that the corresponding non-modal fragment is decidable. Moreover, the main deficit of the one-variable modal fragment—its inability to speak about worlds of Kripke models in a reasonably expressive language—can be overcome now by a proper choice of decidable fragments of classical predicate logic. Thus, the obtained decidable fragments are in a sense optimal.

[3] It may be worth reminding the reader that the one-variable fragment of classical first-order logic is equivalent to the propositional modal logic **S5** [31]. More generally, it is easy to see that the one-variable fragment of a modal predicate logic with constant domains and the propositional part L is equivalent to the product $L \times \mathbf{S5}$; see [12].

As in the case of the guarded fragment of classical first-order logic [1], the idea to consider our fragments of modal predicate logics came from *propositional modal logic*. In the series of papers [24, 19, 4, 3, 32, 33, 34] a number of expressive and decidable concept description languages with modal operators were designed and investigated with the aim of representing and processing knowledge in dynamic application domains (where knowledge may depend on interacting agents, time, actions, etc.). Although those formalisms appeared in the framework of knowledge representation in artificial intelligence, from the modal logic point of view they can be regarded as multi-dimensional polymodal systems whose semantical structures resemble Kripke models for modal predicate logics (see Section 5 for more details). And in fact those systems turn out to be embeddable in the fragments constructed in this paper, which may serve as an evidence of that these fragments really posses enough expressive power to be practically applicable.

It is not our aim in this paper to prove the decidability of fragments of as many modal predicate logics as possible (actually, that might be quite boring; modal logic has already passed through this experience). Instead we deal with only one sufficiently representative polymodal logic $\mathbf{QK}^*$ with two kinds of necessity operators: $\Box_1, \ldots, \Box_l$ interpreted in models (with constant and expanding domains) by arbitrary binary relations $R_1, \ldots, R_l$, and $\Box_1^*, \ldots, \Box_l^*$ interpreted by the transitive and reflexive closures of $R_1, \ldots, R_l$. The reason for selecting $\mathbf{QK}^*$ is purely technical. First, it is complex enough to show how our method works. On the other hand, it is not too sophisticated to involve too many technical details and special cases (for example, we could have taken quantified **PDL**, but this would require to explain details that are not necessary to understand the method; the reader can restore them following [32]). Finally, as an easy consequence we obtain the decidability of the corresponding fragments of some standard modal predicate logics such as **QK**, **QT**, **QK4**, **QS4** simply by embedding them into $\mathbf{QK}^*$.

The paper is organized in the following way. The next section defines the syntax and semantics of the first-order modal logics we consider in this paper. It contains also two technical results: we show that the two-variable monadic fragment of $\mathbf{QK}^*$ is not recursively enumerable, and that in many cases satisfiability in models with expanding domains can be reduced to satisfiability in models with constant domains. In Section 3 we codify Kripke models satisfying a given formula φ, the "boxed" subformulas in which contain at most one free variable, in certain structures called quasimodels and intended to "take care" of the modal contexts in φ. In Section 4 we use the mosaic method to prove a rather general satisfiability criterion for such formulas, and in Section 5 we apply it to single out several decidable fragments of first-order modal logics.

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2 First-order modal logic

In this paper, we deal with the *first-order polymodal language* $\mathcal{ML}$ constructed in the standard way from the following alphabet:

- *predicate symbols* $P_0, P_1, \ldots,$
- *individual variables* $x_0, x_1, \ldots,$
- *individual constants* $c_0, c_1, \ldots,$
- the *booleans* $\wedge$, $\neg$,
- the *universal quantifier* $\forall x$ for each individual variable x,
- the *necessity operators* $\Box_1, \ldots, \Box_l$, $l \geq 1$.

As usual, we assume that the set of predicate symbols in $\mathcal{ML}$ is nonempty and that each of them is of some fixed arity; 0-ary predicates are called *propositional variables* and denoted by $p_0, p_1, \ldots$. It will be assumed that we have a sufficient supply of those variables, unary predicate symbols, and an infinite set *var* of individual variables. Formulas in $\mathcal{ML}$ will be denoted by small Greek letters φ, ψ, etc. Finally, by $\mathcal{L}$ we mean the classical (non-modal) first-order language that results from $\mathcal{ML}$ by omitting all formulas containing necessity operators.

$\mathcal{ML}$ is interpreted in *first-order Kripke models*, which are structures of the form $\mathfrak{M} = \langle \mathfrak{F}, D, I \rangle$, where $\mathfrak{F} = \langle W, R_1, \ldots, R_l \rangle$ is the underlying (propositional) *polymodal Kripke frame* (R_i a binary relation on a nonempty set of worlds W), D is a nonempty set, the *domain* of $\mathfrak{M}$, and I is a function associating with every world $w \in W$ a first-order $\mathcal{L}$-structure

$$I(w) = \left\langle D, P_0^{I,w}, \ldots, c_0^{I,w}, \ldots \right\rangle,$$

in which $P_i^{I,w}$, for each i, is a predicate on D of the same arity as P_i (for a propositional variable p_i, the predicate $p_i^{I,w}$ is simply one of the propositional constants $\top$, "truth", or $\bot$, "falsehood"), and $c_i^{I,w}$ is an element in D such that $c_i^{I,u} = c_i^{I,v}$ for any $u, v \in W$. To simplify notation we will omit the superscript I and write P_i^w, p_i^w, c_i^w, etc., if this does not involve ambiguity.

An *assignment* in D is a function $\mathfrak{a}$ from *var* to D. The *truth-relation* $(\mathfrak{M}, w) \models^{\mathfrak{a}} \varphi$ (or simply $w \models^{\mathfrak{a}} \varphi$, if $\mathfrak{M}$ is understood) in the model $\mathfrak{M}$ under the assignment $\mathfrak{a}$ is defined inductively in the standard manner:

- $w \models^{\mathfrak{a}} P_i(y_1, \ldots, y_n)$ iff $P_i^w(\mathfrak{a}(y_1), \ldots, \mathfrak{a}(y_n))$ is true in $I(w)$ (this fact will also be written as $I(w) \models^{\mathfrak{a}} P_i(y_1, \ldots, y_n)$ or $I(w) \models P_i[\mathfrak{a}(y_1), \ldots, \mathfrak{a}(y_n)]$);
- $w \models^{\mathfrak{a}} \psi \wedge \chi$ iff $w \models^{\mathfrak{a}} \psi$ and $w \models^{\mathfrak{a}} \chi$;
- $w \models^{\mathfrak{a}} \neg\psi$ iff $w \not\models^{\mathfrak{a}} \neg\psi$;
- $w \models^{\mathfrak{a}} \forall x \psi(x, y_1, \ldots, y_n)$ iff $w \models^{\mathfrak{b}} \psi(x, y_1, \ldots, y_n)$ for every assignment $\mathfrak{b}$ in D that may differ from $\mathfrak{a}$ only on x;

- $w \models^{\mathfrak{a}} \Box_i \varphi$ iff $v \models^{\mathfrak{a}} \varphi$ for all v such that wR_iv.

The set of $\mathcal{ML}$-formulas that are true in all models under all assignments will be denoted by **QK**.

Remark 1. (i) The models introduced above are known as *models with constant domains*. To define *models with expanding domains* we need one more function, say $\mathfrak{d}$, such that $\mathfrak{d}(w) \subseteq D$ and $\mathfrak{d}(u) \subseteq \mathfrak{d}(v)$ whenever uR_iv. The only difference from the models with constant domains is in the truth-definition of $\forall x\psi$ which now looks as follows:

- $w \models^{\mathfrak{a}} \forall x\psi(x, y_1, \ldots, y_n)$ iff $w \models^{\mathfrak{b}} \psi(x, y_1, \ldots, y_n)$ for every assignment $\mathfrak{b}$ that may differ from $\mathfrak{a}$ only on x, provided that $\mathfrak{b}(x) \in \mathfrak{d}(w)$.

Thus, $\mathfrak{d}(w)$ is regarded as the true domain of $I(w)$. (For more details consult [17].) Actually, later in this section we will show that expanding domains can be reduced to constant ones, at least as far as the decidability of the fragments we are interested in is concerned.

(ii) We deal with models having *rigid designators* (in the sense that for any $u, v \in W$, we have $c_i^u = c_i^v$). But again, everything we prove in this paper holds for the non-rigid case as well.

Syntactically the logic **QK** can be represented by the calculus QK containing all the axiom schemata and inference rules of classical predicate calculus, the Barcan formulas

$$\forall x \Box_i \varphi \to \Box_i \forall x \varphi,$$

the modal axiom schemata

$$\Box_i(\varphi \to \psi) \to (\Box_i \varphi \to \Box_i \psi)^4$$

and the necessitation rules $\varphi / \Box_i \varphi$, for all $i = 1, \ldots, l$. (The Barcan formulas are not axioms if we consider models with expanding domains.)

By adding to QK the standard modal axiom schemata of **T**, **K4**, **S4** we obtain modal predicate calculi QT, $QK4$, $QS4$ representing the corresponding logics **QT**, **QK4**, **QS4** that are known to be characterized by the classes of all reflexive, transitive, and quasi-ordered models, respectively (see e.g. [17]).

One more modal predicate logic, $\mathbf{QK}^*$, we consider in this paper is formulated in the language $\mathcal{ML}$ with $2l$ necessity operators $\Box_i$, $\Box_i^*$, for $i = 1, \ldots, l$. As before, $\Box_i$ is interpreted in models by an arbitrary accessibility relation R_i, while $\Box_i^*$ is interpreted by the reflexive and transitive closure R_i^* of R_i. Since R_i^* is uniquely determined by R_i, we may denote frames for this language as before: $\mathfrak{F} = \langle W, R_1, \ldots, R_l \rangle$. To simplify notation we will assume that $\circ$ is either blank or $*$, so that $\Box_i^\circ$ ranges over $\{\Box_i, \Box_i^*\}$ and R_i° over $\{R_i, R_i^*\}$.

The logic $\mathbf{QK}^*$ is the set of all formulas that are true in all models for this language. It is easy to see that **QK**, **QT**, **QK4**, **QS4** are embeddable into $\mathbf{QK}^*$. So from now on we will be concentrating mainly on $\mathbf{QK}^*$.

[4] Here $\chi_1 \to \chi_2$ is the standard abbreviation for $\neg(\chi_1 \wedge \neg\chi_2)$.

It is worth noting that the logic $\mathbf{QK}^*$ is very complex. Even its two-variable monadic fragment turns out to be not recursively enumerable. One can show this by reducing the recurrent tiling problem for $\mathbb{N} \times \mathbb{N}$ (which is known to be Σ^1_1-complete; see [16]) to the satisfiability problem in that fragment. Here is a sketch of the proof based actually on the same idea as the reductions of [29, 20].

We remind the reader that a tile $\mathfrak{t}$ is a 1×1 square with fixed orientation and colored edges $right(\mathfrak{t})$, $left(\mathfrak{t})$, $up(\mathfrak{t})$, and $down(\mathfrak{t})$. The $\mathbb{N} \times \mathbb{N}$ recurrent tiling problem is formulated as follows: given a finite set $\mathcal{T}$ of tiles and a tile $\mathfrak{t}_0 \in \mathcal{T}$, to determine whether there is a tiling of $\mathbb{N} \times \mathbb{N}$ by $\mathcal{T}$ such that $\mathfrak{t}_0$ occurs infinitely often in the first row. More precisely, the problem is to find out whether there exists a function f from $\mathbb{N} \times \mathbb{N}$ into $\mathcal{T}$ such that, for all $m, n \in \mathbb{N}$,

- $right(f(n,m)) = left(f(n+1,m))$,
- $up(f(n,m)) = down(f(n,m+1))$,
- the set $\{n \in \mathbb{N} : f(n,0) = \mathfrak{t}_0\}$ is infinite.

Suppose we are given a set $\mathcal{T} = \{\mathfrak{t}_0, \dots, \mathfrak{t}_n\}$ of tiles. Let R be a binary predicate and $P_0, \dots, P_n$ unary predicates. Define a first-order modal formula $\varphi_\mathcal{T}$ in this language as the conjunction of the following formulas:

$$
\begin{array}{l}
\exists x (P_0(x) \wedge \Box^* \Diamond^* (P_0(x) \wedge \Diamond \top)), \\
\Box^* \forall x \exists y R(x,y), \\
\Box^* \forall x,y ((R(x,y) \to \Box^* R(x,y)) \wedge (\neg R(x,y) \to \Box^* \neg R(x,y))), \\
\Box^* \forall x (\bigvee_{i=0}^n P_i(x) \wedge \bigwedge_{i \neq j} (P_i(x) \to \neg P_j(x))), \\
\Box^* \forall x (P_i(x) \to \forall y (R(x,y) \to \bigvee_{right(\mathfrak{t}_i) = left(\mathfrak{t}_j)} P_j(y))), \\
\Box^* \forall x (P_i(x) \to \Box \bigvee_{up(\mathfrak{t}_i) = down(\mathfrak{t}_j)} P_j(x)).
\end{array}
$$

(Here $\Diamond^\circ \psi = \neg \Box^\circ \neg \psi$.)

Lemma 2. *The formula $\varphi_\mathcal{T}$ is satisfiable in a model with constant or expanding domains iff there is a recurrent tiling of $\mathbb{N} \times \mathbb{N}$ by $\mathcal{T}$.*

Proof It should be clear how to satisfy $\varphi_\mathcal{T}$ if $\mathcal{T}$ recurrently tiles $\mathbb{N} \times \mathbb{N}$. Conversely, suppose $\varphi_\mathcal{T}$ is satisfied at the root w_0 in a model $\langle \mathfrak{F}, D, \mathfrak{d}, I \rangle$ (with constant or expanding domains), where $\mathfrak{F} = \langle W, S \rangle$ and

$$
I(w) = \left\langle D, R^{I,w}, P_0^{I,w}, \dots, P_n^{I,w} \right\rangle,
$$

for $w \in W$. Without loss of generality we may assume that $\langle W, S \rangle$ is an intransitive tree[5] and $\langle D, R^{I,w_0} \rangle$ is an intransitive forest (if this is not the case, we can use the standard unravelling technique). By the first conjunct of $\varphi_\mathcal{T}$, we have $a_0 \in \mathfrak{d}(w_0)$ for which $w_0 \models P_0[a_0]$, and an infinite sequence of worlds $w_0 S w_1 S w_2 \dots$ such that $\{w_i : w_i \models P_0[a_0]\}$ is infinite. According to the second conjunct, we have an infinite sequence $a_0 R^{I,w_0} a_1 R^{I,w_0} a_2 \dots$ of elements in

[5] A definition of an intransitive tree can be found at the beginning of Section 4.

$\mathfrak{d}(w_0)$. By the third conjunct, for all $w \in W$ and $i, j \in \mathbb{N}$, we have $a_i R^{I,w} a_j$ iff $a_i R^{I,w_0} a_j$. Now define a function f by putting, for all $i, j \in \mathbb{N}$, $f(i,j) = \mathfrak{t}_k$, whenever $I(w_i) \models P_k[a_j]$. It is straightforward to check that f is a recurrent tiling of $\mathbb{N} \times \mathbb{N}$. ❑

The formula $\varphi_{\mathcal{T}}$ contains only two variables, and the binary predicate R can be simulated by means of two unary predicates Q_1, Q_2 (see e.g. [17]):

$$R(x,y) = \Diamond^*(Q_1(x) \wedge Q_2(y)).$$

This yields us

Theorem 3. *The two-variable monadic fragment of* $\mathbf{QK}^*$ *(no matter with constant or expanding domains) is not recursively enumerable.*

Let us see now how satisfiability in models with expanding domains can be reduced to satisfiability in models with constant domains. Denote by $md(\varphi)$ the *modal depth* of φ, i.e., the maximal number of nested necessity operators in φ. Put $\Box^{\leq 0}\varphi = \varphi$ and, for $m \geq 0$,

$$\Box^{\leq m+1}\varphi = \Box^{\leq m}\varphi \wedge \bigwedge_{i=1}^{l} \Box_i^* \Box^{\leq m}\varphi.$$

Let $E(x)$ be a unary predicate which does not occur in φ. By induction on the construction of φ we define its *relativization* $\varphi{\downarrow}E$:

$$\begin{aligned}
P_i(s_1,\dots,s_n){\downarrow}E &= E(s_1) \wedge \dots \wedge E(s_n) \wedge P_i(s_1,\dots,s_n),\\
(\psi \wedge \chi){\downarrow}E &= (\psi{\downarrow}E) \wedge (\chi{\downarrow}E),\\
(\neg\psi){\downarrow}E &= \bigwedge\{E(y) : y \text{ is free in } \psi\} \wedge \neg(\psi{\downarrow}E),\\
(\forall x\psi){\downarrow}E &= \forall x(E(x) \to (\psi{\downarrow}E)),\\
(\Box_i^\circ\psi){\downarrow}E &= \bigwedge\{E(y) : y \text{ is free in } \psi\} \wedge \Box_i^\circ(\psi{\downarrow}E).
\end{aligned}$$

Here each s_j is either an individual variable or a constant.

Proposition 4. *For every Kripke frame* $\mathfrak{F} = \langle W, R_1, \dots, R_l\rangle$*, a formula* φ *is satisfied in a model on* $\mathfrak{F}$ *with expanding domains iff the formula*

$$\varphi' = \varphi{\downarrow}E \wedge \Box^{\leq md(\varphi)}\forall x(E(x) \to \bigwedge_{i=1}^{l} \Box_i^* E(x))$$

is satisfied in a model based on $\mathfrak{F}$ *and having constant domain.*

Proof Without loss of generality we may consider φ to be a sentence.

Assuming that φ is satisfied in a model $\mathfrak{M} = \langle \mathfrak{F}, D, \mathfrak{d}, I\rangle$ with expanding domains and that

$$I(w) = \left\langle D, P_0^{I,w}, \dots, c_0^{I,w}, \dots \right\rangle$$

for $w \in W$, we construct a model $\mathfrak{N} = \langle \mathfrak{F}, D, J \rangle$ with constant domain D by taking

$$J(w) = \left\langle D, E^{J,w}, P_0^{I,w}, \ldots, c_0^{I,w}, \ldots \right\rangle,$$

where $E^{J,w} = \mathfrak{d}(w)$. It is readily checked by induction that $(\mathfrak{M}, w) \models^{\mathfrak{a}} \psi$ iff $(\mathfrak{N}, w) \models^{\mathfrak{a}} \psi \downarrow E$, for every $w \in W$, every subformula ψ of φ, and every assignment $\mathfrak{a} : var \mapsto \mathfrak{d}(w)$. By the truth-definition for $\forall x$, the definition of the relativization to E, and since φ is a sentence, this should be enough to conclude that $\varphi \downarrow E$ is satisfied in $\mathfrak{N}$. For its truth-value (as well as that of φ in $\mathfrak{M}$) can be computed in every world w using only assignments into $\mathfrak{d}(w)$. It follows that φ' is satisfied in $\mathfrak{N}$.

Conversely, suppose φ' is satisfied in a world v in a model $\mathfrak{N} = \langle \mathfrak{F}, D, J \rangle$ (with constant domain D) under an assignment $\mathfrak{a}$, and

$$J(w) = \left\langle D, E^{J,w}, P_0^{J,w}, \ldots, c_0^{J,w}, \ldots \right\rangle,$$

for $w \in W$. Consider the model $\mathfrak{M} = \langle \mathfrak{F}, D, \mathfrak{d}, I \rangle$ such that $\mathfrak{d}(w) = E^{J,w}$ and

$$I(w) = \left\langle D, P_0^{I,w}, \ldots, c_0^{J,w}, \ldots \right\rangle,$$

where each $P_i^{I,w}$ is the restrictions of $P_i^{J,w}$ to $E^{J,w}$, for every w accessible from v in $\leq md(\varphi)$ steps, and $I(w) = J(w)$ for all the other worlds w in $\mathfrak{F}$. Now, using the fact that the truth-value of φ in v depends only on the worlds that are accessible in $\leq md(\varphi)$ steps from v, one can easily show that $(\mathfrak{M}, v) \models^{\mathfrak{a}} \varphi$. ❑

Remark 5. It is to be noted that φ and φ' are of different forms. So when proving decidability of a fragment in $\mathcal{ML}$ we should always check that φ and φ' belong to it simultaneously. Fortunately that will always be the case for all the fragments we deal with in this paper. From now on we will be considering only models with constant domains.

Given a set of $\mathcal{ML}$-formulas Γ, denote by $sub_n\Gamma$ the closure under negation of the set of all subformulas of formulas in Γ containing precisely n free variables; $sub\varphi$ is the set of all subformulas in a formula φ, and $con\varphi$ the set of all constants in φ. Without loss of generality we may identify ψ and $\neg\neg\psi$; so $sub_n\Gamma$ is finite whenever Γ is finite.

Definition 6. Denote by $\mathcal{ML}_1$ the set of all $\mathcal{ML}$-formulas φ such that every subformula of φ of the form $\Box_i^\circ \psi$ contains at most one free variable. In other words, $\mathcal{ML}_1$ allows quantification into modal contexts only with one free variable. From now on we will be assuming that *all our formulas are in* $\mathcal{ML}_1$.

Remark 7. It is worth noting that, in contrast to the one-variable fragment of first-order modal logic, only modalities *de re* are restricted in $\mathcal{ML}_1$, while modalities *de dicto* are arbitrary.

For every formula $\psi(x) = \Box_i^\circ\varphi(x)$ with one free variable x, we reserve a unary predicate and denote it by $P_\psi(x)$. Likewise, for every sentence $\psi = \Box_i^\circ\varphi$ we fix a propositional variable p_ψ. $P_\psi(x)$ and p_ψ will be called the *surrogates* for $\psi(x)$ and ψ, respectively.

Given a formula φ, denote by $\overline{\varphi}$ the formula that results from φ by replacing all subformulas of the form $\Box_i^\circ\psi(x)$ and $\Box_i^\circ\chi$, which are not within the scope of another box, with their surrogates. Thus, $\overline{\varphi}$ contains no occurrences of modal operators, i.e., it is an $\mathcal{L}$-formula; we will call $\overline{\varphi}$ the $\mathcal{L}$-*reduct* of φ. For a set of $\mathcal{ML}_1$-formulas Γ, we put $\overline{\Gamma} = \{\overline{\psi} : \psi \in \Gamma\}$.

3 Quasimodels

Suppose we are given an $\mathcal{ML}_1$-sentence φ and asked whether it is satisfiable. Being only modal logicians, we would like to assume, at least for a moment, that we have an access to a classical oracle that is capable of providing us with an $\mathcal{L}$-structure satisfying the $\mathcal{L}$-reduct of φ, if it exists. Suppose we have got such a structure. Our task then is to try to expand it to a Kripke model satisfying φ. This means, in particular, that we should take care of only the "boxed" subformulas of φ, which, recall, have at most one free variable.

The aim of this section is to show that modulo those subformulas every such Kripke model can be codified in a structure, called a *quasimodel*, that may be thought of as a model the worlds in which have disjoint domains with a bounded number of elements (depending on φ), each element satisfies some specified set of subformulas of φ, and the correspondence between those elements in different worlds is established by special functions called *runs*.

Fix a variable x not occurring in φ and put

$$sub_x\varphi = \{\psi\{x/y\} : \psi(y) \in sub_1\varphi\}.$$

Definition 8 (types). By a *1-type* for φ we mean any subset t of $sub_x\varphi$ such that $\overline{t}$ is a maximal $\mathcal{L}$-consistent subset of $\overline{sub_x\varphi}$. A *0-type* for φ is a subset Φ of $sub_0\varphi$ such that $\overline{\Phi}$ is a maximal $\mathcal{L}$-consistent subset of $\overline{sub_0\varphi}$.[6] Given a 1-type t for φ and a constant $c \in con\varphi$, the pair $\langle t, c\rangle$ will be called an *indexed 1-type* for φ (indexed by c) and denoted by $t_c(x)$ or simply t_c.

For example, if

$$\varphi = \exists x_1\exists x_2\exists x_3(P(x_1, x_2) \wedge \Box\exists x_4 P(x_4, x_3))$$

then the set

$$\{\exists x_2\exists x_3(P(x, x_2) \wedge \Box\exists x_4 P(x_4, x_3)),\ \Box\exists x_4(x_4, x),\ \neg\exists x_4 P(x_4, x)\}$$

is a 1-type and $\{\varphi\}$ is a 0-type for φ.

[6] In principle, for our purposes it would be enough to define t and Φ as boolean-saturated subsets of $sub_x\varphi$ and $sub_0\varphi$.

In what follows we will not be distinguishing between a finite set t of formulas and the conjunction $\bigwedge t$ of formulas in it. Thus, a 1-type may be thought of as a formula with one free variable and a 0-type as a sentence. There are only finitely many types for φ. To a certain extent, every world w in a model under a given assignment can be characterized (modulo φ, of course) by the set of 1-types that hold in this world, the set of 1-types that hold on the constants, and the 0-type that is true in w. This motivates the following definition.

Definition 9 (world candidate). Suppose that T is a set of 1-types for φ, $T^{con} = \{t_c : c \in con\varphi\}$ a set of indexed 1-types such that $\{t : t_c \in T^{con}\} \subseteq T$, and let Φ be a 0-type for φ. Then the triple $\mathfrak{T} = \langle T, T^{con}, \Phi\rangle$ is called a *world candidate* for φ.

Not all world candidates can represent worlds in models. To single out those that can, we require one more definition.

Definition 10 (realizable world candidate). Consider a first-order $\mathcal{L}$-structure

$$\mathfrak{D} = \langle D, P_0^{\mathfrak{D}}, \ldots, c_0^{\mathfrak{D}}, \ldots\rangle \tag{1}$$

and suppose that $a \in D$. The set

$$t^{\mathfrak{D}}(a) = \{\psi \in sub_x\varphi : \mathfrak{D} \models \overline{\psi}[a]\}$$

is clearly a 1-type for φ; we call it the *1-type for φ on a* in $\mathfrak{D}$. Say that $\mathfrak{D}$ *realizes* a world candidate $\langle T, T^{con}, \Phi\rangle$ if the following conditions hold:

- $T = \{t^{\mathfrak{D}}(a) : a \in D\}$,
- $T^{con} = \{\langle t^{\mathfrak{D}}(c), c\rangle : c \in con\varphi\}$,[7]
- $\mathfrak{D} \models \overline{\Phi}$.

Denote by $\sharp(\varphi)$ the number of all pairwise non-isomorphic realizable world candidates for φ. It should be clear that

$$\sharp(\varphi) \leq 2^{2^{|sub_1\varphi|}} \cdot |con\varphi| \cdot 2^{|sub_1\varphi|} \cdot 2^{|sub_0\varphi|}.$$

Lemma 11. *A world candidate $\mathfrak{T} = \langle T, T^{con}, \Phi\rangle$ for φ is realizable iff the $\mathcal{L}$-formula*

$$\alpha_{\mathfrak{T}} = \bigwedge \overline{\Phi} \wedge \bigwedge\{\exists x\, \overline{t}(x) : t \in T\} \wedge \forall x \bigvee\{\overline{t}(x) : t \in T\} \wedge \bigwedge\{\overline{t}[c] : t_c \in T^{con}\}$$

is satisfiable in some $\mathcal{L}$-structure.

Proof Follows directly from the definitions. ❑

[7] Without loss of generality we may identify $c_i \in con\varphi$ and $c_i^{\mathfrak{D}} \in D$.

Lemma 12. *Let κ be a cardinal, $\kappa \geq \aleph_0$. Then every realizable world candidate $\langle T, T^{con}, \Phi\rangle$ is realized in an $\mathcal{L}$-structure $\mathfrak{D}$ of the form* (1) *such that for every $t \in T$ the set*

$$\mathfrak{D}_t = \{a \in D : \mathfrak{D} \models \overline{t}[a]\}$$

is of cardinality κ.

Proof Follows from classical model theory, since the language $\mathcal{L}$ does not contain equality. ❑

We are in a position now to define the central notion of this section, that of a quasimodel. Consider a frame $\mathfrak{F} = \langle W, R_1, \ldots, R_l\rangle$ and a map f associating with each $w \in W$ a realizable world candidate $f(w) = \langle T_w, T_w^{con}, \Phi_w\rangle$.

Definition 13 (run). A *run* in $\langle \mathfrak{F}, f\rangle$ is a function r from W into the set of realizable 1-types for φ such that

- $r(w) \in T_w$, for all $w \in W$, and
- for every $\Box_i^\circ \psi \in sub_x\varphi$ and every $w \in W$, we have $\Box_i^\circ \psi \in r(w)$ iff $\psi \in r(v)$ for all $v \in W$ such that $wR_i^\circ v$.

Definition 14 (quasimodel). The pair $\langle \mathfrak{F}, f\rangle$ is called a *quasimodel* for φ (based on $\mathfrak{F}$) if the following conditions hold:

- for every $c \in con\varphi$, the function r_c defined by $r_c(w) = t$, for $\langle t, c\rangle \in T_w^{con}$, $w \in W$, is a run in $\langle \mathfrak{F}, f\rangle$;
- for every pair $w \in W$ and $t \in T_w$, there exists a run r in $\langle \mathfrak{F}, f\rangle$ such that $r(w) = t$;
- for all $w \in W$ and $\Box_i^\circ \psi \in sub_0\varphi$, we have $\Box_i^\circ \psi \in \Phi_w$ iff $\psi \in \Phi_v$ for all $v \in W$ such that $wR_i^\circ v$.

Say that φ is *satisfied* in the quasimodel $\langle \mathfrak{F}, f\rangle$ if there is $w \in W$ such that $\varphi \in \Phi_w$. Instead of $\psi \in \Phi_w$ or $\psi \notin \Phi_w$, for $\psi \in sub_0\varphi$, we will often write $w \models \psi$ or, respectively, $w \not\models \psi$.

Example 15. Consider the formula

$$\varphi = \exists y(\Box P(y) \wedge \neg\Box\forall z P(z)).$$

Let $\mathfrak{F} = \langle W, R\rangle$ contain two worlds, say u and v, such that uRv. Define realizable world candidates $\mathfrak{T}_u = \langle T_u, T_u^{con}, \Phi_u\rangle$ and $\mathfrak{T}_v = \langle T_v, T_v^{con}, \Phi_v\rangle$ by taking

$$\begin{gathered}
T_u = \{t_1, t_2\},\ T_v = \{t_3, t_4\},\ T_u^{con} = T_v^{con} = \emptyset,\\
t_1 = \{\Box P(x) \wedge \neg\Box\forall z P(z),\ \Box P(x),\ \neg P(x)\},\\
t_2 = \{\neg(\Box P(x) \wedge \neg\Box\forall z P(z)),\ \neg\Box P(x),\ \neg P(x)\},\\
\Phi_u = \{\varphi,\ \neg\Box\forall z P(z),\ \neg\forall z P(z)\},\\
t_3 = \{\neg(\Box P(x) \wedge \neg\Box\forall z P(z)),\ \Box P(x),\ P(x)\},\\
t_4 = \{\neg(\Box P(x) \wedge \neg\Box\forall z P(z)),\ \Box P(x),\ \neg P(x)\},\\
\Phi_v = \{\neg\varphi,\ \Box\forall z P(z),\ \neg\forall z P(z)\},
\end{gathered}$$

and let $f(u) = \mathfrak{T}_u$, $f(v) = \mathfrak{T}_v$. The functions r_1 and r_2 defined by

$$r_1(u) = t_1,\ r_1(v) = t_2,\ r_2(u) = t_3,\ r_2(v) = t_4$$

are clearly runs in $\langle \mathfrak{F}, f\rangle$. So this pair is a quasimodel satisfying φ.

Theorem 16. *A sentence φ is satisfied in a model based on a frame $\mathfrak{F}$ iff it is satisfied in a quasimodel for φ based on $\mathfrak{F}$.*

Proof Suppose φ is satisfied in a model $\mathfrak{M} = \langle \mathfrak{F}, D, I\rangle$, $\mathfrak{F} = \langle W, R_1, \ldots, R_l\rangle$. For every $w \in W$, define $f(w) = \langle T_w, T_w^{con}, \Phi_w\rangle$ by taking

$$T_w = \{t^{I(w)}(a) : a \in D\},$$
$$T_w^{con} = \{\left\langle t^{I(w)}(c), c\right\rangle : c \in con\varphi\},$$
$$\Phi_w = \{\psi \in sub_0\varphi : I(w) \models \overline{\psi}\}.$$

It is easy to see that $\langle \mathfrak{F}, f\rangle$ is a quasimodel satisfying φ (for every $a \in D$, the function $r(w) = t^{I(w)}(a)$, $w \in W$, is clearly a run in $\langle \mathfrak{F}, f\rangle$).

Conversely, suppose that φ is satisfied in a quasimodel $\langle \mathfrak{F}, f\rangle$ for φ. Take a cardinal $\kappa \geq \aleph_0$ exceeding the cardinality of the set Ω of all runs in $\langle \mathfrak{F}, f\rangle$ and put

$$D = \{\langle r, \xi\rangle : r \in \Omega, \xi < \kappa\}.$$

By Lemma 12, for every $w \in W$ there exists an $\mathcal{L}$-structure $I(w)$ with domain D realizing $f(w)$ and such that $c^w = \langle r_c, 0\rangle$, for every $c \in con\varphi$, and

$$r(w) = \{\psi \in sub_x\varphi : I(w) \models \overline{\psi}[\langle r, \xi\rangle]\},$$

for all $r \in \Omega$ and $\xi < \kappa$. Let $\mathfrak{M} = \langle \mathfrak{F}, D, I\rangle$ and let $\mathfrak{a}$ be an arbitrary assignment in D. We show by induction that, for all $\psi \in sub\varphi$ and $w \in W$,

$$I(w) \models^{\mathfrak{a}} \overline{\psi} \text{ iff } (\mathfrak{M}, w) \models^{\mathfrak{a}} \psi.$$

The basis of induction, i.e., the case when $\psi = P_i(x_1, \ldots, x_n)$ is clear; for then $\psi = \overline{\psi}$. The induction step for $\psi = \psi_1 \wedge \psi_2$, $\psi = \neg\psi_1$, and $\psi = \forall x\psi_1$ follows by the induction hypothesis from the obvious equations:

$$\overline{\psi_1 \wedge \psi_2} = \overline{\psi_1} \wedge \overline{\psi_2},\ \overline{\neg\psi_1} = \neg\overline{\psi_1},\ \overline{\forall x\psi_1} = \forall x\overline{\psi_1}.$$

Let $\psi = \Box_i^\circ\chi(y)$ and assume $\mathfrak{a}(y) = \langle r, \xi\rangle$. We then have $\overline{\psi} = P_{\Box_i^\circ\chi}$, and so

$$\begin{aligned}
I(w) \models^{\mathfrak{a}} P_{\Box_i^\circ\chi}(y) \quad &\text{iff} \quad \Box_i^\circ\chi \in r(w)\\
&\text{iff} \quad \forall v\ (wR_i^\circ v \to \chi \in r(v))\\
&\text{iff} \quad \forall v\ (wR_i^\circ v \to I(v) \models^{\mathfrak{a}} \overline{\chi})\\
&\text{iff} \quad \forall v\ (wR_i^\circ v \to (\mathfrak{M}, v) \models^{\mathfrak{a}} \chi(y))\\
&\text{iff} \quad (\mathfrak{M}, w) \models^{\mathfrak{a}} \Box_i^\circ\chi(y).
\end{aligned}$$

Finally, for a sentence $\psi = \Box_i^\circ \chi$, we have $\overline{\psi} = p_{\Box_i^\circ \chi}$ and

$$\begin{array}{lll} I(w) \models^{\mathfrak{a}} p_{\Box_i^\circ \chi} & \text{iff} & \Box_i^\circ \chi \in \Phi_w \\ & \text{iff} & \forall v\ (wR_i^\circ v \to \chi \in \Phi_v) \\ & \text{iff} & \forall v\ (wR_i^\circ v \to I(v) \models^{\mathfrak{a}} \overline{\chi}) \\ & \text{iff} & \forall v\ (wR_i^\circ v \to (\mathfrak{M}, v) \models^{\mathfrak{a}} \chi) \\ & \text{iff} & (\mathfrak{M}, w) \models^{\mathfrak{a}} \Box_i^\circ \chi. \end{array}$$

Since $\varphi \in \Phi_w$, for some $w \in W$, it follows that $(\mathfrak{M}, w) \models \varphi$. ❑

4 Satisfiability criterion

In this section we prove an effective satisfiability criterion for those sentences $\varphi \in \mathcal{ML}_1$ that have an oracle capable of deciding the realizability problem for the world candidates for φ. The idea is to represent quasimodels for φ as possibly infinite mosaics consisting of repeating finite patterns we call *blocks*.

Say that a frame $\mathfrak{F} = \langle W, R_1, \ldots, R_l \rangle$ is an *intransitive tree* if $R_i \cap R_j = \emptyset$, for $i \neq j$, and the frame $\langle W, R \rangle$ with $R = \bigcup_{i=1}^{l} R_i$ is rooted, cycle-free, and contains no distinct paths of the form $xRy_1R \ldots Ry_nRy$ and $xRz_1R \ldots Rz_mRy$. By a *tree quasimodel* for φ we mean a quasimodel for φ based on an intransitive tree. Using the standard unravelling technique of modal logic (see e.g. [9]) one can prove the following theorem.

Theorem 17 (tree quasimodel completeness). *A formula φ is satisfiable iff it is satisfiable in a tree quasimodel for φ.*

Fix again a sentence $\varphi \in \mathcal{ML}_1$. To introduce the notion of a block for φ, we require two auxiliary definitions. Suppose that $\mathfrak{F}$ is a frame as above and f is a map from W into the set of realizable world candidates for φ, so that $f(w) = \langle T_w, T_w^{con}, \Phi_w \rangle$. It will be convenient for us not to distinguish between w and $f(w)$, i.e., we will assume that the worlds w in $\mathfrak{F}$ *are* the realizable world candidates $\langle T_w, T_w^{con}, \Phi_w \rangle$.

Definition 18 (weak run). A *weak run* in $\langle \mathfrak{F}, f \rangle$ is a function r from W into $\bigcup_{w \in W} T_w$ such that

- $r(w) \in T_w$, for all $w \in W$, and
- for every $\Box_i^\circ \psi \in sub_x \varphi$ and every $w \in W$, if $\Box_i^\circ \psi \in r(w)$ then $\psi \in r(v)$ for all v such that $wR_i^\circ v$.

The difference between runs and weak runs is only that the latter do not necessarily contain witnesses for formulas of the form $\neg\Box_i^\circ \psi \in r(w)$ (in a run we must have a "witness" v such that $wR_i^\circ v$ and $\neg\psi \in r(v)$). The difference between quasimodels and weak quasimodels to be defined below is of the same sort.

Definition 19 (weak quasimodel). The pair $\langle \mathfrak{F}, f \rangle$ is called a *weak quasimodel* for φ if the following holds:

- for every $c \in con\varphi$, the function r_c is a weak run in $\langle \mathfrak{F}, f \rangle$;
- for all $w \in W$ and $t \in T_w$, there exists a weak run r in $\langle \mathfrak{F}, f \rangle$ such that $r(w) = t$;
- for every $w \in W$ and every $\Box_i^\circ \psi \in sub_0\varphi$, if $\Box_i^\circ \psi \in \Phi_w$ then $\psi \in \Phi_v$ for all v such that $wR_i^\circ v$.

Say that an intransitive tree $\mathfrak{F} = \langle W, R_1, \ldots, R_l \rangle$ is a *bouquet* if no point in W different from its root has more than one R successor, $R = \bigcup_{i=1}^{l} R_i$.

Definition 20 (block). Let $\mathfrak{B} = \langle \mathfrak{F}, f \rangle$ be a weak quasimodel based on a finite bouquet $\mathfrak{F}$ with root w_0. A weak run r in $\mathfrak{B}$ is called *root-saturated* if, for every $\Box_i^\circ \psi(x) \in sub_x\varphi$, whenever $\Box_i^\circ \psi(x) \notin r(w_0)$ then there is $w \in W$ such that $w_0 R_i^\circ w$ and $\psi(x) \notin r(w)$.

Now, we say $\mathfrak{B}$ is a *block* for φ if it satisfies conditions (a)–(c) below:

(a) for every $w \in W$ and every $t \in T_w$ there is a root-saturated weak run r in $\mathfrak{B}$ such that $r(w) = t$;

(b) every weak run r_c in $\mathfrak{B}$, for $c \in con\varphi$, is root-saturated;

(c) for every $\Box_i^\circ \psi \in sub_0\varphi$, if $w_0 \not\models \Box_i^\circ \psi$ then there exists $w \in W$ such that $w_0 R_i^\circ w$ and $w \not\models \psi$.

The following notion will be used for organizing blocks in a mosaic-like structure that is intended to represent a quasimodel satisfying φ.

Definition 21 (satisfying set). A set $\mathcal{S}$ of blocks for φ is called a *satisfying set* for φ if (i) it contains a block with root w_0 such that $w_0 \models \varphi$ (i.e, $\varphi \in \Phi_{w_0}$) and (ii) for every realizable world candidate $\mathfrak{T}$ in every block in $\mathcal{S}$, there exists a block in $\mathcal{S}$ having $\mathfrak{T}$ as its root.

We are going to show now that φ is satisfiable iff there is a satisfying set for φ whose blocks contain at most N worlds, where $N < \omega$ is effectively determined by φ.

Theorem 22 (satisfiability criterion). *An $\mathcal{ML}_1$-sentence φ is satisfiable iff there is a satisfying set for φ, each block in which contains at most*

$$N = \sharp(\varphi) \cdot \max\{1, |sub_x\varphi|\} \cdot (|sub_0\varphi| + 2|sub_x\varphi| \cdot 2^{|sub_x\varphi|})$$

worlds.

Proof To simplify notation, we will confine ourself to considering the language with only one necessity operator $\Box$ and its "star" version $\Box^*$. The general case is treated in absolutely the same way.

($\Rightarrow$) Suppose φ is satisfiable. Then, by Theorem 17, there is a tree quasimodel $\mathfrak{m} = \langle \mathfrak{F}, f \rangle$ for φ satisfying φ at its root. Moreover, without loss of generality we may also assume that if wRu in $\mathfrak{F}$ then there are sufficiently many worlds $v \in W$, called *twins of* u, such that wRv, $f(u) = f(v)$ and the subquasimodel of $\mathfrak{m}$ generated by u and v are isomorphic. In any case, we can achieve this by duplicating (as many times as we need) the subtree of $\mathfrak{m}$ generated by u and connecting it with w by R. The resulting structure will clearly be again a tree quasimodel satisfying φ (for it can be "p-morphically" mapped onto the original quasimodel).

We begin our construction of a satisfying set for φ by associating with each world $w \in W$ a block $\mathfrak{B}_w = \langle \mathfrak{F}_w, f_w \rangle$, $\mathfrak{F}_w = \langle W_w, R_w \rangle$.

First, for every formula $\Box^\circ \psi \in sub_0 \varphi$ such that $w \not\models \Box^\circ \psi$ we select a world $w' \in W$ for which $wR^\circ w'$, $w' \not\models \psi$, and put it into an auxiliary set $Sel(w)$ (which at the very beginning is empty). For every 1-type $t \in T_w$ we fix a run r in $\mathfrak{m}$ coming through t; if $t_c \in T_w^{con}$, $c \in con\varphi$, then we take $r = r_c$. Then, for every $\Box^\circ \psi(x) \in sub_x \varphi$ such that $\Box^\circ \psi(x) \notin r(w)$, we select a world $w' \in W$ for which $wR^\circ w'$, $\psi(x) \notin r(w')$ and put it in $Sel(w)$ *together with one of its twins* w''. The number of the selected worlds does not exceed

$$|sub_0(\varphi)| + 2 \cdot |sub_x(\varphi)| \cdot 2^{|sub_x(\varphi)|}.$$

Without loss of generality we may assume all of them to be pairwise distinct. For each selected w' there is a unique path from w to w', namely the set $(w, w') = \{w_1 : wR^* w_1 R^* w'\}$. Again, without loss of generality we assume that distinct paths (w, w') and (w, w'') (for $w' \neq w''$) have no common worlds save w (otherwise the duplication technique will do the job).

Finally, we define W_w to be the set of all worlds in the paths (w, w'), for $w' \in Sel(w)$, R_w to be the restriction of R to W_w, and f_w the restriction of f to W_w (taking into account all the duplications, of course).

The constructed structure $\mathfrak{B}_w$ is a block. Indeed, it is clearly a weak quasimodel for φ based on a finite bouquet, having root w (for it can be regarded as a subquasimodel of $\mathfrak{m}$) and satisfying conditions (b) and (c) by the construction. To show that it satisfies (a) as well, suppose that $u \in W_w$, $t \in T_u$, and let r be a weak run in $\mathfrak{B}_w$ coming through t, i.e., $r(u) = t$. Consider the 1-type $r(w)$ and the set $\mathcal{B}$ of formulas $\chi = \Box^\circ \psi \in sub_x \varphi$ for which $\chi \notin r(w)$. For each of them there is a weak run r_χ such that $r_\chi(w) = r(w)$, $\psi \notin r_\chi(w_\chi)$, for some $w_\chi \in Sel(w)$, and $u \notin (w, w_\chi)$. Using these weak runs and r we can define a function r' by taking, for every $w' \in W_w$,

$$r'(w') = \begin{cases} r(w') & \text{if } w' \notin (w, w_\chi), \text{ for any } \chi \in \mathcal{B} \\ r_\chi(w') & \text{otherwise} \end{cases}$$

Clearly, r' is a root-saturated weak run in $\mathfrak{B}_w$ coming through t, which establishes (a).

The problem now is that the branches in $\mathfrak{B}_w$ may be too long. So our next step is to extract from $\mathfrak{B}_w$ a substructure $\mathfrak{A}_w$ which is still a block for φ and

whose branches are of length $\leq \sharp(\varphi)\cdot\max\{1,|sub_x\varphi|\}$. We will do this by cutting out certain fragments of branches in $\mathfrak{B}_w$.

Consider a branch (w,w') and suppose that w' was selected to "saturate" $\Box\psi \in sub_0\varphi$ or $\Box\psi(x) \in sub_x\varphi$ in w by means of a weak run r. Since wRw', the branch $(w,w') = \{w,w'\}$ does not need "truncation": we simply leave it as it is.

Suppose now that w' in the branch (w,w') was selected to "saturate" $\Box^*\psi \in sub_0\varphi$ or $\Box^*\psi(x) \in sub_x\varphi$ by means of a weak run r.

The "truncation" of (w,w') can be done in the following way. Let $w_1,\ldots,w_n$ be all the worlds in (w,w') such that $w = w_1R^ww_2R^w\ldots R^ww_n = w'$. If $n \leq \sharp(\varphi)\cdot\max\{1,|sub_x\varphi|\}$ then we do nothing with this branch. Otherwise let r be the weak run in $\mathfrak{B}_w$ such that $r(w) = t$ and $\psi(x) \notin r(w')$; if w' "saturates" $\Box^*\psi \in sub_0\varphi$ then r may be any weak run in $\mathfrak{B}_w$. Since $n > \sharp(\varphi)\cdot\max\{1,|sub_x\varphi|\}$, there must be two worlds w_i and w_j, $1 \leq i < j \leq n$ such that $f(w_i) = f(w_j)$ and $r(w_i) = r(w_j)$. Then we cut out from (w,w') all the worlds in the interval (w_i,w_j) save w_i and put $w_iRw^\dagger$ iff $w_jRw^\dagger$, for all $w^\dagger \in W^w$. It should be clear that the resulting structure is still a block for φ, and so by deleting repeating worlds in the branches of $\mathfrak{B}_w$ we can construct a block $\mathfrak{A}_w$ for φ whose branches are of length $\leq \sharp(\varphi)\cdot\max\{1,|sub_x(\varphi)|\}$.

The satisfying set for φ we are looking for can be constructed now by taking the blocks $\mathfrak{A}_w$ for all $w \in W$.

($\Leftarrow$) Let $\mathcal{S}$ be a satisfying set for φ. We are going to construct a quasimodel $\mathfrak{m}$ satisfying φ as the limit of a sequence of weak quasimodels

$$\mathfrak{m}_n = \langle\mathfrak{F}_n, f_n\rangle\,,\ \mathfrak{F}_n = \langle W_n, R_n\rangle\,,\ n = 1,2,\ldots,$$

the first of which, $\mathfrak{m}_1$, is a block in $\mathcal{S}$ satisfying φ at its root. Suppose we have already constructed a weak quasimodel $\mathfrak{m}_n$. For every world $w \in W_n - W_{n-1}$ ($W_0 = \emptyset$) select a block $\mathfrak{B}_{w'} \in \mathcal{S}$ such that $f_n(w) = f_{w'}(w')$. Without loss of generality we may assume all the selected blocks and the weak quasimodel $\mathfrak{m}_n$ to be disjoint. The weak quasimodel $\mathfrak{m}_{n+1}$ is then the result of hooking the selected blocks $\mathfrak{B}_{w'}$ to $\mathfrak{m}_n$ by identifying their roots w' with $w \in W_n - W_{n-1}$.

Define the limit $\mathfrak{m} = \langle\mathfrak{F}, f\rangle$, $\mathfrak{F} = \langle W, R\rangle$, of the constructed sequence by taking

$$W = \bigcup\{W_n : n \geq 1\},\ R = \bigcup\{R_n : n \geq 1\},\ f = \bigcup\{f_n : n \geq 1\},$$

and show that $\mathfrak{m}$ is a quasimodel for φ indeed.

First let us prove that for all $\Box^\circ\psi \in sub_0\varphi$ and $w \in W$, we have:

$$\Box^\circ\psi \in \Phi_w \text{ iff } \forall w' \in W\ (wR^\circ w' \Rightarrow \psi \in \Phi_{w'}). \tag{2}$$

Case 1: $\Box^\circ = \Box$. Take the (uniquely determined) block $\mathfrak{B}_w = \langle\mathfrak{F}_w, f_w\rangle$ such that wR_ww' iff wRw', for all $w' \in W$. Then (2) follows immediately from the definition of blocks.

Case 2: $\Box^\circ = \Box^*$. Suppose $\Box^*\psi \notin \Phi_w$ and consider the block $\mathfrak{B}_w$. By (c), there exists a world w' in $\mathfrak{B}_w$ such that wR^*_ww' and $\psi \notin \Phi_{w'}$. But then wR^*w' and $\psi \notin \Phi_{w'}$.

It remains to show that there are enough runs in $\mathfrak{m}$. To this end it suffices to observe that any r constructed in the following manner is a run in $\mathfrak{m}$. Let r^1 be an arbitrary weak root-saturated run in $\mathfrak{m}_1$, and suppose that we have already defined r^m. For any $w \in W_m - W_{m-1}$ take a weak root-saturated run r_w in $\mathfrak{B}_w$ such that $r^m(w) = r_w(w)$ and put for $w' \in W_{m+1}$

$$r^{m+1}(w') = \begin{cases} r^m(w') & \text{if } w' \in W_m; \\ r_w(w') & \text{if } w' \in W_w - W_m. \end{cases}$$

Finally, let $r = \bigcup\{r^m : m > 0\}$. Similarly to the proof above it can be shown that r is a run in $\mathfrak{m}$. ❑

As an immediate consequence we obtain the following satisfiability criterion.

Theorem 23. *Let $\mathcal{ML}' \subseteq \mathcal{ML}_1$ and suppose that there is an algorithm that is capable of deciding, given $\varphi \in \mathcal{ML}'$ and a world candidate $\mathfrak{T} = \langle T, T^{con}, \Phi\rangle$ for φ, whether $\mathfrak{T}$ is realizable, or equivalently, whether the formula $\alpha_{\mathfrak{T}}$ is satisfiable. Then the fragment $\mathbf{QK}^* \cap \mathcal{ML}'$ is decidable.*

In other words, this theorem reduces the satisfiability problem for formulas in $\mathbf{QK}^* \cap \mathcal{ML}'$ to the satisfiability problem for $\mathcal{L}$-formulas of the form $\alpha_{\mathfrak{T}}$, $\mathfrak{T} = \langle T, T^{con}, \Phi\rangle$, where T and T^{con} are just (arbitrary) sets of $sub_x\varphi$ and Φ a subset of $sub_0\varphi$. A few examples of the use of this reduction will be given in the next section.

Remark 24. In principle it is not hard to extend Theorem 23 to some other standard modal logics, e.g. **D**, **KD45**, **GL**, **S4.3** (see [33]). Using the technique developed in [34, 32] one can prove similar criteria for first-order temporal logics (with $\langle \mathbb{N}, <\rangle$, $\langle \mathbb{Z}, <\rangle$, or $\langle \mathbb{Q}, <\rangle$ as the time structures) and first-order dynamic logic.

5 Decidable fragments

Now we apply the criterion of Theorem 23 to single out a number of decidable fragments of $\mathbf{QK}^*$, **QK**, **QT**, **QK4**, and **QS4**. But before that let us observe that **QK** is just a fragment of $\mathbf{QK}^*$, **QS4** is embedded into $\mathbf{QK}^*$ by the translation replacing every $\Box_i$ with $\Box_i^*$, **QT** is embedded into $\mathbf{QK}^*$ by the translation replacing every $\Box_i\varphi$ with $\varphi \wedge \Box_i\varphi$, and **QK4** by the translation replacing every $\Box_i$ with $\Box_i\Box_i^*$.

Denote by $\mathcal{ML}^2$ the set of all $\mathcal{ML}$-formulas containing at most two individual variables, and let $\mathcal{ML}_1^2 = \mathcal{ML}_1 \cap \mathcal{ML}^2$. That is to say, $\mathcal{ML}_1^2$ consists of all $\mathcal{ML}$-formulas with at most two variables, the modal operators in which are applied to subformulas with at most one free variable.

Theorem 25. *The fragments $L \cap \mathcal{ML}_1^2$, for $L \in \{\mathbf{QK}^*, \mathbf{QK}, \mathbf{QT}, \mathbf{QK4}, \mathbf{QS4}\}$, are decidable.*

Proof The $\mathcal{L}$-formula $\alpha_{\mathfrak{T}}$, corresponding to a world candidate $\mathfrak{T}$ for a formula $\varphi \in \mathcal{ML}_1^2$, contains not more than two individual variables. As was mentioned in the introduction, the satisfiability problem for such formulas is decidable. It remains to use the criterion of Theorem 23. ❑

And since $\mathcal{ML}_1^2$ contains the set $\mathcal{ML}^1$ of $\mathcal{ML}$-formulas with at most one variable, we also have:

Corollary 26. *The fragments $L \cap \mathcal{ML}^1$, for $L \in \{\mathbf{QK}^*, \mathbf{QK}, \mathbf{QT}, \mathbf{QK4}, \mathbf{QS4}\}$, are decidable.*

We have obtained this result using the mosaic technique, which in general constructs only an *infinite* model satisfying a given formula. The more familiar way of establishing decidability in (propositional) modal logic—proving the finite model property, that is the existence of finite models for satisfiable formulas—does not work for the fragments mentioned in Corollary 26. To show this, let us note first that in the predicate modal case one should distinguish between two types of finite model property.

Say that a fragment L' of a first-order modal logic L has the *finite frame property* if every formula in L' satisfiable in some model for L can be satisfied in a model for L based on a finite Kripke frame. L' has the *finite domain property* if every satisfiable formula in L' can be satisfied in a model for L with finite domain.

Theorem 27. *The one-variable fragments of* $\mathbf{QK}^*$, $\mathbf{QK4}$, *and* $\mathbf{QS4}$ *with a single unary predicate have neither the finite frame nor the finite domain property.* !!!

Proof Consider the conjunction φ of the following formulas:

$$\exists x P(x),\ \forall x \Box (P(x) \to \Box P(x)),\ \Box \exists x \neg P(x),\ \Box \forall x (\neg P(x) \to \Diamond P(x)).$$

One can readily check that φ is satisfied in the model $\mathfrak{M} = \langle \mathfrak{F}, \mathbb{N}, I \rangle$ based on $\mathfrak{F} = \langle \mathbb{N}, < \rangle$ or $\mathfrak{F} = \langle \mathbb{N}, \leq \rangle$, where for each $n \in \mathbb{N}$,

$$I(n) = \left\langle \mathbb{N}, P^{I,n} \right\rangle, \quad P^{I,n} = \{0, \dots, n \dots\}.$$

It is not hard to see, however, that every transitive model satisfying φ has both an infinite underlying Kripke frame and an infinite domain. ❑

Remark 28. The same result holds, of course, in the case of expanding domains (the finite domain property means then that the universe D is finite).

The transitivity of the underlying frames was essential in the proof above. For we have the following:

Theorem 29. *The fragments $\mathcal{ML}_1 \cap \mathbf{QK}$ and $\mathcal{ML}_1 \cap \mathbf{QT}$ have the finite frame property.*

Proof The proof of Theorem 22 shows that every satisfiable formula $\varphi \in \mathcal{ML}_1$ is satisfied at the root of a tree quasimodel, the set of R_i-successors of every world w in which is finite, $1 \leq i \leq l$. By transforming this quasimodel (in accordance with Theorem 16) into a real model and then taking the submodel of the latter consisting of the worlds accessible from the root in $\leq md(\varphi)$ steps, we construct a model based on a finite Kripke frame and still satisfying φ at its root. ❑

$\mathcal{ML}_1 \cap \mathbf{QK}$ and $\mathcal{ML}_1 \cap \mathbf{QT}$ contain full classical first-order logic. So they cannot have the finite domain property. The following observation shows how one can extract fragments of these logics with both the finite frame and the finite domain properties. !!!

Theorem 30. *Let $\mathcal{ML}' \subseteq \mathcal{ML}_1$ and suppose that for every $\varphi \in \mathcal{ML}'$ and every realizable world candidate $\mathfrak{T}$ for φ, the formula $\alpha_{\mathfrak{T}}$ is satisfiable in a finite first-order structure. Then $\mathbf{QK} \cap \mathcal{ML}'$ and $\mathbf{QT} \cap \mathcal{ML}'$ have both the finite frame and the finite domain properties.*

Proof The result follows immediately from Theorem 29 and the fact that if $\alpha_{\mathfrak{T}}$, $\mathfrak{T} = \langle T, T^{con}, \Phi \rangle$, is satisfiable in a finite first-order structure then there exists $m \in \omega$ such that, for every $n \geq m$, $\mathfrak{T}$ is realized in an $\mathcal{L}$-structure $\mathfrak{D}$ of the form (1) in which $|\mathfrak{D}_t| = n$, for every $t \in T$. ❑

Corollary 31. $\mathbf{QK} \cap \mathcal{ML}_1^2$, $\mathbf{QT} \cap \mathcal{ML}_1^2$, $\mathbf{QK} \cap \mathcal{ML}^1$, *and* $\mathbf{QT} \cap \mathcal{ML}^1$ *have both the finite frame and the finite domain properties.*

Proof It is well known that the two-variable fragment and the monadic fragment of first-order logic have the finite model property (see [6]). ❑

One more interesting fragment of $\mathcal{ML}$ is the set of *monadic formulas*, all predicate symbols in which are at most unary. Denote this fragment by $\mathcal{ML}^m$, and let $\mathcal{ML}_1^m = \mathcal{ML}_1 \cap \mathcal{ML}^m$. In this case the formula $\alpha_{\mathfrak{T}}$, corresponding to a world candidate $\mathfrak{T}$ for $\varphi \in \mathcal{ML}_1^m$, is a monadic $\mathcal{L}$-formula. As is well-known, the monadic fragment of first-order logic is decidable, which yields us:

Theorem 32. *The fragments $L \cap \mathcal{ML}_1^m$, for $L \in \{\mathbf{QK}^*, \mathbf{QK}, \mathbf{QT}, \mathbf{QK4}, \mathbf{QS4}\}$, are decidable.*

J. van Benthem (private communication) suggested the following natural generalization of guarded formulas of [1] to the modal case.

Definition 33 (guarded fragment). Denote by **MGF** the smallest set of $\mathcal{ML}$-formulas such that

- every atomic formula is in **MGF**;
- if φ and ψ are in **MGF**, then so are $\varphi \wedge \psi$, $\neg\varphi$, $\Box_i\varphi$ and $\Box_i^*\varphi$;
- if $\overline{x}, \overline{y}$ are tuples of variables, $G(\overline{x}, \overline{y})$ is atomic, $\varphi(\overline{x}, \overline{y}) \in \mathbf{MGF}$, and every free variable occurring in $\varphi(\overline{x}, \overline{y})$ occurs in $G(\overline{x}, \overline{y})$ as well, then $\exists \overline{y}(G(\overline{x}, \overline{y}) \wedge \varphi(\overline{x}, \overline{y}))$ is in **MGF**.

The set **MGF** is called the *guarded fragment* of first-order modal logic.

Let $\mathbf{MGF}_1 = \mathbf{MGF} \cap \mathcal{ML}_1$.

Theorem 34. *The fragments* $L \cap \mathbf{MGF}_1$*, for* $L \in \{\mathbf{QK}^*, \mathbf{QK}, \mathbf{QT}, \mathbf{QK4}, \mathbf{QS4}\}$*, are decidable.*

Proof It suffices to observe that if $\varphi \in L \cap \mathbf{MGF}_1$ and $\mathfrak{T}$ is a world candidate for φ then $\alpha_{\mathfrak{T}}$ belongs to the packed fragment of $\mathcal{L}$ defined by Marx [21] who proved that this fragment is decidable.[8] ❑

Remark 35. We do not know, however, whether the whole fragment **MGF** is decidable.

Another kind of decidable fragments of first-order modal logics can be associated with the so-called *modal description logics*, that is extensions of concept description logics with various modal operators. (see e.g. [3, 15, 19, 32, 33, 34, 35].)

From the technical point of view pure (non-modal) description logics can be characterized as variable free fragments of first-order predicate logics (sometimes extended with fixed-point operators). They originate from practical knowledge representation systems which, in turn, can be traced back to semantic networks and Minsky frames, see e.g. [7]. An application domain is represented in the language of a description logic in terms of concepts (unary predicates), roles (binary predicates), and object names (constants). The expressive power of such a language depends on the available concept and role constructs, for instance, conjunction and negation of concepts, and composition, union, and the reflexive transitive closure of roles. Modal operators are added to description logics in order to capture various dynamic features of application domains.

Let us consider first a modal extension of the basic description logic called $\mathcal{ALC}$ (see [25]).

Definition 36. The *modal description language* $\mathcal{ALC}_{\mathcal{M}}$ is based on a list of

- *concept names* $C_0, C_1, \ldots,$
- *role names* $R_0, R_1, \ldots,$
- *object names* $a_0, a_1, \ldots,$
- and the *modal operators* $\Box_i$ and $\Box_i^*$, for $i = 1, \ldots, l$.

Starting from these, we define complex *concepts* and *formulas* using the following constructs. Atomic concepts are simply concept names, and

- if C and D are concepts, then so are $\top$, $C \wedge D$, $\neg C$,
- if C is a concept and R a role name, then $\exists R.C$ is a concept,
- if C is a concept then so are $\Box_i C$ and $\Box_i^* C$.

[8] Note that $\alpha_{\mathfrak{T}}$ does not necessarily belong to the guarded fragment of $\mathcal{L}$ (i.e., to $\mathcal{L} \cap \mathbf{MGF}$).

Atomic formulas are expressions of the form $\top$, $C = D$, $a : C$, aRb, where C and D are concepts, R is a role name and a, b are object names. Every atomic formula is a formula, and if φ and ψ are formulas then so are $\varphi \wedge \psi$, $\neg\varphi$, $\Box_i\varphi$, and $\Box_i^*\varphi$.

The intended meaning of the concepts and formulas will become clear from the following translation T of $\mathcal{ALC}_{\mathcal{M}}$ into the language of first-order modal logic with constants $a_0, a_1, \ldots$.

Fix two different variables x, y. The translation C^T of a concept C is a formula with at most one free variable defined by taking

$$\begin{aligned} C_i^T &= P_i(x) \\ \top^T &= \top \\ (C \wedge D)^T &= C^T \wedge D^T \\ (\neg C)^T &= \neg C^T \\ (\exists R.C)^T &= \exists y(R(x,y) \wedge C^T\{y/x\}) \\ (\Box_i^\circ C)^T &= \Box_i^\circ C^T \end{aligned}$$

The translation φ^T of an $\mathcal{ALC}_{\mathcal{M}}$-formula φ is an $\mathcal{ML}$-sentence defined in the following way:

$$\begin{aligned} (C = D)^T &= \forall x(C^T \leftrightarrow D^T) \\ (a : C)^T &= C^T\{a/x\} \\ (aR_ib)^T &= R_i(a,b) \\ (\varphi \wedge \psi)^T &= \varphi^T \wedge \psi^T \\ (\neg\varphi)^T &= \neg\varphi^T \\ (\Box_i^\circ\varphi)^T &= \Box_i^\circ\varphi^T \end{aligned}$$

It follows from the definition that $\varphi^T \in \mathcal{ML}_1^2$, for every formula $\varphi \in \mathcal{ALC}_{\mathcal{M}}$. Hence we obtain:

Theorem 37. *The set $\{\varphi \in \mathcal{ALC}_{\mathcal{M}} : \varphi^T$ is satisfiable$\}$ is decidable.*

This theorem covers various decidability results for modal description logics, e.g. those of [3, 15, 19] and some of [32].

Remark 38. In [35] we extend the language $\mathcal{ALC}_{\mathcal{M}}$ by allowing applications of the modal operators $\Box_i$ and $\Diamond_i$ not only to concepts and formulas, but to roles as well, i.e., if R is a role then so are $\Box_i R$ and $\Diamond_i R$. The intended meaning of the modalized roles is described by the clauses:

$$\begin{aligned} (\Box_i R)^T &= \Box_i R(x,y) \\ (\Diamond_i R)^T &= \Diamond_i R(x,y). \end{aligned}$$

It is shown in [35] that the satisfiability problem for this language is decidable in the classes of arbitrary frames, **S5**-frames and **KD45**-frames. However, it becomes undecidable in arbitrary frames if we add the operators $\Box_i^*$.

The reader must have already noticed that $\mathcal{ALC}$ is in principle nothing else but the polymodal propositional logic **K** (this fact was first observed in [23]). There are, however, much more expressive (and yet decidable) concept description languages; see e.g. [8]. Here is only one example, the logic $\mathcal{CI}$ introduced in [13, 14]. In its syntax it is allowed to form the union $R \vee S$ of roles R and S, their composition $R \circ S$, the reflexive and transitive closure R^*, and the inversion R^- of R; moreover, every concept C gives rise to the role $C?$ (the set of all pairs $\langle x, x \rangle$ such that $x \in C$). This language may be regarded as a terminological variant of propositional dynamic logic **PDL**.

The first-order modal language we deal with in this paper does not have enough expressive power to represent all these constructs, say that in each world of a model R^* should be the transitive reflexive closure of R, or that R^- is the converse of R. But we can interpret $\mathcal{CI}$ in models whose worlds are suitable first-order $\mathcal{L}$-structures, and use the following generalization of Theorem 23, which is proved in precisely the same way.

Theorem 39. *Let $\mathcal{FO}$ be a class of first-order structures and $\mathcal{ML}' \subseteq \mathcal{ML}_1$. Suppose also that:*

(i) *there is an algorithm that is capable of deciding, given a world candidate for $\varphi \in \mathcal{ML}'$, whether it is realizable in a structure from $\mathcal{FO}$, and*

(ii) *there is a cardinal $\kappa \geq \aleph_0$ such that for any $\kappa' \geq \kappa$, every world candidate $\langle T, T^{con}, \Phi \rangle$ realizable in a structure from $\mathcal{FO}$ is realizable in a structure $\mathfrak{D} \in \mathcal{FO}$ in which $|\mathfrak{D}_t| = \kappa'$ for every $t \in T$.*

Then the satisfiability problem for $\mathcal{ML}'$-formulas in first-order modal models, the worlds of which are structures from $\mathcal{FO}$, is decidable.

Corollary 40. *The satisfiability problem for the modal description logic $\mathcal{CI}_{\mathcal{M}}$ (the modal operators $\Box_i^\circ$ are applied to both concepts and formulas) is decidable.*

Proof That (i) holds was actually shown in [13], and (ii) follows from the fact that the class of first-order structures for $\mathcal{CI}$ is closed under the disjoint unions. ❑

For more decidability results of this sort see [32, 33, 34, 35].

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